Transverse momentum dependence in structure functions in hard scattering processes

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Chapter 1

High energy processes with hadrons

1.1 Diagrammatic approach

The basic degrees of freedom that feel the strong interactions, quarks and gluons, are confined into hadrons, strongly interacting particles. Considering the nucleons (light hadrons), the characteristic energy and distance scales are given by the nucleon mass, $\Lambda \sim M_N$, or taking into account the color degrees of freedom one may prefer a scale $\Lambda \sim M_N/N_c \sim 300$ MeV. We refer to this as $\mathscr{O}(M)$ or $\mathscr{O}(Q^0)$ if we consider high-energy processes. Such processes are characterised by hard kinematical variables that are of order Q with $Q^2 \gg \Lambda^2$. Depending on details, the high-energy scale Q can be the CM energy, $Q \sim \sqrt{s}$ or it can be a measure of the exchanged momentum.

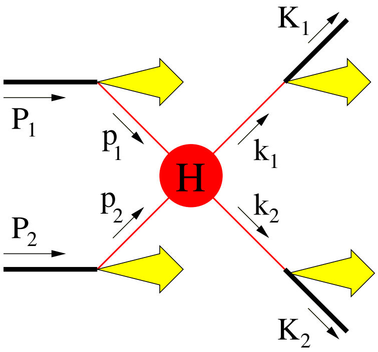


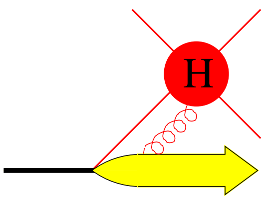
Figure 1.1: Schematic illustration of the contribution of a hard subprocess, parton (p_1) + parton (p_2) $\to$ parton (k_1) + parton (k_2), to the (2-particle inclusive) scattering process hadron (P_1) + hadron (P_2) $\to$ hadron (K_1) + hadron (K_2) + X, at the level of the amplitude. The process being hard implies for the hadronic momenta $P_1 \cdot P_2 \sim P_1 \cdot K_1 \sim Q^2$, etc.

The basic framework for the strong interactions is QCD. Hadrons, however, do not correspond to free particle states created via the quark and gluon operators in QCD. The situation thus differs from that of QED with physical electrons and photons. In the latter case one knows how in the calculation of an S-matrix element contraction of annihilation and creation operator in the field and particle state lead to the spinor wave function. For positive times $\xi^0 = t > 0$ one has

$$\langle 0|\psi_i(\xi)|p,s\rangle = \langle 0|\psi_i(\xi)\, b^\dagger(\boldsymbol{p},s)|0\rangle = \langle 0|\psi_i(0)|p,s\rangle\, e^{-i\,p\cdot\xi} = u_i(\boldsymbol{p},s)\, e^{-i\,p\cdot\xi}, \tag{1.1}$$

with $p^0 = +E_p = +\sqrt{\boldsymbol{p}^2 + m^2}$. Such a matrix element is 'untruncated' as seen e.g. from

$$\langle 0|\psi_i(\xi)|p,s\rangle\,\theta(t) = \theta(t)\int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot\xi}\, \frac{i(\not{k}+m)}{k^2 - m^2 + i\epsilon}\, \frac{u_i(\boldsymbol{p},s)}{2m}\,(2\pi)^3\, 2E_p\, \delta^3(\boldsymbol{k}-\boldsymbol{p}). \tag{1.2}$$



In a process involving a composite hadronic state $|P\rangle$, contractions with one or several of the quark and gluon operators may be involved, leading to nonzero matrix elements for a quark between the hadron state and a remainder, but also for nonzero matrix elements involving multi-parton field combinations,

$$\langle X|\psi_i(\xi)|P\rangle,\ \langle X|A^\mu(\eta)\,\psi(\xi)|P\rangle, \dots\ .$$

Correlators, describing parton distributions

For a particular hadron and a parton field combination, one may collect those operators that involve hadron $|P\rangle$ into (distribution) correlators

$$\begin{aligned}\Phi_{ij}(p;P) &= \sum_X \int \frac{d^3P_X}{(2\pi)^3\,2E_X}\;\langle P|\overline{\psi}_j(0)|X\rangle\,\langle X|\psi_i(0)|P\rangle\,\delta^4(p+P_X-P) \\ &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle,\end{aligned} \tag{1.3}$$

where a summation over color indices is understood. It is often convenient to use momentum space fields, $\psi(p) \equiv \int d^4x\; e^{i\,p\cdot x}\,\psi(x)$, for a free field expansion leading to

$$\psi_i(\boldsymbol{p},t) = \sum_s \left(u_i(\boldsymbol{p},s)\,b(\boldsymbol{p},s)\frac{e^{-iE_pt}}{2E_p} + v_i(-\boldsymbol{p},s)\,d^\dagger(-\boldsymbol{p},s)\frac{e^{iE_pt}}{2E_p}\right), \tag{1.4}$$

$$\begin{aligned}\psi_i(p) = \sum_s \Big(&u_i(\boldsymbol{p},s)\,b(\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(p^0) \\ &+ v_i(-\boldsymbol{p},s)\,d^\dagger(-\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(-p^0)\Big).\end{aligned} \tag{1.5}$$

For the correlator we have

$$(2\pi)^4\delta^4(p-p')\,\Phi_{ij}(p;P) = \frac{1}{(2\pi)^4}\,\langle P|\overline{\psi}_j(p')\,\psi_i(p)|P\rangle. \tag{1.6}$$

This latter form is convenient for interpretation of the nature of the correlators because for a free field we have

$$\begin{aligned}\langle X|\psi_i(p)|P\rangle &= \langle P_X|b(\boldsymbol{p},s)|P\rangle\,u_i(\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(p^0) \\ &+ \langle P_X|d^\dagger(-\boldsymbol{p},s)|P\rangle\,v_i(-\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(-p^0).\end{aligned}$$

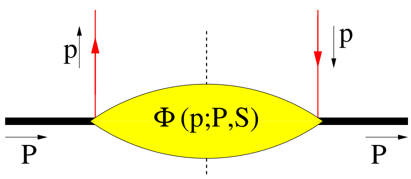
One also encounters correlators involving matrix elements of the form

$$\Phi^{\mu}_{A\,ij}(p,p_1;P) = \frac{1}{(2\pi)^8}\int d^4\xi\,d^4\eta\;e^{i\,(p-p_1)\cdot\xi}\;e^{i\,p_1\cdot\eta}\;\langle P|\overline{\psi}_j(0)\,A^\mu(\eta)\,\psi_i(\xi)|P\rangle, \tag{1.7}$$

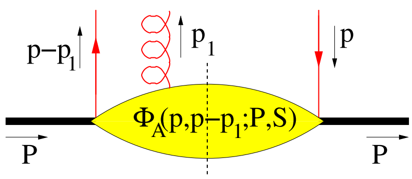
or with momentum space operators (for gluons $A_\mu(p) \equiv \int d^4x\; e^{i\,p\cdot x}\,A_\mu(x)$) one has

$$(2\pi)^4\delta^4(p-p')\,\Phi^{\mu}_{A\,ij}(p,p_1;P) = \frac{1}{(2\pi)^8}\,\langle P|\overline{\psi}_j(p')\,A^\mu(p_1)\,\psi_i(p-p_1)|P\rangle. \tag{1.8}$$

Pictorially one has for the correlators,



or



.

We will not attempt to calculate these, but leave them as the soft parts, requiring nonperturbative QCD methods to calculate them. In particular, although being 'untruncated' in the quark legs, they will no longer exhibit poles corresponding to free quarks. These are fully unintegrated parton correlators for initial state hadrons, in general quite problematic quantities. For example, they are by themselves not even color gauge-invariant, an issue to be discussed below. We will later also discuss similar correlators for final state hadrons. When more hadrons are involved, one needs to consider two-hadron correlators, involving two-hadron states (or correlators involving hadronic states in initial and final state), etc. If the hadrons are well-separated in momentum phase-space with $P_i\cdot P_j \sim Q^2$, one expects on dimensional grounds that incoherent contributions are suppressed by $1/(P_i-P_j)^2 \sim 1/Q^2$. Such a separation in momentum space requires a hard inclusive scattering process ($Q^2 \sim s$), which then at high energy and/or for large momentum transfer still can be factorized into forward correlators. The inclusive character is needed to assure that partons originate from *one* hadron, leaving a (target) jet. In turn, partons decay into a jet in which we limit ourselves to the consideration of an identified hadronic state (which could in principle also be a multi-particle, e.g. two-pion, state). In all of the hadronic states mentioned before one can also consider polarized hadronic states. The spin of quarks is contained in Dirac structure and that of gluons in the Lorentz structure of correlators.

The basic idea in the diagrammatic approach is to realize that the correlator involves hadronic states and quark and gluon operators. The correlators can be studied independent from the hard process, provided we have dealt with the issue of color gauge invariance. The correlator is the Fourier transform in the space-time arguments of the quark and gluon fields. In the correlators, all momenta of hadrons *and* quarks and gluons (partons) inside the hadrons are soft which means that $p^2 \sim p\cdot P \sim P^2 = M_N^2 \ll Q^2 \sim s$. The *off-shellness* being of hadronic order implies that in the hard process partons are in essence *on-shell*. Consistency of this may be checked by using QCD interactions to give partons a large off-shellness of $\mathscr{O}(Q)$ and check the behavior as a function of the momenta. In these considerations one must also realize that beyond tree-level one has to distinguish bare and renormalized fields.

Correlators, describing parton fragmentation

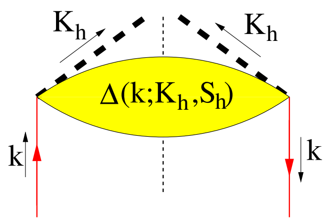
In the hard process final state partons decay into a jet, in the discussion of which we limit ourselves to the consideration of an identified hadronic state (which could in principle also be a multi-particle, e.g. two-pion, state). For the fragmentation process of a parton (with momentum k) into hadrons (with momentum P_h) we combine the decay matrix elements in the (fragmentation) correlator, for quarks

$$\begin{aligned}\Delta_{ij}(k;K_h) &= \sum_X \frac{1}{(2\pi)^4}\int d^4\xi\; e^{ik\cdot\xi}\,\langle 0|\psi_i(\xi)|K_h,X\rangle\langle K_h,X|\overline{\psi}_j(0)|0\rangle \\ &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{ik\cdot\xi}\,\langle 0|\psi_i(\xi)a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle, \qquad (1.9)\end{aligned}$$

where an averaging over color indices is implicit. In a momentum space representation for the operators, we have

$$(2\pi)^4\delta^4(k-k')\,\Delta_{ij}(k;K_h) = \frac{1}{(2\pi)^4}\sum_X \langle 0|\psi_i(k)|K_h,X\rangle\langle K_h,X|\overline{\psi}_j(k')|0\rangle. \qquad (1.10)$$

Pictorially we have



In particular, we note that in fragmentation correlators, one no longer deals with plane wave hadronic states, but with out states $|K_h,X\rangle$. There are a a number of other subtleties in these definitions. The use of intermediate states X and in addition one specified state with momentum K_h needs some explanation. First note that the unit operator can be written as

$$\hat{I} \equiv \sum_X |X\rangle\langle X| = \sum_{n=0}^{\infty}\hat{I}_n \qquad (1.11)$$

with (assuming at this stage one type/flavor of hadrons)

$$\hat{I}_n = \frac{1}{n!}\int d\widetilde{K}_1\ldots d\widetilde{K}_n\; a^\dagger(K_1)\ldots a^\dagger(K_n)|0\rangle\langle 0|a(K_1)\ldots a(K_n) \qquad (1.12)$$

containing the n-particle states (with $d\widetilde{K}$ being the invariant one-particle phase-space). Thus the summation appearing in the definition of the fragmentation correlator is for a given hadron sector

$$\begin{aligned}\sum_X |P_h, X\rangle\langle P_h, X| &= |P_h\rangle\langle P_h| + \int d\widetilde{K}_1\; |P_h, K_1\rangle\langle P_h, K_1| \\ &\quad + \frac{1}{2!}\int d\widetilde{K}_1 d\widetilde{K}_2\; |P_h, K_1, K_2\rangle\langle P_h, K_1, K_2| + \ldots \\ &= a_h^\dagger \hat{I} a_h = a_h^\dagger a_h. \end{aligned} \tag{1.13}$$

After integrating over P_h one obtains

$$\int d\tilde{P}_h \sum_X |P_h, X\rangle\langle P_h, X| = \sum_{n=0}^{\infty} n\,\hat{I}_n = \hat{N}_h, \tag{1.14}$$

which is the number operator for hadrons h. This will become relevant when one integrates over the phase-space of particles in the final state to go from 1-particle inclusive to inclusive scattering processes.

Other useful operators are the momentum operator

$$\begin{aligned}\hat{P}^\mu &= \int d\widetilde{K}_1\; |K_1\rangle K_1^\mu \langle K_1| + \frac{1}{2!}\int d\widetilde{K}_1 d\widetilde{K}_2\; |K_1, K_2\rangle (K_1^\mu + K_2^\mu)\langle K_1, K_2| + \ldots \\ &= \sum_{h,X}\int d\widetilde{K}_h |K_h, X\rangle K_h^\mu \langle K_h, X|, \end{aligned} \tag{1.15}$$

or the operator

$$\begin{aligned}\hat{P}^\mu \hat{P}^\nu &= \sum_{h_1,X}\int d\widetilde{K}_1\; |K_1, X\rangle K_1^\mu K_1^\nu \langle K_1, X| \\ &\quad + \sum_{h_1,h_2,X}\int d\widetilde{K}_1 d\widetilde{K}_2\; |K_1, K_2, X\rangle K_1^\mu K_2^\nu \langle K_1, K_2, X|. \end{aligned} \tag{1.16}$$

Inclusion of spin

In principle hadrons could be polarized, having additional degrees of freedom, $|P,\alpha\rangle$, etc. In order to treat the spin of initial states, one then can explicitly work with distribution correlators in the hadron spin-space,

$$\Phi_{ij,\beta\alpha}(p;P) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,p\cdot\xi}\;\langle P,\beta|\overline{\psi}_j(0)\,\psi_i(\xi)|P,\alpha\rangle. \tag{1.17}$$

It is convenient to include the off-diagonal elements in the definition. Having a non-pure initial state described by a spin density matrix $\rho(P,S) = \sum_\alpha |P,\alpha\rangle \text{Prob}_\alpha \langle P,\alpha|$ one then finds the spin-dependent correlator

$$\Phi_{ij}(p;P,S) \equiv \rho_{\alpha\beta}(P,S)\,\Phi_{ij,\beta\alpha}(p;P). \tag{1.18}$$

A single spin vector S is sufficient to parameterize the density matrix for a spin 1/2 hadron. For hadrons with higher spins one needs additional parameters (e.g. a spin vector *and* a symmetric traceless tensor to describe spin 1).

For fragmentation correlators, the role of spin is different. In that case not only the specific kind of hadron in the final state, but also its spin state is fixed. This means that besides having Dirac structure, we also include spin states (off-diagonal to remain as general as possible)

$$\Delta_{ij,\beta\alpha}(k;P_h) = \sum_X \frac{1}{(2\pi)^4}\int d^4\xi\; e^{ik\cdot\xi}\;\langle 0|\psi_i(\xi)|P_h,\alpha,X\rangle\langle P_h,\beta,X|\overline{\psi}_j(0)|0\rangle. \tag{1.19}$$

In many applications we will use a spin-dependent fragmentation correlator $\Delta_{ij}(k;P_h,S_h)$ by defining

$$\Delta_{ij}(k;P_h,S_h) \equiv (2s_h+1)\,\Delta_{ij,\beta\alpha}(k;P_h)\,\rho_{\alpha\beta}(P_h,S_h), \tag{1.20}$$

where $\rho(P_h,S_h)$ is the usual spin density matrix. The factor $(2s_h+1)$ assures that for a spin vector $S_h = 0$ one ends up with a sum over spins for the produced hadron. Depending on the spin the parametrization of density matrix may require beside the spin vector polarization tensors of higher rank. Note that in most applications S_h (or other tensors) will be replaced by analyzing power $A_h(P_h, f)$ of the decay channel (with f representing the final state variables) of the produced particle, e.g. in the case of production of Λ's or ρ's, rather than the tunable polarization for initial states (see section on spin vectors).

1.2 Sudakov decompositions and n-dependence

In a hard process, the parton fields that appear in the different correlators correspond to partons in the subprocess for which the momenta satisfy $p_i \cdot p_j \sim Q^2$. In the study of a particular correlator it implies the presence of a 'hard' environment. To connect the correlator to the hard part of the process, it is useful to introduce for each correlator with hadron momentum P, a null-vector n, such that $P \cdot n \sim Q$. Using this relation, n would be dimensionless. It is actually more convenient to replace $n/(P\cdot n)$ by a dimensionful null-vector $n \sim 1/Q$, such that $P\cdot n = 1$. The vectors P and n can be used to keep track of the importance of various terms in the correlators and in the components of momentum and spin vectors[1]. The n-vector will acquire a meaning in the explicit applications or play an intermediary role. At leading order, it will turn out that the precise form of n doesn't matter, but at subleading $(1/Q)$ order one needs to be careful.

For parton momenta we write

$$p = x\,P + p_T + \underbrace{(p\cdot P - x\,M^2)}_{\sigma}\,n, \tag{1.21}$$

where the term $x\,P \sim Q$, $p_T \sim M$ and $\sigma\,n \sim M^2/Q$. We have the exact relations $p\cdot p_T = p_T^2 = (p - x\,P)^2$. The momentum fraction

$$x = p\cdot n \tag{1.22}$$

is $\mathscr{O}(1)$. Note that one can construct *two* conjugate null-vectors,

$$n_+ = P - \tfrac{1}{2}\,M^2\,n \quad \text{and} \quad n_- = n, \tag{1.23}$$

satifying $n_+\cdot n_- = 1$ and $n_+^2 = n_-^2 = 0$, that can be used to define light-cone components[2] $a^\pm = a\cdot n_\mp$. The symmetric and antisymmetric 'transverse' projectors are defined as

$$g_T^{\mu\nu} = g^{\mu\nu} - n_+^{\{\mu}n_-^{\nu\}} = g^{\mu\nu} - P^{\{\mu}n^{\nu\}} + M^2\,n^\mu n^\nu \approx g^{\mu\nu} - P^{\{\mu}n^{\nu\}} \tag{1.24}$$

$$\epsilon_T^{\mu\nu} = \epsilon^{n_+ n_- \mu\nu} = \epsilon^{-+\mu\nu} = \epsilon^{Pn\mu\nu}. \tag{1.25}$$

The decomposition of spin vectors is discussed at the end of this chapter.

Different n-vectors

A choice of a different null-vector n', in principle leads to different x', σ' and $p_{T'}$ as well as different transverse projectors. With $P\cdot n' = 1$, implying $P\cdot \Delta n = 0$, one easily finds that the differences vanish at order Q^0,

$$\mathscr{O}(Q^0): \qquad \Delta x = \Delta p_T^2 = \Delta n = 0. \tag{1.26}$$

With $\Delta n = n' - n$ arbitrary (of order $1/Q$) one easily finds that the changes Δx and Δp_T are related,

$$\mathscr{O}(Q^0): \qquad \Delta p_T = -\Delta x\,P \tag{1.27}$$

(corresponding to the validity of $xP + p_T \approx x'P + p_{T'}$ up to $\mathscr{O}(Q^0)$). By looking at various contractions order by order one finds

$$\mathscr{O}(1/Q): \qquad \Delta x = -\Delta p_T\cdot n = p_T\cdot \Delta n = -\frac{\Delta p_T^2}{2\,\sigma_p} = -\frac{\Delta\sigma_p}{M^2} \tag{1.28}$$

(Note that $\Delta p_T \cdot n = p_{T'}\cdot n$ and $p_T\cdot \Delta n = p_T\cdot n'$). We note also the following relations valid at $\mathscr{O}(Q^0)$ for the transverse momenta, $p_T \approx p_{T'} - (p_{T'}\cdot n)P \approx p_{T'T}$ and similarly $p_{T'} \approx p_T - (p_T\cdot n')P \approx p_{TT'}$.

[1] If one prefers a dimensionless vector, one must make a choice $P\cdot n \sim Q$. In that case all further appearances of n in this section should simply be replaced by $n/(P\cdot n)$.

[2] There is an arbitrariness in the definition of these vectors, allowing $n_+ \to \alpha\,n_+$ and $n_- \to n_-/\alpha$. In this way one can make dimensionless vectors. In the explicit appearance of vectors such a rescaling corresponds to a boost.

Given two (hard) hadronic momenta P and P' one thus can (disregarding $1/Q^2$ mass corrections) use $P'/(P\cdot P')$ as the null-vector n for the hadron with momentum P. At $\mathscr{O}(Q^0)$ the momentum fractions $x \approx (p\cdot P')/(P\cdot P')$ are the same for any of the hard (hadronic) momenta P' involved in the process. We note that the integration

$$\int d^4p = \int dx\, d^2p_T\, d\sigma = \int d(p\cdot n)\, d^2p_T\, d(p\cdot P), \tag{1.29}$$

is insensitive to the choice of n-vector.

Another way to study the n-dependence is by applying ∂_n, using $\partial_n^\mu\, x = p^\mu$,

$$\partial_n^\mu\, p^\nu = p^\mu\, P^\nu + \partial_n^\mu\, p_T^\nu + g^{\mu\nu}(p\cdot P - x\, M^2) - M^2\, p^\mu n^\nu.$$

Having $\partial_n^\mu\, p^\nu = 0$ implies

$$\partial_n^\mu\, p_T^\nu = -g^{\mu\nu}(p\cdot P - x\, M^2) - p^\mu(P^\nu - M^2\, n^\nu).$$

Explicit components are

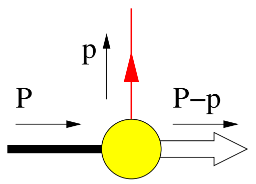
$$\begin{aligned}
n\cdot\partial_n\, p_T^\nu &= -x\, P^\nu - (p\cdot P - 2x\, M^2)\, n^\nu, \\
P\cdot\partial_n\, p_T^\nu &= -(2\, p\cdot P - x\, M^2)\, P^\nu + M^2\,(p\cdot P)\, n^\nu, \\
(P - M^2 n)\cdot\partial_n\, p_T^\nu &= -2(p\cdot P - x\, M^2)(P^\nu - M^2\, n^\nu), \\
\partial_{nT}^\mu\, p_T^\nu &= -g_T^{\mu\nu}(p\cdot P - x\, M^2) - p_T^\mu(P^\nu - M^2\, n^\nu).
\end{aligned}$$

For the vector Δn (orthogonal to P) we can write $\Delta n^\mu = \Delta n\cdot n(P^\mu - M^2\, n^\mu) + \Delta n_T^\mu$ to get

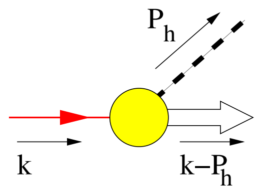
$$\begin{aligned}
&\Delta x = p_T\cdot\Delta n_T + \Delta n\cdot n\,(p\cdot P - x\, M^2), \\
&\Delta p_T^2 = -2(p\cdot P - x\, M^2)\, p_T\cdot\Delta n_T.
\end{aligned}$$

Regions of importance in parton kinematics

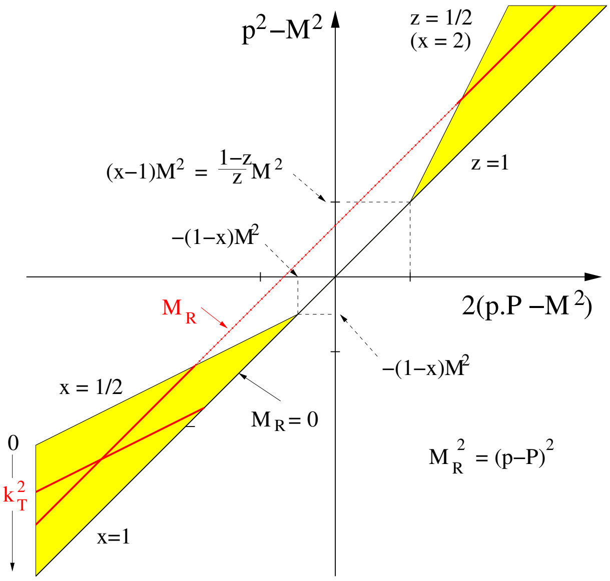
We want to illustrate the kinematics for partons and translate it to physically intuitive quantities, the off-shellness p^2 for partons or the invariant mass squared $M_R^2 = (P-p)^2$ of the residual (spectator) hadronic system and the (spacelike) transverse momentum squared $p_T^2 = -\boldsymbol{p}_T^2$. We can do this for partons in hadrons (distributions), but also for the production of hadrons from partons (fragmentation),



$p = x\, P + p_T + \sigma_p\, n$
$P^+ = P\cdot n \equiv 1$
$x = p\cdot n = p^+$
$\sigma_p = p\cdot P - x\, M^2$
$n_+ \equiv P - \frac{1}{2}M^2\, n$
$p^- = p\cdot n_+ = p\cdot P - \frac{1}{2}\, xM^2$



$k = z^{-1}\, P_h + k_T + \sigma_h\, n_h$
$P_h\cdot n_h = P_h^- \equiv 1$
$z^{-1} = k\cdot n_h = k^-$
$\sigma_k = k\cdot P_h - z^{-1}\, M_h^2$
$n_- \equiv P_h - \frac{1}{2}M_h^2\, n_h$
$k^+ = k\cdot n_- = k\cdot P_h - \frac{1}{2}\, z^{-1}\, M_h^2$



The kinematic freedom for the partons is illustrated in a plot of the off-shellness p^2 versus $2p\cdot P$, where it is convenient to slightly shift the zero-points to get a nice symmetry. The relations

$$\begin{aligned} p^2 - M^2 &= 2x\left(p\cdot P - M^2\right) + p_T^2 \\ &\quad - (1-x)^2\, M^2, \qquad (1.30) \\ p^2 - M^2 &= 2\left(p\cdot P - M^2\right) + M_R^2, \quad (1.31) \end{aligned}$$

determine lines of constant M_R or p_T^2. They are used to map out the physical region determined by $p_T^2 = -\boldsymbol{p}_T^2 \le 0$ and $M_R^2 \ge 0$. We note that for $x = 1$ the region disappears shrinking to the line $M_R = 0$ or $x = 1$. We note the symmetry $x = 1/z$ in jumping from distributions of partons into hadrons to fragmentation of partons into hadrons. We assume that the dominant contributions come from the regions where $p^2 \sim p\cdot P \sim M_R^2 \sim M^2$, thus Φ disappearing as a function of these variables.

1.3 Distribution correlators in high-energy processes

In a hard process, the different importance of the various components allows up to specific orders in $1/Q$, an integration over some components of the parton momenta. The fact that the main contribution in $\Phi(p;P)$ is assumed to come from regions where $p\cdot P \le M^2$, whereas the momenta have characteristic scale Q, allows performing the σ-integration up to M^2/Q^2 contributions (and possible contributions from non-integrable tails). Transverse momentum dependent (TMD) correlators are light-front correlators, integrated over $\sigma \equiv p\cdot P - xM^2$,

$$\Phi_{ij}(x, p_T; n) = \int d(p\cdot P)\;\Phi_{ij}(p;P) = \int d\sigma_p\, d\tau_p\, \delta(\tau_p - 2x\,\sigma_p - p_T^2 - x^2M^2)\,\Phi_{ij}(p;P) \qquad (1.32)$$

$$= \int \frac{d(\xi\cdot P)\, d^2\xi_T}{(2\pi)^3}\; e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle\Big|_{LF}\,, \qquad (1.33)$$

where we have suppressed the dependence on hadron momentum P and made the off-shellness $\tau = p^2$ explicit. The argument of the delta function is useful when one wants to study which regions in off-shellness or in the residual mass spectrum $M_R^2 = (P-p)^2$ contribute for given x and p_T^2. The subscript LF refers to light-front, implying $\xi\cdot n = 0$. The light-cone correlators are the correlators containing the parton distribution functions depending only on the light-cone momentum fraction x,

$$\Phi_{ij}(x; n) = \int d(p\cdot P)\, d^2p_T\;\Phi_{ij}(p;P) = \int d\sigma_p\, d\tau_p\, \theta(2x\,\sigma_p - \tau_p + x^2M^2)\,\Phi_{ij}(p;P) \qquad (1.34)$$

$$= \int \frac{d(\xi\cdot P)}{(2\pi)}\; e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle\Big|_{LC}\,, \qquad (1.35)$$

where the subscript LC refers to light-cone, implying $\xi\cdot n = \xi_T = 0$. This integration is generally allowed (again up to M^2/Q^2 contributions and contributions coming from tails, e.g. logarithmic corrections from $1/p_T^2$ tails) if we are interested in hard processes, in which only hard scales (large invariants $\sim Q^2$ or ratios thereof, angles, rapidities) are measured. If one considers hadronic scale observables (correlations or transverse momenta in jets, slightly off-collinear configurations) one will need the TMD correlators.

Twist expansion

The correlators encompass the information on the soft parts. They depend on the hadron and (contained) quark momenta P and p (and spin vectors). The structure of the correlator is reproduced from these momenta incorporating the required Dirac and Lorentz structure. Clearly, it is advantageous to maximize the number of components along the momentum (collinear). For the *soft* scalar objects this means maximizing the number of contractions with n. This leads for nonlocal operators to the dominance of the twist-2 operators

$$\overline{\psi}(0)\not{n}\psi(\xi) \qquad \text{and} \qquad G^{n\alpha}(0)G^{n\beta}(\xi), \tag{1.36}$$

(the latter with transverse indices α and β). Twist in this case is just equal to the canonical dimension of the operator combination (remember that dim(n) = -1).

The fact that the matrix elements involve operators on the light-front, allows for specific operators, the socalled *good* operators in front-form quantization, an easy interpretation in terms of *partons*. These partons are the quanta created by the good fields. The good fields are

$$\psi_+ = \frac{1}{2}\not{n}_+\not{n}_-\,\psi = \frac{1}{2}\not{P}\not{n}\,\psi \quad \text{and} \quad A^\mu_T = g_T^{\mu\nu}A_\nu. \tag{1.37}$$

In front-form quantization the other components of the fields can be expressed in the good fields using equations of motion, at least after imposing the gauge $A^+ = A\cdot n = 0$ (where $G^{n\alpha} = \partial^n A_T^\alpha$). More important, however, for the description of hard processes is that matrix elements involving these good fields turn out to be the leading ones in an expansion in the inverse hard scale $1/Q$.

As argued above (and made more explicit in applications to hard scattering processes) the correlators involve collinear momenta (soft with respect to each other), but for use within the hard process an external direction shows up, represented by a null vector n, which will acquire a meaning in the explicit applications or play an intermediary role. It can be used in the projection of components or in an expansion of fields or field combinations, e.g.

$$\psi = \psi_+ + \psi_- = \frac{1}{2}\not{P}\not{n}\,\psi + \frac{1}{2}\not{n}\not{P}\,\psi, \tag{1.38}$$

$$A^\mu = (A\cdot n)P^\mu + A_T^\mu + (A\cdot P)n^\mu. \tag{1.39}$$

$$\overline{\psi}\gamma^\mu\psi = (\overline{\psi}\not{n}\psi)P^\mu + \overline{\psi}\gamma_T^\mu\psi + (\overline{\psi}\not{P}\psi)n^\mu. \tag{1.40}$$

Examples of orders of magnitude of the fields within the matrix elements appearing in hard processes are

$$\langle\, n\cdot A\,\rangle \sim n\cdot P = 1, \tag{1.41}$$

$$\langle\, A_T^\alpha\,\rangle \sim \langle\, G_T^{n\alpha}\,\rangle \sim p_T^\alpha \sim M, \tag{1.42}$$

$$\langle\, \psi\,\rangle \sim M^{3/2}, \tag{1.43}$$

$$\langle\, \overline{\psi}\,\not{n}\,\psi\,\rangle \sim \langle\, \overline{\psi}\,[\not{n},\gamma_T^\mu]\,\psi\,\rangle \sim \langle\, G_T^{n\alpha}\,G_T^{n\beta}\,\rangle \sim M^2. \tag{1.44}$$

$$\langle\, \overline{\psi}\psi\,\rangle \sim \langle\, \overline{\psi}\,D^n\,\psi\,\rangle \sim \langle\, \overline{\psi}\,\gamma_T^\mu\,\psi\,\rangle \sim M^3\ , \tag{1.45}$$

$$\langle\, \overline{\psi}\,D_T^\alpha\,\psi\,\rangle \sim \langle\, \overline{\psi}\,\not{P}\,\psi\,\rangle \sim M^4\ , \tag{1.46}$$

These results are obvious because $\langle\, \overline{\psi}\gamma^\mu\,\psi\,\rangle$ in a matrix element of the form $\Phi(p;P)$ must involve the (relevant) momenta p^μ or P^μ, it must be a vector and it must have dimension 3, leaving $M^2\,p^\mu$ or $M^2\,P^\mu$ as possibilities. Knowing the order of magnitude of the momenta as appearing in a hard scattering process, we obtain the above results. The above integrated or the TMD correlators also can depend on n^μ, e.g. appearing via the transverse tensors $g_T^{\mu\nu}$ or $\epsilon_T^{\mu\nu}$ (both being of order unity).

Gauge choices and n-dependence

We already remarked that for a given hadron, it is certainly convenient to also use n as a vector that fixes a light-like gauge $A^+ = A\cdot n = 0$. This is essential when one wants to discuss or interpret the correlator and the fields appearing in it in front form quantization. Such a treatment, however, is considered problematic in the treatment of the hard process to which the soft parts couple. Here a light-like gauge choice produces additional poles for which one must introduce prescriptions. Moreover if one has several soft parts one can only make *one* gauge choice in the calculation.

The freedom in choosing n, even choosing n different for each hadron involved, also allows the treatment of an arbitrary axial gauge $A\cdot v = 0$ with $v^2 \neq 0$, applied in the treatment of the full process. We simply could take n as[3]

$$v \equiv (v\cdot n_+)\,n + \frac{v^2}{2\,(v\cdot n_+)}\,n_+ \approx (P\cdot v)\,n + \frac{v^2}{2\,(P\cdot v)}\,P. \tag{1.47}$$

[3]Note that the length of v is irrelevant; making it dimensionless one has $P\cdot v/\sqrt{|v^2|} \sim Q$ or we could give it dimension energy^{-1}, allowing setting $P\cdot v = 1$. It is possible to include a small transverse piece v_T on the righthandside of the expression for v, which implies (taking $v\cdot n_+ = 1$) writing $n \to n + v_T - \frac{1}{2}\,v_T^2\,n_+$. If one assumes $v_T \sim M/Q^2$ (thus smaller than $v \sim 1/Q$) it might serve as a regulator.

Consider a gluon with polarization μ and momentum $p = x\,P + p_T + (p\cdot P - x\,M^2)\,n$ connecting a soft part and the truncated hard part $H_\mu(p,\ldots;v)$, which we assume to be $\mathcal{O}(Q^d)$. For an on-shell gluon with momentum $p_0 = x\,P + p_T - (p_T^2/2x)\,n$ (differing from p by a vector proportional to $M^2\,n$) one has $p_0\cdot H(p_0,\ldots;v) = 0$. This implies for a gluon attached to the correlator

$$p\cdot H(p,\ldots;v) \;\propto\; \underbrace{M^2\,n\cdot H(p,\ldots;v)}_{\mathscr{O}(Q^{d-1})},$$

showing that for Ward identities the gluons in the soft part can, up to $\mathscr{O}(1/Q^2)$ corrections (compared to the expectation Q^{d+1}), be considered to behave as on-shell partons. With the above n-choice one sees, moreover, that

$$\begin{aligned}\frac{n\cdot H(p,\ldots;v)}{P\cdot n} &\approx \frac{v\cdot H(p,\ldots;v)}{P\cdot v} - \frac{v^2}{2(P\cdot v)^2}\,P\cdot H(p,\ldots;v)\\ &\approx \frac{v\cdot H(p,\ldots;v)}{P\cdot v} + \underbrace{\frac{v^2}{2x\,(P\cdot v)^2}\,p_T\cdot H(p,\ldots;v)}_{\mathscr{O}(1/Q^{d-2})},\end{aligned}$$

which means that for the soft part omitting the $n\cdot A$ gluons (putting $n\cdot H = 0$) implies[4] at leading order also the omission of the $v\cdot A$ gluons ($v\cdot H = 0$).

Color gauge invariance

The field combinations considered sofar in the correlators are not color gauge-invariant since they involve the A-fields and, more important, because they involve nonlocal field combinations. At each specific order in Q one of course expects gauge-invariant combinations. Along the light-cone, the leading combinations involve the 'parton fields'

$$\not{n}\,\psi(\xi) = \not{n}\,\psi_+(\xi) \quad\text{and}\quad G^{n\alpha}(\xi),$$

while $A^+ = A^n = n\cdot A$ operators appear in gauge links along the light-cone ($\xi^+ = n\cdot\xi = \xi_T = 0$),

$$U^{[n]}_{[0,\xi]} = \mathscr{P}\,\exp\left(-i\int_0^\xi d(\eta{\cdot}P)\,n{\cdot}A(\eta)\right), \tag{1.48}$$

which are needed to connect colored parton fields. Which n appears in a correlator is fixed by the hard process, although some freedom in n may remain. We note that the exponent in the gauge link is in essence built from 'operators' $(n\cdot\partial)^{-1}n\cdot A$, which are $\mathscr{O}(1)$. Actually, the gauge-invariant correlators will in some cases appear multiplied with the parton momentum, $p^\mu\,\Phi(p;P)$, etc., which implies a derivative ∂_ξ^μ in the matrix element, which is e.g. standard in the matrix elements involving gluon fields $G^{\mu\nu}$. The color gauge-invariant light-cone correlators for quarks and gluons are

$$\Phi_{ij}(x;n) = \int \frac{d(\xi\cdot P)}{(2\pi)}\;e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,U^{[n]}_{[0,\xi]}\,\psi_i(\xi)|P\rangle\Big|_{LC}, \tag{1.49}$$

$$\Gamma^{\alpha\beta}(x;n) = \int \frac{d(\xi\cdot P)}{(2\pi)}\;e^{i\,p\cdot\xi}\;\langle P|\mathrm{Tr}\left(G^{n\beta}(0)\,U^{[n]}_{[0,\xi]}\,G^{n\alpha}(\xi)\,U^{[n]}_{[\xi,0]}\right)|P\rangle\Big|_{LC}, \tag{1.50}$$

while for the TMD light-front correlators

$$\Phi_{ij}(x,p_T;n,C) = \int \frac{d(\xi\cdot P)\,d^2\xi_T}{(2\pi)^3}\;e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,U^{[n,C]}_{[0,\xi]}\,\psi_i(\xi)|P\rangle\Big|_{LF}, \tag{1.51}$$

$$\Gamma^{\alpha\beta}(x,p_T;n,C,C') = \int \frac{d(\xi\cdot P)\,d^2\xi_T}{(2\pi)^3}\;e^{i\,p\cdot\xi}\;\langle P|\mathrm{Tr}\left(G^{n\beta}(0)\,U^{[n,C]}_{[0,\xi]}\,G^{n\alpha}(\xi)\,U^{[n,C']}_{[\xi,0]}\right)|P\rangle\Big|_{LF}, \tag{1.52}$$

where we in passing mention that path dependence (indicated by the arguments C and C') will arise because of the (necessary) transverse piece(s) in the gauge link.

$$\Phi_{\text{quark}}(x,p_T;n,C) = \int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\;e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,U^{[n,C]}_{[0,\xi]}\,\psi_i(\xi)|P\rangle\Big|_{\xi\cdot n=0},$$

$$\Phi^{\alpha\beta}_{\text{gluon}}(x,p_T;n,C,C') = \int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\;e^{i\,p\cdot\xi}\;\langle P|\mathrm{Tr}\left(G^{n\beta}(0)\,U^{[n,C]}_{[0,\xi]}\,G^{n\alpha}(\xi)\,U^{[n,C']}_{[\xi,0]}\right)|P\rangle\Big|_{\xi\cdot n=0},$$

[4]Here the condition on the smallness of a possible v_T in defining n becomes important.

Sum rules

Completing the integration over the correlators, one ends up with a local matrix element

$$\begin{aligned} \mathrm{Tr}\,(\Gamma\Phi) &= \int dx\, d^2p_T\; \Gamma_{ji}(x,p_T)\,\Phi_{ij}(x,p_T) \\ &= \langle P|\overline{\psi}_j(0)\,\Gamma_{ji}(i\partial_\xi)\,U^{[n,C]}_{[0,\xi]}\,\psi_i(\xi)|P\rangle\Big|_{\xi=0}. \end{aligned} \tag{1.53}$$

In particular when the operator $\overline{\psi}\Gamma U\psi$ is an operator with simple or known expectation values between plane wave states (including possibly spin dependence) this provides interesting sum rules for the functions appearing in the correlator.

1.4 Fragmentation correlators in high-energy processes

In high-energy processes, it is useful to employ the Sudakov decomposition of the momenta. We write for the parton momentum

$$k = \frac{1}{z}\,P_h + k_T + \left(k\cdot P_h - \frac{M_h^2}{z}\right)\frac{n}{P_h\cdot n}, \tag{1.54}$$

where $z = P_h\cdot n/k\cdot n$. The above equation defines the coordinates of k for fixed P_h. One can consider variations of P_h for fixed k, in which case we write

$$P_h = z\,k + P_{h\perp} - \left(k\cdot P_h - \frac{M_{h\perp}^2}{z}\right)\frac{n}{k\cdot n} \tag{1.55}$$

with $M_{h\perp}^2 = M_h^2 - P_{h\perp}^2$. For a fixed set of light-like vectors $n_\pm$ identifying the lightlike vector in both above equations as $n = n_+$ (and thus $n_- = P_h/(P_h\cdot n)$ or $k/(k\cdot n)$, respectively), we write the momenta in either one of the following forms

$$\boxed{\begin{aligned} P_h &= \left[P_h^-, \frac{M_h^2}{2P_h^-}, 0_T\right] \\ k &= \left[\frac{P_h^-}{z}, \frac{k\cdot P_h - M_h^2/2z}{P_h^-}, k_T\right] \end{aligned}} \quad \overset{LT}{\longleftrightarrow} \quad \boxed{\begin{aligned} P_h &= \left[P_h^-, \frac{M_{h\perp}^2}{2P_h^-}, P_{h\perp}\right] \\ k &= \left[k^-, \frac{k\cdot P_h - M_{h\perp}^2/2z}{z\,k^-}, 0\right] \end{aligned}} \tag{1.55}$$

These two representations are connected by a Lorentz transformation[5] that leaves the minus component unchanged (with $b^- = k^- = P_h^-/z$ and $b_T = -k_T = P_{h\perp}/z$) switching between either k or P_h having no transverse component. We note that we have used the invariant $k\cdot P_h$ in the above, but it is also possible to use k^2 related through $z^2(k^2 - k_T^2) = 2z\,k\cdot P_h - M_h^2$.

For integrations this implies

$$\begin{aligned} \int dz\,d^2k_T\,d(k\cdot P_h) &= \int \frac{P_h\cdot n}{(k\cdot n)^2}\,d(k\cdot n)\,d^2k_T\,d(k\cdot P_h) = \int z^2\,d^4k\ldots|_{P_h} = \\ &= \int \frac{1}{z^2\,(k\cdot n)}\,d(P_h\cdot n)\,d^2P_{h\perp}\,d(P_h\cdot k) = \int \frac{1}{z^2}\,d^4P_h\ldots|_k. \end{aligned} \tag{1.56}$$

We consider the TMD correlator integrated only over $k\cdot P_h$,

$$\begin{aligned} \Delta_{ij}(z,k_T;n,C) &= \frac{1}{4z}\int \frac{d(k\cdot P_h)}{P_h\cdot n}\,\Delta_{ij}(k,P_h;C) = \frac{1}{4z}\int \frac{dk^2}{2(k\cdot n)}\,\Delta_{ij}(k,P_h;C) = \int \frac{dk^+}{4\,z}\,\Delta_{ij}(k,P_h;C) \\ &= \sum_X \int \frac{d\xi^+\,d^2\xi_T}{4z\,(2\pi)^3}\,e^{ik\cdot\xi}\,\langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\Big|_{LF} \end{aligned} \tag{1.57}$$

[5] A transformation leaving the minus component unchanged, parametrized by b^- and $\boldsymbol{b}_T$ is

$$\left[a^-, a^+, \boldsymbol{a}_T\right] \to \left[a^-, a^+ + \frac{a_T\cdot b_T}{b^-} - \frac{b_T^2\,a^-}{2(b^-)^2}, a_T - \frac{a^-}{b^-}\,b_T\right].$$

where LF refers to the light-front, $\xi^- = \xi \cdot n = 0$. The integration in coordinate space can be written in a coordinate independent way,

$$\int \frac{d\xi^+\,d^2\xi_T}{4z\,(2\pi)^3}\ e^{ik\cdot\xi}\ldots = \int \frac{(k\cdot n)}{4\,(P_h\cdot n)^2}\,\frac{d(\xi\cdot P_h)\,d^2\xi_T}{(2\pi)^3}\ \exp\left(\frac{i}{z}P_h\cdot\xi + ik_T\cdot\xi_T\right)\ldots\ .$$

Integrating over the transverse momenta, we define

$$\begin{aligned}\Delta_{ij}(z;n) &= \int d^2P_{h\perp}\ \Delta_{ij}(z,k_T;n,C) = \frac{z}{4}\int dk^+\,d^2k_T\ \Delta_{ij}(k,P_h;C)\\ &= \sum_X\int \frac{z}{4}\frac{d\xi^+}{(2\pi)}\ e^{ik\cdot\xi}\ \langle 0|U^{[n]}_{[a,\xi]}\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)U^{[n]}_{[0,a]}|0\rangle\Big|_{LC}\end{aligned} \tag{1.58}$$

where LC refers to the light-cone, $\xi^- = \xi\cdot n = \xi_T = 0$. Instead of ξ^+ one can use

$$\int \frac{z}{4}\frac{d\xi^+}{(2\pi)}\ e^{ik\cdot\xi}\ldots = \int \frac{1}{4\,(k\cdot n)}\,\frac{d(\xi\cdot P_h)}{(2\pi)}\ \exp\left(\frac{i}{z}P_h\cdot\xi\right)\ldots\ .$$

We note that issues on dependence on n can be taken over from the distribution functions, including issues on gauge choices and the possibility to get for time-like axial gauges a natural regulator from the transverse momenta.

Sum rules for fragmentation functions

Since for fragmentation correlators, the hadrons are produced, one can construct observables by summing and integrating over them including particular (in principle spin dependent) weights. To construct the observable, we assume the parton momentum k to be fixed, varying P_h and P_x. The sum rule then is

$$\begin{aligned}&\sum_h\int dz\,d^2P_{h\perp}\ O^h_{\alpha\beta}(z,P_{h\perp})\,\Delta_{ij,\beta\alpha}\left(z,\tfrac{P_{h\perp}}{z}\right)\\ &= \sum_{h,X}\int\frac{d^4P_h}{(2\pi)^4}\,(2\pi)\delta(P_h^2-M_h^2)\,\theta(P_h\cdot n)\int\frac{dk^2}{8\pi\,(k\cdot n)}\\ &\qquad\times\int d^4\xi\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)|P_h,\alpha,X\rangle\,O^h_{\alpha\beta}(P_h)\,\langle P_h,\beta,X|\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\\ &= \int\frac{dk^2}{8\pi\,(k\cdot n)}\int d^4\xi\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)\,\hat{O}\,\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\\ &= \int\frac{dk^+}{4\pi}\int d^4\xi\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)\,\hat{O}\,\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\\ &= \frac{1}{2}\int d\xi^+\,d^2\xi_T\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)\,\hat{O}\,\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\Big|_{LF}\end{aligned} \tag{1.59}$$

where the operator $\hat{O}$ is

$$\hat{O} = \sum_{h,X}\int\frac{dz\,d^2P_{h\perp}}{2z(2\pi)^3}|P_h,\alpha,X\rangle\,O^h_{\alpha\beta}(P_h)\,\langle P_h,\beta,X|. \tag{1.60}$$

If the operator $\hat{O}$ is also known at the parton level,

$$\hat{O} = \sum_s\int\frac{dp^-\,d^2p_T}{2p^-\,(2\pi)^3}\ b^\dagger(p,s)o(p,s)b(p,s), \tag{1.61}$$

we obtain (again at fixed k)

$$\begin{aligned}\sum_h\int dz\,d^2P_{h\perp}\ O^h_{\alpha\beta}(z,P_{h\perp})\,\Delta_{ij,\beta\alpha}\Big|_k &= \frac{1}{2(k\cdot n)}\sum_s u_i(k,s)o(k,s)\overline{u}_j(k,s)\\ &= \left[\frac{\not{k}}{2(k\cdot n)}\left(\frac{o(k,s)+o(k,-s)}{2}+\gamma_5\not{s}\,\frac{o(k,s)-o(k,-s)}{2}\right)\right]_{ij}.\end{aligned} \tag{1.62}$$

The simplest application of this is considering the fragmentation of unpolarized quarks into hadrons with $O^h_{\alpha\beta} = P_h \cdot n\, \delta_{\alpha\beta}$ corresponding to the the operator $\hat{O} = \hat{P}\cdot n$. Using the leading order parametrization

$$\Delta_{ij,\alpha\alpha}(z, k_T; P_h, S_h; n) = \frac{1}{2(P_h \cdot n)}\, D_1(z, k_T^2)\, (\not{P}_h)_{ij}, \tag{1.63}$$

for the correlator (summed over spins), the sumrule becomes

$$\sum_h \int dz\, z D_1^{q\to h}(z) = 1, \tag{1.64}$$

which is precisely the momentum sumrule for fragmentation functions.

1.5 Color treatment

Let's take as an example the Drell-Yan process for which the basic process involves two correlators. Making color explicit, we have in the amplitude the contribution $\langle X|\overline{\psi}^r(\xi_2)\psi^r(\xi_1)|P_1P_2\rangle$ and in the cross section

$$\sigma \propto \sum_{r,s} \langle P_1|\psi^r(\xi_1)\overline{\psi}^s(0_1)|P_1\rangle\, \langle P_2|\psi^s(0_2)\overline{\psi}^r(\xi_2)|P_2\rangle. \tag{1.65}$$

Realizing that the matrix elements have a color triplet projector as intermediate state we have $\Phi^{rs} \propto \langle|\psi^r\overline{\psi}^s|\rangle \propto \delta^{rs}$ and one in fact only can use the trace $\Phi = \sum \Phi^{rr}$. Because $\mathrm{Tr}_c(I\,I) = (1/N_c)\mathrm{Tr}_C(I)\,\mathrm{Tr}_C(I)$, one then finds $\sigma \propto (1/N_c)\Phi_1\overline{\Phi}_2$. Including gauge links coming from collinear gluons attached to the correlators, we get (note the color embedding!)

$$\sigma \propto \langle P_1|\psi^{r_1}(\xi_1)\,\overline{\psi}^{s_1}(0_1)|P_1\rangle\, U^{s_1 s}_{[-\infty,0_2]}\, U^{ss_2}_{[0_1,-\infty]}\, \langle P_2|\psi^{s_2}(0_2)\,\overline{\psi}^{r_2}(\xi_2)|P_2\rangle\, U^{r_2 r}_{[-\infty,\xi_1]}\, U^{rr_1}_{[\xi_2,-\infty]}. \tag{1.66}$$

Identifying $0_1 = 0_2 = 0$ and performing a gauge transformation $V(\xi) \equiv V_{[0,\xi]}$, we get

$$\begin{aligned} \sigma &\propto \mathrm{Tr}_C\big[\langle P_1|V(\xi)\psi(\xi_1)\,\overline{\psi}(0_1)\,V^\dagger(0_1)|P_1\rangle\, V(-\infty)\,U_{[-\infty,0_2]}\,V^\dagger(0_2)\,V(0_1)\,U_{[0_1,-\infty]}\,V^\dagger(-\infty) \\ &\quad \times \langle P_2|V(0_2)\,\psi(0_2)\,\overline{\psi}(\xi_2)V^\dagger(\xi_2)|P_2\rangle\, V(-\infty)\,U_{[-\infty,\xi_1]}\,V^\dagger(\xi_1)\,V(\xi_2)\,U_{[\xi_2,-\infty]}\,V^\dagger(-\infty)\big]. \\ &= \mathrm{Tr}_C\big[\langle P_1|V_{[0,\xi_1]}\psi(\xi_1)\,\overline{\psi}(0)|P_1\rangle\, V_{[0,-\infty]}\,U_{[-\infty,0]}\,V_{[0,-\infty]}\,U_{[-\infty,0]} \\ &\quad \times \langle P_2|\psi_s(0)\,\overline{\psi}(\xi_2)\,V_{[\xi_2,0]}|P_2\rangle\, V_{[0,-\infty]}\,U_{[-\infty,\xi_1]}V_{[\xi_1,0]}\,V_{[0,\xi_2]}\,U_{[\xi_2,-\infty]}\,V_{[-\infty,0]}\big]. \end{aligned} \tag{1.67}$$

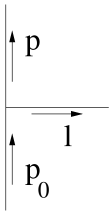
If the path in V is choosen to run via $-\infty$, $V_{[0,\xi]} = U^-_{[0,\xi]}$ (thus a 'staple' via minus infinity), we get

$$\begin{aligned} \sigma &\propto \mathrm{Tr}_C\big[\langle P_1|U^-_{[0,\xi_1]}\psi(\xi_1)\,\overline{\psi}(0)|P_1\rangle\, \langle P_2|\psi(0)\,\overline{\psi}(\xi_2)\,U^-_{[\xi_2,0]}|P_2\rangle\big]. \\ &= \frac{1}{N_c}\langle P_1|\overline{\psi}(0)\,U^-_{[0,\xi_1]}\psi(\xi_1)|P_1\rangle\, \langle P_2|\overline{\psi}(\xi_2)\,U^-_{[\xi_2,0]}\,\psi(0)|P_2\rangle, \end{aligned} \tag{1.68}$$

where the splitting is allowed because the matrix elements are proportional to unit matrices in color space (there are no A-fields in the 'second' part that belong to correlator 1 or the other way around).

1.6 Large transverse momenta

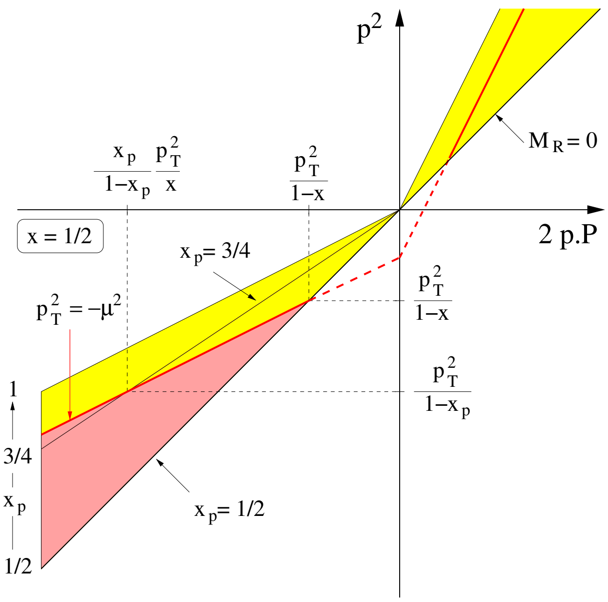
We started out with the assumption that the support of the correlators Φ and Δ is restricted to regions where the scalar products of the momenta involved are of hadronic size, or stated differently the fall-off as a function of these invariants should be sufficiently fast. To check consistency requires consideration of large transverse momenta generated after emission of hard partons. These will produce 'their own' hadrons and can at that stage be treated as on-shell partons. As our first case we look at parton(p_0) $\to$ parton(p)+parton(l) (with $p = p_0 - l$), the emission of an on-shell parton with momentum l by a parton with momentum p_0 (of which we will first neglect p_{0T}). The momentum fraction is reduced from $p_0.n = x/x_p$ ($x \le x_p \le 1$) to the lower value $p.n = x$ and producing at the same time a (moderately large) transverse momentum p_T. We neglect the $\mathcal{O}(M)$ contributions.



$$
\begin{aligned}
p_0 &\approx \frac{x}{x_p}\,P \\
l &\approx (1-x_p)\,p_0 - p_T - \frac{p_T^2}{2\,(1-x_p)\,(p_0\cdot n)}\,n \approx \frac{(1-x_p)}{x_p}\,x\,P - p_T - \frac{x_p}{1-x_p}\,\frac{p_T^2}{2x}\,n, \\
p &\approx x_p\,p_0 + p_T + \frac{p_T^2}{2\,(1-x_p)\,(p_0\cdot n)}\,n \approx x\,P + p_T + \underbrace{\frac{x_p}{1-x_p}\,\frac{p_T^2}{2x}}_{\sigma_p}\,n.
\end{aligned}
$$

For the invariant momenta we have

$$
p_0^2 \approx l^2 \approx 0 \qquad \text{and} \qquad p^2 \approx \frac{p_T^2}{1-x_p}.
$$



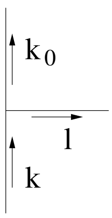
The figure to the left shows the support region as also discussed previously, but now neglecting all $\mathcal{O}(M)$ contributions. Thus $p\cdot P \approx \sigma_p \approx p^-$ and

$$
p^2 \approx 2x\,\sigma_p + p_T^2.
$$

Although we assumed Φ to vanish for large values of the variables $p\cdot P$ and p^2, these can take larger values, e.g. after parton braching, as discussed above. In that case the additional variables entering are the fraction $p_0\cdot n = x/x_p$ of the original parton and (considered below) its transverse momentum p_{0T} with $x \le x_p \le 1$. By varying x_p from 1 (minimal loss of longitudinal momentum along P) to $x_p = x$ (maximal loss) we scan the full physical region of $\Phi(x,p_T)$. For given x and p_T, the values of the invariants are fixed for a given x_p,

$$
\begin{aligned}
\tau_p &\approx p^2 \approx \frac{p_T^2}{1-x_p}, \\
\sigma_p &\approx p\cdot P \approx \frac{x_p}{1-x_p}\,\frac{p_T^2}{2x}.
\end{aligned}
$$

The situation for fragmentation is analogous.



$$
\begin{aligned}
k_0 &\approx \frac{z_k}{z}\,P_h \\
l &\approx \frac{(1-z_k)}{z_k}\,k_0 - k_T - \frac{z_k\,k_T^2}{2\,(1-z_k)\,(k_0\cdot n)}\,n \approx \frac{(1-z_k)}{z}\,P_h - k_T - \frac{z\,k_T^2}{2(1-z_k)}\,n, \\
k &\approx z_k^{-1}\,k_0 + k_T + \frac{z_k\,k_T^2}{2\,(1-z_k)\,(k_0\cdot n)}\,n \approx z^{-1}\,P_h + k_T + \underbrace{\frac{z\,k_T^2}{2(1-z_k)}}_{\sigma_k}\,n.
\end{aligned}
$$

For the invariant momenta, we have

$$
k^2 \approx l^2 \approx 0 \qquad \text{and} \qquad k^2 \approx \frac{z_k}{1-z_k}\,k_T^2.
$$

We will now give the explicit integrations including also small transverse momenta p_{0T} for distributions (and k_{0T} for fragmentation). We get

$$
\Phi(x,p_T) = \int d\sigma_p\,d\tau_p\,\delta(\tau_p - 2x\,\sigma_p - p_T^2)\int d^4l\,\delta(l^2)\ \ldots\Phi_0(p_0) \tag{1.69}
$$

$$
= \int d\sigma_p\,d\tau_p\,\delta(\tau_p - 2x\,\sigma_p - p_T^2)\int d(p_0\cdot n)\,d^2p_{0T}\,d(p_0\cdot P)\,\delta(l^2)\ \ldots\Phi_0(p_0), \tag{1.70}
$$

The integration over $p_0 \cdot n = x/x_p$ is easily turned into an x_p-integration, the integration over $p_0 \cdot P$ can be performed to get $\Phi_0(x/x_p, p_{0T})$ and in the evaluation of l^2 we have (note $l_T = p_{0T} - p_T$). Using

$$l^2 = (p_0 - p)^2 = (p_T - p_{0T})^2 - \frac{1-x_p}{x_p}\, 2x\,\sigma_p, \tag{1.71}$$

we find

$$\begin{aligned}
\Phi(x,p_T) &= \int d\sigma_p\, d\tau_p\, dx_p\, d^2p_{0T}\, \frac{x}{x_p^2}\, \delta\left(\tau_p - 2x\,\sigma_p - p_T^2\right) \\
&\qquad \times \delta\left(p_T^2 - 2p_{0T}\cdot p_T + p_{0T}^2 - \frac{1-x_p}{x_p}\, 2x\,\sigma_p\right)\ \ldots \Phi_0\left(\frac{x}{x_p}, p_{0T}\right) \\
&= \int d\sigma_p\, d\tau_p\, dx_p\, d^2p_{0T}\, \frac{x}{x_p^2}\, \delta\left(\tau_p - 2x\,\sigma_p - p_T^2\right) \\
&\qquad \times \delta\left(\frac{p_T^2}{x_p} - 2p_{0T}\cdot p_T + p_{0T}^2 - \frac{1-x_p}{x_p}\, \tau_p\right)\ \ldots \Phi_0\left(\frac{x}{x_p}, p_{0T}\right) \\
&= \int_x^1 \frac{dx_p\, d^2p_{0T}}{2x_p\,(1-x_p)}\ \ldots \Phi_0\left(\frac{x}{x_p}, p_{0T}\right),
\end{aligned} \tag{1.72}$$

where in the integrand invariants like τ_p and σ_p are fixed,

$$\tau_p = p^2 = \frac{1}{1-x_p}\, p_T^2 - \frac{2x_p}{1-x_p}\, p_{0T}\cdot p_T + \frac{x_p}{1-x_p}\, p_{0T}^2, \tag{1.73}$$

$$2x\,\sigma_p = p^2 - p_T^2 = \frac{x_p}{1-x_p}\, p_T^2 - \frac{2x_p}{1-x_p}\, p_{0T}\cdot p_T + \frac{x_p}{1-x_p}\, p_{0T}^2. \tag{1.74}$$

We note that

$$\frac{1}{p^2} = \frac{(1-x_p)}{p_T^2}\left(1 + 2\,x_p\,\frac{p_{0T}\cdot p_T}{p_T^2} + 4\,x_p^2\,\frac{(p_{0T}\cdot p_T)^2}{(p_T^2)^2} - x_p\,\frac{p_{0T}^2}{p_T^2} + \mathcal{O}(p_{0T}^3)\right) \tag{1.75}$$

$$\frac{1}{(p^2)^2} = \frac{(1-x_p)^2}{(p_T^2)^2}\left(1 + 4\,x_p\,\frac{p_{0T}\cdot p_T}{p_T^2} + 12\,x_p^2\,\frac{(p_{0T}\cdot p_T)^2}{(p_T^2)^2} - 2\,x_p\,\frac{p_{0T}^2}{p_T^2} + \mathcal{O}(p_{0T}^3)\right). \tag{1.76}$$

For fragmentation functions, we get

$$l^2 = (k_0 - k)^2 = (k_T - k_{0T})^2 - (1-z_k)\, 2z^{-1}\,\sigma_k, \tag{1.77}$$

and we find

$$\begin{aligned}
\Delta(z,k_T) &= \int d\sigma_k\, d\tau_k\, dz_k\, d^2k_{0T}\, \frac{1}{z\,z_k}\, \delta\left(\tau_k - 2z^{-1}\,\sigma_k - k_T^2\right) \\
&\qquad \times \delta\left(k_T^2 - 2k_{0T}\cdot k_T + k_{0T}^2 - (1-z_k)\, 2z^{-1}\,\sigma_k\right)\ \ldots \Delta_0\left(\frac{z}{z_k}, k_{0T}\right) \\
&= \int d\sigma_k\, d\tau_k\, dz_k\, d^2k_{0T}\, \frac{1}{z\,z_k}\, \delta\left(\tau_k - 2z^{-1}\,\sigma_k - k_T^2\right) \\
&\qquad \times \delta\left(z_k\,k_T^2 - 2k_{0T}\cdot k_T + k_{0T}^2 - (1-z_k)\,\tau_k\right)\ \ldots \Delta_0\left(\frac{z}{z_k}, k_{0T}\right) \\
&= \int_x^1 \frac{dz_k\, d^2k_{0T}}{2(1-z_k)}\ \ldots \Delta_0\left(\frac{z}{z_k}, k_{0T}\right),
\end{aligned} \tag{1.78}$$

where in the integrand invariants like τ_k and σ_k are fixed,

$$\tau_k = k^2 = \frac{z_k}{1-z_k}\, k_T^2 - \frac{2}{1-z_k}\, k_{0T}\cdot k_T + \frac{1}{1-z_k}\, k_{0T}^2, \tag{1.79}$$

$$2z^{-1}\,\sigma_k = k^2 - k_T^2 = \frac{1}{1-z_k}\, k_T^2 - \frac{2}{1-z_k}\, k_{0T}\cdot k_T + \frac{1}{1-z_k}\, k_{0T}^2. \tag{1.80}$$

We note that

$$\frac{1}{k^2} = \frac{(1-z_k)}{z_k\,k_T^2}\left(1 + 2\,\frac{p_{0T}\cdot p_T}{z_k\,k_T^2} + 4\,\frac{(k_{0T}\cdot k_T)^2}{z_k^2\,(k_T^2)^2} - \frac{k_{0T}^2}{z_k\,k_T^2} + \mathcal{O}(k_{0T}^3)\right) \tag{1.81}$$

$$\frac{1}{(k^2)^2} = \frac{(1-z_k)^2}{z_k^2\,(k_T^2)^2}\left(1 + 4\,\frac{k_{0T}\cdot k_T}{z_k\,k_T^2} + 12\,\frac{(k_{0T}\cdot k_T)^2}{z_k^2\,(k_T^2)^2} - 2\,\frac{k_{0T}^2}{z_k\,k_T^2} + \mathcal{O}(k_{0T}^3)\right). \tag{1.82}$$

Appendix A: The gauge link

As indicated gauge link are essential ingredients to have proper definitions of color gauge-invariant correlators. They arise from insertions of quark-gluon-quark correlators and of these the leading ones are expected to be correlators containing $A \cdot n$ fields. The general $A^\mu(\eta)$ field in the correlator in Eq. 1.7 can be written as

$$A^\mu = \frac{(A\cdot n)}{P\cdot n}\,P^\mu + A_T^\mu + \frac{(P\cdot n)(A\cdot P)-(A\cdot n)M^2}{(P\cdot n)^2}\,n^\mu.$$

This expansion can be written down for $A^\mu(x)$ or $A^\mu(p)$ To see, how quark-gluon-quark correlators are turned into color gauge-invariant objects, it is convenient to look at the momentum space field and rewrite the momentum in terms of the gluon momentum

$$\begin{aligned} A^\mu(p) &= \int d^4\eta\; e^{i\,p\cdot\eta}\,A^\mu(\eta) \\ &= \int d^4\eta\; e^{ip\cdot\eta}\Bigg[\frac{(A(\eta)\cdot n)}{p\cdot n}\,p^\mu \\ &\qquad + \frac{(p\cdot n)\,A_T^\mu(\eta) - p_T^\mu\,(A(\eta)\cdot n)}{p\cdot n} \\ &\qquad + \frac{(p\cdot n)(A(\eta)\cdot P)-(p\cdot P)(A(\eta)\cdot n)}{(p\cdot n)(P\cdot n)}\,n^\mu\Bigg]. \end{aligned}$$

In the correlator the momentum $p^\mu \longrightarrow i\partial^\mu(\eta)$, so

$$\begin{aligned} A^\mu(p) &= \frac{1}{p\cdot n}\int d^4\eta\; e^{i\,p\cdot\eta}\Bigg[A^n(\eta)\,p^\mu \\ &\qquad + i\partial^n(\eta)\,A_T^\mu(\eta) - i\partial_T^\mu(\eta)\,A^n(\eta) \\ &\qquad + \frac{i\partial^n(\eta)A^P(\eta)-i\partial^P(\eta)A^n(\eta)}{P\cdot n}\,n^\mu\Bigg] \\ &= \frac{1}{p\cdot n}\left[A^n(p)\,p^\mu + i\,G_T^{n\mu}(p) + \frac{iG^{nP}(p)}{P\cdot n}\,n^\mu\right] \end{aligned}$$

The first term will lead to gauge links along the n-direction. The introduction of the gluon momentum p^μ, rather than staying with the hadron momentum is done not only because it renders the subleading parts gauge invariant but it provides the necessary and convenient starting point for using Ward identities for the A^n insertions (gluon with polarization along the parton/hadron momentum). It will turn out that the insertions on the external legs of the hard part give the path dependence.

To see this, consider field combinations denoted as

$$\underbracket{U_-^{[n]}\psi}(p^+) = U_-^{[n]}(p^+)\,\psi(p^+) = \int d\xi^-\;e^{i\,p^+\cdot\xi^-}\;U_{[-\infty,\xi^-]}^{[n]}\psi(\xi^-),$$

where $\xi^- = \xi\cdot P$ and $p^+ = p\cdot n$, which are the only relevant components that need to be considered here. Explicitly one has (with $A^+ = A\cdot n$)

$$\underbracket{U_-^{[n]}\psi}(p^+) = \sum_{N=0}^{\infty}(-i)^N\int_{-\infty}^{\infty}d\xi^-\int_{-\infty}^{\xi}d\eta_N^-\int_{\eta_N^-}^{\xi}d\eta_{N-1}^-\cdots\int_{\eta_2^-}^{\xi}d\eta_1^-\;A^+(\eta_N^-)\ldots A^+(\eta_1^-)\psi(\xi)\,e^{i\,p^+\xi^-},$$

in which the arguments run between $-\infty < \eta_N^- < \eta_{N-1}^- < \ldots < \eta_1^- < \xi^-$, implemented through $\theta(\eta_{N-1}^- -$

$\eta_N^-)\ldots\theta(\eta_1^- - \eta_2^-)\theta(\xi^- - \eta_1^-)$, which can be rewritten as momentum-space integrations,

$$\begin{aligned}
\underbrace{U_-^{[n]}}\psi(p^+) &= \sum_{N=0}^{\infty}(-i)^N\int_{-\infty}^{\infty}d\xi^-\int_{-\infty}^{\infty}d\eta_N^-\int_{-\infty}^{\infty}d\eta_{N-1}^-\cdots\int_{-\infty}^{\infty}d\eta_1^- \\
&\quad\times\int\frac{dp_N^+}{-2\pi\, i}\cdots\int\frac{dp_1^+}{-2\pi\, i}\,\frac{e^{-i\,p_N^+(\eta_{N-1}^- - \eta_N^-)}}{p_N^+ + i\epsilon}\cdots\frac{e^{-i\,p_1^+(\xi^- - \eta_1^-)}}{p_1^+ + i\epsilon}\;A^+(\eta_N^-)\ldots A^+(\eta_1^-)\psi(\xi^-)\,e^{i\,p^+\xi^-} \\
&= \sum_{N=0}^{\infty}\int\frac{dp_N^+}{2\pi}\cdots\int\frac{dp_1^+}{2\pi}\,\frac{A^+(p_N^+)}{(p_N^+ + i\epsilon)}\,\frac{A^+(p_{N-1}^+ - p_N^+)}{(p_{N-1}^+ + i\epsilon)}\cdots\frac{A^+(p_1^+ - p_2^+)}{(p_1^+ + i\epsilon)}\,\psi(p^+ - p_1^+). \\
&= \sum_{N=0}^{\infty}\int\frac{dp_N^+}{2\pi}\cdots\int\frac{dp_1^+}{2\pi}\,\frac{A^+(p_N^+)}{(p_N^+ + i\epsilon)}\,\frac{A^+(p_{N-1}^+)}{(p_N^+ + p_{N-1}^+ + i\epsilon)}\cdots\frac{A^+(p_1^+)}{(p_N^+ + \ldots + p_1^+ + i\epsilon)} \\
&\quad\times\psi(p^+ - p_1^+ - \ldots - p_N^+). \\
&= \sum_{N=0}^{\infty}\int\frac{dp_1^+}{2\pi}\cdots\int\frac{dp_N^+}{2\pi}\,\frac{A^+(p_1^+)}{(p_1^+ + i\epsilon)}\,\frac{A^+(p_2^+)}{(p_1^+ + p_2^+ + i\epsilon)}\cdots\frac{A^+(p_N^+)}{(p_1^+ + \ldots + p_N^+ + i\epsilon)} \\
&\quad\times\psi(p^+ - p_1^+ - \ldots - p_N^+).
\end{aligned}$$

Summarizing,

$$\begin{aligned}
\underbrace{U_-^{[n]}}\psi(p) &= \int d^4\xi\;\exp\left(i\,p\cdot\xi\right)\,\mathcal{P}\exp\left(-ig\int_{-\infty}^{\xi\cdot P}d(\eta\cdot P)\;n\cdot A(\eta)\right)\,\psi(x) \\
&= \sum_{N=0}^{\infty}\int\frac{d^4p_N}{(2\pi)^4}\cdots\int\frac{d^4p_1}{(2\pi)^4}\,\frac{A^n(p_N)}{(x_N + i\epsilon)}\,\frac{A^n(p_{N-1} - p_N)}{(x_{N-1} + i\epsilon)}\cdots\frac{A^n(p_1 - p_2)}{(x_1 + i\epsilon)}\,\psi(p - p_1) \\
&= \sum_{N=0}^{\infty}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)}{(x_1 + i\epsilon)}\,\frac{A^n(p_2)}{(x_1 + x_2 + i\epsilon)}\cdots\frac{A^n(p_N)}{(x_1 + \ldots + x_N + i\epsilon)}\,\psi\left(p - \sum_{i=1}^{N}p_i\right),
\end{aligned} \tag{1.83}$$

where $x_i = p_i\cdot n$. We thus can expand

$$\underbrace{U_\pm^{[n]}}\psi(p) = \sum_{M=0}^{\infty}\underbrace{U_\pm^{[n](M)}}\psi(p), \tag{1.84}$$

where $U_\pm^{[n](0)} = 1$. The gauge link and the terms in its expansion, not only have a particular structure in coordinate or momentum space, but they also have a charge structure. In particular for applications in non-abelian gauge theories, it is convenient to expand the gauge link, like the field $A^\mu = A^{a\mu}T_a$, in terms of color matrices,

$$U_\pm^{[n]} = U_\pm^{a[n]}\,T_a, \tag{1.85}$$

which is possible for each of the terms in the expansion of the gauge link, but also works for the full gauge link.

It is possible to use a more symmetric expression for the gauge link in momentum space. The momenta $p_1,\ldots,p_N$ are integration variables. We can use the relations

$$\frac{1}{(x_1 + x_2 + i\epsilon)}\left[\frac{1}{(x_1 + i\epsilon)} + \frac{1}{(x_2 + i\epsilon)}\right] = \frac{1}{(x_1 + i\epsilon)(x_2 + i\epsilon)}, \tag{1.86}$$

$$\begin{aligned}
\frac{1}{(x_1 + x_2 + x_3 + i\epsilon)}\underbrace{\left[\frac{1}{(x_1 + x_2 + i\epsilon)(x_1 + i\epsilon)} + \ldots\right]}_{6\text{ permutations}} &= \frac{1}{(x_1 + x_2 + x_3 + i\epsilon)}\underbrace{\left[\frac{1}{(x_1 + i\epsilon)(x_2 + i\epsilon)} + \ldots\right]}_{3\text{ permutations}} \\
&= \frac{1}{(x_1 + i\epsilon)(x_2 + i\epsilon)(x_3 + i\epsilon)},
\end{aligned} \tag{1.87}$$

and its generalization to more momenta, to symmetrize the result. This simply works in the abelian case, when all permutations of $A^n(p_{\tau(1)})\ldots A^n(p_{\tau(N)})$ are identical, but can also be used in the non-abelian case. In that case one has for two fields (omitting the prescription)

$$\begin{aligned}\frac{A^n(p_1)\,A^n(p_2)}{x_1(x_1+x_2)} &= \frac{1}{2}\frac{A^n(p_1)\,A^n(p_2)}{x_1(x_1+x_2)}+\frac{1}{2}\frac{A^n(p_2)\,A^n(p_1)}{x_2(x_1+x_2)}\\ &= \frac{1}{4}\frac{\{A^n(p_1),A^n(p_2)\}}{x_1\,x_2}+\frac{1}{4}\frac{(x_2-x_1)\,[A^n(p_1),A^n(p_2)]}{x_1\,x_2\,(x_1+x_2)}=\frac{1}{2}\frac{A^n(p_1)\,A^n(p_2)}{x_1\,x_2},\end{aligned}$$

where the commutator term is not important, being proportional to $(x_2-x_1)\,\delta(x_1-x_2)$. Thus we have

$$\begin{aligned}\underbrace{U_-^{[n]}\psi}(p) &= \int d^4\xi\;\exp\left(i\,p\cdot\xi\right)\mathcal{P}\exp\left(-ig\int_{-\infty}^{\xi\cdot P}d(\eta\cdot P)\;n\cdot A(\eta)\right)\psi(x)\\ &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(x_1+i\epsilon)(x_2+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right),\end{aligned}\qquad(1.88)$$

which is now by construction symmetrized in color space. For a link along n coming from $+\infty$ one has

$$\begin{aligned}\underbrace{U_+^{[n]}\psi}(p) &= \int d^4\xi\;\exp\left(i\,p\cdot\xi\right)\mathcal{P}\exp\left(-ig\int_{\infty}^{\xi\cdot P}d(\eta\cdot P)\;n\cdot A(\eta)\right)\psi(x)\\ &= \sum_{N=0}^{\infty}\int\frac{d^4p_N}{(2\pi)^4}\cdots\int\frac{d^4p_1}{(2\pi)^4}\,\frac{A^n(p_N)}{(-x_N+i\epsilon)}\,\frac{A^n(p_{N-1}-p_N)}{(-x_{N-1}+i\epsilon)}\cdots\frac{A^n(p_1-p_2)}{(-x_1+i\epsilon)}\,\psi(p-p_1).\\ &= \sum_{N=0}^{\infty}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)}{(-x_1+i\epsilon)}\cdots\frac{A^n(p_N)}{(-x_1-\ldots-x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right),\\ &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(-x_1+i\epsilon)(-x_2+i\epsilon)\ldots(-x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right).\end{aligned}\qquad(1.89)$$

One also sees that a term $U_\pm^{[n](M)}$ is the consecutive action of M simple $U_\pm^{[n](1)}$-links. The conjugate link is

$$\begin{aligned}\underbrace{\overline{\psi}(p)\,U_-^{[n]\dagger}} &= \int d^4\xi\;\exp\left(-i\,p\cdot\xi\right)\,\overline{\psi}(\xi)\,\mathcal{P}\exp\left(+ig\int_{-\infty}^{\xi\cdot P}d(\eta\cdot P)\;n\cdot A(\eta)\right)\\ &= \sum_{N=0}^{\infty}\frac{(-)^N}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\overline{\psi}\left(p+\sum_{i=1}^{N}p_i\right)\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(p_1^+ + i\epsilon)(p_2^+ + i\epsilon)\ldots(p_N^+ + i\epsilon)}\\ &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\overline{\psi}\left(p-\sum_{i=1}^{N}p_i\right)\,\frac{A^n(-p_1)\,A^n(-p_2)\ldots A^n(-p_N)}{(p_1^+ - i\epsilon)(p_2^+ - i\epsilon)\ldots(p_N^+ - i\epsilon)}.\end{aligned}\qquad(1.90)$$

The links along the n-direction are actually essential to render the insertion of other gluon components gauge-invariant. To see this, one must realize that including the n-links the relevant pieces in the correlator contain

$$\ldots i\partial^\mu(\eta)\ldots U_{[\ldots\eta]}^{[n]}\,A^\alpha(\eta)\,U_{[\eta,\ldots]}^{[n]}\cdots\;.$$

In order to evaluate the pieces we give a number of useful relations for gauge links

$$i\,\partial_\xi^n\,U_{[0,\xi]}^{[n]} = U_{[0,\xi]}^{[n]}\,iD^n(\xi),\qquad(1.91)$$

$$ig\,G^{n\alpha}(\xi) = [iD^n(\xi),iD_T^\alpha(\xi)] = [iD^n(\xi),g\,A_T^\alpha(\xi)]-[i\partial_\xi^\alpha,g\,A^n(\xi)],\qquad(1.92)$$

$$[i\partial_\xi^n,U_{[\eta,\xi]}^{[n]}\,gA_T^\alpha(\xi)\,U_{[\xi,\eta]}^{[n]}] = U_{[\eta,\xi]}^{[n]}\left(ig\,G^{n\alpha}(\xi)+[i\partial_\xi^\alpha,gA^n(\xi)]\right)U_{[\xi,\eta]}^{[n]},\qquad(1.93)$$

$$iD_T^\alpha(\eta)\,U_{[\eta,\xi]}^{[n]}-U_{[\eta,\xi]}^{[n]}\,iD_T^\alpha(\xi) = \int_{\eta\cdot P}^{\xi\cdot P}d(\zeta\cdot P)\;U_{[\eta,\zeta]}^{[n]}\;\underbrace{[iD^n(\zeta),iD_T^\alpha(\zeta)]}_{ig\,G^{n\alpha}(\zeta)}\;U_{[\zeta,\xi]}^{[n]}.\qquad(1.94)$$

In particular one sees that (with α a transverse component),

$$i\partial^n(\eta)U_{[\ldots\eta]}^{[n]}\,A^\alpha(\eta)\,U_{[\eta,\ldots]}^{[n]}\quad\Longrightarrow\quad U_{[\ldots\eta]}^{[n]}\,[iD^n(\eta),A^\alpha(\eta)]U_{[\eta,\ldots]}^{[n]}\qquad(1.95)$$

while
$$-i\partial^\alpha(\eta)U^{[n]}_{[\ldots\eta]}\,A^n(\eta)\,U^{[n]}_{[\eta,\ldots]}\quad\Longrightarrow\quad -U^{[n]}_{[\ldots\eta]}\,[i\partial^\alpha(\eta),A^n(\eta)]U^{[n]}_{[\eta,\ldots]}\tag{1.96}$$
and for the combination
$$i\partial^n(\eta)U^{[n]}_{[\ldots\eta]}\,A^\alpha(\eta)\,U^{[n]}_{[\eta,\ldots]}-i\partial^\alpha(\eta)U^{[n]}_{[\ldots\eta]}\,A^n(\eta)\,U^{[n]}_{[\eta,\ldots]}\quad\Longrightarrow\quad U^{[n]}_{[\ldots\eta]}\,iG^{n\alpha}(\eta)\,U^{[n]}_{[\eta,\ldots]}.\tag{1.97}$$
Similarly
$$i\partial^n(\eta)U^{[n]}_{[\ldots\eta]}\,A^P(\eta)\,U^{[n]}_{[\eta,\ldots]}-i\partial^P(\eta)U^{[n]}_{[\ldots\eta]}\,A^n(\eta)\,U^{[n]}_{[\eta,\ldots]}\quad\Longrightarrow\quad U^{[n]}_{[\ldots\eta]}\,iG^{nP}(\eta)\,U^{[n]}_{[\eta,\ldots]}.\tag{1.98}$$
It is useful to have the above relations also in momentum space. We have (everywhere absorbing g in A)
$$\int d^4\xi\;e^{i\,p\cdot\xi}\,\psi(\xi)=\psi(p),\tag{1.99}$$
$$\int d^4\xi\;e^{i\,p\cdot\xi}\,i\partial^\mu_\xi\psi(\xi)=p^\mu\,\psi(p),\tag{1.100}$$
$$\int d^4\xi\;e^{i\,p\cdot\xi}\,A^\mu(\xi)\psi(\xi)=\int\frac{d^4p_0}{(2\pi)^4}\,A^\mu(p_0)\,\psi\left(p-p_0\right)=\underbrace{A^\mu\,\psi}(p),\tag{1.101}$$
$$\int d^4\xi\;e^{i\,p\cdot\xi}\,A^\mu(\xi)\,A^\nu(\xi)\,\psi(\xi)=\int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,A^\mu(p_1)\,A^\nu(p_2)\,\psi\left(p-p_1-p_2\right),\tag{1.102}$$
$$\begin{aligned}\int d^4\xi\;e^{i\,p\cdot\xi}\,A^\mu(\xi)\,A^\nu(\xi)&=\int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,A^\mu(p_1)\,A^\nu(p_2)\,(2\pi)^4\,\delta^4\left(p-p_1-p_2\right),\\&=\int\frac{d^4p_1}{(2\pi)^4}\,A^\mu(p_1)\,A^\nu(p-p_1),\end{aligned}\tag{1.103}$$
$$\begin{aligned}\int d^4\xi\;e^{i\,p\cdot\xi}\,iG^{n\alpha}(\xi)&=iG^{n\alpha}(p)\\&=x\,A^\alpha_T(p)-p^\alpha_T\,A^n(p)+\int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,[A^n(p_1),A^\alpha_T(p_2)]\,(2\pi)^4\,\delta^4\left(p-p_1-p_2\right),\end{aligned}\tag{1.104}$$
For the Fourier transforms of the fields including a gauge link one has besides Eqs 1.83 or 1.88,
$$\begin{aligned}\underbrace{U^{[n]}_-i\partial^n\psi}(p)&=\sum_{N=0}^\infty\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(x_1+i\epsilon)(x_2+i\epsilon)\ldots(x_N+i\epsilon)}\left(x-\sum_{i=1}^N x_i\right)\psi\left(p-\sum_{i=1}^N p_i\right)\\&=x\,\underbrace{U^{[n]}_-\psi}(p)\\&\quad-\sum_{N=1}^\infty\sum_{i=1}^N\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)}{(x_1+i\epsilon)}\ldots A^n(p_i)\ldots\frac{A^n(p_N)}{(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^N p_i\right)\\&=x\,\underbrace{U^{[n]}_-\psi}(p)\\&\quad-\sum_{N=1}^\infty\frac{1}{(N-1)!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\ldots A^n(p_{N-1})}{(x_1+i\epsilon)\ldots(x_{N-1}+i\epsilon)}\,A^n(p_N)\,\psi\left(p-\sum_{i=1}^N p_i\right).\\&=x\,\underbrace{U^{[n]}_-\psi}(p)-\underbrace{U^{[n]}_-A^n\psi}(p).\end{aligned}$$
Using Eq. 1.83 one can of course also obtain this result. Thus one finds
$$\underbrace{U^{[n]}_-iD^n\psi}(p)=i\partial^n\,\underbrace{U^{[n]}_-\psi}(p)\tag{1.105}$$
For a transverse derivative one obtains
$$\begin{aligned}\underbrace{U^{[n]}_-i\partial^\alpha_T\psi}(p)&=p^\alpha_T\,\underbrace{U^{[n]}_-\psi}(p)-\sum_{N=1}^\infty\sum_{i=1}^N\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\\&\qquad\frac{A^n(p_1)\ldots p^\alpha_{iT}A^n(p_i)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_i+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^N p_i\right).\end{aligned}$$

Using

$$\begin{aligned}
[A^n(p_1)\ldots A^n(p_N), A_T^\alpha(p_0)] &= A^n(p_1)\ldots A^n(p_N)\,A_T^\alpha(p_0) - A_T^\alpha(p_0)\,A^n(p_1)\ldots A^n(p_N) \\
&= \sum_{i=1}^{N} A^n(p_1)\ldots A^n(p_{i-1})\,[A^n(p_i), A_T^\alpha(p_0)]\,A^n(p_{i+1})\ldots A^n(p_N),
\end{aligned}$$

we get

$$\begin{aligned}
&\underline{U_-^{[n]}\,A_T^\alpha}\,\psi(p) - \underline{A_T^\alpha\,U_-^{[n]}}\,\psi(p) \\
&= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{d^4p_0}{(2\pi)^4}\left[\frac{A^n(p_1)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_N+i\epsilon)}, A_T^\alpha(p_0)\right]\psi\left(p-p_0-\sum_{i=1}^{N}p_i\right) \\
&= \sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{d^4p_0}{(2\pi)^4}\,\frac{A^n(p_1)\ldots[A^n(p_i),A_T^\alpha(p_0)]\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_i+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-p_0-\sum_{i=1}^{N}p_i\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
p_T^\alpha\,\underline{U_-^{[n]}}\psi(p) &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,p_T^\alpha\,\frac{A^n(p_1)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right) \\
&= \sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_{i-1}}{(2\pi)^4}\,\frac{d^4p_{i+1}}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4} \\
&\qquad \frac{A^n(p_1)\ldots A^n(p_{i-1})}{(x_1+i\epsilon)\ldots(x_{i-1}+i\epsilon)}\,p_T^\alpha\,\frac{A^n(p_{i+1})\ldots A^n(p_N)}{(x_{i+1}+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right),
\end{aligned}$$

and

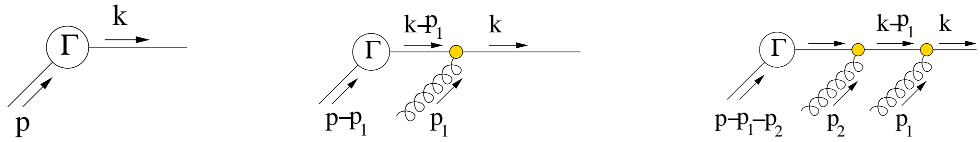
$$\begin{aligned}
&\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\ldots A^n(p_{i-1})}{(x_1+i\epsilon)\ldots(x_{i-1}+i\epsilon)}\,A_T^\alpha(p_i)\,\frac{A^n(p_{i+1})\ldots A^n(p_N)}{(x_{i+1}+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right) \\
&= \sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\ldots x_i\,A_T^\alpha(p_i)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_i+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right).
\end{aligned}$$

Combining the results,

$$\begin{aligned}
&\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\ldots iG^{n\alpha}(p_i)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_i+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right) \\
&= \sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\ldots A^n(p_{i-1})}{(x_1+i\epsilon)\ldots(x_{i-1}+i\epsilon)}\,A_T^\alpha(p_i)\,\frac{A^n(p_{i+1})\ldots A^n(p_N)}{(x_{i+1}+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right) \\
&+ \underline{U_-^{[n]}\,A_T^\alpha}\,\psi(p) - \underline{A_T^\alpha\,U_-^{[n]}}\,\psi(p) + \underline{U_-^{[n]}i\partial_T^\alpha}\psi(p) - p_T^\alpha\,\underline{U_-^{[n]}}\psi(p). \\
&= \sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\ldots A^n(p_{i-1})}{(x_1+i\epsilon)\ldots(x_{i-1}+i\epsilon)}\,A_T^\alpha(p_i)\,\frac{A^n(p_{i+1})\ldots A^n(p_N)}{(x_{i+1}+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right) \\
&+ \underline{U_-^{[n]}iD_T{}^\alpha}\psi(p) - \underline{iD_T^\alpha\,U_-^{[n]}}\psi(p).
\end{aligned}$$

Appendix B: Calculation of gauge link

We want to consider a hard proces and see which gluon correlators need to be resummed to get a gauge-invariant correlator including a gauge link. We first look at the insertions onto *one* fermion line. At this stage nonabelian effects do not matter. Working with momentum representation of the fields, we consider of the $A^\mu(p)$ insertion only the collinear term, to be precise $A^n(p)p^\mu/(p.n)$. Furthermore, first consider the situation that the hard process is a simple (constant) vertex (like a γ^α in deep inelastic scattering). Thus, we are going to resum the diagrams



The first term gives as relevant contribution

$$A_0 = \overline{\psi}(k)\Gamma\psi(p),$$

where the $\psi(p)$ and $\overline{\psi}(k)$ are fields belonging to the correlators of initial (momentum p) and final state quark (momentum k) respectively. The first gauge link contribution is

$$A_1 = \int \frac{d^4p_1}{(2\pi)^4}\,\overline{\psi}(k)\,\frac{-i\,\not{p}_1\,A^k(p_1)}{p_1\cdot k}\,\frac{i(\not{k}-\not{p}_1)}{(-2k\cdot p_1+i\epsilon)}\,\Gamma\,\psi(p-p_1).$$

The numerator becomes $\not{p}_1(\not{k}-\not{p}_1) = \not{p}_1\not{k} = \{\not{k},\not{p}_1\} = 2\,k\cdot p_1$. The added term is zero since $\overline{\psi}(k)\not{k}\approx 0$. Thus one has (note that the sign of $k\cdot p_1$ is plus),

$$A_1 \;=\; \overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{(-k\cdot p_1+i\epsilon)}\,\Gamma\,\psi(p-p_1) = \overline{\psi}(k)\,\underbrace{U_+^{[k](1)}\,\Gamma\psi}(p) \tag{1.106}$$

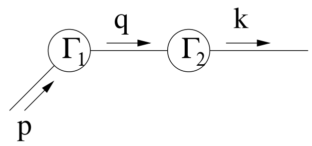
The 2-gluon term becomes

$$\begin{aligned}
A_2 &= \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\;\overline{\psi}(k)\,\frac{A^k(p_1)}{p_1\cdot k}\,\frac{A^k(p_2)}{p_2\cdot k}\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\not{p}_2\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2}\,\Gamma\,\psi(p-p_1-p_2)\\
&= \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\;\overline{\psi}(k)\,\frac{A^k(p_1)}{p_1\cdot k}\,\frac{A^k(p_2)}{p_2\cdot k}\,\frac{\not{p}_1\,(\not{k}-\not{p}_1)}{(-2k\cdot p_1+i\epsilon)}\,\frac{\not{p}_2\,(\not{k}-\not{p}_1-\not{p}_2)}{(-2k\cdot p_1-2k\cdot p_2+i\epsilon)}\,\Gamma\,\psi(p-p_1-p_2)\\
&= \overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,\frac{A^k(p_1)}{(-k\cdot p_1+i\epsilon)}\,\frac{A^k(p_2)}{(-k\cdot p_1-k\cdot p_2+i\epsilon)}\,\Gamma\,\psi(p-p_1-p_2)\\
&= \overline{\psi}(k)\,\underbrace{U_+^{[k](2)}\,\Gamma\psi}(p),
\end{aligned} \tag{1.107}$$

etc. and the total result becoming

$$\sum_{N=0}^{\infty} A_N = \overline{\psi}(k)\,\underbrace{U_+^{[k]}\,\Gamma\psi}(p) = \overline{\psi}(k)\,U_+^{a[k]}(p)\,\Gamma\psi(p). = \overline{\psi}(k)\,T_a\,\Gamma\,U_+^{a[k]}\psi(p). \tag{1.108}$$

The next situation to be investigated is the momentum dependence if there is momentum dependence in the basic diagram,



$$A_0 = i\,\overline{\psi}(k)\,\Gamma_2\frac{1}{\not{q}}\,\Gamma_1\,\psi(p). \tag{1.109}$$

The two one-gluon insertions are

$$\begin{aligned}
A_1 &= i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1+i\epsilon}\,\Gamma_2\,\frac{1}{\not{q}-\not{p}_1}\,\Gamma_1\,\psi(p-p_1)\\
&\quad + i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\Gamma_2\,\frac{A^k(p_1)}{k\cdot p_1}\,\frac{1}{\not{q}}\,\not{p}_1\frac{1}{\not{q}-\not{p}_1}\,\Gamma_1\,\psi(p-p_1)\\
&= i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1+i\epsilon}\,\Gamma_2\,\frac{1}{\not{q}}\,\Gamma_1\,\psi(p-p_1)\\
&\quad + i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1+i\epsilon}\,\Gamma_2\,\left[\frac{1}{\not{q}-\not{p}_1}-\frac{1}{\not{q}}\right]\,\Gamma_1\,\psi(p-p_1)\\
&\quad + i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\Gamma_2\,\frac{A^k(p_1)}{k\cdot p_1}\,\left[\frac{1}{\not{q}-\not{p}_1}-\frac{1}{\not{q}}\right]\,\Gamma_1\,\psi(p-p_1).
\end{aligned}$$

As before we have

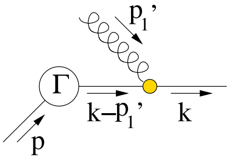
$$\not{p}_1 \frac{1}{\not{k} - \not{p}_1 + i\epsilon} \approx \frac{2k \cdot p_1}{-2k \cdot p_1 + i\epsilon},$$

which actually becomes equal to -1 if it is multiplied by a function that vanishes when $p_1 \to 0$, so the second and third line in the above expression cancel and provided[6] $[A^k(p_1), \Gamma_2] = 0$, we are left with

$$A_1 = i\,\overline{\psi}(k) \int \frac{d^4p_1}{(2\pi)^4} \frac{A^k(p_1)}{-k \cdot p_1 + i\epsilon}\,\Gamma_2\,\frac{1}{\not{\partial}}\,\Gamma_1\,\psi(p - p_1) = i\,\overline{\psi}(k)\,\underbrace{U_+^{[k](1)}\Gamma_2\frac{1}{\not{\partial}}}\,\Gamma_1\psi(p). \qquad (1.110)$$

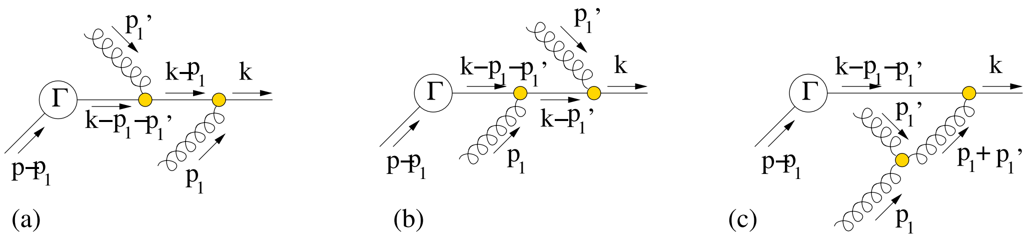
There are three two-gluon insertions, yielding the second order term in the link, etc.

Going back, we consider the insertion on a line coming from a different correlator in the basic diagram with just one vertex Γ. Starting with $A_0 = [\overline{\psi}(k)\,\Gamma\,\psi(p)] \ldots \psi(p')$, we find



$$\begin{aligned} A_1 &= \left[\overline{\psi}(k) \int \frac{d^4p_1'}{(2\pi)^4} \frac{A^k(p_1')}{k \cdot p_1'}\,\not{p}_1' \frac{1}{\not{k} - \not{p}_1' + i\epsilon}\,\Gamma\,\psi(p)\right] \ldots \psi(p' - p_1'), \\ &= \left[\overline{\psi}(k) \int \frac{d^4p_1'}{(2\pi)^4} \frac{A^k(p_1')}{-k \cdot p_1' + i\epsilon}\,\Gamma\,\psi(p)\right] \ldots \psi(p' - p_1'), \\ &= \left[\overline{\psi}(k)\,\underbrace{U_+^{[k](1)}\,\Gamma\,\psi(p)\right] \ldots \psi}(p') = \left[\overline{\psi}(k)\,U_+^{[k](1)}(p)\,\Gamma\,\psi(p)\right] \ldots \psi(p'). \end{aligned} \qquad (1.111)$$

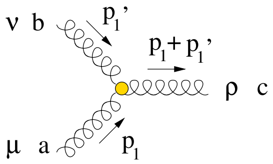
Next we consider the situation of two insertions onto a final state leg with momentum k coming from different correlators, for which there are two (or in general for a nonabelian case, three) contributions,



For the third diagram (relevant in the non-abelian case) we use the propagator

$$D^{\rho\sigma}(k) = \frac{i\,g_s^{\rho\sigma}(k)}{k^2 + i\,\epsilon},$$

where $g_s^{\mu\nu}(k) = -g^{\mu\nu} + k^\mu\,k^\nu/k^2$. Furthermore, we need the QCD vertex



$$\begin{aligned} &V_{abc}^{\mu\nu\rho}(p_1, p_1', p_1 + p_1') \\ &= i\,f_{abc}\left[(p_1 - p_1')^\rho\,g^{\mu\nu} + (2p_1 + p_1')^\nu\,g^{\mu\rho} - (p_1 + 2p_1')^\mu\,g^{\nu\rho}\right]. \end{aligned}$$

The contraction with $p_{1\mu}$ yields

$$p_{1\mu}\,V_{abc}^{\mu\nu\rho}(p_1, p_1', p_1 + p_1') = i\,f_{abc}\left[(p_1 + p_1')^2\,g_s^{\nu\rho}(p_1 + p_1') - p_1'^2\,g_s^{\nu\rho}(p_1')\right],$$

Contraction with $p_{1\mu}$ and $p_{1\nu}'$ gives

$$p_{1\mu}\,p_{1\nu}'\,V_{abc}^{\mu\nu\rho}(p_1, p_1', p_1 + p_1') = \frac{i}{2}\,f_{abc}\,(p_1 + p_1')^2\,(p_1 - p_1')^\rho$$

and including the color contractions with $A_a^n(p_1)$ and $A_b^n(p_1')$ as well as the propagator for leg $p_1 + p_1'$, we get the matrix-valued result

$$p_{1\mu}\,p_{1\nu}'\,T_c\,A_a^n(p_1)\,A_b^n(p_1')V_{abc}^{\mu\nu\rho'}(p_1, p_1', p_1 + p_1')\,D^{\rho'\rho}(p_1 + p_1') = -\frac{i}{2}\,[A^n(p_1), A^n(p_1')]\,(p_1 - p_1')^\rho.$$

[6] If this commutator is not zero, there must be gluon insertions into Γ_2

For the three diagrams we then obtain

$$\begin{aligned} A_{11} &= \Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &+ \Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1'}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &- \Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{[A^k(p_1),A^k(p_1')]}{2\,(k\cdot p_1)(k\cdot p_1')}\,(\not{p}_1-\not{p}_1')\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \end{aligned}$$

Let us investigate the singularity structure. Using $x_1 \equiv k\cdot p_1$, $x_1' \equiv k\cdot p_1'$, and $\alpha_1 \equiv p_1\cdot p_1'$ one has a part (coming from diagrams (a) and (c) above) containing

$$\begin{aligned} &\frac{A^k(p_1)A^k(p_1')}{x_1\,x_1'}\,\frac{\left[-x_1\,x_1'\not{p}_1' - \frac{1}{2}\,x_1\,x_1'(\not{p}_1-\not{p}_1')\right]}{(-x_1+i\epsilon)(-x_1'+i\epsilon)}\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} \\ &\quad = \frac{1}{2}\,\frac{A^k(p_1)A^k(p_1')}{(-x_1+i\epsilon)(-x_1'+i\epsilon)}\,(-\not{p}_1-\not{p}_1')\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} = \frac{1}{2}\,\frac{A^k(p_1)A^k(p_1')}{(-x_1+i\epsilon)(-x_1'+i\epsilon)}, \end{aligned}$$

and similarly a part with the fields in opposite order. Hence the result for A_{11} can be rewritten as

$$\begin{aligned} A_{11} &= \frac{1}{2}\Big[\overline{\psi}(k)\int\frac{d^4p_1'}{(2\pi)^4}\,\underbrace{U_+^{[k](1)}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1'}}\,\Gamma\psi(p)\Big]\ldots\psi(p'-p_1') \\ &- \frac{1}{2}\Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1)}{k\cdot p_1}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1'\left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}-\frac{1}{\not{k}-\not{p}_1'}\right]\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &+ \frac{1}{2}\Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1)}{k\cdot p_1}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1\left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}-\frac{1}{\not{k}-\not{p}_1}\right]\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &+ \frac{1}{2}\Big[\overline{\psi}(k)\int\frac{d^4p_1}{(2\pi)^4}\,\underbrace{U_+^{[k](1)}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p')} \\ &- \frac{1}{2}\Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1\left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}-\frac{1}{\not{k}-\not{p}_1}\right]\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &+ \frac{1}{2}\Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1'\left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}-\frac{1}{\not{k}-\not{p}_1'}\right]\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &- \frac{1}{2}\Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{[A^k(p_1),A^k(p_1')]}{(k\cdot p_1)(k\cdot p_1')}\,(\not{p}_1-\not{p}_1')\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \end{aligned}$$

We note that in the lines 2, 3 and 5, 6 of the above equation, the prescription doesn't matter because the part multiplying the product of A-fields vanishes for $p_1 \to 0$ and $p_1' \to 0$. Thus including line 7 all terms cancel and we have

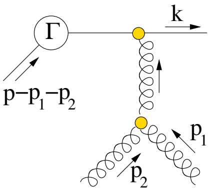
$$A_{11} = \overline{\psi}(k)\,U_+^{[k](11)}(p,p')\,\Gamma\,\psi(p)\ldots\psi(p') = \frac{1}{2}\Big[\overline{\psi}(k)\{U_+^{[k](1)},U_+^{[k](1)}\}\,\Gamma\,\psi(p)\Big]\ldots\psi(p'), \tag{1.112}$$

i.e. the gaugelinks on (fermion) leg with momentum k is the (color) symmetrized product of gaugelinks. The abelian case (without third diagram)

$$\begin{aligned} A_{11} &= \Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1)}{k\cdot p_1}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1'}\,\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &+ \Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1)}{k\cdot p_1}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,\not{p}_1'\left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}-\frac{1}{\not{k}-\not{p}_1'}\right]\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1') \\ &+ \Big[\overline{\psi}(k)\int\frac{d^4p_1\,d^4p_1'}{(2\pi)^8}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\frac{A^k(p_1)}{k\cdot p_1}\,\not{p}_1'\left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}-\frac{1}{\not{k}-\not{p}_1'}\right]\Gamma\,\psi(p-p_1)\Big]\ldots\psi(p'-p_1'), \end{aligned}$$

where the first two lines come from the first contribution. Because $[A^k(p_1),A^k(p_1')] = 0$ the result is

$$A_{11} = [\overline{\psi}(k)\,\Gamma\,U_+^{[k](1)}\psi(p)]\ldots U_+^{[k](1)}\psi(p') \tag{1.113}$$



Returning to the A_2-term of a link for $\psi(p)$ we note that the contribution of the nonabelian diagram in which two legs are attached to *one* soft part is actually already included in the soft part with one collinear gluon. Its contribution

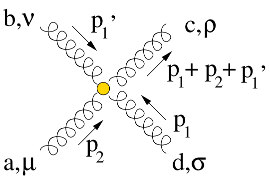
$$-\frac{1}{2}\left[\frac{A^k(p_1)}{k\cdot p_1},\frac{A^k(p_2)}{k\cdot p_2}\right](\not p_1-\not p_2)\,\frac{1}{\not k-\not p_1-\not p_2}=-\frac{1}{2}\,\frac{[A^k(p_1),A^k(p_2)]}{(-x_1+i\epsilon)(-x_2+i\epsilon)}\,\frac{x_2-x_1}{x_1+x_2},$$

is actually just the vanishing antisymmetric term discussed before (see Eq. 1.88). Without further calculation, the result for the contribution from gluons from 3 different soft parts will be

$$\begin{aligned}A_{111} &= \overline{\psi}(k)\,U_+^{[k](111)}(p,p',p'')\,\ldots\,\psi(p)\ldots\psi(p')\,\ldots\,\psi(p'')\\ &= \frac{1}{3!}\,\overline{\psi}(k)\,\left\{U_+^{[k](1)}(p),U_+^{[k](1)}(p'),U_+^{[k](1)}(p'')\right\}\,\ldots\,\psi(p)\,\ldots\,\psi(p')\,\ldots\,\psi(p''),\end{aligned}\qquad(1.114)$$

where A,B,C indicates the symmetric combination. For this we would actually also need the four-point gluon vertex.

This four-point gluon vertex also is needed in the final case we will explicitly investigate, which is two gluons collinear with momentum p and one collinear with momentum p'. We now need a number of diagrams and the four-point vertex, contracted with three momenta. The latter is given by

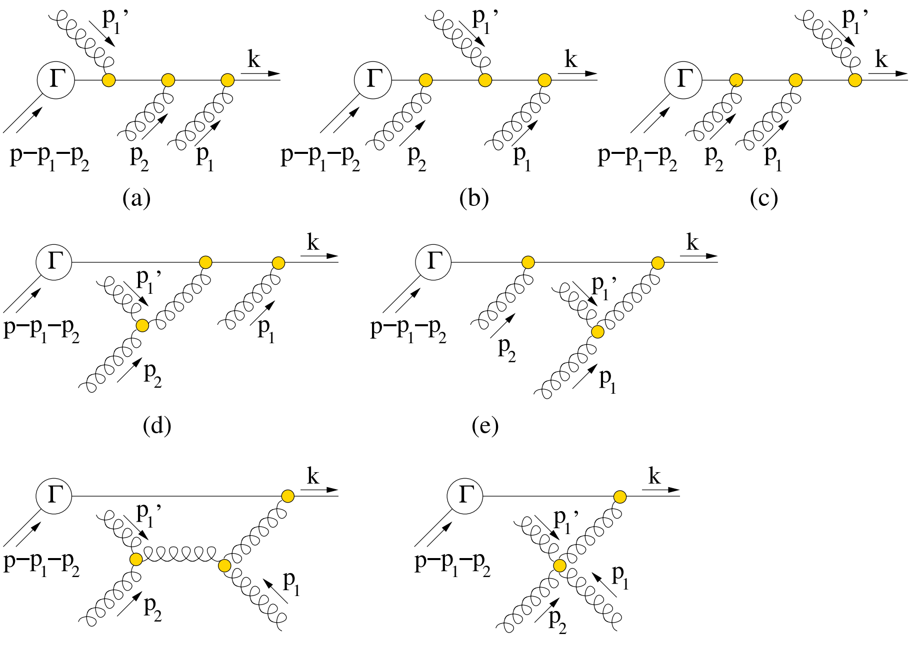


$$\begin{aligned}V^{\mu\nu\rho\sigma}_{abcd} &= -i\,g^2\big[f_{abe}\,f_{cde}\,(g^{\mu\rho}\,g^{\nu\sigma}-g^{\mu\sigma}\,g^{\nu\rho})\\ &\qquad+f_{ace}\,f_{bde}\,(g^{\mu\nu}\,g^{\rho\sigma}-g^{\mu\sigma}\,g^{\nu\rho})\\ &\qquad+f_{ade}\,f_{bce}\,(g^{\mu\nu}\,g^{\rho\sigma}-g^{\mu\rho}\,g^{\nu\sigma})\big]\\ &= -i\,g^2\big[f_{abe}\,f_{cde}\,(2\,g^{\mu\rho}\,g^{\nu\sigma}-g^{\mu\sigma}\,g^{\nu\rho}-g^{\mu\nu}\,g^{\rho\sigma})\\ &\qquad+f_{ace}\,f_{bde}\,(2\,g^{\mu\nu}\,g^{\rho\sigma}-g^{\mu\sigma}\,g^{\nu\rho}-g^{\mu\rho}\,g^{\nu\sigma})\big]\end{aligned}$$

The contraction with $p_1'^{\nu}\,A^n(p_1')$, $p_2^\mu\,A^n(p_2)$ and $p_1^\sigma\,A^n(p_1)$ yields

$$\begin{aligned}&p_1^\sigma p_2^\mu p_1'^{\nu}\,A^n_d(p_1)\,A^n_a(p_2)\,A^n_b(p_1')\,T_c\,V^{\mu\nu\rho\sigma}_{abcd} =\\ &\quad ig^2\bigg[[A^n(p_1),[A^n(p_2),A^n(p_1')]]\,\big(2p_2^\rho\,p_1\cdot p_1'-p_1^\rho\,p_2\cdot p_1'-p_1'^{\rho}\,p_2\cdot p_1\big)\\ &\qquad+[A^n(p_2),[A^n(p_1),A^n(p_1')]]\,\big(2p_1^\rho\,p_2\cdot p_1'-p_2^\rho\,p_1\cdot p_1'-p_1'^{\rho}\,p_2\cdot p_1\big)\bigg].\end{aligned}$$

The contributions to the gauge-link are the following



For the seven diagrams involving three collinear gluons, two collinear to p, one to p', we obtain the contributions

$$A_{21} = \left[\overline{\psi}(k)\int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,\frac{d^4p_1'}{(2\pi)^4}\,\ldots\,\Gamma\,\psi(p-p_1)\right]\ldots\psi(p'-p_1')$$

with three basically 'abelian' contributions,

$$\begin{aligned}(a) &= \frac{A^k(p_1)}{k\cdot p_1}\,\frac{A^k(p_2)}{k\cdot p_2}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,\not{p}_2\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}\\ (b) &= \frac{A^k(p_1)}{k\cdot p_1}\,\frac{A^k(p_1')}{k\cdot p_1'}\,\frac{A^k(p_2)}{k\cdot p_2}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}\\ (c) &= \frac{A^k(p_1')}{k\cdot p_1'}\,\frac{A^k(p_1)}{k\cdot p_1}\,\frac{A^k(p_2)}{k\cdot p_2}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1'}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}\end{aligned}$$

of which the singularity structure is kept in contributions (a_0), (b_0) and (c_0) and we get of which the singularity structure is (using $D = 1/(\not{k}-\not{p}_1-\not{p}_2-\not{p}_1')$)

$$\begin{aligned}(a) &= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{x_1\,x_2\,\not{p}_1'\,D}{(-x_1+i\epsilon)(-x_1-x_2+i\epsilon)}\\ &= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{x_2}{(x_1+x_2)}\,\not{p}_1'\,D\\ (b) &= \frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{x_1}{(-x_1+i\epsilon)}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D\\ &= \frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}(-\not{p}_1')\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D\\ &= \frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{x_1'}{(x_1+x_1'-\alpha_1)}\,\not{p}_2\,D\\ (c) &= \frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{x_1'}{(-x_1'+i\epsilon)}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D\\ &= \frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}(-\not{p}_1)\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D\\ &= \frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{(x_1-\alpha_1)}{(x_1+x_1'-\alpha_1)}\,\not{p}_2\,D\end{aligned}$$

We have here introduced $\alpha_1 = p_1'\cdot p_1$, $\alpha_2 = p_1'\cdot p_2$, and $\alpha = \alpha_1+\alpha_2$. A useful relation is $\not{p}_1\not{p}_1'\not{p}_2 = \alpha_1\,\not{p}_2 = \alpha_2\,\not{p}_1$, also implying $\alpha_1\,x_2 = \alpha_2\,x_1$ or $x_1/x_2 = \alpha_1/\alpha_2$. We note that summing these three diagrams in the abelian case simply give

$$\begin{aligned}[(a)+(b)+(c)]_{\rm abelian} &= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x_1'+i\epsilon)}\left[-\not{p}_1'-(x_1+x_2)\not{p}_2\right]\,D\\ &= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x_1'+i\epsilon)} = \frac{1}{2}\,\frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)},\end{aligned} \tag{1.115}$$

giving $A_{21} = [\overline{\psi}(k)\,\Gamma\,U_+^{[k](2)}\psi(p)]\ldots U_+^{[k](1)}\psi(p')$. In the nonabelian case, we also have two effectively 'three-gluon' contributions,

$$\begin{aligned}(d) &= -\frac{1}{2}\,\frac{A^k(p_1)}{k\cdot p_1}\left[\frac{A^k(p_2)}{k\cdot p_2},\frac{A^k(p_1')}{k\cdot p_1'}\right]\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,(\not{p}_2-\not{p}_1')\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}\\ &= +\frac{1}{2}\,\frac{A^k(p_1)\,[A^k(p_2),A^k(p_1')]}{x_1\,x_2\,x_1'}\,(\not{p}_2-\not{p}_1')\,D\\ (e) &= -\frac{1}{2}\left[\frac{A^k(p_1)}{k\cdot p_1},\frac{A^k(p_1')}{k\cdot p_1'}\right]\frac{A^k(p_2)}{k\cdot p_2}\,(\not{p}_1-\not{p}_1')\,\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}\\ &= +\frac{1}{2}\,\frac{[A^k(p_1),A^k(p_1')]\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{(x_1-x_1'-\alpha_1)}{(x_1+x_1'-\alpha_1)}\,\not{p}_2\,D\end{aligned}$$

The sum of these two gives

$$\begin{aligned}(d)+(e) &= \frac{1}{2}\frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{x_1\,x_2\,x_1'}\,(\not{p}_2-\not{p}_1')\,D\\ &+\frac{1}{2}\frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\left(\not{p}_1'-\frac{2\,x_1'}{(x_1+x_1'-\alpha_1)}\not{p}_2\right)\,D\\ &-\frac{1}{2}\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{(x_1-x_1'-\alpha_1)}{(x_1+x_1'-\alpha_1)}\,\not{p}_2\,D.\end{aligned}$$

An alternative in this calculation is to not add (d) and (e), but rather to add (e) to (b) and (c) to avoid having to deal with poles in $1/(\not{k}-\not{p}_1-\not{p}_1')$. One gets

$$\begin{aligned}(b)+(c)+(e) &= \frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\left[-\not{p}_1'-\frac{1}{2}\,(\not{p}_1-\not{p}_1')\right]\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D\\ &+\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\left[-\not{p}_1+\frac{1}{2}\,(\not{p}_1-\not{p}_1')\right]\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D\\ &= \frac{1}{2}\frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_2\,D+\frac{1}{2}\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_2\,D.\end{aligned}$$

Including the contributions of (a) and (d), we have

$$\begin{aligned}(a)+\ldots+(e) &= \frac{1}{2}\frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{x_1\,x_2\,x_1'}\left[\not{p}_2\,D+\frac{x_2-x_1}{x_1+x_2}\,\not{p}_1'\,D\right]\\ &+\frac{1}{2}\frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_1'\,D+\frac{1}{2}\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_2\,D.\end{aligned}$$

Finally, there are two types of 'four-gluon' contributions,

$$\begin{aligned}(f) &= \frac{1}{2}\left[\frac{A^k(p_1)}{k\cdot p_1},\left[\frac{A^k(p_2)}{k\cdot p_2},\frac{A^k(p_1')}{k\cdot p_1'}\right]\right]\,(p_2-p_1')_\lambda\\ &\times\left[(p_1+p_2+p_1')^2\,g_s^{\lambda\tau}(p_1+p_2+p_1')-(p_2+p_1')^2\,g_s^{\lambda\tau}(p_2+p_1')\right]\\ &\times\,iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}\\ &= -\left[\frac{A^k(p_1)}{k\cdot p_1},\left[\frac{A^k(p_2)}{k\cdot p_2},\frac{A^k(p_1')}{k\cdot p_1'}\right]\right]\,(\alpha_1-\alpha_2)\,(p_2-p_1')^\tau\\ &\times\,iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'},\end{aligned}$$

$$\begin{aligned}(g) &= -\left[\frac{A^k(p_1)}{k\cdot p_1},\left[\frac{A^k(p_2)}{k\cdot p_2},\frac{A^k(p_1')}{k\cdot p_1'}\right]\right](2\,\alpha_1\,p_2^\tau-\alpha_2\,p_1^\tau)\\ &\times\,iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}\\ &-\left[\frac{A^k(p_2)}{k\cdot p_2},\left[\frac{A^k(p_1)}{k\cdot p_1},\frac{A^k(p_1')}{k\cdot p_1'}\right]\right](2\,\alpha_2\,p_1^\tau-\alpha_1\,p_2^\tau)\\ &\times\,iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}.\end{aligned}$$

Using the contractions

$$\begin{aligned}-\alpha\,p_2^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \alpha\,\not{p}_2-\tfrac{1}{2}\,\alpha_2\,(\not{p}_1+\not{p}_2+\not{p}_1')\\ &= \alpha_1\,\not{p}_2-\alpha_2\,\not{p}_1+\tfrac{1}{2}\,\alpha_2\,(\not{p}_1+\not{p}_2-\not{p}_1')=\tfrac{1}{2}\,\alpha_2\,(\not{p}_1+\not{p}_2-\not{p}_1')\\ -\alpha\,p_1^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \alpha_2\,\not{p}_1-\alpha_1\,\not{p}_2+\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1')=\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1')\\ -\alpha\,p_1'^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \tfrac{1}{2}(\not{p}_1+\not{p}_2-\not{p}_1'),\\ -\alpha(p_2-p_1')^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \alpha_1\,\not{p}_2-\alpha_2\,\not{p}_1-\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1'),=-\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1'),\end{aligned}$$

realizing that with p_1 and p_2 being integration variables,

$$
\begin{aligned}
[A^k(p_1),[A^k(p_2),A^k(p_1')]]\,F(p_1,p_2) &= A^k(p_1)A^k(p_2)A^k(p_1')\,F(p_1,p_2)+A^k(p_1')A^k(p_1)A^k(p_2)\,F(p_2,p_1)\\
&\quad -A^k(p_1)A^k(p_1')A^k(p_2)\,\left(F(p_1,p_2)+F(p_2,p_1)\right),\\
[A^k(p_2),[A^k(p_1),A^k(p_1')]]\,F(p_1,p_2) &= A^k(p_1)A^k(p_2)A^k(p_1')\,F(p_2,p_1)+A^k(p_1')A^k(p_1)A^k(p_2)\,F(p_1,p_2)\\
&\quad -A^k(p_1)A^k(p_1')A^k(p_2)\,\left(F(p_1,p_2)+F(p_2,p_1)\right),
\end{aligned}
$$

and using

$$
C=(\not{p}_1+\not{p}_2-\not{p}_1')\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}=\frac{(x_1+x_2-x_1'+[\not{p}_1',\not{p}_1]+[\not{p}_1',\not{p}_2]}{(-x_1-x_2-x_1'+\alpha)},
$$

we can rewrite the last two contributions as

$$
\begin{aligned}
(f) &= -\frac{1}{4}\,\frac{[A^k(p_1),[A^k(p_2),A^k(p_1')]]}{x_1\,x_2\,x_1'}\,\frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha^2}\,C\\
&= -\frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha^2}\,C\\
&\quad +\frac{1}{4}\,\frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{(\alpha_1-\alpha_2)^2}{\alpha^2}\,C\\
&\quad -\frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{\alpha_2(\alpha_2-\alpha_1)}{\alpha^2}\,C\\
(g) &= -\frac{1}{4}\,\frac{[A^k(p_1),[A^k(p_2),A^k(p_1')]]}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C-\frac{1}{4}\,\frac{[A^k(p_2),[A^k(p_1),A^k(p_1')]]}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C\\
&= -\frac{1}{2}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C\\
&\quad +\frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C\\
&\quad -\frac{1}{2}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C
\end{aligned}
$$

Adding the contributions and using $x_1/x_2\approx\alpha_1/\alpha_2$ we get

$$
\begin{aligned}
(f)+(g) &= -\frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{x_1}{(x_1+x_2)}\,(\not{p}_1+\not{p}_2-\not{p}_1')\,D\\
&\quad +\frac{1}{4}\,\frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'}\,(\not{p}_1+\not{p}_2-\not{p}_1')\,D\\
&\quad -\frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{x_2}{(x_1+x_2)}\,(\not{p}_1+\not{p}_2-\not{p}_1')\,D.
\end{aligned}
$$

The full result becomes

$$
\begin{aligned}
(a)+\ldots+(g) &= -\frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\left[\frac{x_2}{(x_1+x_2)}+\frac{x_2-x_1}{x_1+x_2}\right]\\
&\quad -\frac{1}{4}\,\frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'}\\
&\quad -\frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{x_2}{(x_1+x_2)}\\
&= \frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x_1'+i\epsilon)}\\
&\quad \frac{1}{4}\,\frac{A^k(p_1)\,A^k(p_1'),A^k(p_2)}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)}\\
&\quad \frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x_1'+i\epsilon)}\\
&= \frac{1}{8}\,\frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)}\\
&\quad \frac{1}{4}\,\frac{A^k(p_1)\,A^k(p_1'),A^k(p_2)}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)}\\
&\quad \frac{1}{8}\,\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)}.
\end{aligned}
$$

So we get

$$
A_{21} = [\overline{\psi}(k)\,U_+^{[k](21)}(p,p')\,\Gamma\ldots\psi(p)]\ldots\psi(p') \tag{1.116}
$$

$$
\begin{aligned}
&= [\overline{\psi}(k)\Bigg[\frac{1}{4}\,U_+^{[k](2)}(p)\,U_+^{[k](1)}(p')+\frac{1}{4}\,U_+^{[k](1)}(p)\,U_+^{[k](1)}(p')\,U_+^{[k](1)}(p)\\
&\qquad +\frac{1}{4}\,U_+^{[k](1)}(p')\,U_+^{[k](2)}(p)\Bigg]\Gamma\ldots\psi(p)]\ldots\psi(p')
\end{aligned} \tag{1.117}
$$

$$
\begin{aligned}
&= [\overline{\psi}(k)\Bigg[\frac{1}{8}\,U_+^{[k](1)}(p)\,U_+^{[k](1)}(p)\,U_+^{[k](1)}(p')+\frac{1}{4}\,U_+^{[k](1)}(p)\,U_+^{[k](1)}(p')\,U_+^{[k](1)}(p)\\
&\qquad +\frac{1}{8}\,U_+^{[k](1)}(p')\,U_+^{[k](1)}(p)\,U_+^{[k](1)}(p)\Bigg]\Gamma\ldots\psi(p)]\ldots\psi(p')
\end{aligned} \tag{1.118}
$$

This reproduces the correct (expected) results in the abelian case, and in the limit that $U^{[k](1)}(p')$ is replaced by a $F^{\alpha k}(p_1')$, or that $U^{[k](1)}(p)$ is replaced by a $F^{\alpha k}(p)$.

1.7 Gauge links in the Drell-Yan process

In this section we discuss the application of the general procedure (outlined in the paper of Bomhof, Mulders and Pijlman, EPJ C47 (2006), 417) to the Drell-Yan process. Figure 4 in that paper then reduces to the study of gluon insertions from one of the hadrons (momentum P) into a hard process, illustrated in Fig. 1.2

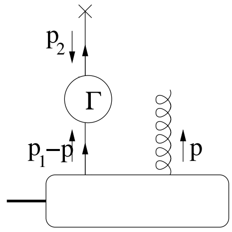


Figure 1.2: The quark-gluon correlator $\Phi_A(P;p_1,p)$ that needs to be combined with the hard part of the Drell-Yan process. We will consider DY-like processes, allowing the hard part Γ to contain any hard process (not only lepton pair production).

For the gauge link, one needs in particular the quark gluon propagator containing an A^+ gluon, which gives a leading contribution in the expansion in inverse hard scale. The leading contribution is proportional to P^+. In particular we thus look at the part $\Phi^n(P;p_1,p)\,P^+$. We need to calculate the contributions in Fig. 1.3.

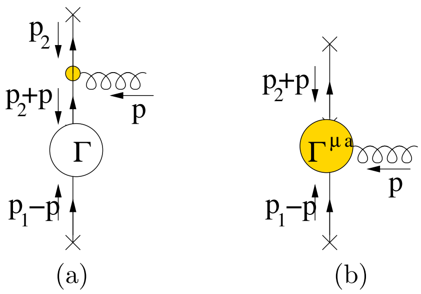


Figure 1.3: The two leading quark-gluon contributions. The first one (a) is the one that gives the familiar piece along the n_- direction, in this case running from minus infinity to the position of the quark field. If color plays a role in the hard part Γ the part (b) needs to be included. It contains all insertions into the truncated vertex Γ, starting with for one gluon $\Gamma^{\mu a}(p_1-p,p_2,p)$.

The contribution in Fig. 1.3(a) gives

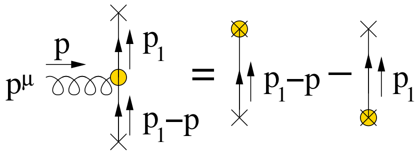
$$\begin{aligned}\text{Fig. 1.3(a)} \quad \Longrightarrow \quad & (-i\not{P}\,T^a)\frac{i}{\not{p}_2+\not{p}}\Gamma(p_1-p,p_2+p) = \frac{\not{P}(\not{p}_2+\not{p})}{2\,p_2{\cdot}p+i\epsilon}T^a\Gamma(p_1-p,p_2+p)\\ = \quad & \frac{T^a}{x+i\epsilon}\,\Gamma(p_1-p,p_2+p),\end{aligned} \tag{1.119}$$

where $x = p\cdot p_2/P\cdot p_2$ (thus $p = xP+\ldots$). Note that a vanishing contribution proportional to $\not{p}_2\not{P}$ needs to be added. The prescription $+i\epsilon$ originates from the hard propagator with momentum p_2+p. The result can be rewritten as the well-known eikonal contribution and a second term,

$$\text{Fig. 1.3(a)} \Longrightarrow \frac{T^a}{x+i\epsilon}\,\Gamma(p_1,p_2) + \frac{T^a}{x}\left(\Gamma(p_1-p,p_2+p) - \Gamma(p_1,p_2)\right). \tag{1.120}$$

In the second term the prescription no longer matters, since the numerator goes to zero for $x\to 0$ In the situation that there is no color involved in Γ and the vertex does not depend on p (like for a $q\bar{q}\to\gamma^*$ vertex), we are ready with the first gluon. Let us allow complications, because those will start playing a role with intertwined gluons from both hadrons or in situations of more complex vertices, such as quark-antiquark or gluon-gluon production.

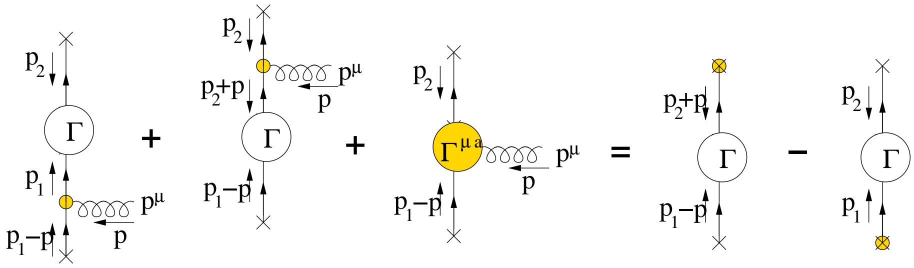
To calculate the internal insertions we use Ward identities. A simple one, which we will actually need is the insertion into the propagator,



It reads

$$\frac{i}{\not{p}_1}(-i\not{p}\,T^a)\frac{i}{\not{p}_1-\not{p}} = T^a\frac{i}{\not{p}_1-\not{p}} - \frac{i}{\not{p}_1}T^a. \tag{1.121}$$

Including the vertex and external propagators, the Ward identity that we need for the internal insertions is the following one,



It reads

$$\frac{-i}{\not{p}_2}\Gamma(p_1,p_2)\frac{i}{\not{p}_1}(-i\not{p}T^a)\frac{i}{\not{p}_1-\not{p}} + \frac{-i}{\not{p}_2}(-i\not{p}T^a)\frac{-i}{\not{p}_2+\not{p}}\Gamma(p_1-p,p_2+p)\frac{i}{\not{p}_1-\not{p}} + \frac{-i}{\not{p}_2}(-p_\mu\Gamma^{\mu a})\frac{i}{\not{p}_1-\not{p}}$$
$$= \frac{-i}{\not{p}_2+\not{p}}T^a\Gamma(p_1-p,p_2+p)\frac{i}{\not{p}_1-\not{p}} - \frac{-i}{\not{p}_2}\Gamma(p_1,p_2)T^a\frac{i}{\not{p}_1}.$$

Using

$$\frac{i}{\not{p}_1}(-i\not{p})\frac{i}{\not{p}_1-\not{p}} = \frac{i}{\not{p}_1-\not{p}} - \frac{i}{\not{p}_1} \quad \text{and} \quad \frac{-i}{\not{p}_2}(-i\not{p})\frac{-i}{\not{p}_2+\not{p}} = \frac{-i}{\not{p}_2+\not{p}} - \frac{-i}{\not{p}_2},$$

we get for the truncated amplitude $\Gamma^{\mu a}$ the Ward identity

$$p_\mu\Gamma^{\mu a} = -T^a\Gamma(p_1-p,p_2+p) + \Gamma(p_1,p_2)\,T^a. \tag{1.122}$$

Collecting now the pieces for Fig. 1.3 we get for the combined result (using $P = p/x$),

$$\begin{aligned}
\text{Fig. 1.3} \quad \Longrightarrow \quad & \frac{T^a}{x+i\epsilon}\,\Gamma(p_1,p_2) + \frac{T^a}{x+i\epsilon}\,(\Gamma(p_1-p,p_2+p) - \Gamma(p_1,p_2)) + \frac{p_\mu\Gamma^{\mu a}}{x} \\
= \quad & \frac{T^a}{x+i\epsilon}\,\Gamma(p_1,p_2) + \frac{T^a}{x}\,(\Gamma(p_1-p,p_2+p) - \Gamma(p_1,p_2)) - \frac{T^a}{x}\Gamma(p_1-p,p_2+p) + \Gamma(p_1,p_2)\frac{T^a}{x} \\
= \quad & \frac{T^a}{x+i\epsilon}\,\Gamma(p_1,p_2) + \frac{1}{x}\,(\Gamma(p_1,p_2)T^a - T^a\Gamma(p_1,p_2)) \\
= \quad & \frac{T^a}{x+i\epsilon}\,\Gamma(p_1,p_2).
\end{aligned} \tag{1.123}$$

The vanishing of the second term in the one but last line is in essence the Ward identity in Eq. 1.122 for $p = 0$ (equivalent to color charge conservation). Note, however, that we could as well have simply written

$$\begin{aligned}
\text{Fig. 1.3} \quad \Longrightarrow \quad & \frac{T^a}{x+i\epsilon}\Gamma(p_1-p,p_2+p) + \frac{p_\mu\Gamma^{\mu a}}{x+i\epsilon} \\
= \quad & \frac{T^a}{x+i\epsilon}\Gamma(p_1-p,p_2+p) - \frac{T^a}{x+i\epsilon}\Gamma(p_1-p,p_2+p) + \Gamma(p_1,p_2)\frac{T^a}{x+i\epsilon} \\
= \quad & \Gamma(p_1,p_2)\frac{T^a}{x+i\epsilon},
\end{aligned} \tag{1.124}$$

Conclusion of this exercise is that the color charge can either remain on the incoming antiquark side of the vertex Γ or it can be moved to the incoming quark side of the vertex, which looks most attractive since it then can be absorbed into Φ as the first order contribution to the gauge link.

Note, however, that in the case of transverse momentum dependence one also needs the transverse pieces in the gauge link. They originate when higher twist contributions involving $A_T^\alpha(p)$ fields in the correlator are included. First the $\overline{\psi}(0)\,A_T^\alpha(\eta)\psi(\xi)$ operators need to be combined with gauge links along the n-direction, then they need to be rewritten into a $G^{n\alpha}$ and $\overline{\psi}(0)\,A_T^\alpha(\pm\infty)\psi(\xi)$ matrix elements. The latter give the transverse gauge pieces of the gauge link. These higher twist contributions have their own color factors, which are evaluated by considering the color structure of the diagram with the color charges at the places where the transverse gluons couple. In order to properly recover all factors when one has different type of color flows in a particular process, one should therefore keep for staple-shaped gauge links all the color charges on the appropriate legs, i.e. for the gauge link in correlator 1 on the antiquark leg from correlator 2 and for the gauge link of correlator 2 on the quark leg of correlator 1.

Appendix B: Spin vectors

For the hadron spin vector satisfying $S^2 = -1$ and $P \cdot S = 0$, we can write

$$S = S_L \frac{P}{M} + S_T - S_L\, M\, n \;\approx\; S_L \frac{P}{M} + S_T, \tag{1.125}$$

satisfying $S_L^2 + \boldsymbol{S}_T^2 = 1$. The quantity S_L is the light-cone helicity, S_T the transverse spin vector. One has

$$S_L = M\, S \cdot n \qquad \text{and} \qquad S_T^\mu = g_T^{\mu\nu} S_\nu. \tag{1.126}$$

The spin vector is used in the parametrization of the density matrix. Often, therefore, the transverse spin will be defined with respect to a hadron momentum P' using

$$g_\perp^{\mu\nu} = g^{\mu\nu} - \frac{P'^{\{\mu} P^{\nu\}}}{P \cdot P'}. \tag{1.127}$$

If the hadron momentum P' is hard with respect to the original hadron, i.e. $P \cdot P'$ large, one still has a useful expansion in which P' has the role as an (approximate) null-vector. One has (first expression exact)

$$S = S_{||} \frac{P}{M\sqrt{1-\delta}} + S_\perp - S_{||}\, M \frac{P'}{(P \cdot P')\sqrt{1-\delta}} \;\approx\; S_{||} \frac{P}{M} + S_\perp, \tag{1.128}$$

with $\delta = M^2 M'^2/(P \cdot P')^2$, which satisfies $S_{||}^2 + \boldsymbol{S}_\perp^2 = 1$ and where

$$S_{||} = \frac{1}{\sqrt{1-\delta}} \frac{M\, S \cdot P'}{P \cdot P'} \;\approx\; \frac{M\, S \cdot P'}{P \cdot P'} \qquad \text{and} \qquad S_\perp = g_\perp^{\mu\nu} S_\nu. \tag{1.129}$$

Comparing both expansions order by order in $(1/Q)$, we find

$$S_{||} - S_L = \frac{M\, S \cdot P'}{P \cdot P'} - M\, S \cdot n = \frac{M\, S_T \cdot P'}{P \cdot P'} = -M\, S_\perp \cdot n,$$

and one finds using that $S \approx S_L \frac{P}{M} + S_T \approx S_{||} \frac{P}{M} + S_\perp$ that

$$S_L \;\approx\; S_{||} \;\approx\; M\, S \cdot n \;\approx\; \frac{M\, S \cdot P'}{P \cdot P'}\;, \tag{1.130}$$

$$S_T \;\approx\; S_\perp - (S_\perp \cdot n)\, P \;\approx\; S_{\perp T}\;, \tag{1.131}$$

$$S_\perp \;\approx\; S_T - \frac{S_T \cdot P'}{P \cdot P'}\, P \;\approx\; S_{T\perp}\;. \tag{1.132}$$

Using

$$p \cdot S = p_T \cdot S_T + \frac{p \cdot P - xM^2}{M}\, S_L = p_{T'} \cdot S_{T'} + \frac{p \cdot P - x' M^2}{M}\, S_{L'}, \tag{1.133}$$

and the fact that the differences $x - x'$ and $S_L - S_{L'}$ are $\mathcal{O}(1/Q)$ we have

$$p_T \cdot S_T \approx p_{T'} \cdot S_{T'}, \tag{1.134}$$

which also holds if we use an approximate $n' \approx P'/P \cdot P'$.

Appendix C: polarization sums for gluons

A useful feature is the fact that the polarization sum for on-shell gluons satisfying $v \cdot A = 0$ is approximately equal to the sum for gluons satisfying $n \cdot A = 0$, at least if n is constructed from v and P as in Eq. 1.47. This is important because at some point on-shell (cut) gluons in the hard part need to be considered to study the large transverse momentum dependence of correlators. In the section on 'Moderate transverse momenta' we have shown the kinematics for the branching parton(p_0) $\to$ parton(p) + parton(l) (with $p = p_0 - l$) in case of the emission of an on-shell parton with momentum l.

Looking at the polarization sum for the cases that either l or p are gluons, we need the products of these vectors with the gauge vector (assume $P\cdot v = 1$),

$$\begin{aligned} l\cdot v &= \frac{x}{x_p}\left[1 - x_p + \eta\,\frac{x_p^2}{1-x_p}\right], \\ p\cdot v &= x\left[1 - \eta\,\frac{x_p}{1-x_p}\right] \approx x. \end{aligned}$$

with

$$\eta = -v^2\,\frac{p_T^2}{4x^2} = v^2\,\frac{|p_T^2|}{4x^2},$$

serving as a small regulator. We note that in $p\cdot v$ the η-term is not harmful (no extra poles),

$$\eta\,\frac{x_p}{1-x_p} = -\frac{x_p}{1-x_p}\,\frac{v^2\,p_T^2}{4\,x^2} \approx -\frac{v^2\,(p\cdot P)}{2x} \approx -\frac{v^2\,(p\cdot P)\,(P\cdot n)}{2\,(p\cdot n)\,(P\cdot v)^2} \sim \mathcal{O}\left(\frac{M^2}{Q^2}\right)$$

(in last step we wrote the expression for the cases that the lengths of n and v are not fixed).

We can write for the polarization sum

$$d^{\mu\nu}(l;v) = -g^{\mu\nu} + \frac{l^\mu\,v^\nu + l^\nu\,v^\mu}{l\cdot v} - \frac{v^2\,l^\mu\,l^\nu}{(l\cdot v)^2} = \sum_i d^{\mu\nu}_{(i)}(l;v) \tag{1.135}$$

with

$$\begin{aligned} d^{\mu\nu}_{(1)}(l;v) &= \frac{1-x_p}{\left(1-x_p+\eta\,\frac{x_p^2}{1-x_p}\right)}\,d^{\mu\nu}(l;n), && (1.136)\\ d^{\mu\nu}_{(2)}(l;v) &= \eta\,\frac{x_p^2}{1-x_p}\,\frac{-g^{\mu\nu}}{\left(1-x_p+\eta\,\frac{x_p^2}{1-x_p}\right)} && (1.137)\\ d^{\mu\nu}_{(3)}(l;v) &= -\eta\,\frac{4\,x_p^2}{|p_T^2|}\,\frac{l^\mu\,l^\nu}{\left(1-x_p+\eta\,\frac{x_p^2}{1-x_p}\right)^2} && (1.138)\\ d^{\mu\nu}_{(4)}(l;v) &= \eta\,\frac{2\,x\,x_p}{|p_T^2|}\,\frac{l^\mu\,P^\nu + l^\nu\,P^\mu}{\left(1-x_p+\eta\,\frac{x_p^2}{1-x_p}\right)}. && (1.139) \end{aligned}$$

We note that for a time-like gauge choice ($v^2 > 0$) one has $\eta > 0$ and one avoids hitting the poles. Similarly we find for the polarization sum when the parton with momentum p is a gluon the sum

$$\frac{d^{\mu\nu}(p;v)}{p^2} = \frac{1}{p^2}\left[-g^{\mu\nu} + \frac{p^\mu\,v^\nu + p^\nu\,v^\mu}{p\cdot v} - \frac{v^2\,p^\mu\,p^\nu}{(p\cdot v)^2}\right] = \frac{1}{p^2}\sum_i d^{\mu\nu}_{(i)}(p;v) \tag{1.140}$$

with

$$\begin{aligned} \frac{d^{\mu\nu}_{(1)}(p;v)}{p^2} &= \frac{1}{p^2\left(1-\eta\,\frac{x_p}{1-x_p}\right)}\,d^{\mu\nu}(p;n) \;\approx\; -\frac{(1-x_p)}{|p_T^2|}\,d^{\mu\nu}(p;n) && (1.141)\\ \frac{d^{\mu\nu}_{(2)}(p;v)}{p^2} &= -\eta\,\frac{x_p}{1-x_p}\,\frac{1}{p^2}\,\frac{-g^{\mu\nu}}{\left(1-\eta\,\frac{x_p}{1-x_p}\right)} \;\approx\; -\eta\,\frac{x_p}{|p_T^2|}\,g^{\mu\nu} && (1.142)\\ \frac{d^{\mu\nu}_{(3)}(p;v)}{p^2} &= -\eta\,\frac{4}{|p_T^2|}\,\frac{1}{p^2}\,\frac{p^\mu\,p^\nu}{\left(1-\eta\,\frac{x_p}{1-x_p}\right)^2} \;\approx\; \eta\,\frac{4(1-x_p)}{|p_T^2|^2}\,p^\mu\,p^\nu && (1.143)\\ \frac{d^{\mu\nu}_{(4)}(p;v)}{p^2} &= \eta\,\frac{2\,x}{|p_T^2|}\,\frac{1}{p^2}\,\frac{p^\mu\,P^\nu + p^\nu\,P^\mu}{\left(1-\eta\,\frac{x_p}{1-x_p}\right)} \;\approx\; -\eta\,\frac{2\,x(1-x_p)}{|p_T^2|^2}\,\left(p^\mu\,P^\nu + p^\nu\,P^\mu\right). && (1.144) \end{aligned}$$

Chapter 2

Specific processes

2.1 General structure

Starting with the annihilation process $\ell_1 + \ell_2 \to X$, we have for the contribution of the process $\ell_1 + \ell_2 \to k_1 + k_2 + \ldots$, the inclusive cross section

$$d\sigma = \frac{1}{F(\ell_1,\ell_2)} \int \prod_i d\tilde{k}_i \, H(\ell_1,\ell_1;k_i)\,\delta^4(\ell_1+\ell_2-\sum_i k_i) \tag{2.1}$$

$$= \frac{1}{F(\ell_1,\ell_2)} \, L(\ell_1,q) \otimes \int \prod_i d\tilde{k}_i \, H(q;k_i)\,\delta^4(q-\sum_i k_i) \tag{2.2}$$

Here $d\tilde{k}_i = d^3k_i/(2\pi)^3\,2E_i$ is the one-particle phase space and it is customary to split of the leptonic part in which $q = \ell_1 + \ell_2$. For the *one-hadron inclusive* process $\ell_1 + \ell_2 \to H(K) + X$ the contribution from the subprocess $\ell_1 + \ell_2 \to k + k_1 + k_2 + \ldots$ is

$$d\sigma = \frac{1}{F(\ell_1,\ell_2)} \, d\widetilde{K} \int d\widetilde{K}_X \int \prod_i d\tilde{k}_i \, \Sigma(\ell_1,\ell_2;K,K_X,k_i)\,\delta^4(\ell_1+\ell_2-K-K_X-\sum_i k_i). \tag{2.3}$$

$$= \frac{1}{F(\ell_1,\ell_2)} d\widetilde{K} \int d^4k \int d\widetilde{K}_X \; \delta^4(k-K-K_X) \times \int \prod_i d\tilde{k}_i \, H(\ell_1,\ell_2;k,k_i) \otimes \Delta(K,k,K_X)\,\delta^4(q-k-\sum_i k_i). \tag{2.4}$$

Making the Sudakov expansion for k using a light-like vector n (outlined elsewhere) we have $d^4k = dz^{-1}\,d^2k_T\,d(k\cdot K) = dz^{-1}\,d\tilde{k}, d(k\cdot K)$ with $z^{-1} = k\cdot n/K\cdot n$ and defining

$$\Delta\left(z^{-1},k_T\right) = \int d(k\cdot K) \int d\widetilde{K}_X \; \delta^4(k-K-K_X)\,\Delta(K,k,K_X), \tag{2.5}$$

$$d\hat{\sigma} = \frac{1}{F(\ell_1,\ell_2)} \, d\tilde{k} \int \prod_i d\tilde{k}_i \, H(\ell_1,\ell_2;k,k_i)\,\delta^4(q-k-\sum_i k_i). \tag{2.6}$$

we find

$$d\sigma = \frac{1}{F(\ell_1,\ell_2)} \, L(\ell_1,q) \otimes d\widetilde{K} \int dz^{-1}\,d^2k_T \times \int \prod_i d\tilde{k}_i \, H(q;k,k_i) \otimes \Delta\left(z^{-1},k_T\right)\,\delta^4(q-k-\sum_i k_i). \tag{2.7}$$

2.2 Introduction to electroweak processes

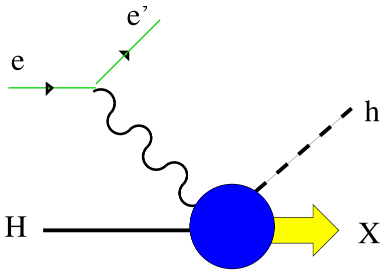
We will consider the following three types of processes,

- The *lepton-hadron* scattering process $\ell H \to \ell' X$

- The *annihilation* process $\ell\bar{\ell} \to X$
- The *lepton-pair production* process $AB \to \ell\bar{\ell}X$ (Drell-Yan process).

All these processes involve electroweak currents, coupling to the leptons in a known way. The basic advantage of electroweak processes lies in the fact that the process is accurately described in terms of the exchange of one photon (for electromagnetic processes), since the coupling, $\alpha = e^2/4\pi \approx 1/137$, is weak. The same is true for the weak vector bosons. On the hadronic side, the coupling to the quarks is known, but the structure of hadrons in terms of quarks and gluons is the unknown part. The fact that the coupling to the quarks is known, however, enables the study of hadron structure.

For lepton-hadron scattering we consider the inclusive measurement $\ell H \to \ell' X$ and the *1-particle inclusive* or *semi-inclusive* measurement $\ell H \to \ell' h X$. The invariants are defined,



$$q^2 = (k - k')^2 \equiv -Q^2 \leq 0 \tag{2.8}$$

$$2P \cdot q \equiv 2M\,\nu \equiv \frac{Q^2}{x_B} \tag{2.9}$$

$$2P_h \cdot q \equiv -z_h\, Q^2 \tag{2.10}$$

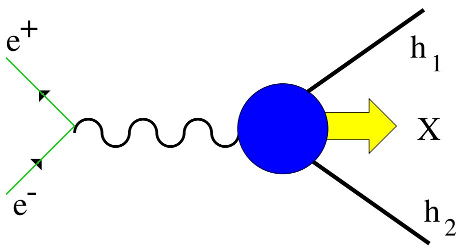
$$P \cdot P_h \equiv z\, P \cdot q \tag{2.11}$$

The variable x_B is the Bjorken scaling variable. Since the invariant mass squared of the hadronic final state satisfies

$$W_X^2 = (P+q)^2 = \frac{1 - x_B}{x_B}\, Q^2 + M^2 \geq M^2, \tag{2.12}$$

one has $0 \leq x_B \leq 1$, with $x_B = 1$ corresponding to elastic scattering, i.e. $W_X^2 = M^2$. In this case a hadron is probed with a spacelike (virtual) photon, for which one can consider a frame in which the momentum only has a spatial component, from which it is clear that the resolving power of the probing photon is of the order $\lambda \approx 1/Q$. Roughly spoken one probes a nucleus (1 - 10 fm) with $Q \approx 10 - 100$ MeV, baryon or meson structure (with sizes in the order of 1 fm) with $Q \approx 0.1 - 1$ GeV and one probes deep into the nucleon (< 0.1 fm) with $Q > 2$ GeV. As we will see, the invariants $z \approx z_h$ for the case of *one* leading jet (to which h belongs) in the limit that $Q^2 \to \infty$.

For the annihilation process we distinguish the *inclusive* measurement $\ell\bar{\ell} \to X$, the *1-particle inclusive* measurements $\ell\bar{\ell} \to hX$ and the *2-particle inclusive* measurements $\ell\bar{\ell} \to h_1 h_2 X$ (hadrons belonging to back-to-back jets). The invariants are defined



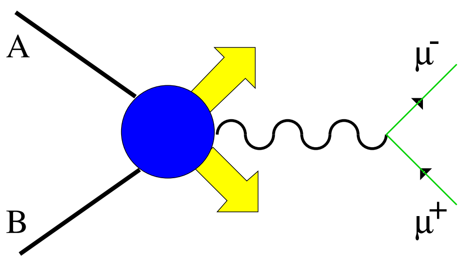
$$q^2 = (k + k')^2 \equiv Q^2 \geq 0 \tag{2.13}$$

$$2\,P_1 \cdot q \equiv z_1\, Q^2 \tag{2.14}$$

$$2\,P_2 \cdot q \equiv z_2\, Q^2 \tag{2.15}$$

In the case of production of hadrons with a timelike (virtual) photon one can consider the rest-frame of the virtual photon, in which case it is clear that Q is a measure of the excitation energy.

For the Drell-Yan process we only consider the *inclusive* case, which already involves two hadrons. We restrict ourselves to lepton pairs with small transverse momentum (compared to Q), for which we have the invariants



$$q^2 = (k + k')^2 \equiv Q^2 \geq 0 \tag{2.16}$$

$$2P_A \cdot q \equiv \frac{Q^2}{x_A} \tag{2.17}$$

$$2P_B \cdot q \equiv \frac{Q^2}{x_B} \tag{2.18}$$

2.3 The hadronic tensor

2.3.1 Lepton-hadron scattering

Consider the process $\ell + H \to \ell' + h + X$, in which a lepton with momentum k scatters off a hadron H with momentum P and one hadron h with momentum P_h is measured in coincidence with the scattered lepton with momentum k'. The lepton emits a highly virtual photon with momentum

$$q^\mu = k^\mu - k'^\mu, \tag{2.19}$$

with $Q^2 \equiv -q^2 > 0$. The unobserved outstate will be denoted by $|P_X\rangle$, having a total momentum P_X^μ. We will consider the most general case of a pure incoming spin state, characterized by the spin vectors S^μ, an observed hadronic spin state characterized by the spin vector S_h and lepton helicities λ and λ'.

We have the following relations

$$P^2 = M^2, \quad P_h^2 = M_h^2, \tag{2.20}$$
$$k^2 = m^2 \approx 0, \quad k'^2 = m'^2 \approx 0, \tag{2.21}$$
$$(k+P)^2 = s, \tag{2.22}$$
$$S^2 = S_h^2 = -1, \tag{2.23}$$
$$P \cdot S = P_h \cdot S_h = 0. \tag{2.24}$$

We will work in the limit where Q^2, $P \cdot q$ and $P_h \cdot q$ are large keeping the ratios $Q^2/2P\cdot q$ and $2P_h \cdot q/Q^2$ finite. The invariant ampitude for the process is given by

$$\mathscr{M} = \bar{u}(k',\lambda')\gamma^\mu u(k,\lambda)\frac{e^2}{Q^2}\langle P_X; P_h S_h|J_\mu(0)|PS\rangle. \tag{2.25}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathscr{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(\ell H)} H^{\mu\nu(\ell H)}, \tag{2.26}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(\ell H)}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma\right). \tag{2.27}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(\ell H)}(P_X; PS; P_h S_h) = \langle PS|J_\mu(0)|P_X; P_h S_h\rangle\langle P_X; P_h S_h|J_\nu(0)|PS\rangle, \tag{2.28}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathscr{M}|^2 d\mathscr{R}, \tag{2.29}$$

with the flux factor

$$F = 4\sqrt{(P\cdot k)^2 - M^2 m^2} \;\approx\; 2s \tag{2.30}$$

and the Lorentz invariant phase space

$$d\mathscr{R} = (2\pi)^4\delta^4(k + P - k' - P_X - P_h)\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3k'}{(2\pi)^3 2k'^0}\frac{d^3P_h}{(2\pi)^3 2P_h^0}. \tag{2.31}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$2M\mathscr{W}_{\mu\nu}^{(\ell H)}(q; PS; P_h S_h) = \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q + P - P_X - P_h)H_{\mu\nu}^{(\ell H)}(P_X; PS; P_h S_h). \tag{2.32}$$

The phase space for the scattered lepton can be rewritten as

$$\frac{d^3k'}{(2\pi)^3 2k'^0} = \frac{E'\,dE'\,d\Omega}{16\pi^3}, \tag{2.33}$$

where Ω is the lepton scattering angle and E' the energy of the scattered lepton. Thus one gets

$$\frac{2E_h\, d\sigma^{(\ell H)}}{d^3P_h\, d\Omega dE'} = \frac{2ME'}{s}\,\frac{\alpha^2}{Q^4} L_{\mu\nu}^{(\ell H)} \mathscr{W}^{\mu\nu(\ell H)}, \tag{2.34}$$

where the lepton tensor is given by the expression between brackets in Eq. (2.27) and the hadron tensor by Eq. (2.32).

Note that for inclusive scattering one obtains the familiar result

$$\frac{d\sigma^{(\ell H)}}{d\Omega dE'} = \frac{2ME'}{s}\,\frac{\alpha^2}{Q^4} L_{\mu\nu}^{(\ell H)} W^{\mu\nu(\ell H)}, \tag{2.35}$$

with the hadron tensor given by

$$\begin{aligned} 2M\, W_{\mu\nu}^{(\ell H)}(q;PS) &= \frac{1}{2\pi}\int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q+P-P_X)\langle PS|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|PS\rangle \\ &= \frac{1}{2\pi}\int d^4x\; e^{iq\cdot x}\;\langle PS|[J_\mu(x),J_\nu(0)]|PS\rangle. \end{aligned} \tag{2.36}$$

2.3.2 Electron-positron annihilation

Consider the process $e^- + e^+ \to h_1 + h_2 + X$, where two hadrons belonging to opposite jets emerge with momenta P_1^μ and P_2^μ. The annihilating incoming leptons with momenta k^μ and k'^μ produce a high mass photon with momentum

$$q^\mu = k^\mu + k'^\mu, \tag{2.37}$$

with $Q^2 \equiv q^2 > 0$. The unobserved outstate will be denoted by $|P_X\rangle$. We will consider the general case of polarized leptons with helicities λ and λ' and production of hadrons of which the spin states are characterized by spin vectors S_1^μ and S_2^μ, respectively. We have the following relations

$$P_1^2 = M_1^2, \quad P_2^2 = M_2^2, \tag{2.38}$$

$$k^2 = k'^2 = m^2 \approx 0, \tag{2.39}$$

$$S_1^2 = S_2^2 = -1, \tag{2.40}$$

$$P_1\cdot S_1 = P_2\cdot S_2 = 0. \tag{2.41}$$

We will work in the limit where Q^2, $P_1\cdot q$ and $P_2\cdot q$ are large, keeping the ratios $2P_1\cdot q/Q^2$ and $2P_2\cdot q/Q^2$ finite. The invariant ampitude for the process is given by

$$\mathscr{M} = \bar{v}(k',\lambda')\gamma^\mu u(k,\lambda)\frac{e^2}{Q^2}\langle P_X;P_1S_1;P_2S_2|J_\mu(0)|0\rangle. \tag{2.42}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathscr{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(e^+e^-)} H^{\mu\nu(e^+e^-)}, \tag{2.43}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(e^+e^-)}(k\lambda;k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}k^\rho k'^\sigma\right). \tag{2.44}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(e^+e^-)}(P_X;P_1S_1;P_2S_2) = \langle 0|J_\mu(0)|P_X;P_1S_1;P_2S_2\rangle\langle P_X;P_1S_1;P_2S_2|J_\nu(0)|0\rangle, \tag{2.45}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathscr{M}|^2 d\mathscr{R}, \tag{2.46}$$

with the flux factor

$$F = 4\sqrt{(k\cdot k')^2 - k^2k'^2}\;\approx\; 2Q^2 \tag{2.47}$$

and the Lorentz invariant phase space

$$d\mathscr{R} = (2\pi)^4\delta^4(k + k' - P_X - P_1 - P_2)\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3P_1}{(2\pi)^3 2P_1^0}\frac{d^3P_2}{(2\pi)^3 2P_2^0}. \tag{2.48}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$\mathscr{W}_{\mu\nu}^{(e^+e^-)}(q; P_1S_1; P_2S_2) = \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q - P_X - P_1 - P_2)H_{\mu\nu}^{(e^+e^-)}(P_X; P_1S_1; P_2S_2). \tag{2.49}$$

One obtains the cross section (including a factor 1/2 from averaging over incoming polarizations)

$$\frac{P_1^0\,P_2^0\,d\sigma^{(e^+e^-)}}{d^3P_1\,d^3P_2} = \frac{\alpha^2}{4\,Q^6}L_{\mu\nu}^{(e^+e^-)}\mathscr{W}^{\mu\nu(e^+e^-)}, \tag{2.50}$$

where the lepton tensor is given by the expression between brackets in Eq. (2.44) and the hadron tensor by Eq. (2.49).

Note that for a single produced hadron one finds

$$E_h\,\frac{d\sigma}{d^3P_h} = \frac{\alpha^2}{2\,Q^6}\,L_{\mu\nu}^{(e^+e^-)}W^{\mu\nu(e^+e^-)}, \tag{2.51}$$

where the hadron tensor is given by

$$W_{\mu\nu}^{(e^+e^-)}(q; P_hS_h) = \frac{1}{(2\pi)}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q{-}P_X{-}P_h)\langle 0|J_\mu(0)|P_X; P_hS_h\rangle\langle P_X; P_hS_h|J_\nu(0)|0\rangle. \tag{2.52}$$

For the annihilation cross section one finds

$$\sigma(e^+e^- \rightarrow hadrons) = \frac{4\pi^2\,\alpha^2}{Q^6}\,L_{\mu\nu}^{(e^+e^-)}R^{\mu\nu(e^+e^-)}, \tag{2.53}$$

where the tensor $R_{\mu\nu}$ is given by

$$\begin{aligned} R_{\mu\nu}^{(e^+e^-)}(q) &= \int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q - P_X)\langle 0|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|0\rangle \\ &= \int d^4x\; e^{iq\cdot x}\,\langle 0|[J_\mu(x), J_\nu(0)]|0\rangle. \end{aligned} \tag{2.54}$$

2.3.3 The Drell-Yan process

Consider the process $A + B \rightarrow \ell + \bar{\ell} + X$, where two spin-$\frac{1}{2}$ hadrons with momenta P_A^μ and P_B^μ interact and two outgoing leptons (considered massless) are measured with momenta k^μ and k'^μ. The leptons are assumed to originate from a high mass photon with momentum

$$q^\mu = k^\mu + k'^\mu, \tag{2.55}$$

with $Q^2 \equiv q^2 > 0$. The unobserved outstate will be denoted by $|P_X\rangle$, having a total momentum P_X^μ. We will consider the case of pure incoming spin states, characterized by the spin vectors S_A^μ and S_B^μ, respectively, and observed lepton helicities λ and λ'. We have the following relations

$$P_A^2 = M_A^2, \quad P_B^2 = M_B^2, \tag{2.56}$$

$$(P_A + P_B)^2 = s, \tag{2.57}$$

$$k^2 = k'^2 = m^2 \approx 0, \tag{2.58}$$

$$S_A^2 = S_B^2 = -1, \tag{2.59}$$

$$P_A \cdot S_A = P_B \cdot S_B = 0. \tag{2.60}$$

In the deep inelastic limit $Q^2, s \rightarrow \infty$, with the ratio $\tau = Q^2/s$ fixed. The invariant amplitude for the process is given by

$$\mathscr{M} = \bar{u}(k,\lambda)\gamma^\mu v(k',\lambda')\frac{e^2}{Q^2}\langle P_X|J_\mu(0)|P_AS_A; P_BS_B\rangle. \tag{2.61}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathscr{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(DY)} H^{\mu\nu(DY)}, \tag{2.62}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(DY)}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'} \left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\, \epsilon_{\mu\nu\rho\sigma} q^\rho k^\sigma\right). \tag{2.63}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(DY)}(P_X; P_A S_A; P_B S_B) = \langle P_A S_A; P_B S_B | J_\mu(0) | P_X\rangle \langle P_X | J_\nu(0) | P_A S_A; P_B S_B\rangle, \tag{2.64}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F} |\mathscr{M}|^2 d\mathscr{R}, \tag{2.65}$$

with the flux factor

$$F = 4\sqrt{(P_A \cdot P_B)^2 - M_A^2 M_B^2} \;\approx\; 2s \tag{2.66}$$

and the Lorentz invariant phase space

$$d\mathscr{R} = (2\pi)^4 \delta^4(P_A + P_B - P_X - k - k') \frac{d^3 P_X}{(2\pi)^3 2P_X^0} \frac{d^3 k}{(2\pi)^3 2k^0} \frac{d^3 k'}{(2\pi)^3 2k'^0}. \tag{2.67}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$\begin{aligned} \mathscr{W}_{\mu\nu}^{(DY)}(q; P_A S_A; P_B S_B) &= \frac{1}{(2\pi)^4} \int \frac{d^3 P_X}{(2\pi)^3 2P_X^0} (2\pi)^4 \delta^4(P_A + P_B - P_X - q) \\ &\qquad \times H_{\mu\nu}^{(DY)}(P_X; P_A S_A; P_B S_B) \\ &= \frac{1}{(2\pi)^4} \int d^4x \;\; e^{iq\cdot x} \langle P_A S_A; P_B S_B | [J_\mu(0), J_\nu(x)] | P_A S_A; P_B S_B\rangle. \end{aligned} \tag{2.68}$$

For the last equality completeness of the out states and causality has been used. The remaining phase space is conveniently written as

$$\frac{d^3 k}{(2\pi)^3 2k^0} \frac{d^3 k'}{(2\pi)^3 2k'^0} = \frac{d^4 q}{(2\pi)^4} \frac{d\Omega}{32\pi^2}, \tag{2.69}$$

where the angles are those of the lepton axis in the rest frame of the two leptons. In terms of the fine-structure constant $\alpha \equiv e^2/4\pi$, we then obtain the Drell-Yan cross section (including a factor 2 from the summation over the lepton polarizations)

$$\frac{d\sigma^{(DY)}}{d^4q\, d\Omega} = \frac{\alpha^2}{2s\, Q^4} L_{\mu\nu}^{(DY)} \mathscr{W}^{\mu\nu(DY)}, \tag{2.70}$$

where the lepton tensor is given by the symmetric part in Eq. (2.63) and the hadron tensor by Eq. (2.68).

2.4 Deep inelastic kinematics

In order to deal with the hard processes, it is convenient to consider a Cartesian set of vectors constructed from the momenta. These start with defining q^μ as a spacelike or timelike direction depending on the process. Then one proceeds using vectors that are orthogonal to q. Such vectors $\tilde{a}$ are obtained subtracting from a the projection along q,

$$\tilde{a}^\mu \equiv \tilde{g}^{\mu\nu} a_\nu = a^\mu - \frac{a \cdot q}{q^2}\, q^\mu, \tag{2.71}$$

where

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}. \tag{2.72}$$

Another set of vectors that is useful, in particular for the theoretical description of the structure of hadrons, are a set of light-like vectors, n_-^μ and n_+^μ ($n_+^2 = n_-^2 = 0$, $n_+\cdot n_- = 1$) that are in essence hadronic momenta divided by the large scale Q. If $P/Q = a\,n_+ + b\,n_-$ and $a \sim 1$ then $b \sim M^2/Q^2 \ll a$. Hadronic momenta divided by Q are thus in essence proportional to one light-like vector, the hard momentum q/Q, however, involves two light-like vectors. We will use for four vectors the notations $p = (p^0, p^1, p^2, p^3)$ or $p = [p^-, p^+, p^1, p^2]$ where $p^\pm \equiv (p^0 \pm p^3)/\sqrt{2}$, depending on the fact if we use a Cartesian set of basisvectors or a set with two light-like vectors. In the latter case one must be aware of the metric, having e.g. $p^\pm = p\cdot n_\mp$.

We will consider two different sets of frames, the first set (type I) has $\boldsymbol{q}_\perp \equiv (q^1, q^2) = \boldsymbol{0}_\perp$, i.e. the virtual photon has no transverse components. We note that there is still freedom to parametrize q, e.g. in lepton-hadron scattering,

$$q \;\stackrel{I}{=}\; \left[-\frac{1}{A}\,\frac{Q}{\sqrt{2}}, A\,\frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right].$$

The quantity A specifies a particular frame. Frames with different A are connected via a simple boost along the z-axis. The second set of frames (type II) are those where the hadrons have no perpendicular momentum, relevant in cases where two hadrons play a role. In these frames the transverse momenta are indicated with $\boldsymbol{p}_T$ and thus $\boldsymbol{P}_T = \boldsymbol{P}_{hT} = \boldsymbol{0}_T$. Note that q in such a frame in general does have transverse components,

$$q \;\stackrel{II}{=}\; \left[-\frac{1}{A}\,\frac{\tilde{Q}}{\sqrt{2}}, A\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{q}_T\right],$$

with $\boldsymbol{q}_T^2 = Q_T^2$ and $Q^2 = \tilde{Q}^2 + Q_T^2$. The connection between frames of type I and II can be made by a Lorentz transformation, e.g. one that leaves the minus component unchanged [1] and involves a parameter b^- and a two-component vector $\boldsymbol{b}$

$$[a^-, a^+, \boldsymbol{a}] \;\longrightarrow\; \left[a^-, a^+ - \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{b^-} + \frac{\boldsymbol{b}^2\,a^-}{2(b^-)^2}, \boldsymbol{a} - \frac{a^-}{b^-}\,\boldsymbol{b}\right]. \tag{2.73}$$

In most of the following we will assume that all hadrons and the virtual photon are in essence parallel, i.e. $Q_T \ll Q$ and up to $\mathcal{O}(1/Q^2)$ corrections $\tilde{Q} \approx Q$. This implies for semi-inclusive lepton-hadron just one leading jet containing the produced hadron, for 2-particle inclusive lepton annihilation just two back-to-back jets and for Drell-Yan only lepton-pairs with transverse momentum $\ll Q$.

2.4.1 Lepton-hadron scattering

For lepton-hadron scattering the starting point of defining a Cartesian set is a spacelike direction defined by the momentum transfer q^μ. Using the target hadron momentum P^μ one can construct an orthogonal four vector $\tilde{P}^\mu = P^\mu - (P\cdot q/q^2)\,q^\mu$, which is timelike and satisfies $\tilde{P}^2 = \kappa\,P\cdot q$ with

$$\kappa = 1 + \frac{M^2\,Q^2}{(P\cdot q)^2} = 1 + \frac{4\,M^2 x_B^2}{Q^2}. \tag{2.74}$$

taking into account mass corrections $\propto M^2/Q^2$ which will vanish for large Q^2 ($\kappa \to 1$). Defining

$$Z^\mu \equiv -q^\mu, \tag{2.75}$$

$$T^\mu \equiv \frac{Q^2}{P\cdot q}\,\frac{\tilde{P}^\mu}{\sqrt{\kappa}} = \frac{q^\mu + 2x_B\,P^\mu}{\sqrt{\kappa}}, \tag{2.76}$$

we have $Z^2 = -Q^2$ and $T^2 = Q^2$ and will mostly consider normalized vectors $\hat{z}^\mu = Z^\mu/Q$ and $\hat{t}^\mu = T^\mu/Q$. Note that $P\cdot T = \sqrt{\kappa}\,P\cdot q$ and $\tilde{P}^2 = (P\cdot\hat{t}\,)^2 = \kappa\,(P\cdot\hat{z})^2$. In the space orthogonal to $\hat{z}$ and $\hat{t}$ one has the tensors

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu, \tag{2.77}$$

$$\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{q}_\sigma \;=\; \frac{1}{(P\cdot q)\sqrt{\kappa}}\,\epsilon^{\mu\nu\rho\sigma}\,P_\rho q_\sigma. \tag{2.78}$$

A relevant vector in the perpendical space appears if we have more than one hadron, e.g. in 1-particle inclusive leptoproduction. For instance $P_{h\perp} = g_\perp^{\mu\nu}\,P_{h\nu}$, defining the orientation of the production plane in semi-inclusive leptoproduction, $\hat{h}^\mu = P_{h\perp}^\mu/|\boldsymbol{P}_{h\perp}|$ (see figure 2.1).

[1] To do this one needs in the two parametrizations boost factors A differing by a factor $Q/\tilde{Q}$.

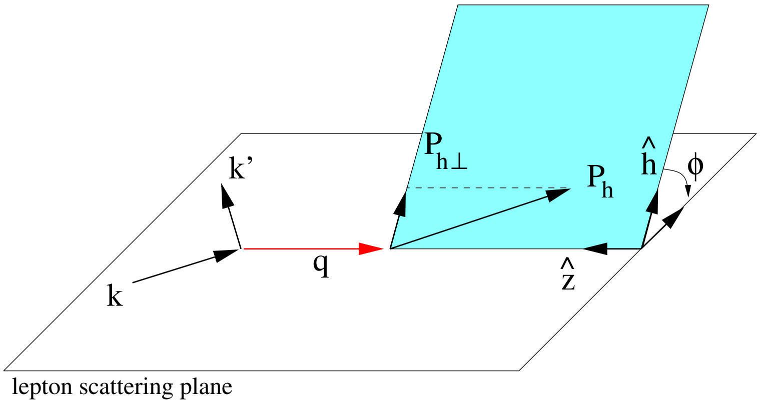


Figure 2.1: Kinematics of lepton hadron scattering in a frame where target hadron and virtual photon momentum are parallel (including target rest frame).

We can also construct a timelike direction using the vector vector P_h. The vector $\tilde{P}_h = P_h^\mu - (P_h \cdot q/q^2)\, q^\mu$ satifies $\tilde{P}_h^2 = \kappa_h\, (P_h \cdot q)^2$ with

$$\kappa_h = 1 + \frac{M_h^2 Q^2}{(P_h \cdot q)^2} = 1 + \frac{4\, M_h^2}{z_h^2\, Q^2}. \tag{2.79}$$

One has $\tilde{P} \cdot \tilde{P}_h = (P \cdot \hat{t}\,)(P_h \cdot \hat{t}\,)$ and $\tilde{P}_h^2 = M_h^2 + (P_h \cdot \hat{z})^2 = (P_h \cdot \hat{t}\,)^2 - \boldsymbol{P}_{h\perp}^2 = \kappa_h\, (P_h \cdot \hat{z})^2$.

One can extend the Cartesian set with

$$X^\mu \equiv \frac{(P_h \cdot T)\, \tilde{P}^\mu - (P \cdot T)\, \tilde{P}_h^\mu}{(P \cdot \hat{t}\,)(P_h \cdot \hat{z}) - (P \cdot \hat{z})(P_h \cdot \hat{t}\,)} \tag{2.80}$$

$$Y^\mu \equiv \frac{\epsilon^{\mu\nu\rho\sigma} P_\nu P_{h\rho} q_\sigma}{(P \cdot \hat{t}\,)(P_h \cdot \hat{z}) - (P \cdot \hat{z})(P_h \cdot \hat{t}\,)} = -\frac{1}{\sqrt{\kappa}}\, \epsilon_\perp^{\mu\nu} X_\nu. \tag{2.81}$$

Explicitly one has

$$\begin{aligned} X^\mu &= q^\mu + x_{\scriptscriptstyle B}\, P^\mu \left[\frac{2}{1 - \frac{(P\cdot T)(P_h\cdot q)}{(P\cdot q)(P_h\cdot T)}} \right] - \frac{P_h^\mu}{z_h} \left[\frac{2}{1 - \frac{(P\cdot q)(P_h\cdot T)}{(P\cdot T)(P_h\cdot q)}} \right] \\ &\approx q^\mu + x_{\scriptscriptstyle B}\, P^\mu - \frac{P_h^\mu}{z_h}, \end{aligned} \tag{2.82}$$

the last expression being without the mass corrections, valid up to $1/Q^2$ corrections.

Introducing

$$\kappa_{h\perp} = 1 + \frac{M_{h\perp}^2 Q^2}{(P_h \cdot q)^2} = 1 + \frac{4\, M_{h\perp}^2}{z_h^2\, Q^2}, \tag{2.83}$$

where $M_{h\perp}^2 = M_h^2 + \boldsymbol{P}_{h\perp}^2$, one can write

$$X^\mu = q^\mu + x_B\, P^\mu \left[\frac{2}{1 + \sqrt{\frac{\kappa}{\kappa_{h\perp}}}} \right] - \frac{P_h^\mu}{z_h} \left[\frac{2}{1 + \sqrt{\frac{\kappa_{h\perp}}{\kappa}}} \right], \tag{2.84}$$

from which one e.g. immediately sees that

$$X^2 = \frac{P_{h\perp}^2}{z_h^2} \left(\frac{2}{1 + \sqrt{\frac{\kappa_{h\perp}}{\kappa}}} \right)^2. \tag{2.85}$$

or using z instead of z_h,

$$X^2 = \frac{P_{h\perp}^2}{z^2}\, \frac{\kappa}{\left(1 - (1-\kappa)\, \frac{M_{h\perp}^2}{z^2\, Q^2}\right)}. \tag{2.86}$$

Next we introduce two simple light-like vectors ($n_+ \cdot n_- = 1$, $n_+^2 = n_-^2 = 0$), such that we have

$$P^\mu \equiv \frac{\tilde{Q}}{\xi\sqrt{2}}\, n_+^\mu + \frac{\xi\, M^2}{\tilde{Q}\sqrt{2}}\, n_-^\mu, \tag{2.87}$$

$$P_h^\mu \equiv \frac{\zeta\tilde{Q}}{\sqrt{2}}\, n_-^\mu + \frac{M_h^2}{\zeta\tilde{Q}\sqrt{2}}\, n_+^\mu, \tag{2.88}$$

$$q^\mu \equiv \frac{\tilde{Q}}{\sqrt{2}}\, n_-^\mu - \frac{\tilde{Q}}{\sqrt{2}}\, n_+^\mu + q_T^\mu, \tag{2.89}$$

where $q_T^2 \equiv -Q_T^2$ and $\tilde{Q}^2 = Q^2 - Q_T^2$. Note that the variables ξ and ζ are equal to the invariants x_B and z_h or z up to corrections of order M^2/Q^2, M_h^2/Q^2 and Q_T^2/Q^2. Furthermore one has that $X^2 = -Q_T^2 = -\boldsymbol{q}_T^2$.

It is possible to take several kinematic corrections into account by starting with the above parametrization in terms of ξ and ζ and calculate the invariants. Inversion gives

$$\xi = x_B\, \frac{\tilde{Q}^2}{Q^2}\, \frac{2}{\left(1+\sqrt{1+\frac{\tilde{Q}^2}{Q^2}\,\frac{4M^2x_B^2}{Q^2}}\right)},$$

$$\zeta = z_h\, \frac{Q^2}{\tilde{Q}^2}\, \frac{\left(1+\sqrt{1+\frac{\tilde{Q}^2}{Q^2}\,\frac{4M_h^2}{z_h^2\,Q^2}}\right)}{2},$$

or instead of the last one in terms of z instead of z_h,

$$\zeta = z\, \frac{\left(1+\sqrt{1+\frac{4x_B^2M^2M_h^2}{z^2\,Q^4}}\right)}{\left(1+\sqrt{1+\frac{\tilde{Q}^2}{Q^2}\,\frac{4x_B^2M^2}{Q^2}}\right)}.$$

Q_T is determined from $\boldsymbol{P}_{h\perp}$ or implicitly from z, z_h and x_B (e.g. equating the two expressions for ζ). For inclusive lepton-hadron scattering one has $Q_T = 0$ and $\xi = 2x_B/(1+\sqrt{\kappa})$. This quantity is referred to as the Nachtmann scaling variable.

We give the hadronic momenta in the frames I and II, including the vector q_T, up to $\mathscr{O}(1/Q^2)$. We do this omitting the 'boost' factors $1/A$ and A multiplying $-$ and $+$ components respectively.

lepton-hadron:			
momentum	frame I	frame II	relations
$q = k - k'$	$\left[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right]$	$\left[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \boldsymbol{q}_T\right]$	$\boldsymbol{q}_T = -\frac{1}{z}\,\boldsymbol{P}_{h\perp}$
P	$\left[\frac{x_BM^2}{Q\sqrt{2}}, \frac{Q}{x_B\sqrt{2}}, \boldsymbol{0}_\perp\right]$	$\left[\frac{x_BM^2}{Q\sqrt{2}}, \frac{Q}{x_B\sqrt{2}}, \boldsymbol{0}_T\right]$	$x_B = -\frac{q^+}{P^+} \approx \frac{Q^2}{2P\cdot q} \approx -\frac{P_h\cdot q}{P_h\cdot P}$
P_h	$\left[\frac{z_hQ}{\sqrt{2}}, \frac{M_{h\perp}^2}{z_hQ\sqrt{2}}, \boldsymbol{P}_{h\perp}\right]$	$\left[\frac{z_hQ}{\sqrt{2}}, \frac{M_h^2}{z_hQ\sqrt{2}}, \boldsymbol{0}_T\right]$	$z_h = \frac{P_h^-}{q^-} \approx -\frac{2P_h\cdot q}{Q^2} \approx \frac{P\cdot P_h}{P\cdot q}$
q_T	$\left[0\ , -2\,\frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\right]$	$\left[0\ , 0\ , \boldsymbol{q}_T\right]$	$\boldsymbol{q}_T^2 = Q_T^2$

The important thing to notice is that the momentum q_T introduced as the transverse part of q in frame II, produces in frame I a term in the +-direction, which will produce effects only suppressed by $1/Q$, rather than mass effects which always will appear suppressed by $1/Q^2$.

The forms of the vectors in the Cartesian set and the vectors $n_\pm$ and q_T including transverse momentum corrections, but neglecting mass corrections (order $1/Q^2$) are explicitly given below.

ℓH and e^-e^+		
vector	frame I	frame II
T	$\left[\frac{\tilde Q}{\sqrt 2}\ ,\ \frac{Q^2}{\tilde Q\sqrt 2}\ ,\ \mathbf{0}_\perp\right]$	$\left[\frac{\tilde Q}{\sqrt 2}\ ,\ \frac{\tilde Q^2+2\,Q_T^2}{\tilde Q\sqrt 2}\ ,\ \boldsymbol{q}_T\right]$
Z	$\left[-\frac{\tilde Q}{\sqrt 2}\ ,\ \frac{Q^2}{\tilde Q\sqrt 2}\ ,\ \mathbf{0}_\perp\right]$	$\left[-\frac{\tilde Q}{\sqrt 2}\ ,\ \frac{\tilde Q}{\sqrt 2}\ ,\ -\boldsymbol{q}_T\right]$
X	$[\ 0\ ,\ 0\ ,\ \boldsymbol{q}_T]$	$\left[\ 0\ ,\ 2\,\frac{Q_T^2}{\tilde Q\sqrt 2},\ \boldsymbol{q}_T\right]$
n_-	$\left[1, \frac{Q_T^2}{\tilde Q^2}, -\frac{\boldsymbol{q}_T\sqrt 2}{\tilde Q}\right]$	$[1\ ,\ 0\ ,\ \mathbf{0}_T]$
n_+	$[0\ ,\ 1\ ,\ \mathbf{0}_T]$	$[0\ ,\ 1\ ,\ \mathbf{0}_T]$
q_T	$\left[\ 0\ ,-2\,\frac{Q_T^2}{\tilde Q\sqrt 2}\ ,\ \boldsymbol{q}_T\right]$	$[\ 0\ ,\ 0\ ,\ \boldsymbol{q}_T]$

Later on we will need to transform from the theoretically useful vectors $n_\pm$ and q_T to the quantities appearing in the expansion of the hadronic tensor, $\hat t$, $\hat z$ and $\hat x$,

$$n_-^\mu = \frac{1}{\tilde Q\sqrt 2}\left(\left(1+\frac{Q_T^2}{Q^2}\right)T^\mu - \left(1-\frac{Q_T^2}{Q^2}\right)Z^\mu - 2\,X^\mu\right) \approx \frac{T^\mu - Z^\mu - 2\,X^\mu}{Q\sqrt 2} \tag{2.90}$$

$$n_+^\mu = \frac{\tilde Q^2}{Q^2}\,\frac{T^\mu + Z^\mu}{\tilde Q\sqrt 2} \approx \frac{T^\mu + Z^\mu}{Q\sqrt 2} \tag{2.91}$$

$$q_T^\mu = X^\mu - \frac{Q_T^2}{Q^2}\,(T^\mu + Z^\mu) \tag{2.92}$$

Again inclusion of mass corrections can be done by using the exact inverse of the expressions for P, P_h and q,

$$n_+ = \frac{\xi\sqrt 2}{\tilde Q}\,\frac{\left(P - \frac{\xi M^2}{\zeta\tilde Q^2}\,P_h\right)}{\left(1-\frac{\xi^2 M^2 M_h^2}{\zeta^2 Q^4}\right)},$$

$$n_- = \frac{\sqrt 2}{\zeta\tilde Q}\,\frac{\left(P_h - \frac{\xi M_h^2}{\zeta\tilde Q^2}\,P\right)}{\left(1-\frac{\xi^2 M^2 M_h^2}{\zeta^2 Q^4}\right)},$$

and rewriting q, P, and P_h in terms of Z, T, and X.

Neglecting the mass and transverse momentum corrections of order $1/Q^2$ (but keeping those of order $1/Q$) we obtain

$$\begin{aligned} g_T^{\mu\nu} &\equiv g^{\mu\nu} - n_+^\mu n_-^\nu - n_-^\mu n_+^\nu \\ &= g_\perp^{\mu\nu} - \frac{Q_T}{Q}\,\hat q^{\{\mu}\hat x^{\nu\}} + \frac{Q_T}{Q}\,\hat t^{\{\mu}\hat x^{\nu\}}, \end{aligned} \tag{2.93}$$

$$g_\perp^{\mu\nu} = g_T^{\mu\nu} - \frac{\sqrt 2\,n_+^{\{\mu}q_T^{\nu\}}}{Q} \tag{2.94}$$

Using normalized vectors is important to see which are the terms containing transverse vectors that should be kept at order $1/Q$. Note that for any vector that in frame II is of the form $[0,0,\boldsymbol{a}_T]$ with $|\boldsymbol{a}_T|\sim 1$, i.e. $\sim Q^0$, one has up to $\mathscr{O}(1/Q^2)$ the relation

$$a_T^\mu \approx g_\perp^{\mu\nu}a_{T\nu} - \frac{a_T\cdot q_T}{Q^2}\,q^\mu + \frac{a_T\cdot q_T}{Q^2}\,T^\mu. \tag{2.95}$$

Note that the first term on the righthand side has *in frame I* the form $[0,0,\boldsymbol{a}_T]$. We will sometimes simply use the notation $a_\perp^\mu = g_\perp^{\mu\nu}a_{T\nu}$ for it, but one must be careful not to confuse this with the vector $g_\perp^{\mu\nu}\,a_\nu$. An example of such a vector is S_T, part of the spin vector characterizing the spin of a spin 1/2 hadron, for which one has

$$P^\mu = \frac{M^2}{2P^+}\,n_-^\mu + P^+\,n_+^\mu, \tag{2.96}$$

$$S^\mu = -S_{hL}\,\frac{M}{2P^+}\,n_-^\mu + S_{hL}\,\frac{P^+}{M}\,n_+^\mu + S_T^\mu, \tag{2.97}$$

such that $P\cdot S = 0$ and $-S^2 = 1 = S_{hL}^2 + \boldsymbol{S}_T^2$. Also for the quark transverse momentum vectors k_T this relation will become important.

Also the lepton momenta k and $k' = k - q$ can be expanded in $\hat{t}$, $\hat{z}$ and a perpendicular component using the scaling variable $y = P\cdot q/P\cdot k$ (in the target restframe reducing to $y = \nu/E$). The result is

$$\begin{aligned} k^\mu = \frac{2-y}{y}\,\frac{1}{\kappa}\,T^\mu - \frac{1}{2}\,Z^\mu + k_\perp^\mu \quad &= \quad \frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)}{2y}\,\frac{Q}{\sqrt{\kappa}}\,\hat{t}^\mu + \frac{\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}}{y}\,\frac{Q}{\sqrt{\kappa}}\,\hat{\ell}^\mu \\ &\overset{Q^2\to\infty}{\longrightarrow} \quad \frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)\,Q}{2y}\,\hat{t}^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}^\mu, \end{aligned} \tag{2.98}$$

where $\hat{\ell}^\mu = k_\perp^\mu/|\boldsymbol{k}_\perp|$, is a spacelike unit-vector in the perpendicular direction lying in the (lepton) scattering plane. The kinematics in the frame where virtual photon and target are collinear (including target rest frame) is illustrated in Fig. 1. With the definition of $\hat{\ell}$, we have for the leptonic tensor[2] neglecting mass corrections ($\kappa = 1$)

$$\begin{aligned} L_{(\ell H)}^{\mu\nu} \quad = \quad \frac{Q^2}{y^2}\Bigg[& -2\left(1-y+\frac{1}{2}\,y^2\right)g_\perp^{\mu\nu} + 4(1-y)\hat{t}^\mu\hat{t}^\nu \\ & +4(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y}\;\hat{t}^{\{\mu}\hat{\ell}^{\nu\}} \\ & -i\lambda_e\,y(2-y)\,\epsilon_\perp^{\mu\nu} - 2i\lambda_e\,y\sqrt{1-y}\;\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\Bigg]. \end{aligned} \tag{2.99}$$

For completeness, we also give the full tensor including mass corrections

$$\begin{aligned} L_{(\ell H)}^{\mu\nu} \quad = \quad \frac{1}{\kappa}\,\frac{Q^2}{y^2}\Bigg[& -2\left(1-y+\frac{1}{4}\,(1+\kappa)\,y^2\right)g_\perp^{\mu\nu} + 4\left(1-y+\frac{1}{4}(1-\kappa)\,y^2\right)\,\hat{t}^\mu\hat{t}^\nu \\ & +4\left(1-y+\frac{1}{4}(1-\kappa)\,y^2\right)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}\;\hat{t}^{\{\mu}\hat{\ell}^{\nu\}} \\ & -i\lambda_e\,\sqrt{\kappa}\;y(2-y)\,\epsilon_\perp^{\mu\nu} - 2i\lambda_e\,\sqrt{\kappa}\;y\,\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}\;\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\Bigg]. \end{aligned}$$

The lepton-momenta also can be written down in the frames discussed above using that up to $\mathscr{O}(1/Q^2)$ corrections

$$y = \frac{q^-}{k^-} \approx \frac{P\cdot q}{P\cdot k} = \frac{Q^2}{x_B\,s}, \tag{2.100}$$

which in the target rest frame ($A = x_B M/Q$) equals $y = \nu/E$.

lepton-hadron:			
momentum	frame I	frame II	relations
k	$\left[\frac{1}{A}\,\frac{Q}{y\sqrt{2}}, A\,\frac{(1-y)Q}{y\sqrt{2}}, \boldsymbol{k}_\perp\right]$	$\left[\frac{1}{A}\,\frac{yQ}{\sqrt{2}}, A\,\frac{\boldsymbol{k}_T^2}{yQ\sqrt{2}}, \boldsymbol{k}_T\right]$	$\boldsymbol{k}_T = \boldsymbol{k}_\perp + \frac{\boldsymbol{q}_T}{y}$
k'	$\left[\frac{1}{A}\,\frac{(1-y)Q}{y\sqrt{2}}, A\,\frac{Q}{y\sqrt{2}}, \boldsymbol{k}_\perp\right]$	$\left[\frac{1}{A}\,\frac{(1-y)Q}{y\sqrt{2}}, A\,\frac{y(\boldsymbol{k}_T^2-2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)}{(1-y)Q\sqrt{2}}, \boldsymbol{k}_T - \boldsymbol{q}_T\right]$	$\boldsymbol{k}_\perp^2 = \frac{1-y}{y^2}Q^2$

[2] A useful relation is

$$\epsilon_{\mu\nu\rho\sigma}\,g_{\alpha\beta} = \epsilon_{\alpha\nu\rho\sigma}\,g_{\mu\beta} + \epsilon_{\mu\alpha\rho\sigma}\,g_{\nu\beta} + \epsilon_{\mu\nu\alpha\sigma}\,g_{\rho\beta} + \epsilon_{\mu\nu\rho\alpha}\,g_{\sigma\beta}$$

or for a vector $a_\perp$ orthogonal to $\hat{t}$ and $\hat{q}$,

$$\epsilon^{\mu\nu\rho\sigma}\hat{z}_\rho a_{\perp\sigma} = \hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho},$$

$$\epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho a_{\perp\sigma} = -\hat{z}^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho}.$$

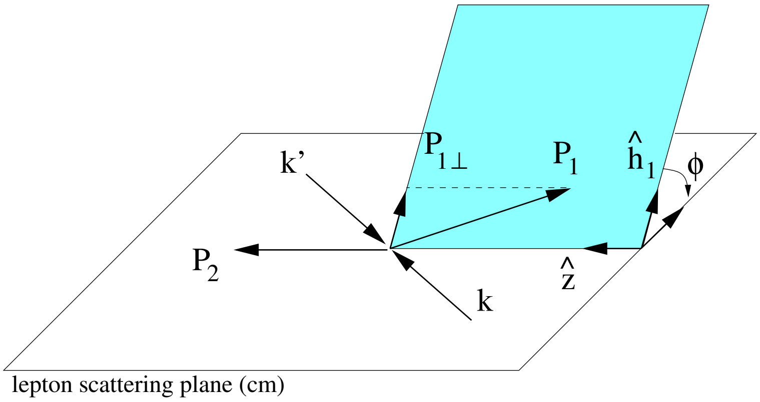


Figure 2.2: Kinematics of the annihilation process in the lepton center of mass frame

The lepton phase space becomes $d^3k'/E' = E'dE'd\Omega = \pi ys dx_B dy$, and one obtains for Eq. 2.34

$$\frac{d\sigma^{(\ell H)}}{dx_B\,dz\,d^2\boldsymbol{q}_T\,dy} = \frac{\pi\,\alpha^2}{2Q^4}\,yz\,L_{\mu\nu}^{(\ell H)}\,2M\mathcal{W}^{\mu\nu(\ell H)}. \tag{2.101}$$

For the inclusive process one finds

$$\frac{d\sigma^{(\ell H)}}{dx_B\,dy} = \frac{\pi\,\alpha^2}{Q^4}\,y\,L_{\mu\nu}^{(\ell H)}\,2MW^{\mu\nu(\ell H)}. \tag{2.102}$$

2.4.2 The annihilation process

For e^-e^+ annihilation the starting point is the timelike direction defined by q. Then it is often convenient to use one of the hadron momenta, say P_2 to construct an orthogonal spacelike vector proportional to $\tilde{P}_2 = P_2 - (P_2\cdot q/q^2)q$.

$$T^\mu \;\equiv\; q^\mu \stackrel{I}{=} \left[\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\mathbf{0}_\perp\right], \tag{2.103}$$

$$Z^\mu \;\equiv\; \frac{q^2}{P_2\cdot q}\,\frac{\tilde{P}_2^\mu}{\sqrt{\kappa_2}} \;=\; \frac{1}{\sqrt{\kappa_2}}\left(\frac{2}{z_2}\,P_2^\mu - q^\mu\right) \stackrel{I}{=} \left[-\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\mathbf{0}_\perp\right], \tag{2.104}$$

For these vectors we have $T^2 = Q^2$ and $Z^2 = -Q^2$ while

$$\kappa_2 = 1 + \frac{M_2^2\,Q^2}{(P\cdot q)^2} = 1 + \frac{4\,M_2^2}{z_2^2\,Q^2}. \tag{2.105}$$

The quantity κ_2 takes into account mass corrections $\propto M_2^2/Q^2$ which will vanish for large Q^2 ($\kappa_2 \to 1$). Note that $P_2\cdot Z = -\sqrt{\kappa_2}\,P_2\cdot q$. We will mostly consider normalized vectors $\hat{t}^\mu = T^\mu/Q$ and $\hat{z}^\mu = Z^\mu/Q$. In the space orthogonal to $\hat{z}$ and $\hat{t}$ one has the tensors

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} - \hat{q}^\mu\hat{q}^\nu + \hat{z}^\mu\hat{z}^\nu, \tag{2.106}$$

$$\epsilon_\perp^{\mu\nu} \equiv -\epsilon^{\mu\nu\rho\sigma}\hat{q}_\rho\hat{z}_\sigma \;=\; \frac{1}{(P_2\cdot q)\sqrt{\kappa_2}}\,\epsilon^{\mu\nu\rho\sigma}P_{2\,\rho}q_\sigma, \tag{2.107}$$

Vectors in the orthogonal space are for instance obtained using the other hadronic momentum P_1 (see figure 2.2). The following Cartesian vectors can be defined,

$$X^\mu \;\equiv\; \frac{-(P_1\cdot Z)\,\tilde{P}_2^\mu + (P_2\cdot Z)\,\tilde{P}_1^\mu}{(P_1\cdot\hat{t}\,)(P_2\cdot\hat{z}) - (P_1\cdot\hat{z})(P_2\cdot\hat{t}\,)}, \tag{2.108}$$

$$Y^\mu \;\equiv\; \frac{\epsilon^{\mu\nu\rho\sigma}P_{2\nu}P_{1\rho}q_\sigma}{(P_1\cdot\hat{t}\,)(P_2\cdot\hat{z}) - (P_1\cdot\hat{z})(P_2\cdot\hat{t}\,)}, \;=\; -\frac{1}{\sqrt{\kappa_2}}\,\epsilon_\perp^{\mu\nu}X_\nu. \tag{2.109}$$

We define in frame I $X = [0, 0, \boldsymbol{q}_T]$, with length $X^2 = -\boldsymbol{q}_T^2 \equiv -Q_T^2$. The proportionality constant for X is chosen such that $X^\mu = q^\mu + \ldots P_1^\mu + \ldots P_2^\mu \approx q^\mu - P_1^\mu/z_1 - P_2^\mu/z_2$. This choice implies that in frame II, in which P_1 and P_2 do not have transverse momentum, one has the transverse component of q precisely equal to $\boldsymbol{q}_T$. In frame II we, moreover, choose two simple light-like vectors ($n_+ \cdot n_- = 1$), such that we have

$$P_1^\mu \equiv \frac{\zeta_1 \tilde{Q}}{\sqrt{2}}\, n_-^\mu + \frac{M_1^2}{\zeta_1 \tilde{Q}\sqrt{2}}\, n_+^\mu, \tag{2.110}$$

$$P_2^\mu \equiv \frac{\zeta_2 \tilde{Q}}{\sqrt{2}}\, n_+^\mu + \frac{M_2^2}{\zeta_2 \tilde{Q}\sqrt{2}}\, n_-^\mu, \tag{2.111}$$

$$q^\mu \equiv \frac{\tilde{Q}}{\sqrt{2}}\, n_-^\mu + \frac{\tilde{Q}}{\sqrt{2}}\, n_+^\mu + q_T^\mu, \tag{2.112}$$

where $\tilde{Q}^2 = Q^2 + Q_T^2$. Note that up to $\mathscr{O}(1/Q^2)$ the variables $\zeta_1 \approx z_1$, $\zeta_2 \approx z_2$ and $\tilde{Q} \approx Q$. Explicitly the hadronic momenta in the frames I and II up to $\mathscr{O}(1/Q^2)$, including the vector q_T are given below.

electron-positron:			
momentum	frame I	frame II	relations
$q = k + k'$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right]$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{q}_T\right]$	$\boldsymbol{q}_T = -\frac{1}{z_1}\boldsymbol{P}_{1\perp}$
P_2	$\left[\frac{M_2^2}{z_2 Q\sqrt{2}}, \frac{z_2 Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right]$	$\left[\frac{M_2^2}{z_2 Q\sqrt{2}}, \frac{z_2 Q}{\sqrt{2}}, \boldsymbol{0}_T\right]$	$z_2 = \frac{P_2^+}{q^+} \approx \frac{2P_2\cdot q}{Q^2} \approx \frac{P_1\cdot P_2}{P_1\cdot q}$
P_1	$\left[\frac{z_1 Q}{\sqrt{2}}, \frac{M_{1\perp}^2}{z_1 Q\sqrt{2}}, \boldsymbol{P}_{1\perp}\right]$	$\left[\frac{z_1 Q}{\sqrt{2}}, \frac{M_1^2}{z_1 Q\sqrt{2}}, \boldsymbol{0}_T\right]$	$z_1 = \frac{P_1^-}{q^-} \approx \frac{2P_1\cdot q}{Q^2} \approx \frac{P_2\cdot P_1}{P_2\cdot q}$
q_T	$\left[0\ , -2\,\frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\right]$	$[0\ , 0\ , \boldsymbol{q}_T]$	$\boldsymbol{q}_T^2 = Q_T^2$

The forms of the vectors T, Z, X, $n_\pm$ and q_T are identical to the case of lepton-hadron scattering. Neglecting the mass and transverse momentum corrections of order $1/Q^2$ (but keeping those of order $1/Q$) we obtain

$$\begin{aligned} g_T^{\mu\nu} &\equiv g^{\mu\nu} - n_+^\mu n_-^\nu - n_-^\mu n_+^\nu \\ &= g_\perp^{\mu\nu} + \frac{Q_T}{Q}\,\hat{z}^{\{\mu}\hat{x}^{\nu\}} + \frac{Q_T}{Q}\,\hat{q}^{\{\mu}\hat{x}^{\nu\}}, \end{aligned} \tag{2.113}$$

$$g_\perp^{\mu\nu} = g_T^{\mu\nu} - \frac{\sqrt{2}\, n_+^{\{\mu} q_T^{\nu\}}}{Q} \tag{2.114}$$

Using normalized vectors is important to see which are the terms containing transverse vectors that should be kept at order $1/Q$. Note that for any vector that in frame II is of the form $[0, 0, \boldsymbol{a}_T]$ with $|\boldsymbol{a}_T| \sim 1$, i.e. Q^0, one has up to $\mathscr{O}(1/Q^2)$ the relation

$$a_T^\mu \approx g_\perp^{\mu\nu} a_{T\nu} + \frac{a_T \cdot q_T}{Q^2}\, Z^\mu + \frac{a_T \cdot q_T}{Q^2}\, q^\mu. \tag{2.115}$$

We will sometimes simply use the notation $a_\perp^\mu = g_\perp^{\mu\nu} a_{T\nu}$, but one must be careful not to confuse it with the vector $g_\perp^{\mu\nu}\, a_\nu$.

Also the leptonic momenta can be expanded in the Cartesian directions. Using the scaling variable $y = P_2 \cdot k / P_2 \cdot q$, we obtain up to $\mathscr{O}(1/Q^2)$ corrections

$$k^\mu = \frac{1}{2}\, q^\mu + \frac{1-2y}{2}\, Z^\mu + k_\perp^\mu \;=\; \frac{Q}{2}\,\hat{q}^\mu + \frac{(1-2y)Q}{2}\,\hat{z}^\mu + Q\sqrt{y(1-y)}\,\hat{\ell}^\mu, \tag{2.116}$$

where $\hat{\ell}^\mu = k_\perp^\mu/|\boldsymbol{k}_\perp|$. This leads to the leptonic tensor

$$
\begin{aligned}
L_{(e^-e^+)}^{\mu\nu} \quad = \quad & Q^2\Bigg[-\left(1-2y+2y^2\right)g_\perp^{\mu\nu}+4y(1-y)\hat{z}^\mu\hat{z}^\nu \\
& -4y(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu+\frac{1}{2}g_\perp^{\mu\nu}\right)-2(1-2y)\sqrt{y(1-y)}\;\hat{z}^{\{\mu}\hat{\ell}^{\nu\}} \\
& +i\lambda\,(1-2y)\,\epsilon_\perp^{\mu\nu}-2i\lambda\,\sqrt{y(1-y)}\;\hat{\ell}_\rho\epsilon_\perp^{\rho\,[\mu}\hat{z}^{\nu]}\Bigg].
\end{aligned} \tag{2.117}
$$

The lepton-momenta also can be written down in the frames I and II using

$$
y=\frac{k^-}{q^-}\approx\frac{P_2\cdot k}{P_2\cdot q}, \tag{2.118}
$$

which in the lepton rest frame equal $y=(1\pm\cos\theta_2)/2$ with θ_2 the angle of hadron (or jet) with respect to the momentum of the incoming leptons.

Electron-positron:			
momentum	frame I	frame II	relations
k	$\left[\frac{yQ}{\sqrt{2}},\frac{(1-y)Q}{\sqrt{2}},\boldsymbol{k}_\perp\right]$	$\left[\frac{yQ}{\sqrt{2}},\frac{\boldsymbol{k}_T^2}{yQ\sqrt{2}},\boldsymbol{k}_T\right]$	$\boldsymbol{k}_T=\boldsymbol{k}_\perp+y\boldsymbol{q}_T$
k'	$\left[\frac{(1-y)Q}{\sqrt{2}},\frac{yQ}{\sqrt{2}},-\boldsymbol{k}_\perp\right]$	$\left[\frac{(1-y)Q}{\sqrt{2}},\frac{(\boldsymbol{k}_T^2-2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)}{(1-y)Q\sqrt{2}},\boldsymbol{q}_T-\boldsymbol{k}_T\right]$	$\boldsymbol{k}_\perp^2=y(1-y)Q^2$

In the e^-e^+ rest frame $d^3P_1d^3P_2/P_1^0P_2^0=(dz_1/z_1)(z_2Q^2\,dz_2/4)d\boldsymbol{P}_{1\perp}d\Omega_2=\pi Q^2\,z_1z_2\,dz_1\,dz_2\,dy\,d^2\boldsymbol{q}_T$, so Eq. 2.50 becomes

$$
\frac{d\sigma^{(e^+e^-)}}{dz_1\,dz_2\,d^2\boldsymbol{q}_T\,dy}=\frac{\pi\,\alpha^2}{2Q^4}\,z_1z_2\,L_{\mu\nu}^{(e^+e^-)}\mathscr{W}^{\mu\nu(e^+e^-)}. \tag{2.119}
$$

For the production of a single hadron

$$
\frac{d\sigma^{(e^+e^-)}}{dz\,dy}=\frac{\pi\,\alpha^2}{Q^4}\,z\,L_{\mu\nu}^{(e^+e^-)}W^{\mu\nu(e^+e^-)}. \tag{2.120}
$$

2.4.3 Drell-Yan scattering

For Drell-Yan scattering, for which q^μ is timelike, one can define the following four orthogonal vectors that can be used to expand any vector. Starting with q^μ defining the timelike vector T^μ,

$$
T^\mu \equiv q^\mu \stackrel{I}{=} \left[\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\boldsymbol{0}_\perp\right], \tag{2.121}
$$

$$
Z^\mu \equiv \frac{P_B\cdot q}{P_B\cdot P_A}\,\tilde{P}_A^\mu-\frac{P_A\cdot q}{P_A\cdot P_B}\,\tilde{P}_B^\mu = \frac{P_B\cdot q}{P_B\cdot P_A}\,P_A^\mu-\frac{P_A\cdot q}{P_A\cdot P_B}\,P_B^\mu \stackrel{I}{=} \left[\frac{Q}{\sqrt{2}},\frac{Q}{\sqrt{2}},\boldsymbol{0}_\perp\right], \tag{2.122}
$$

$$
X^\mu \equiv \frac{(P_B\cdot Z)\,\tilde{P}_A^\mu-(P_A\cdot Z)\,\tilde{P}_B^\mu}{(P_A\cdot\hat{t}\,)(P_B\cdot\hat{z})-(P_A\cdot\hat{t}\,)(P_B\cdot\hat{z})}, \tag{2.123}
$$

$$
Y^\mu \equiv \frac{\epsilon^{\mu\nu\rho\sigma}P_{A\nu}P_{B\rho}q_\sigma}{(P_A\cdot\hat{t}\,)(P_B\cdot\hat{z})-(P_A\cdot\hat{t}\,)(P_B\cdot\hat{z})}. \tag{2.124}
$$

These vectors satisfy $T^2=Q^2$ and up to mass corrections $Z^2\approx-Q^2$. We will use normalized versions $\hat{t}^\mu\equiv T^\mu/Q$ and $\hat{z}^\mu\equiv Z^\mu/\sqrt{-Z^2}$. We note that in this case both hadrons are used to define the spacelike direction, in contrast to e.g. e^+e^- annihilation (compare figs 2.2 and 2.3). In the space transverse to $T=q$ and Z we can use the *perpendicular* tensors

$$
g_\perp^{\mu\nu} \equiv g^{\mu\nu}-\hat{t}^\mu\hat{t}^\nu+\hat{z}^\mu\hat{z}^\nu \tag{2.125}
$$

$$
\epsilon_\perp^{\mu\nu} \equiv -\epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{z}_\sigma=-\frac{1}{Q^2}\epsilon^{\mu\nu\rho\sigma}q_\rho Z_\sigma. \tag{2.126}
$$

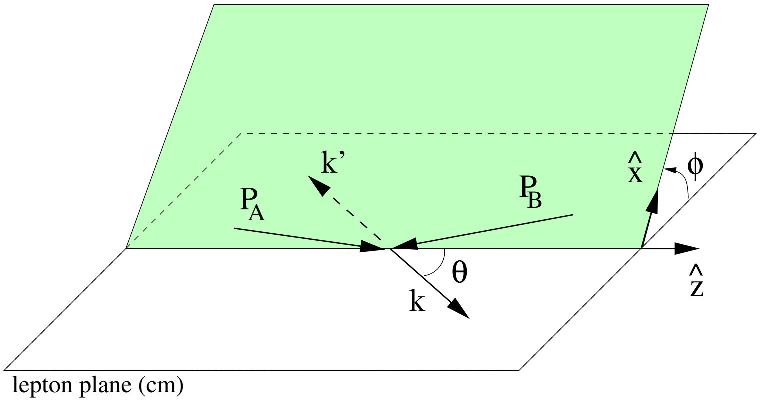


Figure 2.3: Kinematics of the Drell-Yan process in the lepton center of mass frame

We define in frame I $X = [0, 0, \boldsymbol{q}_T]$, with length $X^2 = -\boldsymbol{q}_T^2 \equiv -Q_T^2$. The proportionality constant for X is chosen such that $X^\mu = q^\mu + \ldots P_A^\mu + \ldots P_B^\mu \approx q^\mu - x_A P_A^\mu - x_B P_B^\mu$. This choice implies that in frame II, in which P_A and P_B do not have transverse momentum one has the transverse component of q precisely equal to $\boldsymbol{q}_T$. In frame II we, moreover, choose two simple light-like vectors ($n_+ \cdot n_- = 1$), such that

$$P_A^\mu \equiv \frac{\tilde{Q}}{\xi_A\sqrt{2}}\, n_+^\mu + \frac{\xi_A M_A^2}{\tilde{Q}\sqrt{2}}, n_-^\mu. \tag{2.127}$$

$$P_B^\mu \equiv \frac{\tilde{Q}}{\xi_B\sqrt{2}}\, n_-^\mu + \frac{\xi_B M_B^2}{\tilde{Q}\sqrt{2}}\, n_+^\mu, \tag{2.128}$$

$$q^\mu \equiv \frac{\tilde{Q}}{\sqrt{2}}\, n_-^\mu + \frac{\tilde{Q}}{\sqrt{2}}\, n_+^\mu + q_T^\mu, \tag{2.129}$$

where $\tilde{Q}^2 = Q^2 + Q_T^2$. Note that up to $\mathscr{O}(1/Q^2)$ the variables $\xi_A \approx x_A$, $\xi_B \approx x_B$ and $\tilde{Q} \approx Q$. Next, we explicitly give the hadronic momenta and q_T in the frames I and II, up to $\mathscr{O}(1/Q^2)$.

Drell-Yan:			
momentum	frame I	frame II	relations
$q = k + k'$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp\right]$	$\left[\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{q}_T\right]$	
P_A	$\left[\frac{x_A M_{A\perp}^2}{Q\sqrt{2}}, \frac{Q}{x_A\sqrt{2}}, -\frac{\boldsymbol{q}_T}{2x_A}\right]$	$\left[\frac{x_A M_A^2}{Q\sqrt{2}}, \frac{Q}{x_A\sqrt{2}}, \boldsymbol{0}_T\right]$	$x_A = \frac{q^+}{P_A^+} \approx \frac{Q^2}{2P_A\cdot q} \approx \frac{P_B\cdot q}{P_B\cdot P_A}$
P_B	$\left[\frac{Q}{x_B\sqrt{2}}, \frac{x_B M_{B\perp}^2}{Q\sqrt{2}}, -\frac{\boldsymbol{q}_T}{2x_B}\right]$	$\left[\frac{Q^2}{x_B Q\sqrt{2}}, \frac{x_B M_B^2}{Q\sqrt{2}}, \boldsymbol{0}_T\right]$	$x_B = \frac{q^-}{P_B^-} \approx \frac{Q^2}{2P_B\cdot q} \approx \frac{P_A\cdot q}{P_A\cdot P_B}$
q_T	$\left[-\frac{Q_T^2}{Q\sqrt{2}}, -\frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\right]$	$[0\ , 0\ , \boldsymbol{q}_T]$	$\boldsymbol{q}_T^2 = Q_T^2$

The precise forms of the vectors in the Cartesian set and the vectors $n_\pm$ and q_T are explicitly given below (omitting 'boost' factors $1/A$ and A multiplying $-$ and $+$ components respectively).

vector	DY	
	frame I	frame II
T	$\left[\frac{Q}{\sqrt{2}}\ ,\ \frac{Q}{\sqrt{2}}\ ,\ \boldsymbol{0}_\perp\right]$	$\left[\frac{\tilde{Q}}{\sqrt{2}}\ ,\ \frac{\tilde{Q}}{\sqrt{2}}\ ,\ \boldsymbol{q}_T\right]$
Z	$\left[-\frac{Q}{\sqrt{2}}\ ,\ \frac{Q}{\sqrt{2}}\ ,\ \boldsymbol{0}_\perp\right]$	$\left[-\frac{Q}{\sqrt{2}}\ ,\ \frac{Q}{\sqrt{2}}\ ,\ -\boldsymbol{0}_T\right]$
X	$[\,0\ ,\ 0\ ,\ \boldsymbol{q}_T]$	$\left[\frac{Q_T^2}{Q\sqrt{2}},\ \frac{Q_T^2}{Q\sqrt{2}},\ \frac{\tilde{Q}}{Q}\,\boldsymbol{q}_T\right]$
n_-	$\left[\frac{\tilde{Q}+Q}{2\,Q},\ \frac{\tilde{Q}-Q}{2\,Q},\ -\frac{\boldsymbol{q}_T}{Q\sqrt{2}}\right]$	$[1\ ,\ 0\ ,\ \boldsymbol{0}_T]$
n_+	$\left[\frac{\tilde{Q}-Q}{2\,Q},\ \frac{\tilde{Q}+Q}{2\,Q},\ -\frac{\boldsymbol{q}_T}{Q\sqrt{2}}\right]$	$[0\ ,\ 1\ ,\ \boldsymbol{0}_T]$
q_T	$\left[-\frac{Q_T^2}{Q\sqrt{2}},\ -\frac{Q_T^2}{Q\sqrt{2}}\ ,\ \frac{\tilde{Q}}{Q}\,\boldsymbol{q}_T\right]$	$[\,0\ ,\ 0\ ,\ \boldsymbol{q}_T]$

Later on we will need to transform from the theoretically useful vectors $n_\pm$ and q_T to the quantities appearing in the expansion of the hadronic tensor, $\hat{t}$, $\hat{z}$ and $\hat{x}$,

$$n_-^\mu = \frac{1}{Q\sqrt{2}}\left(\frac{\tilde{Q}}{Q}\,T^\mu - Z^\mu - X^\mu\right) \;\approx\; \frac{T^\mu - Z^\mu - X^\mu}{Q\sqrt{2}} \tag{2.130}$$

$$n_+^\mu = \frac{1}{Q\sqrt{2}}\left(\frac{\tilde{Q}}{Q}\,T^\mu - Z^\mu - X^\mu\right) \;\approx\; \frac{T^\mu + Z^\mu - X^\mu}{Q\sqrt{2}} \tag{2.131}$$

$$q_T^\mu = \frac{\tilde{Q}}{Q}\,X^\mu - \frac{Q_T^2}{Q^2}\,T^\mu \;\approx\; X^\mu - \frac{Q_T^2}{Q^2}\,T^\mu \tag{2.132}$$

The last relation is important to keep track of transverse momentum effects at order $1/Q$. Neglecting the mass and transverse momentum corrections of order $1/Q^2$ (but keeping those of order $1/Q$) we obtain

$$\begin{aligned} g_T^{\mu\nu} &\equiv g^{\mu\nu} - n_+^\mu n_-^\nu - n_-^\mu n_+^\nu \\ &= g_\perp^{\mu\nu} + \frac{Q_T}{Q}\,\hat{q}^{\{\mu}\hat{x}^{\nu\}}, \end{aligned} \tag{2.133}$$

$$g_\perp^{\mu\nu} = g_T^{\mu\nu} - \frac{n_+^{\{\mu}q_T^{\nu\}}}{Q\sqrt{2}} - \frac{n_-^{\{\mu}q_T^{\nu\}}}{Q\sqrt{2}}. \tag{2.134}$$

Using normalized vectors is important to see which are the terms containing transverse vectors that should be kept at order $1/Q$. Note that for any vector that in frame II is of the form $[0, 0, \boldsymbol{a}_T]$ with $|\boldsymbol{a}_T| \sim 1$, i.e. $\sim\ Q^0$, one has up to $\mathscr{O}(1/Q^2)$ the relation

$$a_T^\mu \;\approx\; g_\perp^{\mu\nu} a_{T\nu} + \frac{a_T\cdot q_T}{Q^2}\,q^\mu. \tag{2.135}$$

We will sometimes simply use the notation $a_\perp^\mu = g_\perp^{\mu\nu} a_{T\nu}$, but one must be careful not to confuse it with the vector $g_\perp^{\mu\nu}\,a_\nu$.

Also the lepton momenta can be expressed in the cartesian vectors. For DY we have

$$k^\mu = \frac{1}{2}\,T^\mu + \frac{1-2y}{2}\,Z^\mu + k_\perp^\mu \;=\; \frac{Q}{2}\,\hat{q}^\mu + \frac{(1-2y)Q}{2}\,\hat{z}^\mu + Q\sqrt{y(1-y)}\,\hat{\ell}^\mu, \tag{2.136}$$

where $\hat{\ell}^\mu = k_\perp^\mu/|\boldsymbol{k}_\perp|$. This leads to the leptonic tensor

$$\begin{aligned} L_{(DY)}^{\mu\nu} \;=\; Q^2\Bigg[&-\left(1-2y+2y^2\right) g_\perp^{\mu\nu} + 4y(1-y)\hat{z}^\mu\hat{z}^\nu \\ &-4y(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) - 2(1-2y)\sqrt{y(1-y)}\;\hat{z}^{\{\mu}\hat{\ell}^{\nu\}} \\ &-i\lambda\,(1-2y)\,\epsilon_\perp^{\mu\nu} + 2i\lambda\,\sqrt{y(1-y)}\;\hat{\ell}_\rho\epsilon_\perp^{\rho\,[\mu}\hat{z}^{\nu]}\Bigg]. \end{aligned} \tag{2.137}$$

The lepton momenta can also be written down in frames I and II using

$$y = \frac{k^-}{q^-} \approx \frac{k'^+}{q^+}, \tag{2.138}$$

which in the lepton rest frame equal $y = (1 \pm \cos\theta)/2$ with θ the angle of the leptons with respect to an axis that is approximately parallel to the momentum of hadrons A and B (the z-direction).

Drell-Yan:			
momentum	frame I	frame II	relations
k	$\left[\frac{yQ}{\sqrt{2}}, \frac{(1-y)Q}{\sqrt{2}}, \boldsymbol{k}_\perp\right]$	$\left[\frac{yQ}{\sqrt{2}}, \frac{\boldsymbol{k}_T^2}{yQ\sqrt{2}}, \boldsymbol{k}_T\right]$	$\boldsymbol{k}_T = \boldsymbol{k}_\perp + \frac{1}{2}\boldsymbol{q}_T$
k'	$\left[\frac{(1-y)Q}{\sqrt{2}}, \frac{yQ}{\sqrt{2}}, -\boldsymbol{k}_\perp\right]$	$\left[\frac{(1-y)Q}{\sqrt{2}}, \frac{(\boldsymbol{k}_T^2 - 2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)}{(1-y)Q\sqrt{2}}, \boldsymbol{q}_T - \boldsymbol{k}_T\right]$	$\boldsymbol{k}_\perp^2 = y(1-y)Q^2$

Eq. 2.70 becomes

$$\frac{d\sigma^{(DY)}}{dx_A\, dx_B\, d^2\boldsymbol{q}_T\, dy} = \frac{\pi\,\alpha^2}{Q^4}\, L_{\mu\nu}^{(DY)} \mathscr{W}^{\mu\nu(DY)}. \tag{2.139}$$

2.5 Spin vectors, ...

In the next section it will turn out that the most convenient way to describe the spin vector of the target is via an expansion of the form

$$S^\mu = -S_{hL}\,\frac{Mx_B}{Q\sqrt{2}}\,n_- + S_{hL}\,\frac{Q}{Mx_B\sqrt{2}}\,n_+ + S_T. \tag{2.140}$$

One has up to $\mathscr{O}(1/Q^2)$ corrections $S_L \approx M\,(S\cdot q)/(P\cdot q)$ and $S_T \approx S_\perp$. For a pure state one has $S_L^2 + \boldsymbol{S}_T^2 = 1$, in general this quantity being less or equal than one.

The final state spin vector S_h in the case of detection of a spin 1/2 hadron (e.g. a Λ-baryon) will be expanded in the same way. This vector can e.g. be determined from the decay products (e.g. the $N\pi$ system in case of a Λ). It satisfies $P_h \cdot S_h = 0$ and is written

$$S_h^\mu = S_{hL}\,\frac{z_h\,Q}{M_h\sqrt{2}}\,n_- - S_{hL}\,\frac{M_h}{z_h\,Q\sqrt{2}}\,n_+ + S_{hT}. \tag{2.141}$$

Up to $\mathscr{O}(1/Q^2)$ corrections one has $S_{hL} \approx M_h\,(S_h\cdot q)/(P_h\cdot q)$, but note that one has $S_{hT} \approx S_{h\perp} - S_{hL}\,P_{h\perp}/M_h$. In general one has $S_{hL}^2 + \boldsymbol{S}_{hT}^2 \leq 1$.

Chapter 3

Quark correlation functions

3.1 Distributions: from hadron to quarks

We consider now the most general form of the two-quark correlation function

$$\Phi_{ij}(k,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle, \tag{3.1}$$

where a summation over color indices is implicit, diagrammatically represented in Fig. 3.1. In order to render the definition color gauge-invariant each quark field needs to be accompanied by a path ordered exponential (link operator) of the form

$$\mathscr{U}(a,\xi) = \mathscr{P}\exp\left(-ig\int_a^\xi dz^\mu\, A_\mu(z)\right). \tag{3.2}$$

For the relevant correlation functions Φ in a hard scattering process, we will encounter only those cases in which the link involves gluons of the type $A\cdot n$, where n is a lightlike vector (see Fig. 3.2).

Constraints on the correlation function Φ come from hermiticity, parity and time reversal invariance. We know how the states behave under such transformations and we know how the fields transform. This gives consistency conditions. One finds

$$\Phi^\dagger(k,P,S) = \gamma_0\,\Phi(k,P,S)\,\gamma_0 \quad \text{[Hermiticity]} \tag{3.3}$$

$$\Phi(k,P,S) = \gamma_0\,\Phi(\bar{k},\bar{P},-\bar{S})\,\gamma_0 \quad \text{[Parity]} \tag{3.4}$$

$$\Phi^*(k,P,S) = (-i\gamma_5 C)\,\Phi(\bar{k},\bar{P},\bar{S})\,(-i\gamma_5 C) \quad \text{[Time reversal]} \tag{3.5}$$

where $C = i\gamma^2\gamma_0$, $-i\gamma_5 C = i\gamma^1\gamma^3$ and $\bar{k} = (k^0,-\boldsymbol{k})$.

We will give the explicit proof of these properties. Starting with hermiticity,

$$\begin{aligned}
(\Phi^\dagger)_{ij} = \Phi^*_{ji} &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi_k^\dagger(0)(\gamma_0)_{ki}\psi_j(\xi)|P,S\rangle^* \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi_j^\dagger(\xi)(\gamma_0)_{ik}\psi_k(0)|P,S\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_l(0)(\gamma_0)_{lj}(\gamma_0)_{ik}\psi_k(\xi)|P,S\rangle \\
&= (\gamma_0)_{ik}\,\Phi_{kl}\,(\gamma_0)_{lj},
\end{aligned}$$

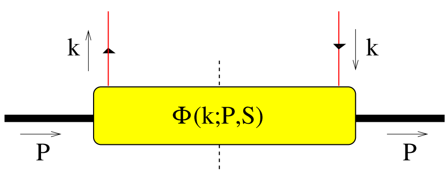


Figure 3.1: The diagrammatic representation of the quark-quark correlation function $\Phi(k,P,S)$.

Figure 3.2: The link as can be obtained from A^+ gluon blobs (see sections 3 and 7) in the case of $\xi^+ = \xi_T = 0$ (left) and the case $\xi^+ = 0, \xi_T \neq 0$ (right).

where from second to third line translation invariance has been used. Next considering parity

$$\begin{aligned}
\Phi_{ij}(k,P,S) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\mathscr{P}^\dagger\mathscr{P}\overline{\psi}_j(0)\mathscr{P}^\dagger\mathscr{P}\psi_i(\xi)\mathscr{P}^\dagger\mathscr{P}|P,S\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle \bar P,-\bar S|\overline{\psi}_l(0)(\gamma_0)_{lj}(\gamma_0)_{ik}\psi_k(\bar\xi)|\bar P,-\bar S\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,\bar k\cdot\xi}\langle \bar P,-\bar S|\overline{\psi}_l(0)(\gamma_0)_{lj}(\gamma_0)_{ik}\psi_k(\xi)|\bar P,-\bar S\rangle \\
&= (\gamma_0)_{ik}\,\Phi_{kl}(\bar p,\bar P,-\bar S)\,(\gamma_0)_{lj},
\end{aligned}$$

where $\mathscr{P}\,\psi(\xi)\mathscr{P}^\dagger = \gamma_0\psi(\bar\xi)$ and from second to third line $k\cdot\xi = \bar k\cdot\bar\xi$ and $d^4\xi = d^4\bar\xi$ has been used. Finally time reversal invariance (with $\mathscr{T}$ anti-unitary),

$$\begin{aligned}
\Phi^*_{ij}(k,P,S) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\mathscr{T}^\dagger\mathscr{T}\overline{\psi}_j(0)\mathscr{T}^\dagger\mathscr{T}\psi_i(\xi)\mathscr{T}^\dagger\mathscr{T}|P,S\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle \bar P,\bar S|\overline{(-i\gamma_5 C\psi)}_j(0)(-i\gamma_5 C\psi)_i(-\bar\xi)|\bar P,\bar S\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,\bar k\cdot\xi}\langle \bar P,\bar S|\overline{(-i\gamma_5 C\psi)}_j(0)(-i\gamma_5 C\psi)_i(\xi)|\bar P,\bar S\rangle \\
&= (-i\gamma_5 C)_{ik}\,\Phi_{kl}(\bar k,\bar P,\bar S)\,(-i\gamma_5 C)_{lj},
\end{aligned}$$

where $\mathscr{T}\,\psi(\xi)\mathscr{T}^\dagger = -i\gamma_5 C\psi(-\bar\xi)$.

Including the link-operator these properties will be different. For the gauge link one has

$$\mathscr{U}^\dagger(a,\xi) = \mathscr{U}(\xi,a), \tag{3.6}$$
$$\mathscr{P}\,\mathscr{U}(a,\xi)\,\mathscr{P}^\dagger = \mathscr{U}(\bar a,\bar\xi), \tag{3.7}$$
$$\mathscr{T}\,\mathscr{U}(a,\xi)\,\mathscr{T}^\dagger = \mathscr{U}(-\bar a,-\bar\xi), \tag{3.8}$$

for which we used $A^\dagger_\mu = A_\mu$, $\mathscr{P}\,A_\mu(\xi)\,\mathscr{P}^\dagger = \bar A_\mu(\bar\xi)$ and $\mathscr{T}\,A_\mu(\xi)\,\mathscr{T}^\dagger = \bar A_\mu(-\bar\xi)$. This means that the space-reversed (time-reversed) correlation function has a different link structure running from $\bar a$ $(-\bar a)$ respectively. However, if the common point is defined with respect to the two fields in the matrix element, no problem arises. For example the straight line link with path $z^\mu(s) = (1-s)\,0^\mu + s\,\xi^\mu$ gives a path $\bar z^\mu$ after applying parity, but after the change of variables one ends up with the same path; similarly for time-reversal.

The most general structure implementing the constraints from hermiticity and parity is

$$\begin{aligned}
\Phi(k,P,S) = \;& M\,A_1 + A_2\,\not{P} + A_3\not{k} + i\,A_4\,\frac{[\not{P},\not{k}]}{2M} + i\,A_5\,(k\cdot S)\gamma_5 + M\,A_6\,\not{S}\gamma_5 \\
&+A_7\,\frac{(k\cdot S)}{M}\,\not{P}\gamma_5 + A_8\,\frac{(k\cdot S)}{M}\,\not{k}\gamma_5 + A_9\,\frac{[\not{P},\not{S}]}{2}\,\gamma_5 + A_{10}\,\frac{[\not{k},\not{S}]}{2}\,\gamma_5 \\
&+A_{11}\,\frac{(k\cdot S)}{M}\,\frac{[\not{P},\not{k}]}{2M}\,\gamma_5 + A_{12}\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu k^\rho S^\sigma}{M},
\end{aligned}\tag{3.9}$$

where the first four terms do not involve the hadron polarization vector. Hermiticity requires all the amplitudes $A_i = A_i(k\cdot P, k^2)$ to be real. The amplitudes A_4, A_5 and A_{12} vanish when also time reversal invariance applies.

One might wonder if the lightlike direction n should not appear in the expansion of the matrix element Φ. We note, however that the matrix element with link, $\Phi^{(n)}$ in the gauge $n\cdot A = 0$ becomes equal to the matrix element $\Phi^{(0)}$ without link. In the expansion of the latter n obviously does not appear. For the fully integrated matrix element, which involves a $d^4\xi$ integration, however, one can consider a different gauge $n'\cdot A = 0$ and perform a change of integration variables such that $\xi\cdot n' = \xi'\cdot n$. One then finds that $\Phi^{(n')}$ $= \Phi^{(n)}$, i.e. the link direction will never appear. Essential in this is the fact that n is not fixed by the momenta k, P or S.

For the applications, it is useful to introduce besides the lightlike vector $n \equiv n_-$ a lightlike vector n_+, such that one has $n_+^2 = n_-^2 = 0$ and $n_+\cdot n_- = 1$. The vector n_+ is fixed by the hadronic momentum such that

$$P = \frac{M^2}{2P^+}\, n_- + P^+\, n_+, \tag{3.10}$$

$$S = -\lambda\, \frac{M}{2P^+}\, n_- + \lambda\, \frac{P^+}{M}\, n_+ + S_T, \tag{3.11}$$

$$k = k^-\, n_- + x\, P^+ n_+ + k_T. \tag{3.12}$$

The parametrization satisfies $P^2 = M^2$ and $P\cdot S = 0$. One immediately deduces $k^- = (k^2 + \boldsymbol{k}_T^2)/2xP^+$, while $2x\, k\cdot P = k^2 + \boldsymbol{k}_T^2 + x^2M^2$. Depending on the use of the soft parts one may need integrations over one or more components of k. At that point the lightlike vector n_- will become relevant.

The fully integrated result leads to a local matrix element, omitting the dependence on hadron momentum and spin vectors (P, S),

$$\Phi_{ij} = \int d^4k\, \Phi_{ij}(k,P,S) = \langle P,S|\overline{\psi}_j(0)\,\psi_i(0)|P,S\rangle, \tag{3.13}$$

It is parametrized as

$$\Phi = \frac{1}{2}\left\{ M\, g_S + g_V\, \not{P} + M\, g_A\, \gamma_5\, \not{S} + g_T\, \frac{[\not{S},\not{P}]\,\gamma_5}{2}\right\}. \tag{3.14}$$

Because the matrix element is local the gauge link will vanish and there will be no dependence on the lightlike vectors. Projecting using a basis of 4×4 Dirac matrices (Γ)and defining

$$\Phi^{[\Gamma]} \equiv \frac{1}{2}\,\mathrm{Tr}\,(\Phi\Gamma)\,, \tag{3.15}$$

one finds

$$2\,\Phi^{[\gamma^\mu]} = \langle P,S|\overline{\psi}(0)\,\gamma^\mu\,\psi(0)|P,S\rangle = g_V\, 2P^\mu, \tag{3.16}$$

$$2\,\Phi^{[\gamma^\mu\gamma_5]} = \langle P,S|\overline{\psi}(0)\,\gamma^\mu\gamma_5\,\psi(0)|P,S\rangle = g_A\, 2M\, S^\mu, \tag{3.17}$$

$$2\,\Phi^{[i\sigma^{\mu\nu}\gamma_5]} = \langle P,S|\overline{\psi}(0)\, i\sigma^{\mu\nu}\gamma_5\,\psi(0)|P,S\rangle = g_T\, 2\, S^{[\mu},P^{\nu]} \tag{3.18}$$

$$2\,\Phi^{[1]} = \langle P,S|\overline{\psi}(0)\,\psi(0)|P,S\rangle = g_S\, 2M, \tag{3.19}$$

$$2\,\Phi^{[i\gamma_5]} = \langle P,S|\overline{\psi}(0)\, i\gamma_5\,\psi(0)|P,S\rangle = 0. \tag{3.20}$$

Note that $g_V = n$ (number of quarks minus antiquarks), g_A is the axial charge for quarks and antiquarks of a particular flavor, g_T is the tensor charge. Multiplying g_S with the quark mass one finds precisely the contribution of the quark mass term to the nucleon mass. The first two matrix elements are special because the operators correspond to conserved currents (for the axial current up to mass terms). The anomalous dimensions of these operators vanish.

In inclusive deep inelastic scattering one needs to consider the correlation functions

$$\begin{aligned}\Phi_{ij}(x) &= \int d^2\boldsymbol{k}_T\, dk^-\;\Phi_{ij}(k,P,S)\Big|_{k^+=xP^+} \\ &= \int \frac{d\xi^-}{2\pi}\; e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathscr{U}(0,\xi)\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}\,,\end{aligned} \tag{3.21}$$

This correlation function is in general the result of a large number of diagrams, as it not only includes matrix elements that contain only quark fields, $\overline{\psi}\psi$, but also matrix elements of the type $\overline{\psi}A^+\ldots A^+\psi$. All of these will contribute in leading order in $1/Q$. It will turn out that in a full calculation gluonic matrix elements containing A^+ fields precisely form the for color gauge invariance necessary link, which runs along the minus direction (see Fig. 3.2). Working in the gauge $A^+ = 0$ none of the gluonic matrix elements will appear and the link in Eq. 3.21 becomes unity.

The quantity $\Phi(x)$ can be parametrized as

$$\begin{aligned}
\Phi(x) &= \frac{1}{2}\left\{f_1(x)\,\not n_+ + \lambda\, g_1(x)\,\gamma_5\,\not n_+ + h_1(x)\,\frac{\gamma_5\,[\not S_T,\not n_+]}{2}\right\}\\
&+\frac{M}{2P^+}\left\{e(x) + g_T(x)\,\gamma_5\,\not S_T + \lambda\, h_L(x)\,\frac{\gamma_5\,[\not n_+,\not n_-]}{2}\right\}\\
&+\frac{M}{2P^+}\left\{-\lambda\, e_L(x)\,i\gamma_5 - f_T(x)\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{T\sigma} + h(x)\,\frac{i\,[\not n_+,\not n_-]}{2}\right\}\\
&+\frac{M^2}{2(P^+)^2}\left\{f_3(x)\,\not n_- + \lambda\, g_3(x)\,\gamma_5\not n_- + h_3(x)\,\frac{\gamma_5\,[\not S_T,\not n_-]}{2}\right\},
\end{aligned} \tag{3.22}$$

The factors of (M/P^+) are the ones required from Lorentz invariance. As we will see, each factor $1/P^+$ gives rise to a suppression factor of $1/Q$ in cross sections. From the structure of the above matrix element, being of the form $\Lambda_H^{t-2}/(P^+)^{t-2}$ one defines the quantity t, the *(operational) twist* of the profile or distribution functions appearing in the parametrization.

To be slightly more specific, using the amplitude expansion for the quark-quark correlation function one can easily analyze the effect of the integration over k^- and $\boldsymbol{k}_T$ for the different Dirac projections of the quark correlation functions (twist analysis). For instance

$$\begin{aligned}
\Phi^{[1]}(x) &= \frac{1}{2}\int dk^-\,d^2\boldsymbol{k}_T\;Tr(1\,\Phi(k,P,S))\Big|_{k^+=xP^+}\\
&= \frac{M}{P^+}\int d(2k\cdot P)\,dk^2\;\pi\,A_1(k^2,k\cdot P)\,\theta(2x\,k\cdot P - x^2M^2 - k^2)\\
&\equiv \frac{M}{P^+}\,e(x),
\end{aligned} \tag{3.23}$$

where $x = k^+/P^+$ and the integration over $\boldsymbol{k}_T$ is rewritten as an integration over k^2 using $\boldsymbol{k}_T^2 = 2x\,k\cdot P - x^2\,M^2 - k^2$. In rewriting the matrix element in this way one has separated it in a function $e(x)$ which we refer to as a distribution or *profile function.* This function, containing only hadron and quark momenta and scalar products of them which are of hadronic size (Λ_H), is of $\mathscr{O}(1)$. It is multiplied with some factor that contains some powers of P^+ and momenta of hadronic size of which the consequence has already been mentioned. The functions e_L, f_T and h are expected to vanish because of time-reversal symmetry. They involve the amplitudes A_4, A_5 and A_{12}. We have kept them here, because they will have potential relevance later and furthermore are useful for comparison with fragmentation functions. The functions are referred to as *T-odd.*

Projecting with the various Dirac matrices one finds the 'leading' (twist two) distribution functions

$$\Phi^{[\gamma^+]}(x) = f_1(x), \tag{3.24}$$
$$\Phi^{[\gamma^+\gamma_5]}(x) = \lambda\, g_1(x), \tag{3.25}$$
$$\Phi^{[i\sigma^{i+}\gamma_5]}(x) = S_T^i\, h_1(x), \tag{3.26}$$

the twist three distribution functions

$$\Phi^{[1]}(x) = \frac{M}{P^+}\, e(x), \tag{3.27}$$

$$\Phi^{[i\gamma_5]} = \frac{M}{P^+}\, e_L(x), \tag{3.28}$$

$$\Phi^{[\gamma^i]}(x) = -\frac{M\,\epsilon_T^{i\rho} S_{T\rho}}{P^+}\, f_T(x), \tag{3.29}$$

$$\Phi^{[\gamma^i\gamma_5]}(x) = \frac{M\,S_T^i}{P^+}\, g_T(x), \tag{3.30}$$

$$\Phi^{[i\sigma^{+-}\gamma_5]}(x) = \frac{M}{P^+}\,\lambda\, h_L(x), \tag{3.31}$$

$$\Phi^{[i\sigma^{ij}\gamma_5]}(x) = \frac{M}{P^+}\,\epsilon_T^{ij}\,\lambda\, h(x), \tag{3.32}$$

and the twist four functions

$$\Phi^{[\gamma^-]}(x) = \left(\frac{M}{P^+}\right)^2 f_3(x), \tag{3.33}$$

$$\Phi^{[\gamma^-\gamma_5]}(x) = \left(\frac{M}{P^+}\right)^2 \lambda\, g_3(x), \tag{3.34}$$

$$\Phi^{[i\sigma^{i-}\gamma_5]}(x) = \left(\frac{M}{P^+}\right)^2 S_T^i\, h_3(x). \tag{3.35}$$

By rewriting Φ in Eq. 3.14 as

$$\begin{aligned}\Phi \;=\; & \frac{P^+}{2}\left\{g_V\, \not{n}_+ + \lambda\, g_A\,\gamma_5\,\not{n}_+ + g_T\,\frac{[\not{S}_T, \not{n}_+]\,\gamma_5}{2}\right\} \\ & + \frac{M}{2}\left\{g_S + g_A\,\gamma_5\,\not{S}_T + \lambda\, g_T\,\frac{[\not{n}_+, \not{n}_-]\,\gamma_5}{2}\right\} \\ & + \frac{M^2}{2P^+}\left\{\frac{1}{2}\, g_V\,\not{n}_- + \frac{1}{2}\,\lambda\, g_A\,\gamma_5\,\not{n}_- + \frac{1}{2}\, g_T\,\frac{[\not{S}_T, \not{n}_-]\,\gamma_5}{2}\right\}. \end{aligned} \tag{3.36}$$

it follows from $\Phi = \int dk^+\ \Phi(x) = P^+ \int dx\ \Phi(x)$ (assuming convergence and integrating over all x-values, which will be discussed later) that

$$\int dx\ f_1(x) = 2\int dx\ f_3(x) = g_V, \tag{3.37}$$

$$\int dx\ g_1(x) = \int dx\ g_T(x) = -2\int dx\ g_3(x) = g_A, \tag{3.38}$$

$$\int dx\ h_1(x) = \int dx\ h_L(x) = 2\int dx\ h_3(x) = g_T, \tag{3.39}$$

$$\int dx\ e(x) = g_S, \tag{3.40}$$

$$\int dx\ e_L(x) = \int dx\ f_T(x) = \int dx\ h(x) = 0. \tag{3.41}$$

Making use of support properties of the distribution functions ($-1 \le x \le 1$) and symmetry relations between quark and antiquark distributions (the latter to be discussed in the next section), $f_1(x) = -\bar{f}_1(-x)$, and finally the fact that the vector charge g_V is in fact the definition of the flavor quantum number, i.e. $g_V \equiv n_q$, the first line turns into the number sum rule

$$\int_0^1 dx\ \left(f_1(x) - \bar{f}_1(x)\right) = n. \tag{3.42}$$

In semi-inclusive deep inelastic scattering or Drell-Yan processes, the matrix elements that are needed for the hadron $\to$ quark pieces in the hard scattering processes are the ones in which the integration over

$\boldsymbol{k}_T$ is not yet performed, namely

$$\begin{aligned}\Phi_{ij}(x,\boldsymbol{k}_T) &= \int dk^-\ \Phi_{ij}(k,P,S)\Big|_{k^+=xP^+,\ k_T} \\ &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\ e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathscr{U}(0,\infty)\,\mathscr{U}(\infty,\xi)\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}. \end{aligned} \tag{3.43}$$

In this case one is sensitive to transverse separation ξ_T. We consider, however, only gauge links attached to each quark along the minus direction, i.e. built from matrix elements with additional A^+ fields, which contribute at the same order in an $1/Q$ expansion (see Fig. 3.2). That this is the result in an actual diagrammatic expansion needs to be proven. It will turn out that the bilocal quark-quark matrix element will be supplemented with links running from 0 to $[0,\infty,\mathbf{0}_T]$ and one running from $[0,\infty,\boldsymbol{\xi}_T]$ to $[0,\xi^-,\boldsymbol{\xi}_T]$ respectively, With the physical condition that any matrix element involving $\overline{\psi}(0)\,A_T(\eta^-=\pm\infty,\eta^+,\eta_T)\,\psi(\xi)$ vanishes, the links can be connected and one has a color gauge invariant quantity, for which after gauge fixing the link becomes unity. At this point it is useful to mention that quark-quark-gluon matrix elements with gluon fields other than A^+ need to be considered separately (see chapter 4).

We write down the expression for $\Phi(x,\boldsymbol{k}_T)$ in terms of n_+, n_- and transverse vectors up to $\mathscr{O}(M/P^+)$, including T-odd parts, but restricting ourselves to twist-two ($\propto 1$) and twist-three ($\propto M/P^+$) parts. Simple kinematic arguments already show that factorization of $\boldsymbol{k}_T$-dependent functions cannot hold beyond twist-three.

$$\begin{aligned}\Phi(x,\boldsymbol{k}_T) = \frac{1}{2}\Bigg\{ & f_1(x,\boldsymbol{k}_T)\,\not{n}_+ + f_{1T}^\perp(x,\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_+^\nu k_T^\rho S_T^\sigma}{M} + g_{1s}(x,\boldsymbol{k}_T)\,\gamma_5\,\not{n}_+ \\ & +h_{1T}(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not{S}_T,\not{n}_+]}{2} + h_{1s}^\perp(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not{k}_T,\not{n}_+]}{2M} + h_1^\perp(x,\boldsymbol{k}_T)\,\frac{i\,[\not{k}_T,\not{n}_+]}{2M}\Bigg\} \\ +\frac{M}{2P^+}\Bigg\{ & e(x,\boldsymbol{k}_T) + f^\perp(x,\boldsymbol{k}_T)\,\frac{\not{k}_T}{M} - f_T(x,\boldsymbol{k}_T)\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{T\sigma} \\ & -\lambda\,f_L^\perp(x,\boldsymbol{k}_T)\,\frac{\epsilon_T^{\rho\sigma}\gamma_\rho k_{T\sigma}}{M} - e_s(x,\boldsymbol{k}_T)\,i\gamma_5 \\ & +g_T'(x,\boldsymbol{k}_T)\,\gamma_5\,\not{S}_T + g_s^\perp(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,\not{k}_T}{M} + h_T^\perp(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not{S}_T,\not{k}_T]}{2M} \\ & +h_s(x,\boldsymbol{k}_T)\,\frac{\gamma_5\,[\not{n}_+,\not{n}_-]}{2} + h(x,\boldsymbol{k}_T)\,\frac{i\,[\not{n}_+,\not{n}_-]}{2}\Bigg\}. \end{aligned} \tag{3.44}$$

We have here use the shorthand notation

$$g_{1s}(x,\boldsymbol{k}_T) \equiv \lambda\,g_{1L}(x,\boldsymbol{k}_T) + g_{1T}(x,\boldsymbol{k}_T)\,\frac{(\boldsymbol{k}_T\cdot\boldsymbol{S}_T)}{M}, \tag{3.45}$$

and similarly for other functions, e.g. $h_{1s}^\perp$, $g_s^\perp$ and h_s. Included are also T-odd functions $f_{1T}^\perp$, $h_1^\perp$, f_T, $f_L^\perp$, e_s and h.

Again we can analyze the Dirac content of the correlation function (twist analysis). For instance for the unit matrix the effect of the integration over k^- is

$$\begin{aligned}\Phi^{[1]}(x,\boldsymbol{k}_T) &= \frac{1}{2}\int dk^-\ Tr(1\,\Phi(k,P,S))\Big|_{k^+=xP^+,\ k_T} \\ &= \frac{M}{P^+}\int d(2k\cdot P)\,dk^2\ A_1(k^2,k\cdot P)\,\delta(\boldsymbol{k}_T^2+k^2-2x\,k\cdot P+x^2M^2) \\ &\equiv \frac{M}{P^+}\,e(x,\boldsymbol{k}_T), \end{aligned} \tag{3.46}$$

where $x=k^+/P^+$ and $\boldsymbol{k}_T$ is the transverse component of the quark momentum k in the frame where P has no transverse components, i.e. frame II discussed in the previous section. The profile function e only depends on x and $\boldsymbol{k}_T^2$. It is expressed as an integral in which all momenta and products thereof are of hadronic size (Λ_H), and is multiplied with a factor of the form $\Lambda_H^{t-4}/(P^+)^{t-2}$, defining the operational twist. It is this factor that will lead in the cross section to a suppression factor $1/Q^{t-2}$.

The various (T-even) Dirac projections $\Phi^{[\Gamma]} = (1/2)\text{Tr}(\Phi\Gamma)$ appearing in here are explicitly

$$\Phi^{[\gamma^+]}(x,\boldsymbol{k}_T) = f_1(x,\boldsymbol{k}_T) - \frac{\epsilon_T^{ij} k_{Ti} S_{Tj}}{M}\, f_{1T}^\perp(x,\boldsymbol{k}_T), \tag{3.47}$$

$$\Phi^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T) = \lambda\, g_{1L}(x,\boldsymbol{k}_T) + g_{1T}(x,\boldsymbol{k}_T)\, \frac{(\boldsymbol{k}_T\cdot\boldsymbol{S}_T)}{M}, \tag{3.48}$$

$$\begin{aligned}\Phi^{[i\sigma^{i+}\gamma_5]}(x,\boldsymbol{k}_T) &= S_T^i\, h_{1T}(x,\boldsymbol{k}_T) + \frac{k_T^i}{M}\, h_{1s}^\perp(x,\boldsymbol{k}_T) - \frac{\epsilon_T^{ij} k_{Tj}}{M}\, h_1^\perp(x,\boldsymbol{k}_T),\\ &= S_T^i\, h_1(x,\boldsymbol{k}_T) + \frac{\lambda k_T^i}{M}\, h_{1L}^\perp(x,\boldsymbol{k}_T)\\ &\quad - \frac{\left(k_T^i k_T^j + \frac{1}{2}\boldsymbol{k}_T^2 g_T^{ij}\right) S_{Tj}}{M^2}\, h_{1T}^\perp(x,\boldsymbol{k}_T) - \frac{\epsilon_T^{ij} k_{Tj}}{M}\, h_1^\perp(x,\boldsymbol{k}_T),\end{aligned} \tag{3.49}$$

and the profile functions that appear multiplied by a factor M/P^+ (twist three) are

$$\Phi^{[1]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\, e(x,\boldsymbol{k}_T) \tag{3.50}$$

$$\Phi^{[\gamma^i]}(x,\boldsymbol{k}_T) = \frac{k_T^i}{P^+}\, f^\perp(x,\boldsymbol{k}_T), \tag{3.51}$$

$$\begin{aligned}\Phi^{[\gamma^i\gamma_5]}(x,\boldsymbol{k}_T) &= \frac{M\, S_T^i}{P^+}\, g_T'(x,\boldsymbol{k}_T) + \frac{k_T^i}{P^+}\, g_s^\perp(x,\boldsymbol{k}_T)\\ &= \frac{M\, S_T^i}{P^+}\, g_T(x,\boldsymbol{k}_T) + \frac{\lambda k_T^i}{P^+}\, g_L^\perp(x,\boldsymbol{k}_T) - \frac{\left(k_T^i k_T^j + \frac{1}{2}\boldsymbol{k}_T^2 g_T^{ij}\right) S_{Tj}}{M^2}\, g_T^\perp(x,\boldsymbol{k}_T),\end{aligned} \tag{3.52}$$

$$\Phi^{[i\sigma^{ij}\gamma_5]}(x,\boldsymbol{k}_T) = \frac{S_T^i k_T^j - k_T^i S_T^j}{P^+}\, h_T^\perp(x,\boldsymbol{k}_T) \tag{3.53}$$

$$\Phi^{[i\sigma^{+-}\gamma_5]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\, h_s(x,\boldsymbol{k}_T). \tag{3.54}$$

Note that sometimes it may be useful to work with the functions projected using $\sigma_{\mu\nu}$ instead of $i\sigma_{\mu\nu}\gamma_5$. These are

$$\Phi^{[\sigma^{i+}]}(x,\boldsymbol{k}_T) = \epsilon_T^{ij}\, S_{Tj}\, h_{1T}(x,\boldsymbol{k}_T) + \frac{\epsilon_T^{ij} k_{Tj}}{M}\, h_{1s}^\perp(x,\boldsymbol{k}_T), \tag{3.55}$$

$$\Phi^{[\sigma^{ij}]}(x,\boldsymbol{k}_T) = \frac{M\,\epsilon_T^{ij}}{P^+}\, h_s(x,\boldsymbol{k}_T), \tag{3.56}$$

$$\Phi^{[\sigma^{+-}]}(x,\boldsymbol{k}_T) = \frac{\epsilon_T^{ij}\, k_{Ti} S_{Tj}}{P^+}\, h_T^\perp(x,\boldsymbol{k}_T). \tag{3.57}$$

The integrated results $f_1(x)$ etc. discussed before are obtained from $f_1(x,\boldsymbol{k}_T^2)$ etc., where one must be aware that $g_1 = g_{1L}$, $h_1 = h_{1T} + (\boldsymbol{k}_T^2/2M^2)h_{1T}^\perp \equiv h_{1T} + h_{1T}^{\perp(1)}$ and $g_T = g_T' + (\boldsymbol{k}_T^2/2M^2)g_T^\perp \equiv g_T' + g_T^{\perp(1)}$. Besides the k_T-integrated functions shown before, it is useful to consider k_T^α-weighted functions,

$$\frac{1}{M}\,\Phi_\partial^\alpha(x) \equiv \int d^2\boldsymbol{k}_T\, \frac{k_T^\alpha}{M}\, \Phi(x,\boldsymbol{k}_T), \tag{3.58}$$

$$\frac{1}{M^2}\,\Phi_{\partial\partial}^{\alpha\beta}(x) \equiv \int d^2\boldsymbol{k}_T\, \frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\, g_T^{\alpha\beta}\right)}{M^2}\, \Phi(x,\boldsymbol{k}_T). \tag{3.59}$$

Note that the operator structure involved is

$$\begin{aligned}\Phi_\partial^\alpha(x,\boldsymbol{k}_T) &= k_T^\alpha\, \Phi(x,\boldsymbol{k}_T)\\ &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ik\cdot\xi}\, \langle P,S|i\partial_T^\alpha \left(\overline{\psi}(0)\,\mathscr{U}(0,\infty)\,\mathscr{U}(\infty,\xi)\,\psi(\xi)\right)|P,S\rangle\bigg|_{\xi^+=0}\\ &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ik\cdot\xi}\, \langle P,S|\overline{\psi}(0)\,\mathscr{U}(0,\infty)\, i\partial_T^\alpha\, \mathscr{U}(\infty,\xi)\,\psi(\xi)|P,S\rangle\bigg|_{\xi^+=0}.\end{aligned} \tag{3.60}$$

Note that in this case choosing the gauge $A^+ = 0$, one cannot just neglect the link operator, because it contains ξ_T dependence. We will return to this in the chapter 4. Note, however, that in the way defined here, the correlation function Φ_∂ is color gauge invariant.

We can use the properties in Appendix A for integrating $\boldsymbol{k}_T^2$ dependent functions over $\boldsymbol{k}_T$. to obtain the nonvanishing projections for twist two

$$\frac{1}{M}\,\Phi_\partial^{\alpha[\gamma^+\gamma_5]}(x) = S_T^\alpha \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M^2}\,g_{1T}(x,\boldsymbol{k}_T) \equiv S_T^\alpha\,g_{1T}^{(1)}(x), \tag{3.61}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[i\sigma^{i+}\gamma_5]}(x) = -g_T^{\alpha i}\,\lambda \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M^2}\,h_{1L}^\perp(x,\boldsymbol{k}_T) \equiv -g_T^{\alpha i}\,\lambda\,h_{1L}^{\perp(1)}(x), \tag{3.62}$$

$$\begin{aligned}\frac{1}{M^2}\,\Phi_{\partial\partial}^{\alpha\beta[i\sigma^{i+}\gamma_5]}(x) &= -\frac{1}{2}\left(g_T^{\alpha i}S_T^\beta + g_T^{\beta i}S_T^\alpha - g_T^{\alpha\beta}S_T^i\right)\int d^2\boldsymbol{k}_T\,\left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^2\,h_{1T}^\perp(x,\boldsymbol{k}_T)\\ &\equiv -\frac{1}{2}\left(g_T^{\alpha i}S_T^\beta + g_T^{\beta i}S_T^\alpha - g_T^{\alpha\beta}S_T^i\right)\,h_{1T}^{\perp(2)}(x),\end{aligned} \tag{3.63}$$

and for twist three,

$$\frac{1}{M}\,\Phi_\partial^{\alpha[\gamma^i]}(x) = -g_T^{\alpha i}\,\frac{M}{P^+}\,f^{\perp(1)}(x), \tag{3.64}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[\gamma^i\gamma_5]}(x) = -g_T^{\alpha i}\,\frac{M}{P^+}\,\lambda\,g_L^{\perp(1)}(x) \tag{3.65}$$

$$\frac{1}{M^2}\,\Phi_{\partial\partial}^{\alpha\beta[\gamma^i\gamma_5]}(x) = -\frac{M}{2P^+}\left(g_T^{\alpha i}S_T^\beta + g_T^{\beta i}S_T^\alpha - g_T^{\alpha\beta}S_T^i\right)\,g_T^{\perp(2)}(x) \tag{3.66}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[i\sigma^{ij}\gamma_5]}(x) = \frac{M}{P^+}\,\left(g_T^{\alpha i}S_T^j - g_T^{\alpha j}S_T^i\right)\,h_T^{\perp(1)}(x), \tag{3.67}$$

$$\frac{1}{M}\,\Phi_\partial^{\alpha[i\sigma^{+-}\gamma_5]}(x) = \frac{M}{P^+}\,S_T^\alpha\,h_T^{(1)}(x). \tag{3.68}$$

We obtain now besides the integrated results, the k_T-weighted results

$$\begin{aligned}\Phi_\partial^\alpha(x) &= \frac{1}{2}\left\{-g_{1T}^{(1)}(x)\,S_T^\alpha\,\not{n}_+\gamma_5 - \lambda\,h_{1L}^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2}\right\}\\ &\quad+\frac{M}{2P^+}\left\{-f^{\perp(1)}(x)\,\gamma^\alpha + \lambda\,g_L^{\perp(1)}(x)\,\gamma^\alpha\gamma_5\right.\\ &\qquad\left.+h_T^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{S}_T]\gamma_5}{2} + h_T^{(1)}(x)\,S_T^\alpha\,\frac{[\not{n}_+,\not{n}_-]\gamma_5}{2}\right\}\end{aligned} \tag{3.69}$$

$$\begin{aligned}\Phi_{\partial\partial}^{\alpha\beta}(x) &= -\frac{1}{2}\,h_{1T}^{\perp(2)}(x)\,\left(\frac{S_T^{\{\alpha}[\gamma^{\beta\}},\not{n}_+]\gamma_5 - g_T^{\alpha\beta}\,[\not{S}_T,\not{n}_+]\gamma_5}{4}\right)\\ &\quad+\frac{M}{2P^+}\,g_T^{\perp(2)}(x)\,\left(\frac{S_T^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not{S}_T\gamma_5}{2}\right)\end{aligned} \tag{3.70}$$

For the matrix elements $\Phi_\partial^+(x)$ no new functions come in. Working in $A^+ = 0$ gauge (or using $\partial^+\mathscr{U}(0,\xi) = \mathscr{U}(0,\xi)D^+$, see chapter 4) one sees that

$$\Phi_\partial^+(x) = xP^+\,\Phi(x) = \int\frac{d\xi^-}{2\pi}\;e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathscr{U}(0,\xi)\,iD^+\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{3.71}$$

and hence using the parametrization of $\Phi^{[\gamma^+]}$,

$$\begin{aligned}2\,\Phi_\partial^{+[\gamma^+]} &= P^+\int dx\;\mathrm{Tr}\left(\Phi_\partial^+(x)\,\gamma^+\right) = (P^+)^2\int dx\;x\,\mathrm{Tr}\left(\Phi\,\gamma^+\right)\\ &= 2(P^+)^2\underbrace{\int dx\;xf_1(x)}_{\epsilon_q} = \langle P,S|\underbrace{\overline{\psi}(0)\,\gamma^+\,iD^+\psi(0)}_{\theta_q^{++}}|P,S\rangle.\end{aligned} \tag{3.72}$$

Realizing that θ_q^{++} is only part of the energy-momentum tensor, it leads to the momentum sumrule $\int dx\;xf_1(x) = \epsilon_q \le 1$. Using the support properties of the distribution functions ($-1 \le x \le 1$) and the symmetry relation $f_1(x) = -\bar{f}_1(-x)$, the sum rule reads

$$\int_0^1 dx\;x\left(f_1(x) + \bar{f}_1(x)\right) = \epsilon_q \le 1. \tag{3.73}$$

3.2 Antiquark distribution functions

The profile functions for antiquarks in a hadron are obtained from the matrix elements

$$\begin{aligned}\overline{\Phi}_{ij}(k,P,S) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\mathscr{U}(0,\xi)\psi_i(\xi)\overline{\psi}_j(0)|P,S\rangle \\ &= -\frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\mathscr{U}(0,\xi)\psi_i(\xi)|P,S\rangle. \end{aligned} \tag{3.74}$$

For a definition of the profile functions that is consistent with the definition of free particle and antiparticle states, one needs the correlation function Φ^c that is defined analogous to Φ but using the conjugate spinors $\psi^c = C\,\overline{\psi}^T$, where $C\,\gamma_\mu^T\,C^\dagger = -\gamma_\mu$,

$$\Phi^c_{ij}(k,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}^c_j(0)\mathscr{U}(0,\xi)\psi^c_i(\xi)|P,S\rangle. \tag{3.75}$$

The relation between these quantities is $\Phi^c = -C\,\overline{\Phi}^T C^\dagger$. Using $\Phi^{c[\Gamma]}$ to define the antiquark profile functions, $\overline{f}(x,\boldsymbol{k}_T)$, etc., one must be aware of the relative sign ($\pm$) between $\overline{\Phi}^{[\Gamma]}$ and $\Phi^{c[\Gamma]}$ depending on $\Gamma = \mp C\Gamma^T C^\dagger$. Explicitly,

$$\overline{\Phi}^{[\Gamma]} = +\Phi^{c[\Gamma]} \qquad \text{for } \Gamma = \gamma_\mu, \sigma_{\mu\nu} \text{ or } i\sigma_{\mu\nu}\gamma_5,$$
$$\overline{\Phi}^{[\Gamma]} = -\Phi^{c[\Gamma]} \qquad \text{for } \Gamma = 1,\ \gamma_\mu\gamma_5 \text{ and } i\gamma_5,$$

We note also that (at the twist two and twist three level) the anticommutation relations for fermions can be used to obtain the symmetry relation

$$\overline{\Phi}_{ij}(k,P,S) = -\Phi_{ij}(-k,P,S). \tag{3.76}$$

For the profile functions this gives the symmetry relations

$$\overline{f}_1(x,\boldsymbol{k}_T^2) = -f_1(-x,\boldsymbol{k}_T^2) \tag{3.77}$$

and identically for g_{1T}, h_{1T}, $h_{1T}^\perp$, $g_L^\perp$ and h_L (C-even functions), while

$$\overline{g}_{1L}(x,\boldsymbol{k}_T^2) = g_{1L}(-x,\boldsymbol{k}_T^2) \tag{3.78}$$

and identically for $h_{1L}^\perp$, e, $f^\perp$, g_T', $g_T^\perp$, $h_T^\perp$ and h_T (C-odd functions).

Explicitly we get (fixing k^- and integrating over k^+)

$$\begin{aligned}&\frac{1}{2}\int dk^+\;\overline{\Phi}(k,P,S)\Big|_{k^-=xP^-,\;k_T} = \\ &\quad\frac{1}{4}\left\{\overline{f}_1(x,\boldsymbol{k}_T)\,\not{n}_- + \overline{g}_{1s}(x,\boldsymbol{k}_T)\,\not{n}_-\gamma_5 - \overline{h}_{1T}(x,\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5 S_T^\mu n_-^\nu - \overline{h}_{1s}^\perp(x,\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5 k_T^\mu n_-^\nu}{M}\right\} \\ &\quad+\frac{M}{4P^-}\left\{-\overline{e}(x,\boldsymbol{k}_T) + \overline{f}^\perp(x,\boldsymbol{k}_T)\,\frac{\not{k}_T}{M} + \overline{g}_T'(x,\boldsymbol{k}_T)\,\not{S}_T\gamma_5 + \overline{g}_s^\perp(x,\boldsymbol{k}_T)\,\frac{\not{k}_T\gamma_5}{M}\right. \\ &\qquad\left. -\overline{h}_T^\perp(x,\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5 S_T^\mu k_T^\nu}{M} - \overline{h}_s(x,\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5 n_-^\mu n_+^\nu\right\}. \end{aligned} \tag{3.79}$$

and the projections for twist two are

$$\overline{\Phi}^{[\gamma^-]}(x,\boldsymbol{k}_T) = \overline{f}(x,\boldsymbol{k}_T), \tag{3.80}$$
$$\overline{\Phi}^{[\gamma^-\gamma_5]}(x,\boldsymbol{k}_T) = -\overline{g}_{1s}(x,\boldsymbol{k}_T), \tag{3.81}$$
$$\overline{\Phi}^{[i\sigma^{i-}\gamma_5]}(x,\boldsymbol{k}_T) = S_T^i\,\overline{h}_{1T}(x,\boldsymbol{k}_T) + \frac{k_T^i}{M}\,\overline{h}_{1s}^\perp(x,\boldsymbol{k}_T), \tag{3.82}$$

while those of twist three are

$$\overline{\Phi}^{[1]}(x, \boldsymbol{k}_T) = -\frac{M}{P^-}\,\overline{e}(x, \boldsymbol{k}_T), \tag{3.83}$$

$$\overline{\Phi}^{[\gamma^i]}(x, \boldsymbol{k}_T) = \frac{k_T^i}{P^-}\,\overline{f}^\perp(x, \boldsymbol{k}_T), \tag{3.84}$$

$$\overline{\Phi}^{[\gamma^i\gamma_5]}(x, \boldsymbol{k}_T) = -\frac{M\,S_T^i}{P^-}\,\overline{g}'_T(x, \boldsymbol{k}_T) - \frac{k_T^i}{P^-}\,\overline{g}_s^\perp(x, \boldsymbol{k}_T), \tag{3.85}$$

$$\overline{\Phi}^{[i\sigma^{ij}\gamma_5]}(x, \boldsymbol{k}_T) = \frac{S_T^i k_T^j - k_T^i S_T^j}{P^-}\,\overline{h}_T^\perp(x, \boldsymbol{k}_T), \tag{3.86}$$

$$\overline{\Phi}^{[i\sigma^{-+}\gamma_5]}(x, \boldsymbol{k}_T) = \frac{M}{P^-}\,\overline{h}_s(x, \boldsymbol{k}_T). \tag{3.87}$$

Using $\sigma_{\mu\nu}$ instead of $i\sigma_{\mu\nu}\gamma_5$ one has

$$\overline{\Phi}^{[\sigma^{i-}]}(x, \boldsymbol{k}_T) = -\epsilon_T^{ij}\,S_{Tj}\,\overline{h}_{1T}(x, \boldsymbol{k}_T) - \frac{\epsilon_T^{ij} k_{Tj}}{M}\,\overline{h}_{1s}^\perp(x, \boldsymbol{k}_T), \tag{3.88}$$

$$\overline{\Phi}^{[\sigma^{ij}]}(x, \boldsymbol{k}_T) = -\frac{M\,\epsilon_T^{ij}}{P^-}\,\overline{h}_s(x, \boldsymbol{k}_T), \tag{3.89}$$

$$\overline{\Phi}^{[\sigma^{-+}]}(x, \boldsymbol{k}_T) = -\frac{\epsilon_T^{ij}\,k_{Ti}S_{Tj}}{P^-}\,\overline{h}_T^\perp(x, \boldsymbol{k}_T). \tag{3.90}$$

The integrated results for antiquarks are

$$\frac{1}{2}\int dk^+\,d^2\boldsymbol{k}_T\;\overline{\Phi}(k,P,S)\Big|_{k^-=xP^-} = \frac{1}{4}\left\{\overline{f}_1(x)\,\not{n}_- + \lambda\,\overline{g}_1(x)\,\not{n}_-\gamma_5 + \overline{h}_1(x)\,\frac{[\not{S}_T,\not{n}_-]\gamma_5}{2}\right\}$$
$$+\frac{M}{4P^-}\left\{-\overline{e}(x) + \overline{g}_T(x)\,\not{S}_T\gamma_5 + \lambda\,\overline{h}_L(x)\,\frac{[\not{n}_-,\not{n}_+]\gamma_5}{2}\right\}, \tag{3.91}$$

$$\frac{1}{2}\int dk^+\,d^2\boldsymbol{k}_T\;\frac{k_T^\alpha}{M}\,\overline{\Phi}(k,P,S)\Big|_{k^-=xP^-} = \frac{1}{4}\left\{\overline{g}_{1T}^{(1)}(x)\,S_T^\alpha\,\not{n}_-\gamma_5 - \lambda\,\overline{h}_{1L}^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{n}_-]\gamma_5}{2}\right\}$$
$$+\frac{M}{4P^-}\left\{-\overline{f}^{\perp(1)}(x)\,\gamma^\alpha - \lambda\,\overline{g}_L^{\perp(1)}(x)\,\gamma^\alpha\gamma_5\right.$$
$$\left. +\overline{h}_T^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{S}_T]\gamma_5}{2} + \overline{h}_T^{(1)}(x)\,S_T^\alpha\,\frac{[\not{n}_-,\not{n}_+]\gamma_5}{2}\right\} \tag{3.92}$$

$$\frac{1}{2}\int dk^+\,d^2\boldsymbol{k}_T\;\frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\,g_T^{\alpha\beta}\right)}{M^2}\,\overline{\Phi}(k,P,S)\Bigg|_{k^-=xP^-} =$$
$$-\frac{1}{4}\,\overline{h}_{1T}^{\perp(2)}(x)\,\left(\frac{S_T^{\{\alpha}[\gamma^{\beta\}},\not{n}_-]\gamma_5 - g_T^{\alpha\beta}\,[\not{S}_T,\not{n}_-]\gamma_5}{4}\right)$$
$$-\frac{M}{4P^-}\,\overline{g}_T^{\perp(2)}(x)\,\left(\frac{S_T^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not{S}_T\gamma_5}{2}\right) \tag{3.93}$$

3.3 Result for an ensemble of free quarks (parton model)

It is instructive to calculate the correlation function for a free quark. This is given by

$$\phi_{ij}(p,s;k) = u_i(k,s)\overline{u}_j(k,s)\delta^4(k-p) = \tfrac{1}{2}\big((\not{k}+m)(1+\gamma_5\not{s})\big)_{ij}\delta^4(k-p), \tag{3.94}$$

where the momentum and spin of the quark are parametrized as

$$k = \left[\frac{\boldsymbol{k}_T^2+m^2}{2k^+},\ k^+,\ \boldsymbol{k}_T\right], \tag{3.95}$$

$$s = \left[-\frac{m\,\lambda_q}{2k^+}+\frac{\boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{k^+}+\frac{\lambda_q\,\boldsymbol{k}_T^2}{2m\,k^+},\ \frac{\lambda_q\,k^+}{m},\ \boldsymbol{s}_{qT}+\frac{\lambda_q}{m}\boldsymbol{k}_T\right] \tag{3.96}$$

in terms of a quark lightcone helicity λ_q and a quark lightcone transverse polarization s_{qT}, such that $k\cdot s = 0$ and $\lambda_q^2+\boldsymbol{s}_{qT}^2 = -s^2 = 1$. Note that this helicity only is a true helicity for a quark with infinite momentum. It is then straightforward to calculate the projections for a free quark target. For twist two

$$\phi^{[\gamma^+]}(k) = \delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = f_{q\uparrow/q\lambda}+f_{q\downarrow/q\lambda}, \tag{3.97}$$

$$\phi^{[\gamma^+\gamma_5]}(k) = \lambda_q\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = f_{q\uparrow/q\lambda}-f_{q\downarrow/q\lambda}, \tag{3.98}$$

$$\phi^{[i\sigma^{i+}\gamma_5]}(k) = \boldsymbol{s}_{qT}^i\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = f_{q\rightarrow/q\boldsymbol{s}_T}-f_{q\leftarrow/q\boldsymbol{s}_T}, \tag{3.99}$$

where $\xi = k^+/p^+$, and we have indicated the intuitive interpretation in terms of probabilities of finding quarks in a quark with spin given by λ and $\boldsymbol{s}_T$ (see below). For twist three we get

$$\phi^{[1]}(k) = \frac{m}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.100}$$

$$\phi^{[\gamma^i]}(k) = \frac{\boldsymbol{k}_T^i}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.101}$$

$$\phi^{[\gamma^i\gamma_5]}(k) = \frac{(m\,\boldsymbol{s}_{qT}^i+\lambda_q\,\boldsymbol{k}_T^i)}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.102}$$

$$\phi^{[i\sigma^{ij}\gamma_5]}(k) = \frac{\boldsymbol{s}_{qT}^i\boldsymbol{k}_T^j-\boldsymbol{k}_T^i\boldsymbol{s}_{qT}^j}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.103}$$

$$\phi^{[i\sigma^{+-}\gamma_5]}(k) = \frac{m\,\lambda_q-\boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{k^+}\,\delta\,(\xi-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T). \tag{3.104}$$

In a parton model description of the target, one uses the expansion for the free quark field to get

$$\Phi_{ij}(k) = 2\,\delta(k^2-m^2)\left[\theta(x)u_i^{(\beta)}(k)\mathscr{P}_{\beta\alpha}(k)\bar{u}_j^{(\alpha)}(k)-\theta(-x)v_i^{(\beta)}(-k)\overline{\mathscr{P}}_{\beta\alpha}(-k)\bar{v}_j^{(\alpha)}(-k)\right], \tag{3.105}$$

where $x = k^+/P^+$. The use of lightcone coordinates is convenient because of the integration over k^- that is needed in deep inelastic processes. The functions $\mathscr{P}$ and $\overline{\mathscr{P}}$ are given by

$$\mathscr{P}_{\beta\alpha}(k) = \mathscr{P}_{\beta\alpha}(x,\boldsymbol{k}_T^2) \equiv \frac{1}{2(2\pi)^3}\int\frac{dx'\,d^2\boldsymbol{k}'_T}{(2\pi)^3\,2x'}\,\langle PS|b_\alpha^\dagger(k')b_\beta(k)|PS\rangle, \tag{3.106}$$

$$\overline{\mathscr{P}}_{\beta\alpha}(k) = \overline{\mathscr{P}}_{\beta\alpha}(x,\boldsymbol{k}_T^2) \equiv \frac{1}{2(2\pi)^3}\int\frac{dx'\,d^2\boldsymbol{k}'_T}{(2\pi)^3\,2x'}\,\langle PS|d_\beta^\dagger(k')d_\alpha(k)|PS\rangle. \tag{3.107}$$

Note that $\mathscr{P}_{\beta\alpha}(k)$ is a production matrix in the quark spin-space of which the trace is the quark density operator evaluated in the target. The Dirac structure can be parametrized as

$$u^{(\beta)}(k,s)\mathscr{P}_{\beta\alpha}(k)\bar{u}^{(\alpha)}(k,s) = \mathscr{P}(k)(\not{k}+m)\left(\frac{1+\gamma_5\not{s}(k)}{2}\right) \tag{3.108}$$

$$v^{(\beta)}(k,s)\overline{\mathscr{P}}_{\beta\alpha}(k)\bar{v}^{(\alpha)}(k,s) = \overline{\mathscr{P}}(k)(\not{k}-m)\left(\frac{1+\gamma_5\overline{\not{s}}(k)}{2}\right) \tag{3.109}$$

in terms of a positive definite quark and antiquark densities $\mathscr{P}(k)$ and $\overline{\mathscr{P}}(k)$ and spin densities $s(k)$ and $\overline{s}(k)$ which satisfy $0\le -s^2(k)\le 1$ or $0\le\lambda_q^2(k)+\boldsymbol{s}_{qT}^2(k)\le 1$. Inserting the free field expansion in the current expectation value $\langle PS|\overline{\psi}(0)\gamma^\mu\psi(0)|PS\rangle = 2P^\mu(N-\overline{N})$, where N and $\overline{N}$ are total number of quarks and antiquarks respectively one obtains from the +-component the normalizations $\int_0^1 dx\int d^2\boldsymbol{k}_T\mathscr{P}(x,\boldsymbol{k}_T^2) = N$ and $\int_0^1 dx\int d^2\boldsymbol{k}_T\overline{\mathscr{P}}(x,\boldsymbol{k}_T^2) = \overline{N}$.

Integrating over k^- one obtains the result

$$\begin{aligned}\frac{1}{2}\int dk^-\,\Phi(k) &= \theta(x)\frac{\mathscr{P}(x,\boldsymbol{k}_T^2)}{2k^+}(\not{k}+m)\left(\frac{1+\gamma_5\not{s}(x,\boldsymbol{k}_T)}{2}\right)\\ &\quad-\theta(-x)\frac{\overline{\mathscr{P}}(-x,\boldsymbol{k}_T^2)}{2k^+}(\not{k}+m)\left(\frac{1+\gamma_5\bar{\not{s}}(-x,-\boldsymbol{k}_T)}{2}\right).\end{aligned} \tag{3.110}$$

This gives (for $x > 0$)

$$\Phi^{[\gamma^+]}(k) = \mathscr{P}(x,\boldsymbol{k}_T^2) = \mathscr{P}_{RR}+\mathscr{P}_{LL} = \mathscr{P}_{\alpha\alpha}+\mathscr{P}_{\beta\beta}, \tag{3.111}$$

$$\Phi^{[\gamma^+\gamma_5]}(k) = \lambda_q(x,\boldsymbol{k}_T)\,\mathscr{P}(x,\boldsymbol{k}_T^2) = \mathscr{P}_{RR}-\mathscr{P}_{LL}, \tag{3.112}$$

$$\Phi^{[i\sigma^{i+}\gamma_5]}(k) = \boldsymbol{s}_{qT}^i(x,\boldsymbol{k}_T)\,\mathscr{P}(x,\boldsymbol{k}_T^2) = \mathscr{P}_{\alpha\alpha}-\mathscr{P}_{\beta\beta}, \tag{3.113}$$

where the indices R/L and α/β are particular (chiral and transverse spin) projections for the fermion fields or spinors, obtained using the projection operators $P_{R/L} = (1\pm\gamma_5)/2$ and $Q^i_{\alpha/\beta} = (1\pm\gamma^i\gamma_5)/2$. The interpretation of the first equation is that of the probability for finding quarks in a target. The second equation is interpreted as the probability for righthanded quarks minus that for lefthanded quarks. The third equation is interpreted as the probability for quarks with spin parallel to the transverse direction i minus that for quarks with spin opposite. The chiral structure of these functions is $\overline{R}L+\overline{L}R$ and they are referred to as *chirally odd* distribution functions. Equating these unpolarized and polarized densities to the distribution functions defined in the previous section,

$$\mathscr{P}(x,\boldsymbol{k}_T^2) = f_1(x,\boldsymbol{k}_T), \tag{3.114}$$

$$\lambda_q(x,\boldsymbol{k}_T)\,\mathscr{P}(x,\boldsymbol{k}_T^2) = \lambda\,g_{1L}(x,\boldsymbol{k}_T)+\frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\,g_{1T}(x,\boldsymbol{k}_T) = g_{1s}(x,\boldsymbol{k}_T), \tag{3.115}$$

$$\begin{aligned}\boldsymbol{s}_{qT}^i(x,\boldsymbol{k}_T)\,\mathscr{P}(x,\boldsymbol{k}_T^2) &= \boldsymbol{S}_T^i\,h_{1T}(x,\boldsymbol{k}_T)+\frac{\boldsymbol{k}_T^i}{M}\left[h_{1L}^\perp(x,\boldsymbol{k}_T)\lambda+h_{1T}^\perp(x,\boldsymbol{k}_T)\frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\right]\\ &= \boldsymbol{S}_T^i\,h_{1T}(x,\boldsymbol{k}_T)+\frac{\boldsymbol{k}_T^i}{M}\,h_{1s}^\perp(x,\boldsymbol{k}_T),\end{aligned} \tag{3.116}$$

shows how the functions g_{1L}, g_{1T}, h_1, $h_{1L}^\perp$ and $h_{1T}^\perp$ are to be interpreted as longitudinal and transverse spin distributions given the spin of the hadron (λ and $\boldsymbol{S}_T$). For the antiquarks the same relations hold between the antiquark helicity $\overline{\lambda}_q$ and transverse polarization $\overline{\boldsymbol{s}}_T$ on the one hand and the antiquark distributions on the other hand. Extending to all x, results are obtained in accordance with the symmetry relations in the previous section, e.g. $f_1(x,\boldsymbol{k}_T) = \theta(x)\mathscr{P}(x,\boldsymbol{k}_T^2)-\theta(-x)\overline{\mathscr{P}}(x,\boldsymbol{k}_T^2)$.

We note that at the twist two level this parton interpretation can be made rigorous as the distribution functions can be expressed as densities involving the socalled good components of ψ, $\psi_+\equiv P_+\psi$ obtained with the projection operator $P_+ = \frac{1}{2}\gamma^-\gamma^+$. In lightfront quantization a Fourier expansion for the good components (at $x^+=0$) can be written down in which the Fourier coefficients can be interpreted as particle and antiparticle creation and annihilation operators The different spin-distributions involve projection operators ($P_{R/L}$ and $Q^1_{\alpha/\beta}$) that commute with P_+. At twist three the analysis of the quark - quark correlation functions lead to a number of new distribution functions. For an ensemble of free quarks they can also be expressed in the quark densities and in this way related to the (six) twist two distribution functions. Explicitly one has for the ensemble of free quarks,

$$e(x,\boldsymbol{k}_T) = \frac{m}{Mx}\,f_1(x,\boldsymbol{k}_T), \tag{3.117}$$

$$f^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\,f_1(x,\boldsymbol{k}_T), \tag{3.118}$$

$$g_T'(x,\boldsymbol{k}_T) = \frac{m}{Mx}\,h_{1T}(x,\boldsymbol{k}_T), \tag{3.119}$$

$$g_s^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\,g_{1s}(x,\boldsymbol{k}_T)+\frac{m}{Mx}\,h_{1s}^\perp(x,\boldsymbol{k}_T), \tag{3.120}$$

$$h_T^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\,h_{1T}(x,\boldsymbol{k}_T), \tag{3.121}$$

$$h_s(x,\boldsymbol{k}_T) = \frac{m}{Mx}\,g_{1s}(x,\boldsymbol{k}_T)+\frac{k_T\cdot S_T}{Mx}\,h_{1T}(x,\boldsymbol{k}_T)+\frac{k_T^2}{M^2x}\,h_{1s}^\perp. \tag{3.122}$$

As we will show in one of the next sections, the above results are not generally true; the presence of nonvanishing quark - quark - gluon correlation functions causes deviations from these naive parton model results.

Summarizing we get for $\Phi^{[\Gamma]}(x,\boldsymbol{k}_T)$ and $\Phi^{[\Gamma]}(x)$, ordered according to twist, chirality and time-reversal behavior for unpolarized (U), longitudinally polarized (L) and transversely polarized (T) (spin 1/2) hadrons:

DISTRIBUTIONS						
		χ-even		χ-odd		
$\Phi^{[\Gamma]}(x,\boldsymbol{k}_T)$		T-even	T-odd	T-even	T-odd	
twist 2	U	f_1			$h_1^\perp$	
	L	g_{1L}		$h_{1L}^\perp$		
	T	g_{1T}	$f_{1T}^\perp$	h_{1T} $h_{1T}^\perp$		
twist 3	U	$f^\perp$	$g^\perp$	e	h	
	L	$g_L^\perp$	$f_L^\perp$	h_L	e_L	
	T	g_T' $g_T^\perp$	f_T' $f_T^\perp$	h_T $h_T^\perp$	e_T $e_T^\perp$	

DISTRIBUTIONS			
		χ-even	χ-odd
$\Phi^{[\Gamma]}(x)$		T-even	T-even
twist 2	U	f_1	
	L	g_1	
	T		h_1
twist 3	U		e
	L		h_L
	T	g_T	

3.4 Fragmentation: from quarks to hadron

For the fragmentation of quarks into hadrons we need the correlation function

$$\begin{aligned}\Delta_{ij}(k,P_h,S_h) &= \sum_X \frac{1}{(2\pi)^4}\int d^4\xi\ e^{ik\cdot\xi}\,\langle 0|\mathscr{U}(0,\xi)\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)|0\rangle \\ &= \frac{1}{(2\pi)^4}\int d^4\xi\ e^{ik\cdot\xi}\,\langle 0|\mathscr{U}(0,\xi)\psi_i(\xi)a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle,\end{aligned} \qquad (3.123)$$

where an averaging over color indices is implicit. We note, however that fragmentation is into a hadron in a specified spin state.

The use of intermediate states X and in addition one specified state with momentum P_h needs some explanation. First note that the unit operator can be written as

$$\mathscr{I} \equiv \sum_X |X\rangle\langle X| = \sum_{p=0}^{\infty}\mathscr{I}_p \qquad (3.124)$$

with

$$\mathscr{I}_p = \frac{1}{p!}\int d\tilde{k}_1\ldots d\tilde{k}_p\ a^\dagger(k_1)\ldots a^\dagger(k_p)|0\rangle\langle 0|a(k_1)\ldots a(k_p) \qquad (3.125)$$

containing the p-particle states (with $d\tilde{k}$ being the invariant one-particle phase-space). Thus

$$\begin{aligned}\sum_X |P_h,X\rangle\langle P_h,X| &= |P_h\rangle\langle P_h| + \int d\tilde{k}_1\ |P_h,k_1\rangle\langle P_h,k_1| \\ &\quad + \frac{1}{2!}\int d\tilde{k}_1 d\tilde{k}_2\ |P_h,k_1,k_2\rangle\langle P_h,k_1,k_2| + \ldots \\ &= a_h^\dagger\mathscr{I}a_h = a_h^\dagger a_h.\end{aligned} \qquad (3.126)$$

After integrating over P_h one obtains

$$\int d\tilde{P}_h\ \sum_X |P_h,X\rangle\langle P_h,X| = \sum_{p=0}^{\infty} p\,\mathscr{I}_p, \qquad (3.127)$$

which is the number operator N_h. This will become relevant when one integrates over the phase-space of particles in the final state to go from 1-particle inclusive to inclusive scattering processes.

For the Dirac structure the same expansion as before can be written down,

$$\begin{aligned}\Delta(k,P_h,S_h) &= M_h\,B_1 + B_2\,\not{P}_h + B_3\not{k} + (B_4/M_h)\,\sigma^{\mu\nu}P_{h\mu}k_\nu \\ &\quad + i\,B_5(k\cdot S_h)\gamma_5 + M_h\,B_6\not{S}_h\gamma_5 + (B_7/M_h)(k\cdot S_h)\not{P}_h\gamma_5 + (B_8/M_h)(k\cdot S_h)\not{k}\gamma_5 \\ &\quad + i\,B_9\,\sigma^{\mu\nu}\gamma_5\,S_{h\mu}P_{h\nu} + i\,B_{10}\,\sigma^{\mu\nu}\gamma_5\,S_{h\mu}k_\nu + i\,(B_{11}/M_h^2)\,(k\cdot S_h)\,\sigma^{\mu\nu}\gamma_5\,k_\mu P_{h\nu} \\ &\quad + (B_{12}/M_h)\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P_h^\nu k^\rho S_h^\sigma,\end{aligned} \qquad (3.128)$$

where the amplitudes B_i depend on $P_h \cdot k$ and k^2. As the states $|P_h, X\rangle$ in the above expression for Δ_{ij} are not plane waves, one cannot apply time-reversal invariance. So the amplitudes B_4, B_5 and B_{12} do not vanish.

The fact that the time-reversal-odd amplitudes do not vanish is a consequence of the final state interactions of the produced hadron h. To see this, it is instructive (even necessary) to treat $|P_h, X\rangle$ as physical states. In that case they should be labeled as outstates. Consider the multi-channel matrix

$$\Delta_{ij}^{(ff')} = \sum_X \int d^4\xi \; e^{ik\cdot\xi} \, \langle 0|\psi_i(\xi)|f;\text{out}\rangle\langle f';\text{out}|\overline{\psi}_j(0)|0\rangle \tag{3.129}$$

(Note that $\Delta^{(f'f)} = \Delta^{(ff')\dagger}$). The behavior of $\Delta^{(ff')}$ under time reversal involves

$$\begin{aligned}\langle f';\text{out}|\overline{\psi}_j(\xi)|0\rangle &= \langle \tilde{f}';\text{in}|(\overline{-i\gamma_5 C\psi})_j(-\bar{\xi})|0\rangle^* \\ &= \langle \tilde{f}';\text{out}|S^\dagger\,(\overline{-i\gamma_5 C\psi})_j(-\bar{\xi})|0\rangle^*\end{aligned} \tag{3.130}$$

from which one finds that the combination

$$\tilde{\Delta}^{ff} = (\sqrt{S})^{ff_1}\Delta^{f_1 f_2}(\sqrt{S^\dagger})^{f_2 f} \tag{3.131}$$

behaves as

$$\tilde{\Delta}^*(k, P_h, S_h) = (-i\gamma_5 C)\,\tilde{\Delta}(\bar{k}, \bar{P}_h, \bar{S}_h)\,(-i\gamma_5 C). \tag{3.132}$$

If the final state would be one unique channel obtains zero because $S = e^{2i\delta}$. For two channels the S-matrix can e.g. be parametrized as

$$S = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix}\begin{pmatrix} e^{2i\delta_1} & 0 \\ 0 & e^{2i\delta_2} \end{pmatrix}\begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}. \tag{3.133}$$

Working in channel space with the diagonal S-matrix one has

$$\tilde{\Delta}^{ff} = \begin{pmatrix} \Delta^{(11)} & e^{i(\delta_1-\delta_2)}\,\Delta^{(12)} \\ e^{-i(\delta_1-\delta_2)}\,\Delta^{(21)} & \Delta^{(22)} \end{pmatrix}. \tag{3.134}$$

From the hermiticity condition one knows that for Δ the amplitude analysis requires real amplitudes $B_i^{(ff')}$ for $i = 1, 12$. The time-reversal invariance condition applied to $\tilde{\Delta}$ requires $\tilde{B}_i^* = \tilde{B}_i$ for $i = 1, 2, 3, 6, \ldots, 11$ and $\tilde{B}_i^* = -\tilde{B}_i$ for $i = 4, 5, 12$. Thus one finds

$$B_i^{(11)} = B_i^{(22)} = 0 \qquad \text{(i = 1, 2, 3, 6,\ldots, 11)} \tag{3.135}$$

$$\sin(\delta_1-\delta_2)\,B_i^{(12)} = 0 \qquad \text{(i = 1, 2, 3, 6,\ldots, 11)} \tag{3.136}$$

$$\cos(\delta_1-\delta_2)\,B_i^{(12)} = 0 \qquad \text{(i = 4, 5, 12)}, \tag{3.137}$$

In general one can make a partial wave expansion of the final state in states $|J, M\rangle$ and one finds

$$\sum_{J,J'} \sin\left(\delta_J - \delta_{J'}\right)\,B_i^{(JJ')} = 0 \qquad \text{(i = 1, 2, 3, 6,\ldots, 11)} \tag{3.138}$$

$$\sum_{J,J'} \cos\left(\delta_J - \delta_{J'}\right)\,B_i^{(JJ')} = 0 \qquad \text{(i = 4, 5, 12)}, \tag{3.139}$$

The first equation is a constraint between partial waves which is trivially satisfied in the absence of final state interactions. The second equation implies in the absence of final state interactions that the amplitudes B_4, B_5 and B_{12} vanish.

For the fragmentation a twist analysis of Δ considering the projections

$$\begin{aligned}\Delta^{[\Gamma]}(z, \boldsymbol{k}_T) &= \frac{1}{4z}\int dk^+ \; Tr(\Delta\,\Gamma)\Big|_{k^-=P_h^-/z,\; k_T} \\ &= \int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{4z\,(2\pi)^3}\; e^{ik\cdot\xi}\, Tr\langle 0|\mathscr{U}(0,\xi)\psi(\xi)a_h^\dagger a_h\overline{\psi}(0)\Gamma|0\rangle\Big|_{\xi^-=0},\end{aligned} \tag{3.140}$$

leads to the following set of twist two profile functions, which depend on $z = P_h^-/k^-$ and $\boldsymbol{k}_T^2 = -k_\perp^2$,

$$\Delta^{[\gamma^-]}(z,\boldsymbol{k}_T) = D_1(z,-z\boldsymbol{k}_T) + \frac{\epsilon_T^{ij} k_{Ti} S_{hTj}}{M_h}\, D_{1T}^{\perp}(z,-z\boldsymbol{k}_T), \tag{3.141}$$

$$\Delta^{[\gamma^-\gamma_5]}(z,\boldsymbol{k}_T) = G_{1s}(z,-z\boldsymbol{k}_T) \tag{3.142}$$

$$\Delta^{[i\sigma^{i-}\gamma_5]}(z,\boldsymbol{k}_T) = S_{hT}^i\, H_{1T}(z,-z\boldsymbol{k}_T) + \frac{k_T^i}{M_h}\, H_{1s}^{\perp}(z,-z\boldsymbol{k}_T) + \frac{\epsilon_T^{ij} k_{Tj}}{M_h}\, H_1^{\perp}(z,-z\boldsymbol{k}_T), \tag{3.143}$$

and those of twist three are

$$\Delta^{[1]}(z,\boldsymbol{k}_T) = \frac{M_h}{P_h^-}\, E(z,-z\boldsymbol{k}_T), \tag{3.144}$$

$$\Delta^{[\gamma^i]}(z,\boldsymbol{k}_T) = \frac{k_T^i}{P_h^-}\, D^{\perp}(z,-z\boldsymbol{k}_T) + \frac{\lambda_h\, \epsilon_T^{ij} k_{Tj}}{P_h^-}\, D_L^{\perp}(z,-z\boldsymbol{k}_T) + \frac{M_h\, \epsilon_T^{ij} S_{hT\,j}}{P_h^-}\, D_T(z,-z\boldsymbol{k}_T), \tag{3.145}$$

$$\Delta^{[i\gamma_5]}(z,\boldsymbol{k}_T) = \frac{M_h}{P_h^-}\, E_s(z,-z\boldsymbol{k}_T), \tag{3.146}$$

$$\Delta^{[\gamma^i\gamma_5]}(z,\boldsymbol{k}_T) = \frac{M_h\, S_{hT}^i}{P_h^-}\, G_T'(z,-z\boldsymbol{k}_T) + \frac{k_T^i}{P_h^-}\, G_s^{\perp}(z,-z\boldsymbol{k}_T), \tag{3.147}$$

$$\Delta^{[i\sigma^{ij}\gamma_5]}(z,\boldsymbol{k}_T) = \frac{S_{hT}^i k_T^j - k_T^i S_{hT}^j}{P_h^-}\, H_T^{\perp}(z,-z\boldsymbol{k}_T) + \frac{M_h\, \epsilon_T^{ij}}{P_h^-}\, H(z,-z\boldsymbol{k}_T), \tag{3.148}$$

$$\Delta^{[i\sigma^{-+}\gamma_5]}(z,\boldsymbol{k}_T) = \frac{M_h}{P_h^-}\, H_s(z,-z\boldsymbol{k}_T). \tag{3.149}$$

Again it may sometimes be useful to work with the functions projected using $\sigma_{\mu\nu}$ instead of $i\sigma_{\mu\nu}\gamma_5$. These are

$$\Delta^{[\sigma^{i-}]}(z,\boldsymbol{k}_T) = -\epsilon_T^{ij}\, S_{hTj}\, H_{1T}(z,-z\boldsymbol{k}_T) - \frac{\epsilon_T^{ij} k_{Tj}}{M_h}\, H_{1s}^{\perp}(z,-z\boldsymbol{k}_T) + \frac{k_T^i}{M_h}\, H_1^{\perp}(z,-z\boldsymbol{k}_T), \tag{3.150}$$

$$\Delta^{[\sigma^{ij}]}(z,\boldsymbol{k}_T) = -\frac{M_h\, \epsilon_T^{ij}}{P_h^-}\, H_s(z,-z\boldsymbol{k}_T), \tag{3.151}$$

$$\Delta^{[\sigma^{-+}]}(z,\boldsymbol{k}_T) = -\frac{\epsilon_T^{ij}\, k_{Ti} S_{hTj}}{P_h^-}\, H_T^{\perp}(z,-z\boldsymbol{k}_T) + \frac{M_h}{P_h^-}\, H(z,-z\boldsymbol{k}_T). \tag{3.152}$$

The shorthand notations G_{1s} etc. stands for

$$G_{1s}(z,-z\boldsymbol{k}_T) = \lambda_h\, G_{1L}(z,-z\boldsymbol{k}_T) + G_{1T}(z,-z\boldsymbol{k}_T)\, \frac{(\boldsymbol{k}_T \cdot \boldsymbol{S}_{hT})}{M_h}. \tag{3.153}$$

The integrated profile functions are defined as

$$\begin{aligned} \Delta^{[\Gamma]}(z) &= z^2 \int d^2\boldsymbol{k}_T \Delta^{[\Gamma]}(z,\boldsymbol{k}_T) = \frac{z}{4}\int dk^+\, d^2\boldsymbol{k}_T\; Tr(\Delta\,\Gamma)\Big|_{k^-=P_h^-/z} \\ &= \frac{z}{8\pi}\int d\xi^+\; e^{ik\cdot\xi}\, Tr\langle 0|\mathscr{U}(0,\xi)\psi(\xi)a_h^\dagger a_h\overline{\psi}(0)\Gamma|0\rangle\Big|_{\xi^-=\xi_T=0}, \end{aligned} \tag{3.154}$$

Besides these we have nonvanishing k_T-weighted functions in analogy to the distribution functions.

The Dirac structure of the fragmentation correlation function integrated over k^+ then becomes up to twist three

$$\frac{1}{4z}\int dk^+\ \Delta(k,P_h,S_h)\Big|_{k^-=P_h^-/z,\ k_T} =$$

$$\frac{1}{4}\Bigg\{D_1(z,-z\boldsymbol{k}_T)\,\not n_- + D_{1T}^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_-^\nu k_T^\rho S_{hT}^\sigma}{M_h} - G_{1s}(z,-z\boldsymbol{k}_T)\,\not n_-\gamma_5$$

$$-H_{1T}(z,-z\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5\,S_{hT}^\mu n_-^\nu - H_{1s}^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\,k_T^\mu n_-^\nu}{M_h} + H_1^\perp(z,-z\boldsymbol{k}_T)\,\frac{\sigma_{\mu\nu}k_T^\mu n_-^\nu}{M_h}\Bigg\}$$

$$+\frac{M_h}{4P_h^-}\Bigg\{E(z,-z\boldsymbol{k}_T) + D^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not k_T}{M_h} + D_T(z,-z\boldsymbol{k}_T)\,\epsilon_{\mu\nu\rho\sigma}n_+^\mu n_-^\nu\gamma^\rho S_{hT}^\sigma$$

$$+\lambda_h\,D_L^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}n_+^\mu n_-^\nu\gamma^\rho k_T^\sigma}{M_h} - E_s(z,-z\boldsymbol{k}_T)\,i\gamma_5$$

$$-G_T'(z,-z\boldsymbol{k}_T)\,\not S_{hT}\gamma_5 - G_s^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not k_T\gamma_5}{M_h} - H_T^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\,S_{hT}^\mu k_T^\nu}{M_h}$$

$$-H_s(z,-z\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5\,n_-^\mu n_+^\nu + H(z,-z\boldsymbol{k}_T)\,\sigma_{\mu\nu}n_-^\mu n_+^\nu\Bigg\}. \tag{3.155}$$

The integrated results are

$$\frac{z}{4}\int dk^+\,d^2\boldsymbol{k}_T\ \Delta(k,P_h,S_h)\Big|_{k^-=P_h^-/z} = \frac{1}{4}\left\{D_1(z)\,\not n_- - \lambda_h\,G_1(z)\,\not n_-\gamma_5 + H_1(z)\,\frac{[\not S_{hT},\not n_-]\gamma_5}{2}\right\}$$

$$+\frac{M_h}{4P_h^-}\Bigg\{D_T(z)\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{hT\sigma} + E(z) - \lambda_h\,E_L(z)\,i\gamma_5$$

$$-G_T(z)\,\not S_{hT}\gamma_5 + \lambda_h\,H_L(z)\,\frac{[\not n_-,\not n_+]\gamma_5}{2} + i\,H(z)\,\frac{[\not n_-,\not n_+]}{2}\Bigg\}, \tag{3.156}$$

$$\frac{z}{4}\int dk^+\,d^2\boldsymbol{k}_T\ \frac{k_T^\alpha}{M_h}\,\Delta(k,P_h,S_h)\Big|_{k^-=P_h^-/z} = \frac{1}{4}\left\{-G_{1T}^{(1)}(z)\,S_{hT}^\alpha\,\not n_-\gamma_5 - \lambda_h\,H_{1L}^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not n_-]\gamma_5}{2}\right\}$$

$$+\frac{1}{4}\left\{D_{1T}^{(1)}(z)\,\epsilon^{\mu\nu\rho\alpha}\gamma_\mu n_{-\nu}S_{hT\rho} + \lambda_h\,H_1^{\perp(1)}(z)\,\frac{i[\not n_-,\gamma^\alpha]}{2}\right\}$$

$$+\frac{M_h}{4P_h^-}\Bigg\{-D^{\perp(1)}(z)\,\gamma^\alpha + \lambda_h\,G_L^{\perp(1)}(z)\,\gamma^\alpha\gamma_5 + H_T^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not S_{hT}]\gamma_5}{2}$$

$$+H_T^{(1)}(z)\,S_{hT}^\alpha\,\frac{[\not n_-,\not n_+]\gamma_5}{2}\Bigg\}, \tag{3.157}$$

$$\frac{z}{4}\int dk^+\,d^2\boldsymbol{k}_T\ \frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\,g_T^{\alpha\beta}\right)}{M_h^2}\,\Delta(k,P_h,S_h)\Bigg|_{k^-=P_h^-/z} =$$

$$-\frac{1}{4}\,H_{1T}^{\perp(2)}(z)\,\left(\frac{S_{hT}^{\{\alpha}[\gamma^{\beta\}},\not n_-]\gamma_5 - g_T^{\alpha\beta}\,[\not S_{hT},\not n_-]\gamma_5}{4}\right)$$

$$+\frac{M_h}{4P_h^-}\,G_T^{\perp(2)}(z)\,\left(\frac{S_{hT}^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not S_{hT}\gamma_5}{2}\right). \tag{3.158}$$

The appropriate normalization of the fragmentation functions can be obtained via a momentum sum rule. For this consider the following integral for the fragmentation function D_1,

$$\int dz\ z\,D_1(z) = \int dz\,d^2P_{h\perp}\ z\,D_1(z,\boldsymbol{P}_{h\perp}) = \int dz\,d^2P_{h\perp}\int\frac{d\xi^+d^2\boldsymbol{\xi}_T}{4\,(2\pi)^3}\ e^{ik\cdot\xi}\,Tr\langle 0|\psi(\xi)a_h^\dagger a_h\overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0} \tag{3.159}$$

(we have omitted the links along the n_+-direction, but including them all subsequent arguments remain valid). Considering the integrand for fixed quark momentum k and choosing this to have no perpendicular component, the $P_{h\perp}$ dependence is only contained in $a_h^\dagger a_h$ and summing over all produced hadrons (for a given quark) one obtains

$$\sum_h\int dz\ z\,D_1(z) = \int\frac{d\xi^+d^2\boldsymbol{\xi}_T}{2k^-}\ e^{ik\cdot\xi}\,\mathrm{Tr}\langle 0|\psi(\xi)\sum_h\int\frac{dP_h^-\,d^2P_{h\perp}}{(2\pi)^3\,2P_h^-}\,a_h^\dagger P_h^- a_h\overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0} = \int\frac{d\xi^+d^2\boldsymbol{\xi}_T}{2k^-}\ e^{ik\cdot\xi}\,\mathrm{Tr}\langle 0|\psi(\xi)P^-\overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0} \tag{3.160}$$

Inserting a complete set of quark states one obtains

$$\sum_h\int dz\ z\,D_1(z) = \int\frac{d\xi^+d^2\boldsymbol{\xi}_T}{2k^-}\ e^{ik\cdot\xi}\,\mathrm{Tr}\langle 0|\psi(\xi)\int\frac{dk'^-\,d^2k'_T}{(2\pi)^3\,2k'^-}\sum_{s'}|k',s'\rangle k'^-\langle k',s'|\overline{\psi}(0)\gamma^-|0\rangle\Big|_{\xi^-=0} = \int\frac{dk'^-\,d^2k'_T}{(2\pi)^3}\int\frac{d\xi^+d^2\boldsymbol{\xi}_T}{4k^-}\ e^{i(k-k')\cdot\xi}\sum_{s'}\bar{u}(k',s')\gamma^-u(k',s')\Big|_{\xi^-=0} = \frac{1}{2}, \tag{3.161}$$

and hence

$$\sum_h\sum_{S_h}\int dz\ z\,D_1(z) = 1. \tag{3.162}$$

3.5 Antiquark fragmentation functions

For the fragmentation of antiquarks the profile functions are obtained from

$$\overline{\Delta}_{ij}(k,P_h,S_h) = \sum_X\frac{1}{(2\pi)^4}\int d^4\xi\ e^{-ik\cdot\xi}\,\langle 0|\overline{\psi}_j(0)|P_h,X\rangle\langle P_h,X|\mathscr{U}(0,\xi)\psi_i(\xi)|0\rangle, \tag{3.163}$$

considering the projections

$$\overline{\Delta}^{[\Gamma]}(z,\boldsymbol{k}_T) = \frac{1}{4z}\int dk^-\ Tr(\overline{\Delta}\,\Gamma)\Big|_{k^+=P_h^+/z,\ \boldsymbol{k}_T} = \pm\int\frac{d\xi^-d^2\boldsymbol{\xi}_T}{4z\,(2\pi)^3}\ e^{-ik\cdot\xi}\,\langle 0|\overline{\psi}(0)\Gamma\,a_h^\dagger a_h\,\mathscr{U}(0,\xi)\psi(\xi)|0\rangle\Big|_{\xi^+=0}, \tag{3.164}$$

and they are denoted $\overline{D}(z,-z\boldsymbol{k}_T)$, etc. The functions $\overline{E}$ and $\overline{G}$ acquire a sign compared with the quark profile functions. For the antiquark fragmentation functions we have

$$\begin{aligned}
&\frac{1}{4z}\int dk^-\;\overline{\Delta}(k,P_h,S_h)\bigg|_{k^+=P_h^+/z,\;k_T} = \\
&\quad\frac{1}{4}\bigg\{\overline{D}_1(z,-z\boldsymbol{k}_T)\,\not{n}_+ + \overline{D}_{1T}^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_-^\nu k_{T\rho}S_{hT\sigma}}{M_h} + \overline{G}_{1s}(z,-z\boldsymbol{k}_T)\,\not{n}_+\gamma_5 \\
&\qquad -\overline{H}_{1T}(z,-z\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5\,S_{hT}^\mu n_+^\nu - \overline{H}_{1s}^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\,k_T^\mu n_+^\nu}{M_h} + \overline{H}_1^\perp(z,-z\boldsymbol{k}_T)\,\frac{\sigma_{\mu\nu}k_T^\mu n_+^\nu}{M_h}\bigg\} \\
&\quad+\frac{M_h}{4P_h^+}\bigg\{-\overline{E}(z,-z\boldsymbol{k}_T) + \overline{D}^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not{k}_T}{M_h} + \overline{D}_T(z,-z\boldsymbol{k}_T)\,\epsilon_{\mu\nu\rho\sigma}n_-^\mu n_+^\nu\gamma^\rho S_{hT}^\sigma \\
&\qquad+\lambda_h\,\overline{D}_L^\perp(z,-z\boldsymbol{k}_T)\,\frac{\epsilon_T^{\rho\sigma}\gamma_\rho k_{T\sigma}}{M_h} + \overline{E}_s(z,-z\boldsymbol{k}_T)\,i\gamma_5 \\
&\qquad+\overline{G}_T'(z,-z\boldsymbol{k}_T)\,\not{S}_{hT}\gamma_5 + \overline{G}_s^\perp(z,-z\boldsymbol{k}_T)\,\frac{\not{k}_T\gamma_5}{M_h} - \overline{H}_T^\perp(z,-z\boldsymbol{k}_T)\,\frac{i\sigma_{\mu\nu}\gamma_5\,S_{hT}^\mu k_T^\nu}{M_h} \\
&\qquad-\overline{H}_s(z,-z\boldsymbol{k}_T)\,i\sigma_{\mu\nu}\gamma_5\,n_+^\mu n_-^\nu + \overline{H}(z,-z\boldsymbol{k}_T)\,\sigma_{\mu\nu}n_+^\mu n_-^\nu\bigg\}.
\end{aligned} \tag{3.165}$$

The symmetry relations in z are $\overline{D}_1(z,\boldsymbol{k}_T'^2) = D_1(-z,\boldsymbol{k}_T'^2)$ and identically for D_T, G_{1T}, H_{1T}, $H_{1T}^\perp$, E_T, $G_L^\perp$, H and H_L, while $\overline{G}_{1L}(z,\boldsymbol{k}_T'^2) = -G_{1L}(-z,\boldsymbol{k}_T'^2)$ and identically for $D_{1T}^\perp$, $H_{1L}^\perp$, $H_1^\perp$, E, $D^\perp$, $D_L^\perp$, E_L, G_T', $G_T^\perp$, $H_T^\perp$ and H_T. The integrated results are

$$\begin{aligned}
&\frac{z}{4}\int dk^-\,d^2\boldsymbol{k}_T\;\overline{\Delta}(k,P_h,S_h)\bigg|_{k^+=P_h^+/z} = \frac{1}{4}\bigg\{\overline{D}_1(z)\,\not{n}_+ + \lambda_h\,\overline{G}_1(z)\,\not{n}_+\gamma_5 + \overline{H}_1(z)\,\frac{[\not{S}_{hT},\not{n}_+]\gamma_5}{2}\bigg\} \\
&\qquad+\frac{M_h}{4P_h^+}\bigg\{-\overline{D}_T(z)\,\epsilon_T^{\rho\sigma}\gamma_\rho S_{hT\sigma} - \overline{E}(z) + \lambda_h\,\overline{E}_L(z)\,i\gamma_5 \\
&\qquad\quad+\overline{G}_T(z)\,\not{S}_{hT}\gamma_5 + \lambda_h\,\overline{H}_L(z)\,\frac{[\not{n}_+,\not{n}_-]\gamma_5}{2} + i\,\overline{H}(z)\,\frac{[\not{n}_+,\not{n}_-]}{2}\bigg\},
\end{aligned} \tag{3.166}$$

$$\begin{aligned}
&\frac{z}{4}\int dk^-\,d^2\boldsymbol{k}_T\;\frac{k_T^\alpha}{M_h}\,\overline{\Delta}(k,P_h,S_h)\bigg|_{k^+=P_h^+/z} = \frac{1}{4}\bigg\{\overline{G}_{1T}^{(1)}(z)\,S_{hT}^\alpha\,\not{n}_+\gamma_5 - \lambda_h\,\overline{H}_{1L}^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2}\bigg\} \\
&\qquad+\frac{M_h}{4P_h^+}\bigg\{-\overline{D}^{\perp(1)}(z)\,\gamma^\alpha - \lambda_h\,\overline{G}_L^{\perp(1)}(z)\,\gamma^\alpha\gamma_5 + \overline{H}_T^{\perp(1)}(z)\,\frac{[\gamma^\alpha,\not{S}_{hT}]\gamma_5}{2} \\
&\qquad\quad+\overline{H}_T^{(1)}(z)\,S_{hT}^\alpha\,\frac{[\not{n}_+,\not{n}_-]\gamma_5}{2}\bigg\},
\end{aligned} \tag{3.167}$$

$$\begin{aligned}
&\frac{z}{4}\int dk^-\,d^2\boldsymbol{k}_T\;\frac{\left(k_T^\alpha k_T^\beta + \frac{1}{2}\boldsymbol{k}_T^2\,g_T^{\alpha\beta}\right)}{M_h^2}\,\overline{\Delta}(k,P_h,S_h)\bigg|_{k^+=P_h^+/z} = \\
&\qquad-\frac{1}{4}\,\overline{H}_{1T}^{\perp(2)}(z)\,\left(\frac{S_{hT}^{\{\alpha}[\gamma^{\beta\}},\not{n}_+]\gamma_5 - g_T^{\alpha\beta}\,[\not{S}_{hT},\not{n}_+]\gamma_5}{4}\right) \\
&\qquad-\frac{M_h}{4P_h^+}\,\overline{G}_T^{\perp(2)}(z)\,\left(\frac{S_{hT}^{\{\alpha}\gamma^{\beta\}}\gamma_5 - g_T^{\alpha\beta}\,\not{S}_{hT}\gamma_5}{2}\right).
\end{aligned} \tag{3.168}$$

3.6 The parton picture for fragmentation functions

Similar as in the case of distribution functions, it is instructive to consider the correlation function for the case of a free quark, given by

$$\delta_{ij}(p,s;k) = u_i(k,s)\overline{u}_j(k,s)\delta^4(k-p) = \tfrac{1}{2}\big((\not k + m)(1+\gamma_5\not s)\big)_{ij}\delta^4(k-p), \tag{3.169}$$

where the momentum and spin of the quark are parametrized as

$$k \;=\; \left[k^-,\ \frac{\boldsymbol{k}_T^2+m^2}{2k^-},\ \boldsymbol{k}_T\right], \tag{3.170}$$

$$s \;=\; \left[\frac{\lambda_q\,k^-}{m},\ -\frac{m\,\lambda_q}{2k^-}+\frac{\boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{k^-}+\frac{\lambda_q\,\boldsymbol{k}_T^2}{2m\,k^-},\ \boldsymbol{s}_{qT}+\frac{\lambda_q}{m}\boldsymbol{k}_T\right] \tag{3.171}$$

in terms of a quark lightcone helicity λ_q and a quark lightcone transverse polarization s_{qT}. The projections become for twist two

$$\delta^{[\gamma^-]}(k) = \tfrac{1}{2}\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = \tfrac{1}{2}D_{q\lambda/q\uparrow}+\tfrac{1}{2}D_{q\lambda/q\downarrow}, \tag{3.172}$$

$$\delta^{[\gamma^-\gamma_5]}(k) = \tfrac{1}{2}\lambda_q\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = \tfrac{1}{2}D_{q\lambda/q\uparrow}-\tfrac{1}{2}D_{q\lambda/q\downarrow}, \tag{3.173}$$

$$\delta^{[i\sigma^{i-}\gamma_5]}(k) = \tfrac{1}{2}\boldsymbol{s}_{qT}^i\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T) = \tfrac{1}{2}D_{q\boldsymbol{s}_T/q\rightarrow}-\tfrac{1}{2}D_{q\boldsymbol{s}_T/q\leftarrow}, \tag{3.174}$$

where $\zeta = p^-/k^-$ and we have given the intuitive interpretation in terms of probabilities for quarks to fragment into quarks with spin characterized by λ and $\boldsymbol{s}_T$. For twist three we get

$$\delta^{[1]}(k) = \frac{m}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.175}$$

$$\delta^{[\gamma^i]}(k) = \frac{\boldsymbol{k}_T^i}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.176}$$

$$\delta^{[\gamma^i\gamma_5]}(k) = \frac{(m\,\boldsymbol{s}_{qT}^i+\lambda_q\,\boldsymbol{k}_T^i)}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.177}$$

$$\delta^{[i\sigma^{ij}\gamma_5]}(k) = \frac{\boldsymbol{s}_{qT}^i\boldsymbol{k}_T^j-\boldsymbol{k}_T^i\boldsymbol{s}_{qT}^j}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{3.178}$$

$$\delta^{[i\sigma^{-+}\gamma_5]}(k) = \frac{m\,\lambda_q-\boldsymbol{k}_T\cdot\boldsymbol{s}_{qT}}{2k^-}\,\delta\,(\zeta-1)\,\delta^2(\boldsymbol{k}_T-\boldsymbol{p}_T). \tag{3.179}$$

Inserting the expansion for the free quark field gives

$$\Delta_{ij}(k) = 4z\,\delta(k^2-m^2)\left[\theta(z)u_i^{(\beta)}(k)\mathscr{D}_{\beta\alpha}(k)\bar{u}_j^{(\alpha)}(k)-\theta(-z)v_i^{(\beta)}(-k)\overline{\mathscr{D}}_{\beta\alpha}(-k)\bar{v}_j^{(\alpha)}(-k)\right], \tag{3.180}$$

where $z = P_h^-/k^-$. The use of lightcone coordinates is convenient because of the integration over k^+ that is needed in deep inelastic processes. The functions $\mathscr{D}$ and $\overline{\mathscr{D}}$ are given by

$$\mathscr{D}_{\beta\alpha}(k) = \mathscr{D}_{\beta\alpha}(z,z^2\boldsymbol{k}_T^2) \equiv \frac{1}{4z(2\pi)^3}\int\frac{dz'\,d^2\boldsymbol{k}_T'}{(2\pi)^3\,2z'}\,\langle 0|b_\beta(k')a_h a_h^\dagger b_\alpha^\dagger(k)|0\rangle, \tag{3.181}$$

$$\overline{\mathscr{D}}_{\beta\alpha}(k) = \overline{\mathscr{D}}_{\beta\alpha}(z,z^2\boldsymbol{k}_T^2) \equiv \frac{1}{4z(2\pi)^3}\int\frac{dz'\,d^2\boldsymbol{k}_T'}{(2\pi)^3\,2z'}\,\langle 0|d_\alpha(k')a_h a_h^\dagger d_\beta^\dagger(k)|0\rangle. \tag{3.182}$$

Note that $\mathscr{D}_{\beta\alpha}(k)$ is a decay matrix in the quark spin-space. Most easily to deal with is the 'momentum sum rule'

$$\begin{aligned}\sum_h\int z dz\,d^2\boldsymbol{P}_{hT}\,\mathscr{D}_{\beta\alpha}(z,\boldsymbol{P}_{hT}^2) &= \frac{1}{2}\int\frac{dz'\,d^2\boldsymbol{k}_T'}{(2\pi)^3\,2z'}\,\frac{1}{k^-}\,\langle 0|b_\beta(k')\sum_h\int\frac{dP_h^-\,d^2\boldsymbol{P}_{hT}}{(2\pi)^3\,2P_h^-}\,a_h P_h^- a_h^\dagger b_\alpha^\dagger(k)|0\rangle\\ &= \frac{1}{2}\int\frac{dz'\,d^2\boldsymbol{k}_T'}{(2\pi)^3\,2z'}\,\frac{1}{k^-}\,\langle 0|b_\beta(k')\,\mathscr{P}_{\rm op}^-b_\alpha^\dagger(k)|0\rangle = \frac{1}{2}\,\delta_{\beta\alpha}.\end{aligned} \tag{3.183}$$

Integrating over k^- one obtains the twist two results (for $z>0$)

$$\Delta^{[\gamma^-]}(k) = \tfrac{1}{2}\mathscr{D}_{RR}+\tfrac{1}{2}\mathscr{D}_{LL} = \tfrac{1}{2}\mathscr{D}_{\alpha\alpha}+\tfrac{1}{2}\mathscr{D}_{\beta\beta}, \tag{3.184}$$

$$\Delta^{[\gamma^-\gamma_5]}(k) = \tfrac{1}{2}\mathscr{D}_{RR}-\tfrac{1}{2}\mathscr{D}_{LL}, \tag{3.185}$$

$$\Delta^{[i\sigma^{i-}\gamma_5]}(k) = \tfrac{1}{2}\mathscr{D}_{\alpha\alpha}-\tfrac{1}{2}\mathscr{D}_{\beta\beta}, \tag{3.186}$$

using the same projection operators as for the distribution functions. In this way an interpretation in terms of quark decay functions is obtained, again rigorous for the twist two functions. In this 'parton picture' the twist three results can also for the fragmentation functions be expressed in the twist two functions.

Summarizing we get for $\Delta^{[\Gamma]}(z,-z\boldsymbol{k}_T)$ and $\Delta^{[\Gamma]}(z)$, ordered according to twist, chirality and time-reversal behavior for production of unpolarized (U), longitudinally polarized (L) and transversely polarized (T) (spin 1/2) hadrons:

FRAGMENTATION						
		χ-even		χ-odd		
$\Delta^{[\Gamma]}(z,-z\boldsymbol{k}_T)$		T-even	T-odd	T-even	T-odd	
twist 2	U	D_1			$H_1^\perp$	
	L	G_{1L}		$H_{1L}^\perp$		
	T	G_{1T}	$D_{1T}^\perp$	H_{1T} $H_{1T}^\perp$		
twist 3	U	$D^\perp$	$G^\perp$	E	H	
	L	$G_L^\perp$	$D_L^\perp$	H_L	E_L	
	T	G_T' $G_T^\perp$	D_T' $D_T^\perp$	H_T $H_T^\perp$	E_T $E_T^\perp$	

FRAGMENTATION					
		χ-even		χ-odd	
$\Delta^{[\Gamma]}(z)$		T-even	T-odd	T-even	T-odd
twist 2	U	D_1			
	L	G_1			
	T			H_1	
twist 3	U			E	H
	L			H_L	E_L
	T	G_T	D_T		

3.7 Structure of $H \to q$ profile functions

In order to study the structure of the profile functions, we will first give the expressions in terms of the amplitudes $A_i(\sigma,\tau)$, where $\sigma = 2\,k\cdot P$ and $\tau = k^2$,

$$f_1(x,\boldsymbol{k}_T^2) = \int \ldots\ [A_2 + xA_3]\,, \tag{3.187}$$

$$g_{1L}(x,\boldsymbol{k}_T^2) = \int \ldots\ \left[-A_6 - \left(\frac{\sigma - 2xM^2}{2M^2}\right)(A_7 + xA_8)\right], \tag{3.188}$$

$$g_{1T}(x,\boldsymbol{k}_T^2) = \int \ldots\ [A_7 + xA_8]\,, \tag{3.189}$$

$$h_{1T}(x,\boldsymbol{k}_T^2) = \int \ldots\ [-(A_9 + xA_{10})]\,, \tag{3.190}$$

$$h_{1L}^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots\ \left[A_{10} - \left(\frac{\sigma - 2xM^2}{2M^2}\right)A_{11}\right], \tag{3.191}$$

$$h_{1T}^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots\ [A_{11}]\,, \tag{3.192}$$

$$e(x,\boldsymbol{k}_T^2) = \int \ldots\ [A_1]\,, \tag{3.193}$$

$$f^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots\ [A_3]\,, \tag{3.194}$$

$$g_T'(x,\boldsymbol{k}_T^2) = \int \ldots\ [-A_6]\,, \tag{3.195}$$

$$g_L^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots\ \left[-\left(\frac{\sigma - 2xM^2}{2M^2}\right)A_8\right], \tag{3.196}$$

$$g_T^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots\ [A_8]\,, \tag{3.197}$$

$$h_T^{\perp}(x,\boldsymbol{k}_T^2) = \int \ldots\ [-A_{10}]\,, \tag{3.198}$$

$$h_L(x,\boldsymbol{k}_T^2) = \int \ldots\ \left[-(A_9 + xA_{10}) - \left(\frac{\sigma - 2xM^2}{2M^2}\right)A_{10} + \left(\frac{\sigma - 2xM^2}{2M^2}\right)^2 A_{11}\right], \tag{3.199}$$

$$h_T(x,\boldsymbol{k}_T^2) = \int \ldots\ \left[-\left(\frac{\sigma - 2xM^2}{2M^2}\right)A_{11}\right], \tag{3.200}$$

where

$$\begin{aligned}\int \ldots\ &= \int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma)\\ &= \int d\sigma d\tau\, \delta(x\eta M^2 + \tau - x\sigma),\end{aligned} \tag{3.201}$$

with

$$\eta = x + \frac{\boldsymbol{k}_T^2}{xM^2}. \tag{3.202}$$

The integrated functions also can be expressed as an integral over amplitudes, e.g.

$$f_1(x) = \pi \int d\sigma d\tau\ \theta(x^2M^2 + \tau - x\sigma)\,[A_2 + xA_3]\,, \tag{3.203}$$

now involving a θ-function to ensure contributions from the physical region $\boldsymbol{p}_T^2 \geq 0$, indicated in the figure. Implicitly, there are several relations which can be traced back to the fact that there are less amplitudes

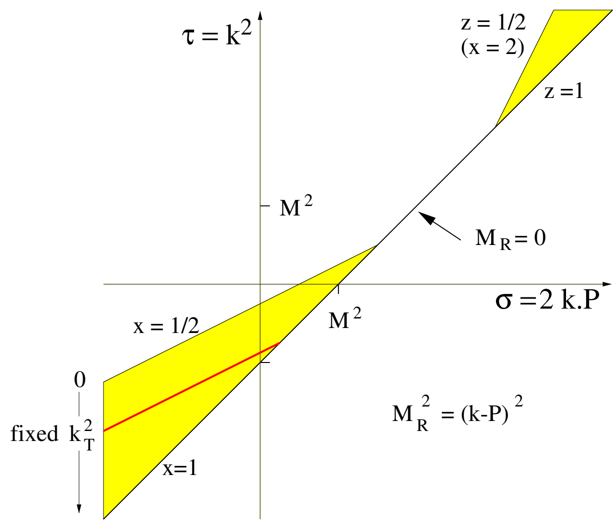


Figure 3.3: The region $\boldsymbol{k}_T^2 \geq 0$ which contributes in the integration over σ and τ in the expressions for the distribution functions and fragmentation functions

than profile functions. Useful are the following relations for $\boldsymbol{k}_T^2$-weighted functions: If we have

$$f^{(n)}(x,\boldsymbol{k}_T^2) = \int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma) \left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^n F(x,\sigma,\tau),$$
$$f'^{(n)}(x,\boldsymbol{k}_T^2) = \int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma) \left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^n \frac{\partial F}{\partial x}(x,\sigma,\tau),$$
$$g^{(n)}(x,\boldsymbol{k}_T^2) = -\int d\sigma d\tau\, \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma) \left(\frac{\boldsymbol{k}_T^2}{2M^2}\right)^n \frac{\sigma - 2x\,M^2}{2M^2}\, F(x,\sigma,\tau), \tag{3.204}$$

one easily proves the relations

$$\frac{d}{dx}\, f^{(1)}(x,\boldsymbol{k}_T^2) = -g(x,\boldsymbol{k}_T^2) + f'^{(1)}(x,\boldsymbol{k}_T^2) + 2M^2\, \frac{d}{d\boldsymbol{k}_T^2}\, g^{(1)}(x,\boldsymbol{k}_T^2),$$
$$\frac{d}{dx}\, f^{(1)}(x) = -g(x) + f'^{(1)}(x) + 2M^2\, g^{(1)}(x,0), \tag{3.205}$$
$$\frac{d}{dx}\, f^{(2)}(x,\boldsymbol{k}_T^2) = -2g^{(1)}(x,\boldsymbol{k}_T^2) + f'^{(2)}(x,\boldsymbol{k}_T^2) + 2M^2\, \frac{d}{d\boldsymbol{k}_T^2}\, g^{(2)}(x,\boldsymbol{k}_T^2),$$
$$\frac{d}{dx}\, f^{(2)}(x) = -2g^{(1)}(x) + f'^{(2)}(x) + 2M^2\, g^{(2)}(x,0). \tag{3.206}$$

For the five g-functions this leads to the two relations (assuming $g^{(n\geq 1)}(x,0)$ to vanish)

$$g_T(x) = g_1(x) + \frac{d}{dx}\, g_{1T}^{(1)}, \tag{3.207}$$
$$g_L^{\perp}(x) = -\frac{d}{dx}\, g_T^{\perp(1)} \qquad \text{and} \qquad 2\, g_L^{\perp(1)}(x) = -\frac{d}{dx}\, g_T^{\perp(2)}, \tag{3.208}$$

while for the six h-functions three relations are found, where the one for $h_T^{\perp}$ is trivial,

$$h_L(x) = h_1(x) - \frac{d}{dx}\, h_{1L}^{\perp(1)}, \tag{3.209}$$
$$h_T(x) = -\frac{d}{dx}\, h_{1T}^{\perp(1)} \qquad \text{and} \qquad 2\, h_T^{(1)}(x) = -\frac{d}{dx}\, h_{1T}^{\perp(2)}, \tag{3.210}$$
$$h_T^{\perp(n)}(x) = h_T^{(n)}(x) - h_{1L}^{\perp(n)}(x). \tag{3.211}$$

Similarly we have for the fragmentation functions expressions of the profile functions in terms of the amplitudes B_1 to B_{12}. This gives relations analogous to the distribution functions,

$$2z\,D_1(z,z^2\boldsymbol{k}_T^2) = \int d\sigma d\tau\;\delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right)\left[B_2 + \frac{1}{z}\,B_3\right], \tag{3.212}$$

etc. with now in addition some *time reversal odd* functions,

$$2z\,D_{1T}^{\perp}(z,-z\boldsymbol{k}_T) = \int \ldots\;[B_{12}]\,, \tag{3.213}$$

$$2z\,H_1^{\perp}(z,-z\boldsymbol{k}_T) = \int \ldots\;[-B_4]\,, \tag{3.214}$$

$$2z\,D_L^{\perp}(z,-z\boldsymbol{k}_T) = \int \ldots\;[-B_{12}]\,, \tag{3.215}$$

$$2z\,D_T(z,-z\boldsymbol{k}_T) = \int \ldots\;\left[-\left(\frac{\sigma - 2M_h^2/z}{2M_h^2}\right)B_{12}\right]. \tag{3.216}$$

$$2z\,E_L(z,-z\boldsymbol{k}_T) = \int \ldots\;\left[-\left(\frac{\sigma - 2M_h^2/z}{2M_h^2}\right)B_5\right]. \tag{3.217}$$

$$2z\,E_T(z,-z\boldsymbol{k}_T) = \int \ldots\;[B_5]\,, \tag{3.218}$$

$$2z\,H(z,-z\boldsymbol{k}_T) = \int \ldots\;\left[\left(\frac{\sigma - 2M_h^2/z}{2M_h^2}\right)B_4\right]. \tag{3.219}$$

The integrated functions are given by

$$\frac{2}{z}\,D_1(z) = \pi\int d\sigma d\tau\;\theta\left(\frac{\sigma}{z} - \tau - \frac{M_h^2}{z^2}\right)\left[B_2 + \frac{1}{z}\,B_3\right], \tag{3.220}$$

with an integration region also indicated in the figure. The following relations can be derived in the case of fragmentation functions: If we have

$$\begin{aligned}
2z\,D^{(n)}(z,z^2\boldsymbol{k}_T^2) &= \int d\sigma d\tau\,\delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right)\left(\frac{\boldsymbol{k}_T^2}{2M_h^2}\right)^n F(z,\sigma,\tau),\\
2z\,D'^{(n)}(z,z^2\boldsymbol{k}_T^2) &= \int d\sigma d\tau\,\delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right)\left(\frac{\boldsymbol{k}_T^2}{2M_h^2}\right)^n \frac{\partial F}{\partial z}(z,\sigma,\tau),\\
2z\,G^{(n)}(z,z^2\boldsymbol{k}_T^2) &= -\int d\sigma d\tau\,\delta\left(\boldsymbol{k}_T^2 + \frac{M_h^2}{z^2} + \tau - \frac{\sigma}{z}\right)\left(\frac{\boldsymbol{k}_T^2}{2M_h^2}\right)^n \frac{\sigma - 2M_h^2/z}{2M_h^2}\,F(z,\sigma,\tau),
\end{aligned} \tag{3.221}$$

one easily proves the relations

$$\begin{aligned}
z^2\frac{d}{dz}\left[z\,D^{(1)}(z,z^2\boldsymbol{k}_T^2)\right] &= z\,G(z,z^2\boldsymbol{k}_T^2) + z^3\,D'^{(1)}(z,z^2\boldsymbol{k}_T^2) + 2M_h^2\,\frac{d}{d\boldsymbol{k}_T^2}\left[z\,G^{(1)}(z,z^2\boldsymbol{k}_T^2)\right],\\
z^2\frac{d}{dz}\left[\frac{D^{(1)}(z)}{z}\right] &= \frac{G(z)}{z} + z\,D'^{(1)}(z) + 2M_h^2\,\frac{G^{(1)}(z,0)}{z},
\end{aligned} \tag{3.222}$$

$$\begin{aligned}
z^2\frac{d}{dz}\left[z\,D^{(2)}(z,z^2\boldsymbol{k}_T^2)\right] &= 2z\,G^{(1)}(z,z^2\boldsymbol{k}_T^2) + z^3\,D'^{(2)}(z,z^2\boldsymbol{k}_T^2) + 2M_h^2\,\frac{d}{d\boldsymbol{k}_T^2}\left[z\,G^{(2)}(z,z^2\boldsymbol{k}_T^2)\right],\\
z^2\frac{d}{dz}\left[\frac{D^{(2)}(z)}{z}\right] &= 2\,\frac{G^{(1)}(z)}{z} + z\,D'^{(1)}(z) + 2M_h^2\,\frac{G^{(1)}(z,0)}{z}.
\end{aligned} \tag{3.223}$$

For the five G-functions this leads to two relations (assuming $G^{(n\geq 1)}(z,0)$ to vanish,

$$\frac{G_T(z)}{z} = \frac{G_1(z)}{z} - z^2\,\frac{d}{dz}\left[\frac{G_{1T}^{(1)}(z)}{z}\right], \tag{3.224}$$

$$\frac{G_L^{\perp}(z)}{z} = z^2\,\frac{d}{dz}\left[\frac{G_T^{\perp(1)}(z)}{z}\right] \quad\text{and}\quad 2\,\frac{G_L^{\perp(1)}(z)}{z} = z^2\,\frac{d}{dz}\left[\frac{G_T^{\perp(2)}(z)}{z}\right]. \tag{3.225}$$

For the eight H-functions this leads to 4 relations,

$$\frac{H_L(z)}{z} = \frac{H_1(z)}{z} + z^2 \frac{d}{dz}\left[\frac{H_{1L}^{\perp(1)}(z)}{z}\right], \tag{3.226}$$

$$\frac{H_T(z)}{z} = z^2 \frac{d}{dz}\left[\frac{H_{1T}^{\perp(1)}(z)}{z}\right] \quad \text{and} \quad 2\,\frac{H_T^{(1)}(z)}{z} = z^2 \frac{d}{dz}\left[\frac{H_{1T}^{\perp(2)}(z)}{z}\right], \tag{3.227}$$

$$H_T^{\perp(n)}(z) = H_T^{(n)}(z) - H_{1L}^{\perp(n)}(z), \tag{3.228}$$

$$\frac{H(z)}{z} = z^2 \frac{d}{dz}\left[\frac{H_1^{\perp(1)}(z)}{z}\right]. \tag{3.229}$$

The last equation relates time reversal odd functions. Other relations between time reversal odd functions are,

$$\frac{E_L(z)}{z} = z^2 \frac{d}{dz}\left[\frac{E_T^{(1)}(z)}{z}\right], \tag{3.230}$$

and

$$\frac{D_T(z)}{z} = z^2 \frac{d}{dz}\left[\frac{D_{1T}^{\perp(1)}(z)}{z}\right], \tag{3.231}$$

$$D_L^{\perp(n)}(z) = -D_{1T}^{\perp(n)}(z). \tag{3.232}$$

Using the splitting of the twist three profile functions in a piece that is expressed in terms of partonic (twist two) functions and a remainder ('true' twist three piece) that as we will see in the next chapter can be expressed in terms of 'good' quark and gluon fields. This is achieved by using the 'free' quark results or equivalently the results for quark-quark-gluon correlation functions using the QCD equations of motion,

$$e(x,\boldsymbol{k}_T) = \frac{m}{Mx}\, f_1(x,\boldsymbol{k}_T) + \tilde{e}(x,\boldsymbol{k}_T), \tag{3.233}$$

$$f^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\, f_1(x,\boldsymbol{k}_T) + \tilde{f}^\perp(x,\boldsymbol{k}_T), \tag{3.234}$$

$$g_T'(x,\boldsymbol{k}_T) = \frac{m}{Mx}\, h_{1T}(x,\boldsymbol{k}_T) + \tilde{g}_T'(x,\boldsymbol{k}_T), \tag{3.235}$$

$$g_L^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\, g_{1L}(x,\boldsymbol{k}_T) + \frac{m}{Mx}\, h_{1L}^\perp(x,\boldsymbol{k}_T) + \tilde{g}_L^\perp(x,\boldsymbol{k}_T), \tag{3.236}$$

$$g_T^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\, g_{1T}(x,\boldsymbol{k}_T) + \frac{m}{Mx}\, h_{1T}^\perp(x,\boldsymbol{k}_T) + \tilde{g}_T^\perp(x,\boldsymbol{k}_T), \tag{3.237}$$

$$g_T(x,\boldsymbol{k}_T) = \frac{1}{x}\, g_{1T}^{(1)}(x,\boldsymbol{k}_T) + \frac{m}{Mx}\, h_1(x,\boldsymbol{k}_T) + \tilde{g}_T(x,\boldsymbol{k}_T), \tag{3.238}$$

$$h_T^\perp(x,\boldsymbol{k}_T) = \frac{1}{x}\, h_{1T}(x,\boldsymbol{k}_T) + \tilde{h}_T^\perp(x,\boldsymbol{k}_T), \tag{3.239}$$

$$h_L(x,\boldsymbol{k}_T) = \frac{m}{Mx}\, g_{1L}(x,\boldsymbol{k}_T) - \frac{\boldsymbol{k}_T^2}{M^2 x}\, h_{1L}^\perp + \tilde{h}_L(x,\boldsymbol{k}_T), \tag{3.240}$$

$$h_T(x,\boldsymbol{k}_T) = \frac{m}{Mx}\, g_{1T}(x,\boldsymbol{k}_T) - \frac{h_{1T}(x,\boldsymbol{k}_T)}{x} - \frac{\boldsymbol{k}_T^2}{M^2 x}\, h_{1T}^\perp + \tilde{h}_T(x,\boldsymbol{k}_T). \tag{3.241}$$

The functions $\tilde{e}$, etc. can be immediately seen as the functions appearing in the twist three projections of the quark-quark-gluon correlation functions.

Comparing the above expression with the second relations found for the g-functions one has

$$g_T = \frac{g_{1T}^{(1)}}{x} + \frac{m}{M}\frac{h_1}{x} + \tilde{g}_T = g_1 + \frac{d}{dx}\, g_{1T}^{(1)}, \tag{3.242}$$

from which one obtains the relation

$$x^2 \frac{d}{dx}\left(\frac{g_{1T}^{(1)}}{x}\right) = -x\, g_1 + \frac{m}{M}\, h_1 + x\, \tilde{g}_T, \tag{3.243}$$

that can be used to eliminate $g_{1T}^{(1)}$,

$$\frac{d}{dx}\,(g_T - g_1) = \frac{d}{dx}\,g_2 = -\frac{1}{x}\,\frac{d}{dx}\,(xg_1) + \frac{m}{M}\,\frac{1}{x}\,\frac{d}{dx}\,h_1 + \frac{1}{x}\,\frac{d}{dx}\,(x\tilde{g}_T), \tag{3.244}$$

or (provided sufficient convergent behavior at the endpoints)

$$g_2(x) = -\left[g_1(x) - \int_x^1 dy\,\frac{g_1(y)}{y}\right] + \frac{m}{M}\left[\frac{h_1(x)}{x} - \int_x^1 dy\,\frac{h_1(y)}{y^2}\right] + \left[\tilde{g}_T(x) - \int_x^1 dy\,\frac{\tilde{g}_T(y)}{y}\right]. \tag{3.245}$$

Similarly we obtain from the expression for h_L above and the second of the h-relations

$$h_L = \frac{m}{M}\,\frac{g_1}{x} - 2\,\frac{h_{1L}^{\perp(1)}}{x} + \tilde{h}_L = h_1 - \frac{d}{dx}\,h_{1L}^{\perp(1)}, \tag{3.246}$$

from which one obtains the relation

$$x^3\,\frac{d}{dx}\left(\frac{h_{1L}^{\perp(1)}}{x^2}\right) = x\,h_1 - \frac{m}{M}\,g_1 - x\,\tilde{h}_L, \tag{3.247}$$

which can be used to eliminate $h_{1L}^{\perp(1)}$,

$$\frac{d}{dx}\left(\frac{h_L - h_1}{x}\right) = \frac{d}{dx}\left(\frac{h_2}{2x}\right) = -\frac{1}{x^2}\,\frac{d}{dx}\,(xh_1) + \frac{m}{M}\,\frac{1}{x^2}\,\frac{d}{dx}\,g_1 + \frac{1}{x^2}\,\frac{d}{dx}\,(x\tilde{h}_L), \tag{3.248}$$

or (provided sufficient convergent behavior at the endpoints)

$$\begin{aligned}\frac{1}{2}\,h_2(x) \quad = \quad & -\left[h_1(x) - 2x\int_x^1 dy\,\frac{h_1(y)}{y^2}\right] + \frac{m}{M}\left[\frac{g_1(x)}{x} - 2x\int_x^1 dy\,\frac{g_1(y)}{y^3}\right] \\ & + \left[\tilde{h}_L(x) - 2x\int_x^1 dy\,\frac{\tilde{h}_L(y)}{y^2}\right].\end{aligned} \tag{3.249}$$

Continuing with the twist-three functions we have

$$h_T = -\frac{h_1}{x} - \frac{h_{1T}^{\perp(1)}}{x} + \frac{m}{M}\,\frac{g_{1T}}{x} + \tilde{h}_T = -\frac{d}{dx}\,h_{1T}^{\perp(1)}, \tag{3.250}$$

from which one obtains the relation

$$x^2\,\frac{d}{dx}\left(\frac{h_{1T}^{\perp(1)}}{x}\right) = h_1 - \frac{m}{M}\,g_{1T} + x\,\tilde{h}_T, \tag{3.251}$$

which can be used to eliminate $h_{1T}^{\perp(1)}$,

$$\frac{d}{dx}\,h_T = -\frac{1}{x}\,\frac{d}{dx}\,h_1 + \frac{m}{M}\,\frac{1}{x}\,\frac{d}{dx}\,g_{1T} + \frac{1}{x}\,\frac{d}{dx}\,(x\tilde{h}_T), \tag{3.252}$$

or (provided sufficient convergent behavior at the endpoints)

$$h_T(x) = -\left[\frac{h_1(x)}{x} - \int_x^1 dy\,\frac{h_1(y)}{y^2}\right] + \frac{m}{M}\left[\frac{g_{1T}(x)}{x} - \int_x^1 dy\,\frac{g_{1T}(y)}{y^2}\right] + \left[\tilde{h}_T(x) - \int_x^1 dy\,\frac{\tilde{h}_T(y)}{y}\right]. \tag{3.253}$$

For the second $\boldsymbol{k}_T^2$-moment one obtains

$$h_T^{(1)} = -\frac{h_1^{(1)}}{x} - \frac{h_{1T}^{\perp(2)}}{x} + \frac{m}{M}\,\frac{g_{1T}^{(1)}}{x} + \tilde{h}_T^{(1)} = -\frac{1}{2}\,\frac{d}{dx}\,h_{1T}^{\perp(2)}, \tag{3.254}$$

from which one obtains the relation

$$x^2\,\frac{d}{dx}\left(\frac{h_{1T}^{\perp(2)}}{x^2}\right) = 2\,\frac{h_1^{(1)}}{x} - 2\,\frac{m}{M}\,\frac{g_{1T}^{(1)}}{x} - 2\,\tilde{h}_T^{(1)}, \tag{3.255}$$

which can be used to eliminate $h_{1T}^{\perp(2)}$,

$$\frac{d}{dx}\left(\frac{h_T^{(1)}}{x}\right) = -\frac{1}{x^2}\,\frac{d}{dx}\,h_{1T}^{(1)} + \frac{m}{M}\,\frac{1}{x^2}\,\frac{d}{dx}\,g_{1T}^{(1)} + \frac{1}{x^2}\,\frac{d}{dx}\,(x\tilde{h}_T^{(1)}), \tag{3.256}$$

or (provided sufficient convergent behavior at the endpoints)

$$\begin{aligned} h_T^{(1)}(x) \quad = \quad & -\left[\frac{h_1^{(1)}(x)}{x} - 2x\int_x^1 dy\,\frac{h_1^{(1)}(y)}{y^3}\right] + \frac{m}{M}\left[\frac{g_{1T}^{(1)}(x)}{x} - 2x\int_x^1 dy\,\frac{g_{1T}^{(1)}(y)}{y^3}\right] \\ & + \left[\tilde{h}_T^{(1)}(x) - 2x\int_x^1 dy\,\frac{\tilde{h}_T^{(1)}(y)}{y^2}\right]. \end{aligned} \tag{3.257}$$

It is sometimes useful to realize that the moments of combinations appearing above are

$$\int_0^1 dx\;x^{n-1}\left[f(x) - k\,x^{k-1}\int_x^1 dy\,\frac{f(y)}{y^k}\right] = \frac{n-1}{n+k-1}\int_0^1 dx\;x^{n-1}\,f(x) \qquad (\text{for } n \geq 1,\ k \geq 1). \tag{3.258}$$

Therefore all first moments ($n = 1$) of the expressions between brackets above vanish, at least if the first moments of the function f is finite.

Chapter 4

Deep inelastic processes

4.1 The point cross sections

In section 1 we have discussed the formalism for three types of hard processes, the Drell-Yan process, e^+e^- annihilation and lepton-hadron scattering,

$$\begin{aligned} A+B &\rightarrow \ell+\overline{\ell}+X, \qquad (DY) \\ e^-+e^+ &\rightarrow h_1+h_2+X, \qquad (e^-e^+) \\ \ell+H &\rightarrow \ell'+h+X. \qquad (\ell H) \end{aligned}$$

The underlying processes are:

$$DY: \qquad q+\overline{q} \rightarrow \ell+\overline{\ell}, \tag{4.1}$$

$$e^-e^+: \quad e^-+e^+ \rightarrow q+\overline{q}, \tag{4.2}$$

$$\ell H: \qquad \ell+q \rightarrow \ell'+q. \tag{4.3}$$

The cross sections for these processes would be quite similar to the (observable) lepton cross sections. The annihilation cross section $e^-e^+ \rightarrow \mu^-\mu^+$, neglecting lepton masses is given by

$$\frac{d\sigma}{dt}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{2\pi\,\alpha^2\,(t^2+u^2)}{s^4}, \tag{4.4}$$

or the equivalent expressions in terms of $y = -t/s = (1+\cos\theta_{cm})/2$,

$$\frac{d\sigma}{d\Omega_{cm}}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{\alpha^2}{4s}\left(1+\cos^2\theta_{cm}\right), \tag{4.5}$$

$$\frac{d\sigma}{dy}(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{4\pi\,\alpha^2}{s}\left(\frac{1}{2}-y+y^2\right). \tag{4.6}$$

The total cross section becomes

$$\sigma(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{4\pi\,\alpha^2}{3\,s}. \tag{4.7}$$

The cross section for $e^-\mu^+ \rightarrow e^-\mu^+$, neglecting lepton masses is given by

$$\frac{d\sigma}{dt}(e^-\mu^+ \rightarrow e^-\mu^+) = \frac{2\pi\,\alpha^2\,(s^2+u^2)}{s^2\,t^2}, \tag{4.8}$$

or using $y = -t/s$,

$$\frac{d\sigma}{dy}(e^-\mu^+ \rightarrow e^-\mu^+) = \frac{4\pi\,\alpha^2\,s}{Q^4}\left(\frac{y^2}{2}+1-y\right). \tag{4.9}$$

Also useful is the cross section for a virtual (transverse) photon, given by

$$\sigma(\gamma_T^*\mu^+ \rightarrow \mu^+) = \frac{4\pi^2\alpha}{Q^2}\,\delta\left(\frac{s}{Q^2}-1\right). \tag{4.10}$$

4.2 The Drell-Yan process

The hadronic tensor in this case is rewritten as

$$\begin{aligned}
\mathscr{W}_{\mu\nu}(q;P_AS_A;P_BS_B) &= \frac{1}{(2\pi)^4}\int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(P_A+P_B-P_X-q) \\
&\qquad \times\langle P_AS_A;P_BS_B|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|P_AS_A;P_BS_B\rangle, \\
&= \frac{1}{(2\pi)^4}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A;P_BS_B|J_\mu(x)\;J_\nu(0)|P_AS_A;P_BS_B\rangle,
\end{aligned}$$

which in tree approximation (Born terms) becomes

$$\begin{aligned}
\mathscr{W}_{\mu\nu} &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A;P_BS_B|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x): \\
&\qquad \times :\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|P_AS_A;P_BS_B\rangle \\
&= \frac{1}{(2\pi)^4}\,\frac{1}{3}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A|\overline{\psi}_j(x)\psi_i(0)|P_AS_A\rangle(\gamma_\mu)_{jk} \\
&\qquad \langle P_BS_B|\psi_k(x)\overline{\psi}_l(0)|P_BS_B\rangle(\gamma_\nu)_{li} \\
&\quad +\frac{1}{(2\pi)^4}\,\frac{1}{3}\int d^4x\; e^{-iq\cdot x}\,\langle P_AS_A|\psi_k(x)\overline{\psi}_l(0)|P_AS_A\rangle(\gamma_\nu)_{li} \\
&\qquad \langle P_BS_B|\overline{\psi}_j(x)\psi_i(0)|P_BS_B\rangle(\gamma_\mu)_{jk}, \\
&= \frac{1}{3}\int d^4p\,d^4k\,\delta^4(p+k-q)\;\text{Tr}\left(\Phi(p)\gamma_\mu\overline{\Phi}(k)\gamma_\nu\right) + \left\{\begin{array}{c} q\leftrightarrow -q \\ \mu\leftrightarrow\nu \end{array}\right\},
\end{aligned} \tag{4.11}$$

where we have used

$$\begin{aligned}
\Phi_{ij}(p) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-ip\cdot x}\,\langle P_AS_A|\overline{\psi}_j(x)\psi_i(0)|P_BS_B\rangle, \\
\overline{\Phi}_{kl}(k) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-ik\cdot x}\,\langle P_BS_B|\psi_k(x)\overline{\psi}_l(0)|P_BS_B\rangle,
\end{aligned}$$

and its symmetry properties (see section 2). Note that since in both Φ (quark production) and $\overline{\Phi}$ (antiquark production) summations over colors are assumed, a factor 1/3 appears in the result in Eq. 4.11

Using the lightcone representation of the momenta in frame II (see section 1) it is easy to see that if the quark momenta in the matrix elements Φ are limited, i.e. p^2, $p\cdot P_A$ are of hadronic scale and similarly in the matrix elements $\overline{\Phi}$ for k^2 and $k\cdot P_B$, one can write the delta function up to $\mathscr{O}(1/Q^2)$ as

$$\delta^4(p+k-q) \approx \delta(p^+-q^+)\,\delta(k^--q^-)\,\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T), \tag{4.12}$$

The result in leading order is then

$$\begin{aligned}
\mathscr{W}_{\mu\nu} &= \frac{1}{3}\int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\text{Tr}\left(\left[\textstyle\int dp^-\Phi(p)\right]\gamma_\mu\left[\textstyle\int dk^+\overline{\Phi}(k)\right]\gamma_\nu\right)\Big|_{\substack{p^+=x_AP_A^+ \\ k^-=x_BP_B^-}} \\
&= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}+\frac{Z_\mu Z_\nu}{Z^2}\right)\frac{1}{3}\,I[f_1\,\overline{f}_1],
\end{aligned} \tag{4.13}$$

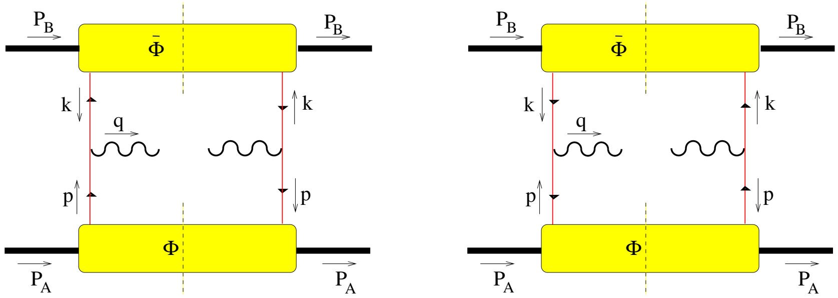


Figure 4.1: Born diagrams for Drell-Yan scattering

where

$$I[f_1\,\overline{f}_1] = \int d^2p_T\, d^2k_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\, f_1(x_A, \boldsymbol{p}_T)\overline{f}_1(x_B, \boldsymbol{k}_T). \tag{4.14}$$

Using the contraction with the leptonic tensor,

$$\begin{aligned} L^{\mu\nu}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} + \frac{Z_\mu Z_\nu}{Z^2}\right) &= 4Q^2\left(\frac{1}{2} - y + y^2\right) \\ &= Q^2\left(1 + \cos^2\theta_{\mu\mu}\right), \end{aligned} \tag{4.15}$$

the cross section becomes

$$\frac{d\sigma(AB\to\mu^+\mu^-X)}{dx_A\,dx_B\,d^2\boldsymbol{q}_T\,d\Omega_{\mu\mu}} = \frac{\alpha^2}{12\,Q^2}\,\left(1+\cos^2\theta_{\mu\mu}\right)\,I[f_1\,\overline{f}_1]. \tag{4.16}$$

Integrated over the transverse momenta of the produced mu pair,

$$\frac{d\sigma(AB\to\mu^+\mu^-X)}{dx_A\,dx_B\,d\Omega_{\mu\mu}} = \frac{\alpha^2}{12\,Q^2}\,\left(1+\cos^2\theta_{\mu\mu}\right)\,f_1(x_A)\,\overline{f}_1(x_B), \tag{4.17}$$

and integrated over the muon angular distribution,

$$\frac{d\sigma}{dx_A\,dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\,f_1(x_A)\,\overline{f}_1(x_B), \tag{4.18}$$

or including the summation over quarks and antiquarks,

$$\begin{aligned} \frac{d\sigma(AB\to\mu^+\mu^-X)}{dx_A\,dx_B} &= \frac{4\pi\,\alpha^2}{9\,Q^2}\sum_a e_a^2\,f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B) \\ &= \frac{1}{3}\sum_a f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\hat{\sigma}(a\overline{a}\to\mu^-\mu^+), \end{aligned} \tag{4.19}$$

where the quark-antiquark annihilation cross section is given by

$$\hat{\sigma}(a\overline{a}\to\mu^-\mu^+) = \frac{4\pi\alpha^2}{3Q^2}\,e_a^2. \tag{4.20}$$

and the factor 1/3 multiplying the summation is the color factor that can be naturally understood because only quarks of the same color can annihilate and we have seen that the definitions of the quark distribution functions included a summation over colors.

Introducing the virtuality of the photon (i.e. the invariant mass $\hat{s}$ of the produced mu pair) as a variable one can consider the Drell-Yan cross section as a function of s. Writing

$$\frac{d\hat{\sigma}}{dQ^2}\,(a\overline{a}\to\mu^-\mu^+) = \frac{4\pi\,\alpha^2}{3\,Q^2}\,e_a^2\,\delta(\hat{s}-Q^2), \tag{4.21}$$

one has

$$\begin{aligned} \frac{d\sigma(AB\to\mu^-\mu^+X)}{dQ^2\,dx_A\,dx_B} &= \frac{1}{3}\sum_a f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\frac{d\hat{\sigma}}{dQ^2}(a\overline{a}\to\mu^-\mu^+) \\ &= \frac{4\pi\alpha^2}{9Q^4}\,\sum_a e_a^2\,f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\delta\left(\frac{s}{Q^2}\,x_Ax_B - 1\right), \end{aligned} \tag{4.22}$$

which exhibits explicitly the scaling in $\tau = s/Q^2$ for the cross section $Q^4\,d\sigma/dQ^2$.

4.3 Electron-positron annihilation

The hadronic tensor in this case is rewritten as

$$\begin{aligned} \mathscr{W}_{\mu\nu}(q;P_1S_1;P_2S_2) &= \frac{1}{(2\pi)^4}\int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q-P_X-P_1-P_2) \\ &\quad\times\langle 0|J_\mu(0)|P_X;P_1S_1;P_2S_2\rangle\langle P_X;P_1S_1;P_2S_2|J_\nu(0)|0\rangle, \\ &= \frac{1}{(2\pi)^4}\int d^4x\;e^{iq\cdot x}\,\langle 0|J_\mu(x)\sum_X|X;P_1S_1;P_2S_2\rangle\langle X;P_1S_1;P_2S_2|J_\nu(0)|0\rangle, \end{aligned}$$

which in tree approximation (Born terms) becomes

$$\begin{aligned}
\mathscr{W}_{\mu\nu} &= \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle 0|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x):\sum_X |X;P_1S_1;P_2S_2\rangle \\
&\qquad\qquad\qquad \times\langle X;P_1S_1;P_2S_2|:\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|0\rangle \\
&= \frac{1}{(2\pi)^4}\,3\int d^4x\; e^{iq\cdot x}\,\langle 0|\overline{\psi}_j(x)\sum_{X_2}|X_2;P_2S_2\rangle\langle X_2;P_2S_2|\psi_i(0)|PS\rangle(\gamma_\mu)_{jk} \\
&\qquad\qquad \langle 0|\psi_k(x)\sum_{X_1}|X_1;P_1S_1\rangle\langle X_1;P_1S_1|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li} \\
&\quad +\frac{1}{(2\pi)^4}\,3\int d^4x\; e^{iq\cdot x}\,\langle 0|\psi_k(x)\sum_{X_2}|X_2;P_2S_2\rangle\langle X_2;P_2S_2|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li} \\
&\qquad\qquad \langle 0|\overline{\psi}_j(x)\sum_{X_1}|X_1;P_1S_1\rangle\langle X_1;P_1S_1|\psi_i(0)|0\rangle(\gamma_\mu)_{jk}, \\
&= 3\int d^4p\,d^4k\,\delta^4(q-k-p)\;\mathrm{Tr}\left(\overline{\Delta}(p)\gamma_\mu\Delta(k)\gamma_\nu\right)+\left\{\begin{array}{c} q\leftrightarrow -q \\ \mu\leftrightarrow\nu\end{array}\right\},
\end{aligned} \tag{4.23}$$

where we have used

$$\Delta_{kl}(k) = \frac{1}{(2\pi)^4}\int d^4x\; e^{ik\cdot x}\,\langle 0|\psi_k(x)\sum_{X_1}|X_1;P_1S_1\rangle\langle X_1;P_1S_1|\overline{\psi}_l(0)|0\rangle,$$

$$\overline{\Delta}_{ij}(p) = \frac{1}{(2\pi)^4}\int d^4x\; e^{ip\cdot x}\,\langle PS|\overline{\psi}_j(x)\sum_{X_2}|X_2;P_2S_2\rangle\langle X_2;P_2S_2|\psi_i(0)|PS\rangle,$$

and its symmetry properties (see section 2). Note that since both in Δ (quark decay) and $\overline{\Delta}$ (antiquark decay) an averaging over colors is assumed, we get a color factor of 3 in Eq. 4.23. We have only considered two hadrons in different jets, i.e. no fragmentation parts involving matrix elements of the form $\langle 0|\psi_j(x)\sum_X|X;P_1S_1;P_2S_2\rangle\;\langle X;P_1S_1;P_2S_2|\psi_i(0)|0\rangle$ are considered.

Using the lightcone representation of the momenta in frame II (see section 1) it is easy to see that if the quark momenta in the matrix elements $\overline{\Delta}$ are limited, i.e. p^2, $p\cdot P_2$ are of hadronic scale and similarly in the matrix elements Δ for k^2 and $k\cdot P_1$, one can write the delta function up to $\mathscr{O}(1/Q^2)$ as

$$\delta^4(q-k-p)\approx\delta(q^- - k^-)\,\delta(q^+ - p^+)\,\delta^2(\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{p}_T), \tag{4.24}$$

The result in leading order is then

$$\begin{aligned}
\mathscr{W}_{\mu\nu} &= 3\int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\delta^2(\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{p}_T)\mathrm{Tr}\left([\textstyle\int dp^-\overline{\Delta}(p)]\gamma_\mu[\int dk^+\Delta(k)]\gamma_\nu\right)\Big|_{\begin{array}{l}k^- = P_1^-/z_1\\ p^+ = P_2^+/z_2\end{array}} \\
&= \left(\frac{q_\mu q_\nu}{q^2}-g_{\mu\nu}+\frac{Z_\mu Z_\nu}{Z^2}\right)12\,z_1z_2\,I[D_1\,\overline{D}_1],
\end{aligned} \tag{4.25}$$

where

$$I[D_1\,\overline{D}_1] = \int d^2k_T\,d^2p_T\,\delta^2(\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{p}_T)\,D_1(z_1,-z_1\boldsymbol{k}_T)\overline{D}_1(z_2,-z_2\boldsymbol{p}_T). \tag{4.26}$$

Using the contraction with the leptonic tensor,

$$\begin{aligned}
L^{\mu\nu}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}+\frac{Z_\mu Z_\nu}{Z^2}\right) &= 4Q^2\left(\frac{1}{2}-y+y^2\right) \\
&= Q^2\left(1+\cos^2\theta\right),
\end{aligned} \tag{4.27}$$

where θ is the angle of the produced hadrons h_2 in the e^-e^+ rest frame. The cross section becomes

$$\frac{d\sigma(e^-e^+\to h_1h_2X)}{d\Omega\,dz_1\,dz_2\,d^2\boldsymbol{q}_T} = \frac{3\,\alpha^2}{4\,Q^2}\left(1+\cos^2\theta\right)z_1^2z_2^2\,I[D_1\,\overline{D}_1]. \tag{4.28}$$

Integrated over transverse momenta of hadron h_1 one finds,

$$\frac{d\sigma(e^-e^+ \to h_1 h_2 X)}{d\Omega\, dz_1\, dz_2} = \frac{3\,\alpha^2}{4\,Q^2}\,\left(1+\cos^2\theta\right) D_1(z_1)\,\overline{D}_1(z_2), \tag{4.29}$$

Including the summation over quarks and antiquarks one obtains

$$\begin{aligned} \frac{d\sigma(e^-e^+ \to h_1 h_2 X)}{d\Omega\, dz_1\, dz_2} &= \frac{3\,\alpha^2}{4\,Q^2}\,\left(1+\cos^2\theta\right)\sum_a e_a^2 D_{1\,h_1/a}(z_1)\,\overline{D}_{1\,h_2/\overline{a}}(z_2) \\ &= 3\sum_a \frac{d\hat{\sigma}}{d\Omega}(e^-e^+ \to a\overline{a})\, D_{1\,h_1/a}(z_1)\,\overline{D}_{1\,h_2/\overline{a}}(z_2), \end{aligned} \tag{4.30}$$

where the annihilation cross section into a quark-antiquark pair is given by

$$\frac{d\hat{\sigma}}{d\Omega}(e^-e^+ \to a\overline{a}) = \frac{\alpha^2}{4\,Q^2}\,\left(1+\cos^2\theta\right). \tag{4.31}$$

The factor 3 multiplying the cross section can be naturally understood as the definitions of each fragmentation function includes an averaging over color and the annihilation can be into a quark-antiquark pair of any of the three colors. The result for the production of a single hadron is obtained by considering hadron 2 as the jet with $\overline{D}_1 = \delta(1-z_2)$, thus

$$\begin{aligned} \frac{d\sigma(e^-e^+ \to hX)}{d\Omega\, dz} &= \frac{3\,\alpha^2}{4\,Q^2}\,\left(1+\cos^2\theta\right)\sum_a e_a^2\, D_{1\,h/a}(z) \\ &= 3\sum_a \frac{d\hat{\sigma}}{d\Omega}(e^-e^+ \to a\overline{a})\, D_{1\,h/a}(z). \end{aligned} \tag{4.32}$$

Integrating over the jet direction gives

$$\begin{aligned} \frac{d\sigma(e^-e^+ \to hX)}{dz} &= \frac{4\pi\,\alpha^2}{Q^2}\sum_a e_a^2\, D_{1\,h/a}(z) \\ &= 3\sum_a \hat{\sigma}(e^-e^+ \to a\overline{a})\, D_{1\,h/a}(z). \end{aligned} \tag{4.33}$$

Finally the jet cross section is found by taking $D_1(z) = \delta(1-z)$,

$$\begin{aligned} \frac{d\sigma(e^-e^+ \to jets)}{d\Omega} &= \frac{3\,\alpha^2}{4\,Q^2}\,\left(1+\cos^2\theta\right)\sum_a e_a^2 \\ &= 3\sum_a \hat{\sigma}(e^-e^+ \to a\overline{a}), \end{aligned} \tag{4.34}$$

and the total cross section

$$\begin{aligned} \sigma(e^-e^+ \to \text{hadrons}) &= \frac{4\pi\,\alpha^2}{Q^2}\sum_a e_a^2 \\ &= 3\sum_a \hat{\sigma}(e^-e^+ \to a\overline{a}) \\ &= \sigma(e^-e^+ \to \mu^-\mu^+)\; 3\sum_a e_a^2. \end{aligned} \tag{4.35}$$

Note that integrating over z the multiplicities of produced particles enter,

$$\int dz\,\frac{d\sigma(e^-e^+ \to hX)}{dz} = \langle n_h^{e^-e^+}\rangle \sigma(e^-e^+ \to \text{hadrons}) \tag{4.36}$$

given by

$$\langle n_h^{e^-e^+}\rangle = \frac{\sum_a e_a^2\, n_{h/a}}{\sum_a e_a^2}, \tag{4.37}$$

where $n_{h/a} = \int dz\, D_{1\,h/a}(z)$.

4.4 lepton-hadron scattering

The hadronic tensor in this case is rewritten as

$$
\begin{aligned}
2M\mathscr{W}_{\mu\nu}(q;PS;P_hS_h) &= \frac{1}{(2\pi)^4}\int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q+P-P_X-P_h) \\
&\qquad \times\langle PS|J_\mu(0)|P_X;P_hS_h\rangle\langle P_X;P_hS_h|J_\nu(0)|PS\rangle, \\
&= \frac{1}{(2\pi)^4}\int d^4x\ e^{iq\cdot x}\,\langle PS|J_\mu(x)\sum_X |X;P_h,S_h\rangle\langle X;P_hS_h|J_\nu(0)|PS\rangle,
\end{aligned}
$$

which in tree approximation (Born terms) becomes

$$
\begin{aligned}
2M\mathscr{W}_{\mu\nu}(q;PS;P_hS_h) &= \frac{1}{(2\pi)^4}\int d^4x\ e^{iq\cdot x}\,\langle PS|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x):\sum_X|X;P_hS_h\rangle \\
&\qquad\qquad \times\langle X;P_hS_h|:\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|PS\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4x\ e^{iq\cdot x}\,\langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle(\gamma_\mu)_{jk} \\
&\qquad\qquad \langle 0|\psi_k(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li} \\
&\quad +\frac{1}{(2\pi)^4}\int d^4x\ e^{iq\cdot x}\,\langle PS|\psi_k(x)\overline{\psi}_l(0)|PS\rangle(\gamma_\nu)_{li} \\
&\qquad\qquad \langle 0|\overline{\psi}_j(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\psi_i(0)|0\rangle(\gamma_\mu)_{jk}, \\
&= \int d^4p\,d^4k\,\delta^4(p+q-k)\ \mathrm{Tr}\left(\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right)+\left\{\begin{array}{c} q\leftrightarrow -q \\ \mu\leftrightarrow\nu\end{array}\right\}, \qquad (4.38)
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\Phi_{ij}(p) &= \frac{1}{(2\pi)^4}\int d^4x\ e^{ip\cdot x}\,\langle PS|\overline{\psi}_j(0)\psi_i(x)|PS\rangle, \\
\Delta_{kl}(k) &= \frac{1}{(2\pi)^4}\int d^4x\ e^{ik\cdot x}\,\langle 0|\psi_k(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle,
\end{aligned}
$$

and its symmetry properties (see section 2). Note that in Φ (quark production) a summation over colors is assumed, while in Δ (quark decay) an averaging over colors is assumed. We have not considered possible target fragmentation parts involving matrix elements of the form $\langle PS|\psi_j(x)\sum_X|X;P_hS_h\rangle$ $\langle X;P_hS_h|\psi_i(0)|PS\rangle$.

Using the lightcone representation of the momenta in frame II (see section 1) it is easy to see that if the quark momenta in the matrix elements Φ are limited, i.e. p^2, $p\cdot P$ are of hadronic scale and similarly in the matrix elements Δ for k^2 and $k\cdot P_h$, one can write the delta function up to $\mathscr{O}(1/Q^2)$ as

$$
\delta^4(p+q-k)\approx\delta(p^+ + q^+)\,\delta(q^- - k^-)\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T), \qquad (4.39)
$$

The result in leading order is then

$$
\begin{aligned}
2M\,\mathscr{W}_{\mu\nu} &= \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\mathrm{Tr}\left([\textstyle\int dp^-\Phi(p)]\gamma_\mu[\int dk^+\Delta(k)]\gamma_\nu\right)\Big|_{\substack{p^+=x_BP^+\\ k^-=P_h^-/z}} \\
&= \left(\frac{q_\mu q_\nu}{q^2}-g_{\mu\nu}+\frac{\tilde{P}_\mu\tilde{P}_\nu}{\tilde{P}^2}\right)2z\,I[f_1\,D_1], \qquad (4.40)
\end{aligned}
$$

where

$$
I[f_1\,D_1]=\int d^2p_T\,d^2k_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\,f_1(x_B,\boldsymbol{p}_T)D_1(z,-z\boldsymbol{k}_T). \qquad (4.41)
$$

Using the contraction with the leptonic tensor,

$$
L^{\mu\nu}\left(\frac{q_\mu q_\nu}{q^2}-g_{\mu\nu}+\frac{\tilde{P}_\mu\tilde{P}_\nu}{\tilde{P}^2}\right)=\frac{4Q^2}{y^2}\left(\frac{y^2}{2}+1-y\right), \qquad (4.42)
$$

the cross section becomes

$$\frac{d\sigma(\ell H \to \ell' hX)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \frac{4\pi\alpha^2\,s}{Q^4}\,\left(\frac{y^2}{2}+1-y\right)\,x_B I[f_1\,D_1], \tag{4.43}$$

or integrated over $\boldsymbol{P}_{h\perp}$,

$$\frac{d\sigma(\ell H \to \ell' hX)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\left(\frac{y^2}{2}+1-y\right)\,x_B f_1(x_B)\,D_1(z). \tag{4.44}$$

The case of inclusive electroproduction is easily obtained using the result $D_1(z) = \delta(1-z)$ for quark production off a quark, giving

$$\frac{d\sigma(\ell H \to \ell' X)}{dx_B\,dy} = \frac{4\pi\alpha^2\,s}{Q^4}\,\left(\frac{y^2}{2}+1-y\right)\,x_B f_1(x_B). \tag{4.45}$$

This leads to the well-known result for electroproduction of hadrons h (now also including the summation over quarks and antiquarks, where the latter come from the second contribution in Eq. 4.38)

$$\frac{\frac{d\sigma(\ell H\to\ell' hX)}{dx\,dy\,dz}}{\frac{d\sigma(\ell H\to\ell' X)}{dx\,dy}} = \frac{N_h^{\ell H}(x,z)}{\sum_a e_a^2\,f_{1\,a/H}(x)} = \frac{\sum_a e_a^2\,f_{1\,a/H}(x)\,D_{1\,h/a}(z)}{\sum_a e_a^2\,f_{1\,a/H}(x)}. \tag{4.46}$$

Upon integration over z one obtains

$$\int dz\,\frac{d\sigma(\ell H \to \ell' hX)}{dx\,dy\,dz} = \langle n_h^{\ell H}(x)\rangle\,\frac{d\sigma(\ell H \to \ell' X)}{dx\,dy} \tag{4.47}$$

where $\langle n_h(x)\rangle$ represents the average number of produced particles as a function of x,

$$\langle n_h^{\ell H}(x)\rangle = \frac{\sum_a e_a^2\,n_{h/a}\,f_{1\,a/H}(x)}{\sum_a e_a^2\,f_{1\,a/H}(x)}, \tag{4.48}$$

where $n_{h/a} = \int dz\,D_{1\,h/a}(z)$.

Finally we note that we can write the inclusive cross section for different values of $s = Q^2/x_B\,y$ in terms of the virtual photon-quark cross section $\sigma\left(\gamma^*(Q^2)\,a \to a\right)$ as

$$\begin{aligned}\frac{d\sigma(\ell H \to \ell' X)}{dx_B\,dy\,dQ^2} &= \frac{1}{Q^2}\,\frac{\alpha}{\pi}\,\frac{1}{y}\,\left(\frac{y^2}{2}+1-y\right)\sum_a f_{1a/H}(x_B)\,\sigma\left(\gamma^*(Q^2)a\to a\right)\\ &= \frac{4\pi\,\alpha^2}{Q^4}\,\frac{1}{y}\,\left(\frac{y^2}{2}+1-y\right)\,f_{1a/H}(x_B)\,\delta\left(\frac{s}{Q^2}x_B y-1\right),\end{aligned} \tag{4.49}$$

which is quite analogous to the situation in the Drell-Yan process.

4.5 Inclusion of longitudinal gluon contributions

We will consider in this section the inclusion of diagrams with gluons connecting the soft and hard part (see fig. 4.2).

$$\Phi^\alpha_{A\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\;e^{i\,k_1\cdot\xi+i(k-k_1)\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^\alpha(\eta)\psi_i(\xi)|P,S\rangle \tag{4.50}$$

(satisfying $\gamma_0\Phi_A^{\alpha\dagger}(k,k_1)\gamma_0 = \Phi_A^\alpha(k_1,k)$), and

$$\Delta^\alpha_{A\,ij}(k,k_1;P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\,d^4\eta\;e^{i\,k_1\cdot(\xi-\eta)+i\,k\cdot\eta}\langle 0|\psi_i(\xi)\,A_T^\alpha(\eta)\,a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle, \tag{4.51}$$

(satisfying $\gamma_0\Delta_A^{\alpha\dagger}(k,k_1)\gamma_0 = \Delta_A^\alpha(k_1,k)$). Performing the integrations $dk^-\,d^2\boldsymbol{k}_T$ and $dk_1^-\,d^2\boldsymbol{k}_{1T}$ one finds, using $k^+ = x\,P^+$ and $k_1^+ = y\,P^+$, the lightcone correlation functions corresponding to multiparton matrix elements, e.g.

$$\begin{aligned}\Phi^\alpha_{A\,ij}(x,y) &\equiv P^+\int dk^-\,d^2\boldsymbol{k}_T\;dk_1^-\,d^2\boldsymbol{k}_{1T}\;\Phi^\alpha_{A\,ij}(k,k_1;P,S)\\ &= P^+\int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^\alpha(\eta)\psi_i(\xi)|P,S\rangle\Big|_{LC},\end{aligned} \tag{4.52}$$

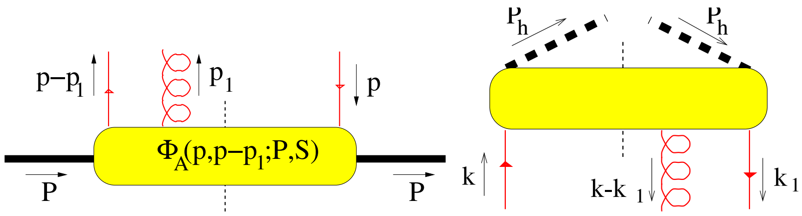


Figure 4.2: The quark-quark-gluon correlation functions

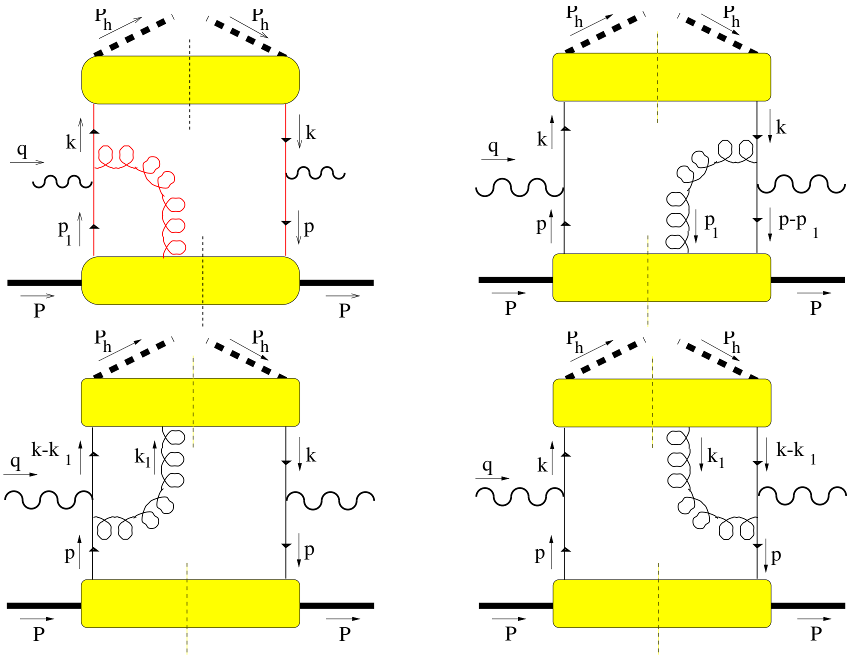


Figure 4.3: Quark-quark-gluon correlation functions in lepton-hadron scattering

where the subscript LC refers to 'lightcone', $\xi^+ = \eta^+ = \xi_T = \eta_T = 0$. Whatever way one parametrizes the above matrix elements, it is clear that in a process the index α being + requires a + component of one of the vectors of the soft part (P, S), which are of order Q after expanding in $n_\pm$ as discussed in the previous chapters. These thus give the dominant contribution. We will analyse them first for lepton-hadron scattering. We obtain four contributions as given in figure 4.3. Two of them have gluons connected to the lower soft part (the hadron → quark part), the others gluons connected to the upper soft part (the quark → hadron part). Including the contribution of the handbag one has

$$\begin{aligned}
2M\,\mathcal{W}_{\mu\nu} \;=\; & \int d^4p\, d^4k\, \delta^4(p+q-k)\ \mathrm{Tr}\left[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right] \\
& - \int d^4p\, d^4k\, d^4p_1\, \delta^4(p+q-k)\left\{\mathrm{Tr}\left[\gamma_\alpha\,\frac{(\not{k}-\not{p}_1+m)}{(k-p_1)^2-m^2+i\epsilon}\,\gamma_\nu\Phi_A^\alpha(p,p-p_1)\gamma_\mu\Delta(k)\right]\right. \\
& \qquad\qquad \left. +\ \mathrm{Tr}\left[\gamma_\mu\,\frac{(\not{k}-\not{p}_1+m)}{(k-p_1)^2-m^2-i\epsilon}\,\gamma_\alpha\Delta(k)\gamma_\nu\Phi_A^\alpha(p-p_1,p)\right]\right\} \\
& - \int d^4p\, d^4k\, d^4k_1\, \delta^4(p+q-k)\left\{\mathrm{Tr}\left[\gamma_\nu\,\frac{(\not{p}-\not{k}_1+m)}{(p-k_1)^2-m^2+i\epsilon}\,\gamma_\alpha\Phi(p)\gamma_\mu\Delta_A^\alpha(k-k_1,k)\right]\right. \\
& \qquad\qquad \left. +\ \mathrm{Tr}\left[\gamma_\alpha\,\frac{(\not{p}-\not{k}_1+m)}{(p-k_1)^2-m^2-i\epsilon}\,\gamma_\mu\Delta_A^\alpha(k,k-k_1)\gamma_\nu\Phi(p)\right]\right\}.
\end{aligned} \tag{4.53}$$

[Note that we have for a quark-quark-gluon blob used momentum p_1 (or k_1) for the gluon and $p-p_1$ (or $k-k_1$) for the quark. This is easier to extend when we consider multiple gluon correlation functions.] In the remainder of this section we will often omit the standard integration $\int d^4p\,d^4k\,\delta^4(p+q-k)$. The momenta p_1 and k_1 connected to the soft hadronic parts are parametrized according to

$$p = \left[p^-, \frac{Q}{\sqrt{2}}, \boldsymbol{p}_T\right], \qquad p_1 = \left[p_1^-, \frac{x_1\,Q}{\sqrt{2}}, \boldsymbol{p}_{1T}\right], \qquad p-k_1 = \left[-z_1\,\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \ldots\right], \tag{4.54}$$

$$k = \left[\frac{Q}{\sqrt{2}}, k^+, \boldsymbol{k}_T\right], \qquad k_1 = \left[\frac{z_1\,Q}{\sqrt{2}}, k_1^+, \boldsymbol{k}_{1T}\right], \qquad k-p_1 = \left[\frac{Q}{\sqrt{2}}, -x_1\,\frac{Q}{\sqrt{2}}, \ldots\right], \tag{4.55}$$

The momentum appearing in the extra fermion propagator is $p-p_1+q = k-p_1$ with $(k-p_1)^2 = -x_1\,Q^2$, or $k-k_1-q = p-k_1$ with $(p-k_1)^2 = -z_1\,Q^2$. Thus one has in leading order in $1/Q$,

$$\frac{\not k - \not p_1 + m}{(k-p_1)^2 - m^2 + i\epsilon} = \frac{\gamma^-}{Q\sqrt{2}} + \frac{\gamma^+}{(-x_1+i\epsilon)\,Q\sqrt{2}} - \frac{\boldsymbol{\gamma}_T\cdot(\boldsymbol{k}_T - \boldsymbol{p}_{1T}) - m}{(-x_1+i\epsilon)\,Q^2}, \tag{4.56}$$

$$\frac{\not p - \not k_1 + m}{(p-k_1)^2 - m^2 + i\epsilon} = \frac{\gamma^+}{Q\sqrt{2}} + \frac{\gamma^-}{(-z_1+i\epsilon)\,Q\sqrt{2}} - \frac{\boldsymbol{\gamma}_T\cdot(\boldsymbol{p}_T - \boldsymbol{k}_{1T}) - m}{(-z_1+i\epsilon)\,Q^2}. \tag{4.57}$$

This can be used to consider separately the contributions of transverse (A_T^α) and longitudinal (A^+) gluons.

For the transverse gluons, the trace of the first gluonic contribution becomes

$$\begin{aligned} &-\int d^4p_1\ \mathrm{Tr}\left[\gamma_\alpha\,\frac{\not k - \not p_1 + m}{(k-p_1)^2 - m^2 + i\epsilon}\,\gamma_\nu \Phi_A^\alpha(p,p-p_1)\gamma_\mu\Delta(k)\right] \\ &= \int \frac{d^4p_1}{(2\pi)^4}\int d^4\xi \int d^4\eta\ e^{i\,(p-p_1)\cdot\xi + i\,p_1\cdot\eta}\ \langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA_T^\alpha(\eta)\,\psi(\xi)|P,S\rangle, \end{aligned}$$

which starts off at order $1/Q$ and at this order requires leading parts from Φ_A^α (proportional to $P_+\Phi_A^\alpha P_-$) and leading parts from Δ (proportional to $P_-\Delta P_+$). As $\{\gamma^-,\gamma_T^\alpha\} = 0$ and $\gamma^- P_+ = P_-\gamma^- = 0$ only the $\gamma^- = P_+\gamma^- P_-$ part in Eq. 4.56 contributes. This term is independent of any of the components of p_1, and we thus can immediately consider the distributions $\int d^4p_1\ \Phi_A^\alpha(p,p-p_1)$, or explicitly

$$\int d^4\xi\ e^{i\,p\cdot\xi}\ \langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA_T^\alpha(\xi)\,\psi(\xi)|P,S\rangle. \tag{4.58}$$

This contribution will be studied in the next section. Note that it can be written in terms of the covariant derivative as

$$\begin{aligned} &\int d^4\xi\ e^{i\,p\cdot\xi}\ \langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, i\,D_T^\alpha(\xi)\,\psi(\xi)|P,S\rangle \\ &- p_T^\alpha \int d^4\xi\ e^{i\,p\cdot\xi}\ \langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\,\psi(\xi)|P,S\rangle. \end{aligned} \tag{4.59}$$

In this section we consider next the contributions of longitudinal gluons (A^+). They lead to traces of the form

$$-\int d^4p_1\ \mathrm{Tr}\left[\gamma^-\,\frac{\not k - \not p_1 + m}{(k-p_1)^2 - m^2 + i\epsilon}\,\gamma_\nu \Phi_A^+(p,p-p_1)\gamma_\mu\Delta(k)\right]$$

The first term in Eq. 4.56 does not contribute. The second term contributes at $\mathscr{O}(1)$ as the dominant contribution in Φ_A^+ is the part projected out by $\int dp_1^-\ P_+\Phi_A^+P_-$ which is of $\mathscr{O}(Q)$. Explicitly, we get for the first correction in Eq. 4.53

$$\begin{aligned} &-\int \frac{d^4p_1}{(2\pi)^4}\int d^4\xi \int d^4\eta\ e^{i\,(p-p_1)\cdot\xi + i\,p_1\cdot\eta}\ \langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^-\ \frac{\gamma^+}{(x_1 - i\epsilon)\,Q\sqrt{2}}\,\gamma_\nu\, gA^+(\eta)\,\psi(\xi)|P,S\rangle \\ &= \int \frac{dx_1}{2\pi}\int d^4\xi \int d\eta^-\ \frac{e^{i\,x_1\,p^+(\eta^- - \xi^-)}}{x_1 - i\epsilon}\ e^{i\,p\cdot\xi}\ \langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\ \frac{\gamma^-\gamma^+}{2}\,\gamma_\nu\, gA^+(\eta)\,\psi(\xi)|P,S\rangle\Big|_{\substack{\eta^+ = \xi^+ \\ \eta_T = \xi_T}} \\ &= \int d^4\xi \int d\eta^-\ \theta(\eta^- - \xi^-)\, e^{i\,p\cdot\xi}\ \langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\,P_+\,\gamma_\nu\, igA^+(\eta^-)\,\psi(\xi)|P,S\rangle \\ &= -\int d^4\xi\ e^{i\,p\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\,P_+\,\gamma_\nu\, ig\int_\infty^{\xi^-} d\eta^-\ A^+(\eta^-)\,\psi(\xi)|P,S\rangle. \end{aligned} \tag{4.60}$$

The second term in Eq. 4.53 gives

$$-\int d^4p_1\,\mathrm{Tr}\left[\gamma_\mu\,\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2-i\epsilon}\,\gamma^-\Delta(k)\gamma_\nu\Phi_A^+(p-p_1,p)\right]$$
$$=\int d^4\xi\;e^{i\,p\cdot\xi}\,\langle P,S|\overline{\psi}(0)\,ig\int_\infty^0 d\eta^-\;A^+(\eta^-)\,\gamma_\mu\,P_-\,\Delta(k)\,\gamma_\nu\,\psi(\xi)|P,S\rangle \tag{4.61}$$

The last two terms in Eq. 4.53 give

$$-\int d^4k_1\Bigg\{\mathrm{Tr}\left[\gamma_\nu\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2+i\epsilon}\,\gamma^+\Phi(p)\gamma_\mu\Delta_A^-(k_1,k)\right]$$
$$+\,\mathrm{Tr}\left[\gamma^+\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2-i\epsilon}\,\gamma_\mu\Delta_A^-(k,k-k_1)\gamma_\nu\Phi(p)\right]\Bigg\}$$
$$=\int d^4\xi\;e^{i\,k\cdot\xi}\Bigg\{\mathrm{Tr}\,\langle 0|\psi(\xi)\,a_h^\dagger a_h\,ig\int_{-\infty}^0 d\eta^+\;A^-(\eta^+)\,\overline{\psi}(0)\,\gamma_\nu\,P_+\,\Phi(p)\,\gamma_\mu|0\rangle$$
$$-\mathrm{Tr}\,\langle 0|\psi(\xi)\,ig\int_{-\infty}^{\xi^+} d\eta^+\;A^-(\eta^+)\,a_h^\dagger a_h\,\overline{\psi}(0)\,\gamma_\nu\,\Phi(p)\,P_-\,\gamma_\mu|0\rangle\Bigg\}.$$

The result of multiple A^+- or A^--gluons together with the tree-level result gives in leading order in $1/Q$ (when the projectors P_+ and P_- don't matter) the exponentiated path-ordered result

$$2M\,\mathscr{W}_{\mu\nu}=\int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\,[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu] \tag{4.62}$$

with

$$\Phi_{ij}(p,P,S)=\frac{1}{(2\pi)^4}\int d^4\xi\;e^{i\,p\cdot\xi}\langle P,S|\overline{\psi}_j(0)\mathscr{U}(0,\infty;0_T)\,\mathscr{U}(\infty,\xi^-;\xi_T)\psi_i(\xi)|P,S\rangle, \tag{4.63}$$
$$\Delta_{ij}(k,P_h,S_h)=\frac{1}{(2\pi)^4}\int d^4\xi\;e^{ik\cdot\xi}\,\langle 0|\mathscr{U}(-\infty,\xi^+;\xi_T)\psi_i(\xi)\,a_h^\dagger a_h\,\overline{\psi}_j(0)\mathscr{U}(0,-\infty;0_T)|0\rangle. \tag{4.64}$$

Provided we assume that matrix elements containing bilocal operators $\overline{\psi}(0)\,A_T(\eta^\pm=\mp\infty,\eta_T)\,\psi(\xi)$ vanish for physical states, the above links can be connected resulting in a color gauge-invariant matrix element that must be used in the definition of the correlation functions.

Before considering the transverse gluons let us check the case of two A^+ gluons. For instance considering a gauge choice $A^-=0$, one needs only to consider the absorption of the A^+ gluons in the 'distribution' part. Dressing the diagram leading to the first of the four terms above with another 'parallel' gluon one obtains a contribution

$$\int d^4p\,d^4k\,\delta^4(p+q-k)\int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\int d^4\xi\int d^4\eta_1\,d^4\eta_2\;e^{i\,(p-p_1-p_2)\cdot\xi+i\,p_1\cdot\eta_1+i\,p_2\cdot\eta_2}$$
$$\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^-\,\frac{\gamma^+}{(x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma^-\,\frac{\gamma^+}{(x_1+x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma_\nu\,gA^+(\eta_2)\,gA^+(\eta_1)\,\psi(\xi)|P,S\rangle$$
$$=\int d^4p\,d^4k\,\delta^4(p+q-k)\int\frac{dx_1}{2\pi}\,\frac{dx_2}{2\pi}\int d^4\xi\int d\eta_1^-\,d\eta_2^-\;\frac{e^{i\,(x_1+x_2)p^+(\eta^--\xi^-)}\,e^{i\,x_2p^+(\eta_2^--\eta_1^-)}}{(x_1+x_2-i\epsilon)\,(x_2-i\epsilon)}\;e^{i\,p\cdot\xi}$$
$$\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)P_+^2\,\gamma_\nu\,gA^+(\eta_2)\,gA^+(\eta_1)\,\psi(\xi)|P,S\rangle. \tag{4.65}$$

The integration over x_1 and x_2 gives

$$i\theta(\eta_1^--\xi^-)\,i\theta(\eta_2^--\eta_1^-), \tag{4.66}$$

leading to the path ordering.

Chapter 5

Gluon fields and correlation functions

5.1 Quark-gluon correlation functions

For the analysis beyond the twist two level, it is necessary to include quark-gluon correlation functions. We define

$$\Phi^{\alpha}_{A\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\eta)\psi_i(\xi)|P,S\rangle \tag{5.1}$$

(satisfying $\gamma_0\Phi^{\alpha\dagger}_{A}(k,k_1)\gamma_0 = \Phi^{\alpha}_{A}(k_1,k)$). Performing the integrations $dk^-\,d^2\boldsymbol{k}_T$ and $dk_1^-\,d^2\boldsymbol{k}_{1T}$ one finds, using $k^+ = x\,P^+$ and $k_1^+ = y\,P^+$, the lightcone correlation functions corresponding to multiparton matrix elements,

$$\begin{aligned}\Phi^{\alpha}_{A\,ij}(x,y) &\equiv P^+\int dk^-\,d^2\boldsymbol{k}_T\;dk_1^-\,d^2\boldsymbol{k}_{1T}\;\Phi^{\alpha}_{A\,ij}(k,k_1;P,S)\\ &= P^+\int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\eta)\psi_i(\xi)|P,S\rangle\Big|_{LC},\end{aligned} \tag{5.2}$$

where the subscript LC refers to 'lightcone', $\xi^+ = \eta^+ = \xi_T = \eta_T = 0$. Up to the twist three level it is often not necessary to consider this general three-field matrix element, but it is sufficient to consider the bilocal matrix elements obtained after integration over d^4k_1,

$$\begin{aligned}\Phi^{\alpha}_{A\,ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\;e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\xi)\psi_i(\xi)|P,S\rangle\\ &= \int d^4k_1\;\Phi^{\alpha}_{A\,ij}(k,k_1;P,S),\end{aligned} \tag{5.3}$$

$$\begin{aligned}(\gamma_0\Phi^{\alpha\dagger}_{A}\gamma_0)_{ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\;e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,gA^{\alpha}_{T}(0)\psi_i(\xi)|P,S\rangle\\ &= \int d^4k_1\;\Phi^{\alpha}_{A\,ij}(k_1,k;P,S).\end{aligned} \tag{5.4}$$

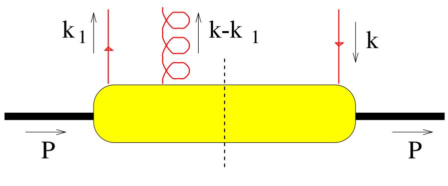


Figure 5.1: The quark-quark-gluon correlation function

We already mention here a third possible bilocal matrix element,

$$\begin{aligned}\Phi'^{\alpha}_{A\,ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\,e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\psi_i(0)\,gA^\alpha_T(\xi)|P,S\rangle \\ &= \int d^4k_1\;\Phi^{\alpha}_{A\,ij}(k+k_1,k_1;P,S).\end{aligned} \tag{5.5}$$

One-argument lightcone correlation functions are obtained after integration over $dk^- d^2\boldsymbol{k}_T$,

$$\Phi^{\alpha}_{A\,ij}(x) \equiv \int dk^-\,d^2\boldsymbol{k}_T\;\Phi^{\alpha}_{A\,ij}(k;P,S). \tag{5.6}$$

A factor P^+ has been included in the definition of $\Phi^\alpha_A(x,y)$, such that one has for the one-argument lightcone correlation functions $\Phi^\alpha_A(x) = \int dy\,\Phi^\alpha_A(x,y)$, etc. First, as before the same constraints as before from parity and time reversal invariance can be used. Including only terms that potentially contribute at the twist three level one obtains using only parity the general form

$$\begin{aligned}F^\alpha(k,P,S) &= C_1\,\not{P}k_\perp^\alpha + M\,C_2\,[\gamma^\alpha,\not{P}] + (C_3/M)\,[\not{P},\not{k}_\perp]\,k_\perp^\alpha + iC_4\,\epsilon^\alpha_{\ \ \nu\rho\sigma}\gamma^\nu\gamma_5 P^\rho k_\perp^\sigma \\ &\quad + M\,C_5\,\not{P}\gamma_5 S_\perp^\alpha + (C_6/M)\,\not{P}\gamma_5 k_\perp^\alpha\,(k\cdot S) + C_7\,[\not{P},\gamma^\alpha]\gamma_5\,(k\cdot S) \\ &\quad + C_8\,[\not{k}_\perp,\not{P}]\gamma_5\,S^\alpha + C_9\,[\not{S}_\perp,\not{P}]\gamma_5\,k_\perp^\alpha + (C_{10}/M^2)\,[\not{k}_\perp,\not{P}]\gamma_5\,k_\perp^\alpha\,(k\cdot S) \\ &\quad + iM\,C_{11}\,\epsilon^\alpha_{\ \ \nu\rho\sigma}\gamma^\nu P^\rho S_\perp^\sigma + i(C_{12}/M)\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu k_\perp^\rho S_\perp^\sigma\,k_\perp^\alpha \\ &\quad + i(C_{13}/M)\,\epsilon^\alpha_{\ \ \nu\rho\sigma}\gamma^\nu P^\rho k_\perp^\sigma\,(k\cdot S) \end{aligned} \tag{5.7}$$

$$F^{\alpha\dagger}(k,P,S) = \gamma_0\Big(C_1^*\,\not{P}k_\perp^\alpha - M\,C_2^*\,[\gamma^\alpha,\not{P}] + \ldots\Big)\gamma_0. \tag{5.8}$$

All amplitudes only depend on $P\cdot k$ and k^2. By choosing our conventions such that the the Dirac structures multiplying C_2, C_3, C_4, and C_{11}, C_{12}, C_{13} are antihermitean ($\Gamma^\dagger = -\gamma_0\Gamma\gamma_0$) and hermitean ($\Gamma^\dagger = \gamma_0\Gamma\gamma_0$) otherwise, all amplitudes are real when time reversal invariance applies.

Before starting with the twist-analysis we will discuss the issue of color gauge invariance. First of all, we of course need to employ correlation functions containing the covariant derivative $i\,D_\mu(\xi) = i\partial_\mu + g\,A_\mu(\xi)$ and field strength tensor $G_{\rho\sigma}(\xi) = (i/g)[D_\rho(\xi), D_\sigma(\xi)]$ instead of the A_μ fields. These are

$$\Phi^{\alpha}_{D\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,iD^\alpha(\eta)\psi_i(\xi)|P,S\rangle, \tag{5.9}$$

and

$$\Phi^{\alpha}_{G\,ij}(k,k_1;P,S) = \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gG^{+\alpha}(\eta)\psi_i(\xi)|P,S\rangle, \tag{5.10}$$

and as before the lightcone correlation functions

$$\Phi^{\alpha}_{D\,ij}(x,y) = \int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,iD^\alpha(\eta)\psi_i(\xi)|P,S\rangle\Big|_{LC}, \tag{5.11}$$

$$\Phi^{\alpha}_{G\,ij}(x,y) = \int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k_1\cdot(\xi-\eta)+ik\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gG^{+\alpha}(\eta)\psi_i(\xi)|P,S\rangle\Big|_{LC}. \tag{5.12}$$

Bilocal correlation functions are again obtained after integration over one momentum,

$$\begin{aligned}\Phi^{\alpha}_{D\,ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\,e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,iD^\alpha_T(\xi)\psi_i(\xi)|P,S\rangle \\ &= \int d^4k_1\;\Phi^{\alpha}_{D\,ij}(k,k_1;P,S).\end{aligned} \tag{5.13}$$

and the function

$$\begin{aligned}\Phi'^{\alpha}_{G\,ij}(k;P,S) &= \int \frac{d^4\xi}{(2\pi)^4}\,e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,\psi_i(0)\,G^{+\alpha}(\xi)|P,S\rangle. \\ &= \int d^4k_1\;\Phi^{\alpha}_{G\,ij}(k+k_1,k_1;P,S) = -ik^+\int d^4k_1\;\Phi^{\alpha}_{A\,ij}(k+k_1,k_1;P,S).\end{aligned} \tag{5.14}$$

The above expressions (without link) are only useful in the gauge $A^+ = 0$, in which case the relation between A_T^α and $G^{+\alpha}$ is simple, $G^{+\alpha} = \partial^+ A_T^\alpha = \partial_- A_T^\alpha$. Possible inversions are (only considering the dependence on the minus component),

$$\begin{aligned} A_T^\alpha(\eta^-) &= A_T^\alpha(\infty) - \int_{-\infty}^{\infty} d\zeta^-\ \theta(\zeta^- - \eta^-)\, G^{+\alpha}(\zeta^-) \\ &= A_T^\alpha(-\infty) + \int_{-\infty}^{\infty} d\zeta^-\ \theta(\eta^- - \zeta^-)\, G^{+\alpha}(\zeta^-) \\ &= \frac{A_T^\alpha(\infty) + A_T^\alpha(-\infty)}{2} - \frac{1}{2}\int_{-\infty}^{\infty} d\zeta^-\ \epsilon(\zeta^- - \eta^-)\, G^{+\alpha}(\zeta^-) \end{aligned} \tag{5.15}$$

or using the representation for the θ-function,

$$\pm i\,\theta(\pm\xi) = \int \frac{dk}{2\pi}\,\frac{e^{ik\xi}}{k \mp i\epsilon}, \qquad i\,\epsilon(\xi) = \int \frac{dk}{2\pi}\,\text{PP}\frac{e^{ik\xi}}{k}, \tag{5.16}$$

one obtains (omitting the hadron momenta and spin vectors)

$$\begin{aligned} \Phi_A^\alpha(k,k_1) &= \delta(k^+ - k_1^+)\,\Phi_{A(\infty)}^\alpha(k,k_1) + \frac{i}{k^+ - k_1^+ - i\epsilon}\,\Phi_G^\alpha(k,k_1) \\ &= \delta(k^+ - k_1^+)\,\Phi_{A(-\infty)}^\alpha(k,k_1) + \frac{i}{k^+ - k_1^+ + i\epsilon}\,\Phi_G^\alpha(k,k_1) \\ &= \delta(k^+ - k_1^+)\,\frac{\Phi_{A(\infty)}^\alpha(k,k_1) + \Phi_{A(-\infty)}^\alpha(k,k_1)}{2} + \text{PP}\frac{i}{k^+ - k_1^+}\,\Phi_G^\alpha(k,k_1), \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} &\delta(k^+ - k_1^+)\,\Phi_{A(\pm\infty)\,ij}^\alpha(k,k_1) \\ &\qquad \equiv \int \frac{d^4\xi}{(2\pi)^4}\,\frac{d^4\eta}{(2\pi)^4}\; e^{i\,k\cdot\xi}\,e^{i\,(k-k_1)\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA_T^\alpha(\pm\infty,\eta^+,\eta_T)\psi_i(\xi)|P,S\rangle, \end{aligned} \tag{5.18}$$

and

$$\delta(k^+ - k_1^+)\,\Big[\Phi_{A(\infty)}^\alpha(k,k_1) - \Phi_{A(-\infty)}^\alpha(k,k_1)\Big] = 2\pi\,\delta(k^+ - k_1^+)\,\Phi_G^\alpha(k,k_1) \tag{5.19}$$

The constraints following from hermiticity, parity and time-reversal are the following,

$$\Phi_D^{\alpha\,\dagger}(k,k_1;P,S) = \gamma_0\,\Phi_D^\alpha(k_1,k;P,S)\gamma_0, \tag{5.20}$$

$$\Phi_D^\alpha(k,k_1;P,S) = \gamma_0\,\Phi_{D\alpha}(\bar{k},\bar{k}_1;\bar{P},-\bar{S})\gamma_0, \tag{5.21}$$

$$\Phi_D^{\alpha\,*}(k,k_1;P,S) = (-i\gamma_5 C)\,\Phi_{D\alpha}(\bar{k},\bar{k}_1;\bar{P},\bar{S})(-i\gamma_5 C). \tag{5.22}$$

Similar properties hold for Φ_A and with a minus sign for the last relation (time reversal) also for Φ_G. We note the following for the boundary condition terms defined in Eq. 5.18 under time-reversal:

$$(2\pi)\,\delta(k^+ - k_1^+)\,\Phi_{A(\infty)}^{\alpha *}(P,S;k,k_1) = (2\pi)\,\delta(\bar{k}^+ - \bar{k}_1^+)\,(-i\gamma_5 C)\,\Phi_{A(-\infty)}^\alpha(\bar{P},\bar{S};\bar{k},\bar{k}_1)\,(-i\gamma_5 C). \tag{5.23}$$

This is the consequence of the fact that the point $\eta^- = \infty$ is defined by $\eta\cdot n_+ = \infty$, which after time reversal transforms into the point $\eta\cdot\bar{n}_+ = -\infty$. Since the component $\bar{\eta}^-$ is not integrated over the minus sign is not removed by a change of variables as is the case in $\Phi_A(k,k_1)$ (analoguously to the case spelled out for Φ in the first paragraph of chapter 2). Thus we see that the left and right sides of Eqs 5.17 are consistent.

Integrating over $dk^-\,d^2\boldsymbol{k}_T$ and $dk_1^-\,d^2\boldsymbol{k}_{1T}$ one finds

$$\begin{aligned} \Phi_A^\alpha(x,y) &= \delta(x-y)\,\Phi_{A(\infty)}^\alpha(x) + \frac{i}{x-y-i\epsilon}\,\Phi_G^\alpha(x,y) \\ &= \delta(x-y)\,\Phi_{A(-\infty)}^\alpha(x) + \frac{i}{x-y+i\epsilon}\,\Phi_G^\alpha(x,y) \\ &= \frac{1}{2}\,\delta(x-y)\,\underbrace{\Phi_{A(\infty)}^\alpha(x) + \Phi_{A(-\infty)}^\alpha(x)}_{\equiv 2\pi\,\Phi_{BC}^\alpha(x)} + \text{PP}\frac{i}{x-y}\,\Phi_G^\alpha(x,y), \end{aligned} \tag{5.24}$$

where

$$\delta(x-y)\,\Phi^{\alpha}_{A(\pm\infty)\,ij}(x) = P^+ \int \frac{d\xi^-}{2\pi}\,\frac{d\eta^-}{2\pi}\;e^{i\,k\cdot\xi}\,e^{i\,(k-k_1)\cdot\eta}\langle P,S|\overline{\psi}_j(0)\,gA^{\alpha}_{T}(\pm\infty,0,\mathbf{0}_T)\psi_i(\xi)|P,S\rangle\Big|_{LC} \qquad (5.25)$$

$$\delta(x-y)\,\underbrace{\left[\Phi^{\alpha}_{A(\infty)}(x)-\Phi^{\alpha}_{A(-\infty)}(x)\right]}_{2\pi\,\Phi^{\alpha}_{G}(x,x)} = 2\pi\,\delta(x-y)\,\Phi^{\alpha}_{G}(x,y). \qquad (5.26)$$

We note that the constraints from hermiticity, parity and time reversal relate functions at different momenta. However, for the lightcone fractions x and y the same fraction appears right and left since they are expansions of quark momenta in hadron momenta, $k = x\,P + \ldots$ and $\bar{k} = x\,\bar{P} + \ldots$, all of which become barred. Further, we note that if $\Phi_G(x,x) \neq 0$ a pole appears in Φ_A, hence the name 'gluonic pole'.

In order to define color gauge invariant functions it is necessary to include the link operator,

$$\mathcal{U}(\eta,\xi) = \mathcal{P}\exp\left(-ig\int_{\eta^-}^{\xi^-} d\zeta^{\mu}\,A_{\mu}(\zeta)\right), \qquad (5.27)$$

where we will implicitly understand that the path runs along the minus direction with $\xi^+ = \eta^+ = 0$ and $\boldsymbol{\xi}_T = \boldsymbol{\eta}_T$. Of course this means that relations hereafter need to be integrated over the minus components of the momenta. The path-ordered integral is defined as

$$\begin{aligned}\mathcal{U}(\eta,\xi) &= \mathcal{P}\exp\left(-ig\int_0^1 ds\,\frac{d\zeta^{\mu}(s)}{ds}\,A_{\mu}(\zeta(s))\right)\\ &\equiv 1 - ig\int_0^1 ds\,\frac{d\zeta^{\mu}(s)}{ds}\,A_{\mu}(\zeta(s))\\ &\quad +(ig)^2\int_0^1 ds_1\,\frac{d\zeta^{\mu}(s)}{ds_1}\,A_{\mu}(\zeta(s_1))\int_{s_1}^1 ds_2\,\frac{d\zeta^{\mu}(s_2)}{ds_2}\,A_{\mu}(\zeta(s_2)) + \mathcal{O}(g^3),\end{aligned} \qquad (5.28)$$

where $\zeta(s)$ is a path running from $\eta = \zeta(0)$ to $\xi = \zeta(1)$. The path ordered exponential is just the (infinite) product of infinitesimal link operators of the form

$$\mathcal{U}(\xi,\xi+d\xi) = 1 - ig\,d\xi^{\mu}A_{\mu}(\xi). \qquad (5.29)$$

From this infinitesimal form one checks that a counter-clockwise plaquette of four links, is given by

$$U(\xi,\xi+d\xi)U(\xi+d\xi,\xi+d\xi+d\eta)U(\xi+d\xi+d\eta,\xi+d\eta)U(\xi+d\eta,\xi) = 1 - ig\,d\xi^{\rho}d\eta^{\sigma}\,G_{\rho\sigma}(\xi). \qquad (5.30)$$

From the infinitesimal forms of links and plaquettes the following properties follow

$$i\partial^+_{\xi}\mathcal{U}(\eta,\xi) = \mathcal{U}(\eta,\xi)\,iD^+(\xi) \qquad (5.31)$$

$$\begin{aligned}iD^{\alpha}_{T}(\eta)\mathcal{U}(\eta,\xi) &= \mathcal{U}(\eta,\xi)\,iD^{\alpha}_{T}(\xi) + \int_{\eta^-}^{\xi^-} d\zeta^-\mathcal{U}(\eta,\zeta)\,[iD^+(\zeta),iD^{\alpha}_{T}(\zeta)]\,\mathcal{U}(\zeta,\xi)\\ &= \mathcal{U}(\eta,\xi)\,iD^{\alpha}_{T}(\xi) + ig\int_{\eta^-}^{\xi^-} d\zeta^-\mathcal{U}(\eta,\zeta)\,G^{+\alpha}(\zeta)\,\mathcal{U}(\zeta,\xi),\end{aligned} \qquad (5.32)$$

where $\alpha = 1{,}2$ (transverse), and hence along the link (where $\zeta_T = \xi_T$) $iD^{\alpha}_{T}(\zeta) = i\partial^{\alpha}_{\xi} + g\,A^{\alpha}_{T}(\zeta)$.

Including links we start with the gauge invariant definition of $\Phi_{ij}(x,\boldsymbol{k}_T)$,

$$\Phi_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^-d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\mathcal{U}(\infty,\xi)\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}. \qquad (5.33)$$

A note can be made at this point about the behavior of the correlation functions with a link (to be discussed in more detail below). Under time reversal the correlation function $\Phi_{ij}(x,\boldsymbol{k}_T)$ will not transform into itself, but the link will run via $\xi^- = -\infty$ instead of $\xi^- = \infty$. Only when the matrix element in Eq. 5.18 vanishes or after integration over transverse momenta time reversal can be used to constrain the parametrization of Φ and similarly other correlation functions. Multiplying the correlation function with the quark momentum we obtain,

$$k^\mu\,\Phi_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^-\,d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,i\partial^\mu\,\mathcal{U}(\infty,\xi)\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}, \tag{5.34}$$

where $i\partial^\mu$ can be read as $i\overleftarrow{\partial}{}_0^\mu$ or $i\overrightarrow{\partial}{}_\xi^\mu$. Depending on the Lorentz index of the derivative, one can use the link relations given before to rewrite the derivative in terms of correlation functions containing the covariant derivative,

$$\left(\Phi_D^\alpha\right)_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{i\,k\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,\mathcal{U}(\infty,\xi)\,iD_T^\alpha(\xi)\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}. \tag{5.35}$$

First of all, one of the correlation functions (with covariant derivative) is trivial. Because of the choice of link, which lies along the minus direction in the point ξ (except for the points $\xi^- = \infty$), one has for the +-component the relation

$$k^+\,\Phi_{ij}(x,\boldsymbol{k}_T) = \int \frac{d\xi^-\,d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,\mathcal{U}(\infty,\xi)\,iD^+\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}. \tag{5.36}$$

For the transverse component one finds

$$\begin{aligned} k_T^\alpha\,\Phi_{ij}(x,\boldsymbol{k}_T) &= (\Phi_\partial^\alpha)_{ij}(x,\boldsymbol{k}_T) \\ &= \int \frac{d\xi^-\,d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ik\cdot\xi}\Big\{\langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,iD_T^\alpha\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0} - (\Phi_{A(\infty)}^\alpha)_{ij}(x,\boldsymbol{k}_T) \\ &\quad - \langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\int_\infty^{\xi^-} d\eta^-\,\mathcal{U}(\infty,\eta)\,g\,G^{+\alpha}(\eta)\,\mathcal{U}(\eta,\xi)\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}\Big\}. \end{aligned} \tag{5.37}$$

Performing the $\boldsymbol{k}_T$ integration this leads to

$$\Phi_\partial^\alpha(x) = \Phi_D^\alpha(x) - \Phi_{A(\infty)}^\alpha(x) - \int dy\;\frac{i}{x-y-i\epsilon}\,\Phi_G^\alpha(x,y) = \Phi_D^\alpha(x) - \Phi_A^\alpha(x). \tag{5.38}$$

The sensitivity to background gluonic fields appears through the boundary terms, i.e. the matrix elements $\Phi_{A(\pm\infty)}^{\alpha[\Gamma]}(x)$. We have already encountered the antisymmetric combination. We define also the symmetric combination. Thus

$$\begin{aligned} 2\pi\,\Phi_{BC}^\alpha(x) &= \left[\Phi_{A(\infty)}^\alpha(x) + \Phi_{A(-\infty)}^\alpha(x)\right] \\ 2\pi\,\Phi_G^\alpha(x,x) &= \left[\Phi_{A(\infty)}^\alpha(x) - \Phi_{A(-\infty)}^\alpha(x)\right] \end{aligned}$$

where the important observation is that these two combinations have opposite behavior under time-reversal (even and odd, respectively). In later calculations we will typically see the following combination showing up,

$$\begin{aligned} &\int dy\,\frac{x-y}{x-y-i\epsilon}\,\Phi_A(x,y) = \int dy\,\frac{i}{x-y-i\epsilon}\,\Phi_G(x,y), \\ &= \Phi_D^\alpha(x) - \Phi_\partial^\alpha(x) - \pi\,\Phi_{BC}^\alpha(x) - \pi\,\Phi_G^\alpha(x,x) \end{aligned}$$

Considering the above as one object, $\Phi_A^{\alpha(eff)}(x)$ one needs in the parametrization T-odd functions if $\Phi_G^\alpha(x,x) \neq 0$. It is consistent with the observation that the presence of links prohibits the use of T-reversal constraints for $\Phi(x,\boldsymbol{k}_T)$.

The projections obtained for the quark-gluon corrrelation functions with transverse gluon fields are not independent from the ones defined for the quark-quark correlation functions, either. They can be connected to quark-quark correlation functions with one good and one bad quark field using the QCD equation of motion, $(i\not{D} - m)\psi(x) = 0$. From this equation it is straightforward to derive the relations

$$iD^\mu\psi + \sigma^{\mu\nu}D_\nu\psi - m\gamma^\mu\psi = 0, \tag{5.39}$$

$$i\gamma^\mu D^\nu\psi - i\gamma^\nu D^\mu\psi + im\,\sigma^{\mu\nu}\psi + i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5\,iD_\rho\psi = 0. \tag{5.40}$$

or explicitly

$$iD^+\psi - i\sigma^{+-}\,iD^+\psi + i\sigma^{\alpha+}\,iD_\alpha\psi - m\,\gamma^+\psi = 0, \tag{5.41}$$
$$\gamma^+\,iD^\alpha\psi - i\epsilon_T^{\alpha\beta}\gamma^+\gamma_5\,iD_\beta\psi = im\,\sigma^{\alpha+}\psi + \gamma^\alpha\,iD^+\psi - i\epsilon_T^{\alpha\beta}\gamma_\beta\gamma_5\,iD^+\psi, \tag{5.42}$$

where $\epsilon_T^{\alpha\beta} = \epsilon^{-+\alpha\beta}$. This immediately leads to the (T-even) relations

$$\Phi_{D\,\alpha}^{[\sigma^{\alpha+}]}(x,\boldsymbol{k}_T) = \epsilon_{T\alpha\beta}\,\Phi_D^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T) = i\,(Mx\,e - m\,f_1) + \epsilon_{Tij}\,k_T^i S_T^j\,xh_T^\perp \tag{5.43}$$
$$\Phi_{D\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T) = Mx\,h_s - m\,g_{1s}, \tag{5.44}$$
$$\Phi_D^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) - i\epsilon_T^{\alpha\beta}\,\Phi_{D\,\beta}^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T) = k_T^\alpha\,xf^\perp - i\epsilon_T^{\alpha\beta}k_{T\beta}\left(xg_s^\perp - \frac{m}{M}\,h_{1s}^\perp\right) - i\epsilon_T^{\alpha\beta}S_{T\beta}\,(Mx\,g_T' - m\,h_{1T}). \tag{5.45}$$

or including the T-odd possibilities

$$\Phi_{D\,\alpha}^{[\sigma^{\alpha+}]}(x,\boldsymbol{k}_T) = \epsilon_{T\alpha\beta}\,\Phi_D^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T) = i\,(Mx\,e - m\,f_1 - iMx\,h) + \epsilon_{Tij}\,k_T^i S_T^j\left(xh_T^\perp + i\,\frac{m}{M}\,f_{1T}^\perp\right) \tag{5.46}$$
$$\Phi_{D\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T) = Mx\,h_s - m\,g_{1s} + i\,Mx\,e_s, \tag{5.47}$$
$$\Phi_D^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) - i\epsilon_T^{\alpha\beta}\,\Phi_{D\,\beta}^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T) = k_T^\alpha\left(xf^\perp + i\,\frac{m}{M}\,h_1^\perp\right) - i\epsilon_T^{\alpha\beta}k_{T\beta}\left(xg_s^\perp - \frac{m}{M}\,h_{1s}^\perp - i\lambda\,x\,f_L^\perp\right) - i\epsilon_T^{\alpha\beta}S_{T\beta}\,(Mx\,g_T' - m\,h_{1T} - iMx\,f_T). \tag{5.48}$$

These relations for Φ_D can actually be considered as defining relations for the twist three correlation functions, again including k_T-dependence. Integrating the $\boldsymbol{k}_T$-dependence the most general form for $\Phi_D^\alpha(x,y)$ actually is

$$\Phi_D^\alpha(x,y) = \frac{M}{2P^+}\Bigg\{G_D(x,y)\,i\epsilon_T^{\alpha\beta}S_{T\beta}\not{n}_+ + \tilde{G}_D(x,y)\,S_T^\alpha\,\gamma_5\not{n}_+ + H_D(x,y)\,\lambda\,\gamma_5\gamma_T^\alpha\not{n}_+ + E_D(x,y)\,\gamma_T^\alpha\not{n}_+\Bigg\}. \tag{5.49}$$

with hermiticity leading to

$$G_D^*(x,y) = -G_D(y,x), \tag{5.50}$$
$$\tilde{G}_D^*(x,y) = \tilde{G}_D(y,x), \tag{5.51}$$
$$H_D^*(x,y) = H_D(y,x), \tag{5.52}$$
$$E_D^*(x,y) = -E_D(y,x). \tag{5.53}$$

The equations of motion are then giving the relations

$$\int dy\;G_D(x,y) = G_D(x) = C(x) + ix\,f_T(x), \tag{5.54}$$
$$\int dy\;\tilde{G}_D(x,y) = \tilde{G}_D(x) = C(x) + x\,g_T(x) - \frac{m}{M}\,h_1(x), \tag{5.55}$$
$$2\int dy\;H_D(x,y) = 2\,H_D(x) = x\,h_L(x) - \frac{m}{M}\,g_1(x) + ix\,e_L(x), \tag{5.56}$$
$$2\int dy\;E_D(x,y) = 2\,E_D(x) = -x\,e(x) + \frac{m}{M}\,f_1(x) + ix\,h(x), \tag{5.57}$$

where the function $C(x)$ cannot be given in terms of quark-quark correlation functions.

From Φ_D, we can get Φ_A, in essence as $\Phi_A = \Phi_D$ - Φ_∂. For the T-even case this leads to

$$
\begin{aligned}
g\Phi_{A\,\alpha}^{[\sigma^{\alpha+}]}(x,\boldsymbol{k}_T) &= \epsilon_{T\alpha\beta}\, g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T)\\
&\equiv iMx\,\tilde{e} + \epsilon_{Tij}\,k_T^i S_T^j\, x\tilde{h}_T^\perp\\
&= i\,(Mx\,e - m\,f_1) - \epsilon_{Tij}\,k_T^i S_T^j\,(h_{1T} - xh_T^\perp)
\end{aligned} \tag{5.58}
$$

or

$$
\begin{aligned}
& g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x,\boldsymbol{k}_T) - g\Phi_A^{\beta[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T)\\
&\qquad \equiv i\epsilon_T^{\alpha\beta}\,Mx\,\tilde{e} - \left(S_T^\alpha k_T^\beta - k_T^\alpha S_T^\beta\right)\, x\tilde{h}_T^\perp,\\
&\qquad = i\epsilon_T^{\alpha\beta}\,(Mx\,e - m\,f_1) + \left(S_T^\alpha k_T^\beta - k_T^\alpha S_T^\beta\right)(h_{1T} - xh_T^\perp),
\end{aligned} \tag{5.59}
$$

$$
g\Phi_{A\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T) \equiv Mx\,\tilde{h}_s = Mx\,h_s - m\,g_{1s} - (k_T\cdot S_T)\,h_{1T} - \frac{k_T^2}{M}\,h_{1s}^\perp, \tag{5.60}
$$

$$
\begin{aligned}
& g\Phi_A^{\alpha[\gamma^+]}(x,\boldsymbol{k}_T) - i\epsilon_T^{\alpha\beta}\, g\Phi_{A\,\beta}^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T)\\
&\qquad \equiv k_T^\alpha\, x\tilde{f}^\perp - i\epsilon_T^{\alpha\beta}k_{T\beta}\,x\tilde{g}_s^\perp - i\epsilon_T^{\alpha\beta}S_{T\beta}\,Mx\,\tilde{g}_T'\\
&\qquad = k_T^\alpha(xf^\perp - f_1) - i\epsilon_T^{\alpha\beta}k_{T\beta}\left(xg_s^\perp - g_{1s} - \frac{m}{M}\,h_{1s}^\perp\right) - i\epsilon_T^{\alpha\beta}S_{T\beta}\,(Mx\,g_T' - m\,h_{1T})\,,
\end{aligned} \tag{5.61}
$$

which are useful objects as they appear as soft quark-quark-gluon parts in a diagrammatic expansion of hard scattering processes. As the operators Γ used in $\Phi_A^{\alpha[\Gamma]}$ are hermitean in the sense that $\Gamma^\dagger = \gamma_0\Gamma\gamma_0$ one has

$$
(\gamma_0\Phi_A^{\alpha\dagger}\gamma_0)^{[\Gamma]} = \left(\Phi_A^{\alpha[\Gamma]}\right)^*. \tag{5.62}
$$

Integrated over $\boldsymbol{k}_T$ the above relations become

$$
g\Phi_{A\,\alpha}^{[\sigma^{\alpha+}]}(x) = \epsilon_{T\alpha\beta}\, g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x) \equiv i\,Mx\,\tilde{e} = i\,(Mx\,e - m\,f_1)\,, \tag{5.63}
$$

or

$$
g\Phi_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(x) - g\Phi_A^{\beta[i\sigma^{\alpha+}\gamma_5]}(x) \equiv i\epsilon_T^{\alpha\beta}\,Mx\,\tilde{e} = i\epsilon_T^{\alpha\beta}\,(Mx\,e - m\,f_1)\,, \tag{5.64}
$$

$$
g\Phi_{A\,\alpha}^{[i\sigma^{\alpha+}\gamma_5]}(x) = \lambda\,Mx\,\tilde{h}_L = \lambda\left(Mx\,h_L - m\,g_1 + 2M\,h_{1L}^{\perp(1)}\right), \tag{5.65}
$$

$$
g\Phi_A^{\alpha[\gamma^+]}(x) - i\epsilon_T^{\alpha\beta}\, g\Phi_{A\,\beta}^{[\gamma^+\gamma_5]}(x) \equiv -i\epsilon_T^{\alpha\beta}S_{T\beta}\,Mx\,\tilde{g}_T = -i\epsilon_T^{\alpha\beta}S_{T\beta}\left(Mx\,g_T - m\,h_1 - M\,g_{1T}^{(1)}\right), \tag{5.66}
$$

where the upper index (1) denotes

$$
f^{(1)}(x) = \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M^2}\,f(x,\boldsymbol{k}_T). \tag{5.67}
$$

The tilde functions are precisely the parts vanishing for the free quark case. This was the way they have been introduced in chapter 2.

With the same parametrization for $\Phi_A^\alpha(x,y)$ as the one for $\Phi_D^\alpha(x,y)$ given above one obtains (including now the T-odd functions)

$$
\int dy\; G_A(x,y) = G_A(x) = C(x) + ix\,f_T(x) + i\,f_{1T}^{\perp(1)}(x), \tag{5.68}
$$

$$
\int dy\; \tilde{G}_A(x,y) = \tilde{G}_A(x) = C(x) + x\,g_T(x) - \frac{m}{M}\,h_1(x) - g_{1T}^{(1)}(x), \tag{5.69}
$$

$$
2\int dy\; H_A(x,y) = 2\,H_A(x) = x\,h_L(x) - \frac{m}{M}\,g_1(x) + 2\,h_{1L}^{\perp(1)}(x) + ix\,e_L(x), \tag{5.70}
$$

$$
2\int dy\; E_A(x,y) = 2\,E_A(x) = -x\,e(x) + \frac{m}{M}\,f_1(x) + ix\,h(x) + 2i\,h_1^{\perp(1)}(x). \tag{5.71}
$$

For the antiquarks one needs to consider matrix elements

$$
\overline{\Phi}_{A\,ij}^\alpha(k,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi_i(\xi)A_T^\alpha(\xi)\overline{\psi}_j(0)|P,S\rangle = -\Phi_{A\,ij}^\alpha(-k,P,S), \tag{5.72}
$$

$$
(\gamma_0\overline{\Phi}_A^{\alpha\dagger}\gamma_0)_{ij}(k,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-i\,k\cdot\xi}\langle P,S|\psi_i(\xi)\,A_T^\alpha(0)\overline{\psi}_j(0)|P,S\rangle. \tag{5.73}
$$

For them one obtains, using for the nonlocal quark-quark matrix element

$$\frac{1}{(2\pi)^4}\int d^4\xi\; e^{-ik\cdot\xi}\,\langle P,S|\psi_i(\xi)\mathcal{U}\,i\partial^\mu\,\mathcal{U}\,\overline{\psi}_j(0)|P,S\rangle = -k^\mu\,\overline{\Phi}_{ij}(k,P,S) \tag{5.74}$$

(where $i\partial^\mu$ can be read as $i\overleftarrow{\partial}^\mu_\xi$ or $i\overrightarrow{\partial}^\mu_0$) and the equations of motion, the relations

$$\begin{aligned} g\overline{\Phi}_{A\,\alpha}^{\,[\sigma^{\alpha-}]}(x,\boldsymbol{k}_T) &= -\epsilon_{T\alpha\beta}\,g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x,\boldsymbol{k}_T)\\ &= i\left(Mx\,\overline{e}-m\,\overline{f}_1\right)-\epsilon_{Tij}\,k_T^iS_T^j\,(\overline{h}_{1T}-x\overline{h}_T^\perp) \end{aligned} \tag{5.75}$$

or

$$\begin{aligned} &g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x,\boldsymbol{k}_T)-g\overline{\Phi}_A^{\beta[i\sigma^{\alpha-}\gamma_5]}(x,\boldsymbol{k}_T)\\ &\qquad= -i\epsilon_T^{\alpha\beta}\left(Mx\,\overline{e}-m\,\overline{f}_1\right)-\left(S_T^\alpha k_T^\beta-k_T^\alpha S_T^\beta\right)(\overline{h}_{1T}-x\overline{h}_T^\perp), \end{aligned} \tag{5.76}$$

$$g\overline{\Phi}_{A\,\alpha}^{\,[i\sigma^{\alpha-}\gamma_5]}(x,\boldsymbol{k}_T) = -Mx\,\overline{h}_s+m\,\overline{g}_{1s}+(k_T\cdot S_T)\,\overline{h}_{1T}+\frac{k_T^2}{M}\,\overline{h}_{1s}^\perp, \tag{5.77}$$

$$\begin{aligned} &g\overline{\Phi}_A^{\alpha[\gamma^-]}(x,\boldsymbol{k}_T)+i\epsilon_T^{\alpha\beta}\,g\overline{\Phi}_{A\,\beta}^{\,[\gamma^-\gamma_5]}(x,\boldsymbol{k}_T)\\ &\qquad= -k_T^\alpha(x\overline{f}^\perp-\overline{f}_1)+i\epsilon_T^{\alpha\beta}k_{T\beta}\left(x\overline{g}_s^\perp-\overline{g}_{1s}-\frac{m}{M}\,\overline{h}_{1s}^\perp\right)+i\epsilon_T^{\alpha\beta}S_{T\beta}\left(Mx\,\overline{g}_T'-m\,\overline{h}_{1T}\right). \end{aligned} \tag{5.78}$$

The $\boldsymbol{k}_T$-integrated result is

$$g\overline{\Phi}_{A\,\alpha}^{\,[\sigma^{\alpha-}]}(x) = -\epsilon_{T\alpha\beta}\,g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x) = i\left(Mx\,\overline{e}-m\,\overline{f}_1\right), \tag{5.79}$$

or

$$g\overline{\Phi}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(x)-g\overline{\Phi}_A^{\beta[i\sigma^{\alpha-}\gamma_5]}(x) = -i\epsilon_T^{\alpha\beta}\left(Mx\,\overline{e}-m\,\overline{f}_1\right), \tag{5.80}$$

$$g\overline{\Phi}_{A\,\alpha}^{\,[i\sigma^{\alpha-}\gamma_5]}(x) = -\lambda\left(Mx\,\overline{h}_L-m\,\overline{g}_1-2M\,\overline{h}_{1L}^{\perp\,(1)}\right), \tag{5.81}$$

$$g\overline{\Phi}_A^{\alpha[\gamma^-]}(x)+i\epsilon_T^{\alpha\beta}\,g\overline{\Phi}_{A\,\beta}^{\,[\gamma^-\gamma_5]}(x) = i\epsilon_T^{\alpha\beta}S_{T\beta}\left(Mx\,\overline{g}_T-m\,\overline{h}_1-M\,g_{1T}^{(1)}\right). \tag{5.82}$$

For the fragmentation part one needs to consider the matrix elements

$$\Delta^\alpha_{A\,ij}(k,k_1;P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\,d^4\eta\;e^{i\,k_1\cdot(\xi-\eta)+i\,k\cdot\eta}\langle 0|\psi_i(\xi)\,A_T^\alpha(\eta)\,a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle, \tag{5.83}$$

or after integration over k_1 the bilocal matrix elements

$$\Delta^\alpha_{A\,ij}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{i\,k\cdot\xi}\langle 0|\psi_i(\xi)\,A_T^\alpha(\xi)\,a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle, \tag{5.84}$$

$$(\gamma_0\Delta_A^{\alpha\dagger}\gamma_0)_{ij}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{i\,k\cdot\xi}\langle 0|\psi_i(\xi)\,a_h^\dagger a_h\,A_T^\alpha(0)\,\overline{\psi}_j(0)|0\rangle. \tag{5.85}$$

The twist analysis for the projections

$$\begin{aligned} \Delta_A^{\alpha[\Gamma]}(z,\boldsymbol{k}_T) &= \frac{1}{4z}\int dk^+\;Tr\,(\Delta_A^\alpha\Gamma)\Big|_{k^-=P_h^-/z,\;\boldsymbol{k}_T}\\ &= \int\frac{d\xi^+d^2\xi_\perp}{4z\,(2\pi)^3}\;e^{i\,k\cdot\xi}\,Tr\,\langle 0|\psi(\xi)\,A_T^\alpha(\xi)\,a_h^\dagger a_h\,\overline{\psi}(0)\Gamma|0\rangle\Big|_{\xi^-=0}, \end{aligned} \tag{5.86}$$

is completely analogous to the distribution part. The equation of motion together with

$$\frac{1}{(2\pi)^4}\int d^4\xi\;e^{ik\cdot\xi}\,\langle 0|\psi_i(\xi)\,i\partial^\mu\,a_h^\dagger a_h\,\overline{\psi}_j(0)|0\rangle = k^\mu\,\Delta_{ij}(k,P_h,S) \tag{5.87}$$

(where $i\partial^\mu$ can be read as $i\overleftarrow{\partial}^\mu_\xi$ or $i\overrightarrow{\partial}^\mu_0$), now give the relations

$$
\begin{aligned}
g\Delta_{A\,\alpha}^{\,[\sigma^{\alpha-}]} = -\epsilon_{\alpha\beta}\, g\Delta_A^{\alpha[i\sigma^{\beta-}\gamma_5]} &= i\left(\frac{M_h}{z}\,\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right)\\
&\quad -\epsilon_{ij}\,k_T^i S_{hT}^j\,\left(\frac{1}{z}\,\tilde{H}_T^\perp + i\,\frac{m}{M_h}\,\tilde{D}_{1T}^\perp\right)\\
&= i\left(\frac{M_h}{z}E - m\,D_1 - i\,\frac{M_h}{z}\,H + i\,\frac{k_T^2}{M_h}\,H_1^\perp\right)\\
&\quad +\epsilon_{ij}\,k_T^i S_{hT}^j\,\left(H_{1T} - \frac{1}{z}H_T^\perp - i\,\frac{m}{M_h}\,D_{1T}^\perp\right), \qquad (5.88)
\end{aligned}
$$

or

$$
\begin{aligned}
g\Delta_A^{\alpha[i\sigma^{\beta-}\gamma_5]} - g\Delta_A^{\beta[i\sigma^{\alpha-}\gamma_5]} &= -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right)\\
&\quad -\left(S_{hT}^\alpha k_T^\beta - k_T^\alpha S_{hT}^\beta\right)\left(\frac{1}{z}\tilde{H}_T^\perp + i\,\frac{m}{M_h}\,\tilde{D}_{1T}^\perp\right)\\
&= -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}E - m\,D_1 - i\,\frac{M_h}{z}\,H + i\,\frac{k_T^2}{M_h}\,H_1^\perp\right)\\
&\quad +\left(S_{hT}^\alpha k_T^\beta - k_T^\alpha S_{hT}^\beta\right)\left(H_{1T} - \frac{1}{z}H_T^\perp - i\,\frac{m}{M_h}\,D_{1T}^\perp\right), \qquad (5.89)
\end{aligned}
$$

$$
\begin{aligned}
g\Delta_{A\,\alpha}^{\,[i\sigma^{\alpha-}\gamma_5]} &= \frac{M_h}{z}\,\tilde{H}_s + i\,\frac{M_h}{z}\,\tilde{E}_s\\
&= \frac{M_h}{z}H_s - m\,G_{1s} + i\,\frac{M_h}{z}E_s - (k_T\cdot S_{hT})H_{1T} - \frac{k_T^2}{M_h}\,H_{1s}^\perp, \qquad (5.90)
\end{aligned}
$$

$$
\begin{aligned}
g\Delta_A^{\alpha[\gamma^-]} + i\epsilon_T^{\alpha\beta}\,g\Delta_{A\,\beta}^{\,[\gamma^-\gamma_5]} &= k_T^\alpha\left(\frac{\tilde{D}^\perp}{z} + i\,\frac{m}{M_h}\,\tilde{H}_1^\perp\right) - \frac{\left(k_T^\alpha k_T^i + \frac{1}{2}\,k_T^2\,g_T^{\alpha i}\right)\epsilon_{T\,ij}S_{hT}^j}{M_h}\,\tilde{D}_{1T}^\perp\\
&\quad +i\epsilon_T^{\alpha\beta}k_{T\beta}\left(\frac{\tilde{G}_s^\perp}{z} - i\,\lambda_h\,\frac{\tilde{D}_L^\perp}{z}\right) + i\epsilon_T^{\alpha\beta}S_{h\,\beta}\left(\frac{M_h}{z}\,\tilde{G}_T' - i\,M_h\,\frac{\tilde{D}_T}{z}\right)\\
&= k_T^\alpha\left(\frac{1}{z}D^\perp - D_1 + i\,\frac{m}{M_h}\,H_1^\perp\right) - \frac{k_T^\alpha}{M_h}\,\epsilon_{ij}k_T^i S_{hT}^j\,D_{1T}^\perp\\
&\quad + i\epsilon_T^{\alpha\beta}k_{T\beta}\left(\frac{1}{z}G_s^\perp - G_{1s} - \frac{m}{M_h}\,H_{1s}^\perp - i\,\lambda_h\,\frac{D_L^\perp}{z}\right)\\
&\quad + i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}G_T' - m\,H_{1T} - i\,M_h\,\frac{D_T}{z}\right), \qquad (5.91)
\end{aligned}
$$

and $\Delta_A^{\alpha\dagger[\Gamma]} = (\Delta_A^{\alpha[\Gamma]})^*$. Note that the twist two profile functions $D_{1T}^\perp$, $H_1^\perp$ and the twist three profile functions E_L, E_T and H, that are odd under time reversal, enter as the imaginary parts in $\Delta_A^{\alpha[\Gamma]}$. Again the tilde functions are the 'interaction-dependent' distribution functions. Note that the time reveral odd functions are interaction-dependent, e.g. $D_{1T}^\perp = \tilde{D}_{1T}^\perp$, etc. We have, however, made the choices $D_T = \tilde{D}_T - zD_{1T}^{\perp(1)}$ and $H = \tilde{H} - zH_1^{\perp(1)}$ which guarantee the absence of $D_{1T}^\perp$ and $H_1^\perp$ in the integrated results.

The $\boldsymbol{k}_T$-integrated results are

$$g\Delta_{A\,\alpha}^{[\sigma^{\alpha-}]}(z) = -\epsilon_{\alpha\beta}\, g\Delta_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(z) = i\left(\frac{M_h}{z}\,\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right)$$
$$= i\left(\frac{M_h}{z}E - m\,D_1 - i\,M_h\,\left(\frac{H}{z} + 2\,H_1^{\perp\,(1)}\right)\right)$$
$$= i\left(\frac{M_h}{z}E - m\,D_1 - i\,M_h\frac{d}{dz}\left[z\,H_1^{\perp(1)}\right]\right) \tag{5.92}$$

or

$$g\Delta_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(z) - g\Delta_A^{\beta[i\sigma^{\alpha-}\gamma_5]}(z) = -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\,\tilde{E} - i\,\frac{M_h}{z}\,\tilde{H}\right),$$
$$= -i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}E - m\,D_1 - i\,M_h\,\left(\frac{H}{z} + 2\,H_1^{\perp(1)}\right)\right), \tag{5.93}$$

$$g\Delta_{A\,\alpha}^{[i\sigma^{\alpha-}\gamma_5]}(z) = \lambda_h\left(\frac{M_h}{z}\,\tilde{H}_L + i\,\frac{M_h}{z}\,\tilde{E}_L\right) = \lambda_h\left(\frac{M_h}{z}H_L - m\,G_1 + i\,\frac{M_h}{z}E_L - 2M_h\,H_{1L}^{\perp\,(1)}\right), \tag{5.94}$$

$$g\Delta_A^{\alpha[\gamma^-]}(z) + i\epsilon_T^{\alpha\beta}\,g\Delta_{A\,\beta}^{[\gamma^-\gamma_5]}(z) = i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,\tilde{G}_T + i\,\frac{M_h}{z}\,\tilde{D}_T\right)$$
$$= i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,G_T - m\,H_1 - M_h\,G_{1T}^{(1)} + i\,M_h\,\left(\frac{D_T}{z} + D_{1T}^{\perp\,(1)}\right)\right)$$
$$= i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,G_T - m\,H_1 - M_h\,G_{1T}^{(1)} - i\,M_h z\frac{d}{dz}\left[D_{1T}^{\perp(1)}\right]\right), \tag{5.95}$$

where

$$D^{(1)}(z) = z^2\int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M_h^2}\,D_1(z,-z\boldsymbol{k}_T). \tag{5.96}$$

For the antiquark fragmentation part one needs to consider the matrix elements

$$\overline{\Delta}_{A\,ij}^{\alpha}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{-i\,k\cdot\xi}\langle 0|\overline{\psi}_j(0)\,A_T^\alpha(\xi)\,a_h^\dagger a_h\psi_i(\xi)|0\rangle = -\Delta_{A\,ij}^{\alpha}(-k,P_h,S_h), \tag{5.97}$$

$$(\gamma_0\overline{\Delta}_A^{\alpha\dagger}\gamma_0)_{ij}(k,P_h,S_h) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{-i\,k\cdot\xi}\langle 0|\overline{\psi}_j(0)\,a_h^\dagger a_h\,A_T^\alpha(0)\,\psi_i(\xi)|0\rangle. \tag{5.98}$$

Using the equation of motion together with

$$\frac{1}{(2\pi)^4}\int d^4\xi\;e^{-ik\cdot\xi}\,\langle 0|\overline{\psi}_j(0)\,i\partial^\mu\,a_h^\dagger a_h\,\psi_i(\xi)|0\rangle = -k^\mu\,\overline{\Delta}_{ij}(k,P_h,S) \tag{5.99}$$

(where $i\partial^\mu$ can be read as $i\,\overleftarrow{\partial}_0^\mu$ or $i\,\overrightarrow{\partial}_\xi^\mu$), now give the relations

$$g\overline{\Delta}_{A\,\alpha}^{\,[\sigma^{\alpha+}]} = \epsilon_{\alpha\beta}\, g\overline{\Delta}_A^{\alpha[i\sigma^{\beta+}\gamma_5]} = i\left(\frac{M_h}{z}\,\overline{E} - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - i\,\frac{k_T^2}{M_h}\,\overline{H}_1^\perp\right) + \epsilon_{ij}\,k_T^i S_{hT}^j\,(\overline{H}_{1T} - \frac{1}{z}\,\overline{H}_T^\perp + i\,\frac{m}{M_h}\,\overline{D}_{1T}^\perp), \tag{5.100}$$

or

$$g\overline{\Delta}_A^{\alpha[i\sigma^{\beta+}\gamma_5]} - g\overline{\Delta}_A^{\beta[i\sigma^{\alpha+}\gamma_5]} = i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\,\overline{E} - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - i\,\frac{k_T^2}{M_h}\,\overline{H}_1^\perp\right) - \left(S_{hT}^\alpha k_T^\beta - k_T^\alpha S_{hT}^\beta\right)(\overline{H}_{1T} - \frac{1}{z}\,\overline{H}_T^\perp + i\,\frac{m}{M_h}\,\overline{D}_{1T}^\perp), \tag{5.101}$$

$$g\overline{\Delta}_{A\,\alpha}^{\,[i\sigma^{\alpha+}\gamma_5]} = -\frac{M_h}{z}\,\overline{H}_s + m\,\overline{G}_{1s} + i\,\frac{M_h}{z}\,\overline{E}_s + (k_T\cdot S_{hT})\overline{H}_{1T} + \frac{k_T^2}{M_h}\,\overline{H}_{1s}^\perp, \tag{5.102}$$

$$g\overline{\Delta}_A^{\alpha[\gamma^+]} - i\epsilon_T^{\alpha\beta}\,g\overline{\Delta}_{A\,\beta}^{\,[\gamma^+\gamma_5]} = -k_T^\alpha\left(\frac{1}{z}\,\overline{D}^\perp - \overline{D}_1 - i\,\frac{m}{M_h}\,\overline{H}_1^\perp\right) - \frac{k_T^\alpha}{M_h}\,\epsilon_{ij}k_T^i S_{hT}^j\,\overline{D}_{1T}^\perp - i\epsilon_T^{\alpha\beta}k_{T\beta}\left(\frac{1}{z}\,\overline{G}_s^\perp - \overline{G}_{1s} - \frac{m}{M_h}\,\overline{H}_{1s}^\perp + i\,\lambda_h\,\frac{\overline{D}_L^\perp}{z}\right) - i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,\overline{G}_T' - m\,\overline{H}_{1T} + i\,\frac{M_h}{z}\,\overline{D}_T\right). \tag{5.103}$$

The $\boldsymbol{k}_T$-integrated results are

$$g\overline{\Delta}_{A\,\alpha}^{\,[\sigma^{\alpha+}]}(z) = \epsilon_{\alpha\beta}\, g\overline{\Delta}_A^{\alpha[i\sigma^{\beta-}\gamma_5]}(z) = i\left(\frac{M_h}{z}\,\overline{E} - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - 2i\,M_h\,\overline{H}_1^{\perp\,(1)}\right) \tag{5.104}$$

or

$$g\overline{\Delta}_A^{\alpha[i\sigma^{\beta+}\gamma_5]}(z) - g\overline{\Delta}_A^{\beta[i\sigma^{\alpha+}\gamma_5]}(z) = i\epsilon_T^{\alpha\beta}\left(\frac{M_h}{z}\,E - m\,\overline{D}_1 + i\,\frac{M_h}{z}\,\overline{H} - 2i\,M_h\,\overline{H}_1^{\perp(1)}\right), \tag{5.105}$$

$$g\overline{\Delta}_{A\,\alpha}^{\,[i\sigma^{\alpha+}\gamma_5]}(z) = -\lambda_h\left(\frac{M_h}{z}\,\overline{H}_L - m\,\overline{G}_1 - i\,\frac{M_h}{z}\,\overline{E}_L - 2M_h\,\overline{H}_{1L}^{\perp\,(1)}\right), \tag{5.106}$$

$$g\overline{\Delta}_A^{\alpha[\gamma^+]}(z) - i\epsilon_T^{\alpha\beta}\,g\overline{\Delta}_{A\,\beta}^{\,[\gamma^+\gamma_5]}(z) = -i\epsilon_T^{\alpha\beta}S_{hT\beta}\left(\frac{M_h}{z}\,\overline{G}_T - m\,\overline{H}_1 - M_h\,\overline{G}_{1T}^{(1)} - i\,M_h\,\overline{D}_{1T}^{\perp(1)} + i\,\frac{M_h}{z}\,\overline{D}_T\right). \tag{5.107}$$

5.2 Gluon distribution functions (new)

The simplest gauge invariant correlator involving two gluon fields are the lightfront correlator

$$\Gamma^{\mu\nu}(x,p_T;n,C,C') = \int \frac{d(\xi\cdot P)\,d^2\xi_T}{(2\pi)^3}\;e^{i\,p\cdot\xi}\,\langle P|G^{n\nu}(0)\,U_{[0,\xi]}^{[n,C]}\,G^{n\mu}(\xi)\,U_{[\xi,0]}^{[n,C']}|P\rangle\Big|_{LF}, \tag{5.108}$$

where a color trace over the operator is understood, and the lightcone correlator

$$\begin{aligned}\Gamma^{\mu\nu}(x;n) &= \int d^2p_T\;\Gamma^{\mu\nu}(x,p_T;n,C,C')\\ &= \int \frac{d(\xi\cdot P)}{(2\pi)}\;e^{i\,xP\cdot\xi}\,\langle P|G^{n\nu}(0)\,U_{[0,\xi]}^{[n]}\,G^{n\mu}(\xi)\,U^{[n]}(\xi,0)|P\rangle\Big|_{LC}\end{aligned} \tag{5.109}$$

For a proper treatment of the transverse moments we need the weighted correlator

$$\Gamma_\partial^{\mu\nu;\alpha}(x;n,C,C') = \int d^2p_T\;p_T^\alpha\,\Gamma^{\mu\nu}(x,p_T;n,C,C') \tag{5.110}$$

and the three-gluon correlators

$$\Gamma_D^{\mu\nu;\alpha}(x;n) = \int \frac{d(\xi\cdot P)}{(2\pi)}\, e^{i\,xP\cdot\xi}\, \langle P|G^{n\nu}(0)\, U_{[0,\xi]}^{[n]}\, [D^\alpha(\xi), G^{n\mu}(\xi)]\, U_{[\xi,0]}^{[n]}|P\rangle\Big|_{LC}, \tag{5.111}$$

$$\Gamma_{[G]}^{\mu\nu;\alpha}(x, x-x';n) = \int \frac{d(\xi\cdot P)}{(2\pi)}\,\frac{d(\eta\cdot P)}{(2\pi)}\, e^{i\,(x-x')P\cdot\xi}\, e^{i\,x'P\cdot\eta}$$
$$\times\ \langle P|G^{n\nu}(0)\, U_{[0,\eta]}^{[n]}\, [G^{n\alpha}(\eta), U_{[\eta,\xi]}^{[n]} G^{n\mu}(\xi) U_{[\xi,\eta]}^{[n]}] U_{[\eta,0]}^{[n]}|P\rangle\Big|_{LC}, \tag{5.112}$$

$$\Gamma_{\{G\}}^{\mu\nu;\alpha}(x, x-x';n) = \int \frac{d(\xi\cdot P)}{(2\pi)}\,\frac{d(\eta\cdot P)}{(2\pi)}\, e^{i\,(x-x')P\cdot\xi}\, e^{i\,x'P\cdot\eta}$$
$$\times\ \langle P|G^{n\nu}(0)\, U_{[0,\eta]}^{[n]}\, \{G^{n\alpha}(\eta), U_{[\eta,\xi]}^{[n]} G^{n\mu}(\xi) U_{[\xi,\eta]}^{[n]}\} U_{[\eta,0]}^{[n]}|P\rangle\Big|_{LC}. \tag{5.113}$$

Of the latter two matrix elements given as multi-parton correlator, we actually need the case $x' = 0$, the gluonic pole matrix elements. Making the (suppressed) color trace in the latter two matrix elements explicit one sees that one deals with fully antisymmetric or fully symmetric matrix elements, in lightcone gauge

$$i\, f_{abc}\, \langle P|G_a^{n\nu}(0)\, G_b^{n\alpha}(\eta)\, G_c^{n\mu}(\xi)|P\rangle\Big|_{LC},$$

and

$$d_{abc}\, \langle P|G_a^{n\nu}(0)\, G_b^{n\alpha}(\eta)\, G_c^{n\mu}(\xi)|P\rangle\Big|_{LC}.$$

5.3 Gluon distribution functions

When one considers QCD corrections to the tree-level results one will also encounter gluon-gluon correlation functions, leading to gluon distributions. In a diagram one needs

$$\frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|A^\nu(0)A^\mu(\xi)|P,S\rangle, \tag{5.114}$$

but the gauge invariant object to consider is

$$\Gamma^{\mu\nu;\rho\sigma}(k;P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,k\cdot\xi}\langle P,S|F^{\mu\nu}(0)\,\mathcal{U}(0,\xi)\, F^{\rho\sigma}(\xi)|P,S\rangle. \tag{5.115}$$

The constraints following from hermiticity, parity and time reversal are

$$\Gamma^{\rho\sigma;\mu\nu\,*}(k,P,S) = \Gamma^{\mu\nu;\rho\sigma}(k,P,S) \qquad \text{[Hermiticity]} \tag{5.116}$$
$$\Gamma^{\mu\nu;\rho\sigma}(k,P,S) = \Gamma_{\mu\nu;\rho\sigma}(\bar{k},\bar{P},-\bar{S}) \qquad \text{[Parity]} \tag{5.117}$$
$$\Gamma^{\mu\nu;\rho\sigma\,*}(k,P,S) = \Gamma_{\mu\nu;\rho\sigma}(\bar{k},\bar{P},\bar{S}) \qquad \text{[Time reversal]} \tag{5.118}$$

where $\bar{k} = (k^0, -\boldsymbol{k})$.

In order to find antisymmetric structures we use the following tensor structures and relations between tensors,

$$
\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma},\\
g^{[\mu}{}_{\rho}g^{\nu]}{}_{\sigma} &= g^{\mu}{}_{[\rho}g^{\nu}{}_{\sigma]} = g^{\mu}{}_{\rho}g^{\nu}{}_{\sigma} - g^{\nu}{}_{\rho}g^{\mu}{}_{\sigma},\\
A^{[\mu}g^{\nu][\rho}B^{\sigma]} &= g^{\nu\rho}A^{\mu}B^{\sigma} + g^{\mu\sigma}A^{\nu}B^{\rho} - g^{\mu\rho}A^{\nu}B^{\sigma} - g^{\nu\sigma}A^{\mu}B^{\rho}.\\
\epsilon^{\mu\nu\alpha\beta}\,\epsilon_{\rho\sigma\alpha\beta} &= -2\,g^{[\mu}{}_{\rho}g^{\nu]}{}_{\sigma},\\
\epsilon^{\mu\nu\alpha A}\,\epsilon^{\rho\sigma}{}_{\alpha}{}^{B} &= \tfrac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,\epsilon^{\rho\sigma}{}_{\alpha\beta}\,g^{AB} - B^{[\mu}g^{\nu][\rho}A^{\sigma]}\\
\epsilon^{\mu\nu AB}\,\epsilon^{\rho\sigma CD} &= \tfrac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,\epsilon^{\rho\sigma}{}_{\alpha\beta}\,(g^{AC}g^{BD} - g^{AD}g^{BC})\\
&\quad -C^{[\mu}g^{\nu][\rho}A^{\sigma]}\,g^{BD} + C^{[\mu}g^{\nu][\rho}B^{\sigma]}\,g^{AD} + D^{[\mu}g^{\nu][\rho}A^{\sigma]}\,g^{BC} - D^{[\mu}g^{\nu][\rho}B^{\sigma]}\,g^{AC}\\
&\quad -C^{[\mu}D^{\nu]}\,A^{[\rho}B^{\sigma]},\\
\epsilon^{\mu\nu A[\sigma}B^{\rho]} + \epsilon^{\rho\sigma A[\nu}B^{\mu]} &= \epsilon^{\mu\nu\rho\sigma}\,g^{AB},\\
\epsilon^{\mu\nu A[\sigma}B^{\rho]} - \epsilon^{\mu\nu B[\sigma}A^{\rho]} &= \epsilon^{AB\nu\sigma}\,g^{\mu\rho} + \epsilon^{AB\mu\rho}\,g^{\nu\sigma} - \epsilon^{AB\mu\sigma}\,g^{\nu\rho} - \epsilon^{AB\nu\rho}\,g^{\mu\sigma}.\\
&= \epsilon^{AB\mu[\rho}g^{\sigma]\nu} - \epsilon^{AB\nu[\rho}g^{\sigma]\mu} = g^{\rho[\mu}\epsilon^{\nu]\sigma AB} - g^{\sigma[\mu}\epsilon^{\nu]\rho AB}.
\end{aligned}
$$

For changing from matrix elements with $F_{\mu\nu}$ to matrix elements containing $\tilde{F}_{\mu\nu} \equiv -(1/2)\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, we note that

$$
\begin{aligned}
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\,\epsilon^{\kappa\lambda\rho\sigma} &= (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}),\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\,\epsilon^{\kappa\lambda\alpha\beta}\,\epsilon^{\rho\sigma}{}_{\alpha\beta} &= 2\,\epsilon^{\mu\nu\rho\sigma}\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\,\epsilon^{\kappa\lambda\alpha A}\,\epsilon^{\rho\sigma}{}_{\alpha}{}^{B} &= \epsilon^{\rho\sigma B[\nu}A^{\mu]},\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\,\epsilon^{\kappa\lambda AB}\epsilon^{\rho\sigma CD} &= A^{[\mu}B^{\nu]}\,\epsilon^{\rho\sigma CD},\\
-\tfrac{1}{2}\,\epsilon^{\mu\nu}{}_{\kappa\lambda}\,\epsilon^{\kappa\lambda A[\sigma}B^{\rho]} &= -A^{[\mu}g^{\nu][\rho}B^{\sigma]}.
\end{aligned}
$$

A possible parametrization of $\Gamma^{\mu\nu;\rho\sigma}$ is

$$\begin{aligned}
\Gamma^{\mu\nu;\rho\sigma}(k;P,S) \;\; = \;\; & M^2\,X_1\,\epsilon^{\mu\nu\alpha\beta}\;\epsilon^{\rho\sigma}{}_{\alpha\beta} \\
& + X_2\,P^{[\mu}g^{\nu][\rho}P^{\sigma]} \\
& + X_3\,k^{[\mu}g^{\nu][\rho}k^{\sigma]} \\
& + (X_4 + i\,X_5)\,P^{[\mu}g^{\nu][\rho}k^{\sigma]} \\
& + (X_4 - i\,X_5)\,k^{[\mu}g^{\nu][\rho}P^{\sigma]} \\
& + (X_6/M^2)\,P^{[\mu}k^{\nu]}\,P^{[\rho}k^{\sigma]} \\
& - 2\,M\,X_7\;\epsilon^{\mu\nu\rho\sigma}\,(k\cdot S) \\
& + i\,M\,X_8\;\epsilon^{\mu\nu P[\sigma}S^{\rho]} \\
& + i\,M\,X_9\;\epsilon^{\mu\nu S[\sigma}P^{\rho]} \\
& + i\,M\,X_{10}\,\epsilon^{\mu\nu k[\sigma}S^{\rho]} \\
& + i\,M\,X_{11}\,\epsilon^{\mu\nu S[\sigma}k^{\rho]} \\
& + i\,(X_{12}/M)\,\epsilon^{\mu\nu P[\sigma}P^{\rho]}\,(k\cdot S) \\
& + i\,(X_{13}/M)\,\epsilon^{\mu\nu k[\sigma}k^{\rho]}\,(k\cdot S) \\
& + i\,(X_{14}/M)\,\epsilon^{\mu\nu P[\sigma}k^{\rho]}\,(k\cdot S) \\
& + i\,(X_{15}/M)\,\epsilon^{\mu\nu k[\sigma}P^{\rho]}\,(k\cdot S) \\
& + ((X_{16} + i\,X_{17})/M)\;\epsilon^{\mu\nu PS}k^{[\rho}P^{\sigma]} \\
& + ((X_{16} - i\,X_{17})/M)\;\epsilon^{\rho\sigma PS}k^{[\mu}P^{\nu]} \\
& + ((X_{18} + i\,X_{19})/M)\;\epsilon^{\mu\nu kS}k^{[\rho}P^{\sigma]} \\
& + ((X_{18} - i\,X_{19})/M)\;\epsilon^{\rho\sigma kS}k^{[\mu}P^{\nu]} \\
& + ((X_{20} + i\,X_{21})/M)\;\epsilon^{\mu\nu kP}P^{[\rho}S^{\sigma]} \\
& + ((X_{20} - i\,X_{21})/M)\;\epsilon^{\rho\sigma kP}P^{[\mu}S^{\nu]} \\
& + ((X_{22} + i\,X_{23})/M)\;\epsilon^{\mu\nu kP}k^{[\rho}S^{\sigma]} \\
& + ((X_{22} - i\,X_{23})/M)\;\epsilon^{\rho\sigma kP}k^{[\mu}S^{\nu]} \\
& + ((X_{24} + i\,X_{25})/M^3)\;\epsilon^{\mu\nu kP}k^{[\rho}P^{\sigma]}\,(k\cdot S) \\
& + ((X_{24} - i\,X_{25})/M^3)\;\epsilon^{\rho\sigma kP}k^{[\mu}P^{\nu]}\,(k\cdot S),
\end{aligned} \tag{5.119}$$

with X_5, X_7, X_{16}, X_{18}, X_{20}, X_{22} and X_{24} being T-odd. The constraints from hermiticity imply that one finds real amplitudes X_i if the tensors symmetric under $\mu\nu \leftrightarrow \rho\sigma$ are multiplied with 1, while the tensors antisymmetric under $\mu\nu \leftrightarrow \rho\sigma$ are multiplied with i; the constraint from parity requires that even numbers of ϵ-tensors are combined only with vectors k and P, while odd numbers of ϵ-tensors are combined with the axial vector S and furthermore vectors k, P; finally the constraint from time-reversal requires that any ϵ-tensor appears multiplied with i and for the rest real stuff.

For the amplitudes with an odd number of ϵ-tensors, it is useful to realize that the quantity

$$\tilde{\Gamma}^{\mu\nu;\rho\sigma}(k;P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{i\,k\cdot\xi}\langle P,S|\tilde{F}^{\mu\nu}(0)\,\mathcal{U}(0,\xi)\,F^{\rho\sigma}(\xi)|P,S\rangle. \tag{5.120}$$

has the structure

$$
\begin{aligned}
\tilde{\Gamma}^{\mu\nu;\rho\sigma}(k;P,S) \quad = \quad & \ldots + M\,X_7\,\epsilon^{\mu\nu\alpha\beta}\;\epsilon^{\rho\sigma}{}_{\alpha\beta}\,(k\cdot S) \\
& - i\,M\,X_8\,P^{[\mu}g^{\nu][\rho}S^{\sigma]} \\
& - i\,M\,X_9\,S^{[\mu}g^{\nu][\rho}P^{\sigma]} \\
& - i\,M\,X_{10}\,k^{[\mu}g^{\nu][\rho}S^{\sigma]} \\
& - i\,M\,X_{11}\,S^{[\mu}g^{\nu][\rho}k^{\sigma]} \\
& - i\,(X_{12}/M)\,P^{[\mu}g^{\nu][\rho}P^{\sigma]}\,(k\cdot S) \\
& - i\,(X_{13}/M)\,k^{[\mu}g^{\nu][\rho}k^{\sigma]}\,(k\cdot S) \\
& - i\,(X_{14}/M)\,P^{[\mu}g^{\nu][\rho}k^{\sigma]}\,(k\cdot S) \\
& - i\,(X_{15}/M)\,k^{[\mu}g^{\nu][\rho}P^{\sigma]}\,(k\cdot S) \\
& + ((X_{16}+i\,X_{17})/M)\,P^{[\mu}S^{\nu]}k^{[\rho}P^{\sigma]} \\
& - ((X_{16}-i\,X_{17})/M)\;\epsilon^{\mu\nu kP}\,\epsilon^{\rho\sigma PS} \\
& + ((X_{18}+i\,X_{19})/M)\,k^{[\mu}S^{\nu]}k^{[\rho}P^{\sigma]} \\
& - ((X_{18}-i\,X_{19})/M)\;\epsilon^{\mu\nu kP}\,\epsilon^{\rho\sigma kS} \\
& + ((X_{20}+i\,X_{21})/M)\,k^{[\mu}P^{\nu]}P^{[\rho}S^{\sigma]} \\
& - ((X_{20}-i\,X_{21})/M)\;\epsilon^{\mu\nu PS}\,\epsilon^{\rho\sigma kP} \\
& + ((X_{22}+i\,X_{23})/M)\,k^{[\mu}P^{\nu]}k^{[\rho}S^{\sigma]} \\
& - ((X_{22}-i\,X_{23})/M)\;\epsilon^{\mu\nu kS}\,\epsilon^{\rho\sigma kP} \\
& + ((X_{24}+i\,X_{25})/M^3)\,k^{[\mu}P^{\nu]}k^{[\rho}P^{\sigma]}\,(k\cdot S) \\
& - ((X_{24}-i\,X_{25})/M^3)\;\epsilon^{\mu\nu kP}\,\epsilon^{\rho\sigma kP}\,(k\cdot S).
\end{aligned} \tag{5.121}
$$

In the next step we try to isolate the dominant parts by expanding the vectors in lightlike and transverse vectors, and the invariants $\sigma \equiv 2k\cdot P$ and $\tau = k^2$,

$$P = A\,n_+^\mu + \frac{M^2}{2A}\,n_-^\mu, \tag{5.122}$$

$$k = xA\,n_+^\mu + k_T^\mu + \frac{\sigma - xM^2}{2A}\,n_-^\mu, \quad \text{or} \quad k - xP = k_T^\mu + \frac{\sigma - 2xM^2}{2\,P\cdot n_-}\,n_-^\mu, \tag{5.123}$$

$$S = \lambda\,\frac{A}{M}\,n_+^\mu + S_T - \lambda\,\frac{M}{2A}\,n_-^\mu, \qquad \text{or} \qquad S - \lambda\,\frac{P}{M} = S_T^\mu - \lambda\,\frac{M}{P\cdot n_-}\,n_-^\mu, \tag{5.124}$$

and we have

$$k\cdot S = \lambda\,\frac{\sigma - 2xM^2}{2M} + k_T\cdot S_T. \tag{5.125}$$

The tensors can be expressed as

$$g^{\mu\nu} \quad = \quad n_+^{\{\mu}n_-^{\nu\}} + g_T^{\mu\nu}, \tag{5.126}$$

$$
\begin{aligned}
\epsilon^{\mu\nu\rho\sigma} \quad = \quad & -n_+^{[\mu}n_-^{\nu]}\,\epsilon_T^{\rho\sigma} + n_+^{[\mu}n_-^{\rho]}\,\epsilon_T^{\nu\sigma} - n_+^{[\mu}n_-^{\sigma]}\,\epsilon_T^{\nu\rho} \\
& - n_+^{[\nu}n_-^{\rho]}\,\epsilon_T^{\mu\sigma} + n_+^{[\nu}n_-^{\sigma]}\,\epsilon_T^{\mu\rho} - n_+^{[\rho}n_-^{\sigma]}\,\epsilon_T^{\mu\nu}.
\end{aligned} \tag{5.127}
$$

The dominant parts are the ones containing the hightest powers of $A = P^+$, e.g. for the unpolarized part we have in order of importance,

$$\Gamma^{+\alpha;+\beta}(k,P,S) = A^2\Big\{-g_T^{\alpha\beta}\left[X_2 + 2x\,X_4 + x^2\,X_3\right] + \frac{k_T^\alpha k_T^\beta}{M^2}\,X_6\Big\}, \tag{5.128}$$

$$\Gamma^{+\alpha;+-}(k,P,S) = A\,k_T^\alpha\Big[(X_4 + x\,X_3) - i\,X_5 + \left(\frac{\sigma - 2xM^2}{2M^2}\right)X_6\Big] \tag{5.129}$$

$$\Gamma^{+\alpha;\beta\gamma}(k,P,S) = A\,g_T^{\alpha[\beta}k_T^{\gamma]}\Big[(X_4 + x\,X_3) + i\,X_5\Big], \tag{5.130}$$

etc. Integrating over k^- we then find that the leading functions (α and β being transverse) are

$$\begin{aligned}
\Gamma_2^{\alpha\beta}(x,\boldsymbol{k}_T) &= \int dk^-\ \Gamma^{+\alpha;+\beta}(k;P,S) \\
&= \frac{x\,P^+}{2}\Bigg(-g_T^{\alpha\beta}\,G(x,\boldsymbol{k}_T) - g_T^{\alpha\beta}\,\frac{\epsilon_T^{ij}k_{Ti}S_{Tj}}{M}\,G_T(x,\boldsymbol{k}_T) \\
&\quad + \left(k_T^\alpha k_T^\beta + \frac{1}{2}\,g_T^{\alpha\beta}\,\boldsymbol{k}_T^2\right)\,\frac{H^\perp(x,\boldsymbol{k}_T)}{M^2} \\
&\quad -i\,\epsilon_T^{\alpha\beta}\left[\lambda\,\Delta G_L(x,\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\Delta G_T(x,\boldsymbol{k}_T)\right] \\
&\quad -\frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{2M^2}\left[\lambda\,\Delta H_L^\perp(x,\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\Delta H_T^\perp(x,\boldsymbol{k}_T)\right] \\
&\quad -\frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}S_{Ti} + S_T^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{4M}\left[\Delta H_T(x,\boldsymbol{k}_T) - \Delta H_T^{\perp(1)}(x,\boldsymbol{k}_T)\right]\Bigg). \qquad (5.131)
\end{aligned}$$

These give the leading (twist two) distribution functions, G_T, $\Delta H_s^\perp$ and ΔH_T being T-odd. Note that the tensor multiplying G_T is actually the antisymmetric version of the tensor multiplying ΔH_T. The use of the combination $\Delta H_T' \equiv \Delta H_T + \Delta H_T^{\perp(1)}$ where $\Delta H_T^{\perp(1)} = (\boldsymbol{k}_T^2/2M^2)\Delta H_T^\perp$ will become clear when we consider explicit representations. At subleading order we have

$$\begin{aligned}
\Gamma_3^{\alpha}(x,\boldsymbol{k}_T) &= \int dk^-\ \Gamma^{+\alpha;+-}(k;P,S) \\
&= \frac{x\,M}{2}\left(i\,\epsilon_T^{\alpha\beta}\,S_{T\beta}\,\Delta G_{3T}'(x,\boldsymbol{k}_T) + \frac{i\,\epsilon_T^{\alpha\beta}\,k_{T\beta}}{M}\,\Delta G_{3s}^\perp(x,\boldsymbol{k}_T) + \frac{k_T^\alpha}{M}\,G_3^\perp(x,\boldsymbol{k}_T)\right), \qquad (5.132)
\end{aligned}$$

where the imaginary parts of these functions are T-odd. Furthermore

$$\begin{aligned}
\Gamma_3^{\alpha;\beta\gamma}(x,\boldsymbol{k}_T) &= \int dk^-\ \Gamma^{+\alpha;\beta\gamma}(k;P,S) \\
&= \frac{x\,M}{2}\left(\frac{g_T^{\alpha[\beta}k_T^{\gamma]}}{M}\,H_3^\perp(x,\boldsymbol{k}_T) + i\,\epsilon_T^{\beta\gamma}\,S_T^\alpha\,\Delta H_{3T}'(x,\boldsymbol{k}_T) + i\,\epsilon_T^{\beta\gamma}\,\frac{k_T^\alpha}{M}\,\Delta H_{3s}^\perp(x,\boldsymbol{k}_T)\right), \qquad (5.133)
\end{aligned}$$

where the imaginary parts of these functions are T-odd.

After integration over the transverse momenta, we obtain

$$\Gamma_2^{\alpha\beta}(x) = \frac{x\,P^+}{2}\left(-g_T^{\alpha\beta}\,G(x) + i\,\epsilon_T^{\alpha\beta}\,\lambda\,\Delta G(x)\right), \qquad (5.134)$$

$$\Gamma_3^{\alpha}(x) = \frac{x\,M}{2}\,i\,\epsilon_T^{\alpha\beta}\,S_{T\beta}\,\Delta G_{3T}(x), \qquad (5.135)$$

$$\Gamma_3^{\alpha;\beta\gamma}(x) = \frac{x\,M}{2}\,i\,\epsilon_T^{\beta\gamma}\,S_T^\alpha\,\Delta H_{3T}(x), \qquad (5.136)$$

where $\Delta G_{3T} = \Delta G_{3T}' + \Delta G_{3T}^{\perp(1)}$ and $\Delta H_{3T} = \Delta H_{3T}' + \Delta H_{3T}^{\perp(1)}$, of which the imaginary parts are T-odd.

Examples of the amplitude expansion for the various functions are

$$xG(x,\boldsymbol{k}_T) = \int \ldots \left[X_2 + 2x\,X_4 + x^2\,X_3 + \frac{\boldsymbol{k}_T^2}{2M^2}\,X_6\right], \qquad (5.137)$$

$$xH^\perp(x,\boldsymbol{k}_T) = \int \ldots [X_6], \qquad (5.138)$$

$$\mathcal{R}e\,xG_3^\perp(x,\boldsymbol{k}_T) = \int \ldots \left[(X_4 + x\,X_3) + \left(\frac{\sigma - 2xM^2}{2M^2}\right)X_6\right], \qquad (5.139)$$

$$\mathcal{I}m\,xG_3^\perp(x,\boldsymbol{k}_T) = \int \ldots [-X_5] \qquad (5.140)$$

where

$$\int \ldots = \int d\sigma d\tau\ \delta(\boldsymbol{k}_T^2 + x^2M^2 + \tau - x\sigma).$$

From the matrix elements and the commutation properties of the gluon fields one immediately gets symmetry relations like $G(-x) = -G(x)$, etc. This differs from the case of quarks, where similar relations connect quark and antiquark distributions at positive and negative x-values. About the normalization, we note that for an unpolarized target

$$\int dk^+\ \Gamma_2^{\alpha\beta}(x) = -g_T^{\alpha\beta}\,\frac{(P^+)^2}{2}\int_{-\infty}^{\infty} dx\ xG(x). \tag{5.141}$$

The left-handside is equal to

$$\int d^4k\ \Gamma^{+\alpha;+\beta}(k;P,S) = \langle P|F^{+\alpha}(0)F^{+\beta}(0)|P\rangle. \tag{5.142}$$

Contracting α and β, using the support restrictions $-1 \le x \le 1$ and the symmetry relation $G(-x) = -G(x)$, one finds that

$$\int d^4k\ \Gamma^{+\alpha;+}{}_{\alpha}(k;P,S) = 2(P^+)^2 \underbrace{\int_0^1 dx\ xG(x)}_{\epsilon_G} = \langle P|\underbrace{F^{+\alpha}(0)F^{+}{}_{\alpha}(0)}_{\theta_G^{++}(0)}|P\rangle \tag{5.143}$$

and realizing that the right-handside is only the gluonic part of the energy momentum tensor, $\int_0^1 dx\,xG(x) = \epsilon_G \le 1$, while the chosen normalization assures the complementarity with the quark part of the energy momentum tensor discussed earlier, $\epsilon_q + \epsilon_G = 1$.

5.4 Explicit spin representation

The correlator $\Gamma_2^{\alpha\beta}$ contains two transverse gluon fields and can be interpreted in terms of gluon distribution functions. To make the gluon spin explicit, it is useful to consider the explicit matrix $M^{\alpha\beta} = (2/xP^+)\Gamma_2^{\alpha\beta}$ with α and β being transverse indices. The result for M is

$$\begin{pmatrix} G + \frac{\boldsymbol{k}_T\wedge\boldsymbol{S}_T}{M}\,G_T + \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\ H^\perp & -i\,S_L\,\Delta G_L - i\,\frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\,\Delta G_T + \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\ H^\perp \\ \quad + \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\ \Delta H_s^\perp + \frac{(k_T^1S_T^2+k_T^2S_T^1)}{2M}\,\Delta H_T' & \quad - \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\ \Delta H_s^\perp - \frac{(k_T^1S_T^1-k_T^2S_T^2)}{2M}\,\Delta H_T' \\ & \\ i\,S_L\,\Delta G_L + i\,\frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\,\Delta G_T + \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\ H^\perp & G + \frac{\boldsymbol{k}_T\wedge\boldsymbol{S}_T}{M}\,G_T - \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\ H^\perp \\ \quad - \frac{\boldsymbol{k}_T^2}{2M^2}\,\cos 2\phi\,\Delta H_s^\perp - \frac{(k_T^1S_T^1-k_T^2S_T^2)}{2M}\,\Delta H_T' & \quad - \frac{\boldsymbol{k}_T^2}{2M^2}\,\sin 2\phi\ \Delta H_s^\perp - \frac{(k_T^1S_T^2+k_T^2S_T^1)}{2M}\,\Delta H_T' \end{pmatrix} \tag{5.144}$$

where (for later convenience given also in spherical vector components and in matrix form in the nucleon spin-space)

$$\boldsymbol{k}_T\cdot\boldsymbol{S}_T = k_T^1S_T^1 + k_T^2S_T^2 = -g_T^{kS} = -(k_T^+S_T^- + k_T^-S_T^+) = \begin{pmatrix} 0 & |k_T|\,e^{-i\phi} \\ |k_T|\,e^{+i\phi} & 0 \end{pmatrix}, \tag{5.145}$$

$$\boldsymbol{k}_T\wedge\boldsymbol{S}_T = k_T^1S_T^2 - k_T^2S_T^1 = \epsilon_T^{kS} = -i(k_T^+S_T^- - k_T^-S_T^+) = \begin{pmatrix} 0 & -i\,|k_T|\,e^{-i\phi} \\ i\,|k_T|\,e^{+i\phi} & 0 \end{pmatrix}, \tag{5.146}$$

$$k_T^1S_T^1 - k_T^2S_T^2 = k_T^+S_T^+ + k_T^-S_T^- = \begin{pmatrix} 0 & |k_T|\,e^{+i\phi} \\ |k_T|e^{-i\phi} & 0 \end{pmatrix}, \tag{5.147}$$

$$k_T^1S_T^2 + k_T^2S_T^1 = -i(k_T^+S_T^+ - k_T^-S_T^-) = \begin{pmatrix} 0 & -i\,|k_T|\,e^{+i\phi} \\ i\,|k_T|\,e^{+i\phi} & 0 \end{pmatrix}. \tag{5.148}$$

We now have used transverse gluon polarizations. Instead we can use circular polarizations,

$$|+\rangle = -\frac{1}{\sqrt{2}}\left(|x\rangle + i\,|y\rangle\right),$$
$$|-\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - i\,|y\rangle\right), \tag{5.149}$$

and we obtain

$$\left(\begin{array}{cc} M^{++} & M^{+-} \\ M^{-+} & M^{--} \end{array}\right) = \left(\begin{array}{cc} \frac{1}{2}(M^{11}+M^{22}) - \mathrm{Im}\,M^{12} & -\frac{1}{2}(M^{11}-M^{22}) + i\,\mathrm{Re}\,M^{12} \\ -\frac{1}{2}(M^{11}-M^{22}) - i\,\mathrm{Re}\,M^{12} & \frac{1}{2}(M^{11}+M^{22}) + \mathrm{Im}\,M^{12} \end{array}\right). \tag{5.150}$$

For the above matrix we find for circularly polarized gluons

$$\left(\begin{array}{cc} G + \frac{\boldsymbol{k}_T\wedge\boldsymbol{S}_T}{M}\,G_T + S_L\,\Delta G_L + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\,\Delta G_T & -\frac{\boldsymbol{k}_T^2}{2M^2}\,e^{-2i\phi}\left[H^\perp + i\,\Delta H_s^\perp\right] - i\,\frac{k_T^- S_T^-}{M}\,\Delta H_T' \\ -\frac{\boldsymbol{k}_T^2}{2M^2}\,e^{+2i\phi}\left[H^\perp - i\,\Delta H_s^\perp\right] + i\,\frac{k_T^+ S_T^+}{M}\,\Delta H_T' & G + \frac{\boldsymbol{k}_T\wedge\boldsymbol{S}_T}{M}\,G_T - S_L\,\Delta G_L - \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\,\Delta G_T \end{array}\right) \tag{5.151}$$

Extended into a 4×4 matrix in gluon $\otimes$ nucleon spin space we obtain

$$\left(\begin{array}{cccc} G+\Delta G & \frac{|k_T|e^{-i\phi}}{M}\left[\Delta G_T - i\,G_T\right] & -e^{-2i\phi}\left[H^{\perp(1)} + i\,\Delta H_L^{\perp(1)}\right] & -i\frac{|k_T|e^{-3i\phi}}{M}\,\Delta H_T^{\perp(1)} \\ \frac{|k_T|e^{+i\phi}}{M}\left[\Delta G_T + i\,G_T\right] & G-\Delta G & -i\frac{|k_T|e^{-i\phi}}{M}\,\Delta H_T & -e^{-2i\phi}\left[H^{\perp(1)} - i\,\Delta H_L^{\perp(1)}\right] \\ -e^{+2i\phi}\left[H^{\perp(1)} - i\,\Delta H_L^{\perp(1)}\right] & i\frac{|k_T|e^{+i\phi}}{M}\,\Delta H_T & G-\Delta G & -\frac{|k_T|e^{-i\phi}}{M}\left[\Delta G_T + i\,G_T\right] \\ i\frac{|k_T|e^{+3i\phi}}{M}\,\Delta H_T^{\perp(1)} & -e^{+2i\phi}\left[H^{\perp(1)} + i\,\Delta H_L^{\perp(1)}\right] & -\frac{|k_T|e^{+i\phi}}{M}\left[\Delta G_T - i\,G_T\right] & G+\Delta G \end{array}\right) \tag{5.152}$$

5.5 Gluon fragmentation functions

For the fragmentation functions one needs the matrix element

$$\hat\Gamma^{\mu\nu;\rho\sigma}(k;P_h,S_h) = \frac{1}{(2\pi)^4}\sum_X\int d^4\xi\; e^{i\,k\cdot\xi}\langle 0|F^{\mu\nu}(\xi)|P_h,X\rangle\,\langle P_h,X|F^{\rho\sigma}(0)|0\rangle. \tag{5.153}$$

In this case no constraints arise from time reversal invariance and the most general expansion of the matrix element becomes,

$$\begin{aligned} \hat\Gamma_2^{\alpha\beta}(z,\boldsymbol{k}_T) &= \int dk^+\;\hat\Gamma^{-\alpha;-\beta}(k;P_h,S_h) \\ &= P_h^-\Bigg(-g_T^{\alpha\beta}\,\hat G(z,-z\boldsymbol{k}_T) - g_T^{\alpha\beta}\,\frac{\epsilon_T^{ij}\,k_{Ti}S_{hT\,j}}{M_h}\,\hat G_T(z,-z\boldsymbol{k}_T) \\ &\quad + \left(k_T^\alpha k_T^\beta + \frac{1}{2}\,g_T^{\alpha\beta}\,\boldsymbol{k}_T^2\right)\,\frac{\hat G^\perp(z,-z\boldsymbol{k}_T)}{M_h^2} \\ &\quad + i\,\epsilon_T^{\alpha\beta}\left[\lambda_h\,\Delta\hat G_L(z,-z\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_{hT}}{M_h}\Delta\hat G_T(z,-z\boldsymbol{k}_T)\right] \\ &\quad + \frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{M_h^2}\left[\lambda_h\,\Delta\hat G_L^\perp(z,-z\boldsymbol{k}_T) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_{hT}}{M_h}\Delta\hat G_T^\perp(z,-z\boldsymbol{k}_T)\right] \\ &\quad + \frac{k_T^{\{\alpha}\epsilon_T^{\beta\}i}S_{hT\,i} + S_{hT}^{\{\alpha}\epsilon_T^{\beta\}i}k_{Ti}}{2M_h}\,\Delta\hat G_T^{\perp\prime}(z,-z\boldsymbol{k}_T)\Bigg). \end{aligned} \tag{5.154}$$

These give the leading (twist two) fragmentation functions. At subleading order we have

$$\begin{aligned} \hat\Gamma_3^{\alpha}(z,-z\boldsymbol{k}_T) &= \int dk^+\;\hat\Gamma_3^{-\alpha;-+}(k;P,S) \\ &= M_h\Bigg(i\,\epsilon_T^{\alpha\beta}\,S_{hT\,\beta}\,\Delta\hat G_{3T}'(z,-z\boldsymbol{k}_T) + \frac{i\,\epsilon_T^{\alpha\beta}\,k_{T\,\beta}}{M}\,\Delta\hat G_{3s}^\perp(z,-z\boldsymbol{k}_T) \\ &\quad + \frac{k_T^\alpha}{M_h}\,\hat G_3^\perp(z,-z\boldsymbol{k}_T)\Bigg). \end{aligned} \tag{5.155}$$

Furthermore

$$
\begin{aligned}
\hat{\Gamma}_3^{\alpha;\beta\gamma}(z,-z\boldsymbol{k}_T) &= \int dk^+\ \hat{\Gamma}^{-\alpha;\beta\gamma}(k;P,S) \\
&= M_h\Bigg(\frac{g_T^{\alpha[\beta}k_T^{\gamma]}}{M_h}\,\hat{H}_3^{\perp}(z,-z\boldsymbol{k}_T) + \frac{k_T^{[\beta}S_{hT}^{\gamma]}}{M_h}\,\frac{i\epsilon_T^{\alpha\delta}k_{T\delta}}{M_h}\,\hat{H}_{3T}^{\perp}(z,-z\boldsymbol{k}_T) \\
&\qquad + i\,\epsilon_T^{\beta\gamma}\,S_{hT}^{\alpha}\,\Delta\hat{H}_{3T}'(z,-z\boldsymbol{k}_T) + i\,\epsilon_T^{\beta\gamma}\,\frac{k_T^{\alpha}}{M_h}\,\Delta\hat{H}_{3s}^{\perp}(z,-z\boldsymbol{k}_T)\Bigg). \qquad (5.156)
\end{aligned}
$$

After integration over the transverse momenta, we obtain

$$
\hat{\Gamma}_2^{\alpha\beta}(z) = \frac{P_h^-}{z^2}\left(-g_T^{\alpha\beta}\,\hat{G}(z) + i\,\epsilon_T^{\alpha\beta}\,\lambda_h\,\Delta\hat{G}(z)\right). \qquad (5.157)
$$

$$
\hat{\Gamma}_3^{\alpha}(z) = \frac{M_h}{z^2}\ \epsilon_T^{\alpha\beta}S_{hT\beta}\,\Delta\hat{G}_{3T}(z), \qquad (5.158)
$$

$$
\hat{\Gamma}_3^{\alpha;\beta\gamma}(x) = \frac{M_h}{z^2}\,i\,\epsilon_T^{\beta\gamma}\,S_{hT}^{\alpha}\,\Delta\hat{H}_{3T}(z), \qquad (5.159)
$$

with $\Delta\hat{G}_{3T} = \Delta\hat{G}_{3T}' + \Delta\hat{G}_{3T}^{\perp(1)}$, and $\Delta\hat{H}_{3T} = \Delta\hat{H}_{3T}' + \Delta\hat{H}_{3T}^{\perp(1)}$.

Chapter 6

Drell-Yan up to $\mathscr{O}(1/Q)$

6.1 The hadron tensor

Up to $\mathscr{O}(1/Q)$ one needs to include the contributions of the handbag diagram, now calculated up to this order with in addition irreducible diagrams with one gluon coupling either to the soft part involving hadron A or the soft part involving hadron B. The expressions thus involve the quark-gluon correlation functions. The full result is neglecting $1/Q^2$ contributions given by

$$
\begin{aligned}
2M\,\mathscr{W}_{\mu\nu} \quad = \quad & \int dk_a^-\,dk_b^+\,d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\,\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T) \\
& \Bigg\{ \text{Tr}\left(\Phi(k_a)\gamma_\mu\overline{\Phi}(k_b)\gamma_\nu\right) \\
& \quad -\text{Tr}\left(\gamma_\alpha\frac{\not{n}_+}{-2q^-}\gamma_\nu\Phi_A^\alpha(k_a)\gamma_\mu\overline{\Phi}(k_b)\right) - \text{Tr}\left(\gamma_\mu\frac{\not{n}_+}{-2q^-}\gamma_\alpha\overline{\Phi}(k_b)\gamma_\nu\Phi_A^{\alpha\dagger}(k_a)\right) \\
& \quad -\text{Tr}\left(\gamma_\nu\frac{\not{n}_-}{(2q^+)}\gamma_\alpha\Phi(k_a)\gamma_\mu\overline{\Phi}_A^{\alpha\dagger}(k_b)\right) - \text{Tr}\left(\gamma_\alpha\frac{\not{n}_-}{(2q^+)}\gamma_\mu\overline{\Phi}_A^{\alpha}(k_b)\gamma_\nu\Phi(k_a)\right)\Bigg\}. \qquad (6.1)
\end{aligned}
$$

Here the terms with $\not{n}_\pm$ arise from fermion propagators in the hard part neglecting contributions that will appear suppressed by powers of Q^2, i.e.

$$
\frac{\not{p}_1-\not{q}+m}{(p_1-q)^2} = \frac{(p_1^+-q^+)\gamma^-}{-2(p_1^+-q^+)q^-} = \frac{\gamma^-}{-2q^-} = \frac{\not{n}_+}{-2q^-} = \frac{x_A\not{P}_A}{Q^2}, \qquad (6.2)
$$

$$
\frac{\not{q}-\not{k}_1+m}{(q-k_1)^2} = \frac{(q^--k_1^-)\gamma^+}{2(q-k_1^-)q^+} = \frac{\gamma^+}{(2q^+)} = \frac{\not{n}_-}{(2q^+)} = -\frac{x_B\not{P}_B}{Q^2}. \qquad (6.3)
$$

The full result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is [Check sign of $g_1\,\bar{g}_1$ and $1/Q$ terms]

$$
\begin{aligned}
\mathcal{W}^{\mu\nu} = \frac{1}{3}\int & d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\;\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\;\times\Bigg\{-g_\perp^{\mu\nu}\Big[f_1\bar{f}_1+g_{1s}\bar{g}_{1s}\Big] \\
& -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1s}^\perp-\frac{k_{b\perp}^{\{\mu}S_{A\perp}^{\nu\}}+(\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp})\,g_\perp^{\mu\nu}}{M_B}\,h_{1T}\bar{h}_{1s}^\perp \\
& -\frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,h_{1s}^\perp\bar{h}_{1T}-\Big(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}\Big)\,h_{1T}\bar{h}_{1T} \\
& +\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\Bigg[-f_1\bar{f}_1+2x_{\scriptscriptstyle A}\,f^\perp\bar{f}_1+g_{1s}\bar{g}_{1s}-2x_{\scriptscriptstyle A}\,g_s^\perp\bar{g}_{1s} \\
& \qquad -\frac{M_B}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1s}^\perp\bar{h}_s+\frac{m}{M_A}\,2\,h_{1s}^\perp\bar{g}_{1s}-\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1s}^\perp \\
& \qquad -\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1s}^\perp-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1s}^\perp\bar{h}_{1T}+\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1s}^\perp \\
& \qquad -\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar{h}_{1T}+\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1T}\Bigg] \\
& +\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\Bigg[f_1\bar{f}_1-2x_{\scriptscriptstyle B}\,f_1\bar{f}^\perp-g_{1s}\bar{g}_{1s}+2x_{\scriptscriptstyle B}\,g_{1s}\bar{g}_s^\perp \\
& \qquad +\frac{M_A}{M_B}\,2x_{\scriptscriptstyle A}\,h_s\bar{h}_{1s}^\perp-\frac{m}{M_B}\,2\,g_{1s}\bar{h}_{1s}^\perp+\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1s}^\perp \\
& \qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1s}^\perp+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1s}^\perp\bar{h}_{1T}-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1s}^\perp\bar{h}_T^\perp \\
& \qquad +\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar{h}_{1T}-\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_T^\perp\Bigg] \\
& +\frac{M_A\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\Bigg[-2x_{\scriptscriptstyle A}\,g_T'\bar{g}_{1s}-\frac{M_B}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_s+\frac{m}{M_A}\,2\,h_{1T}\bar{g}_{1s} \\
& \qquad -\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,h_{1T}\bar{h}_{1s}^\perp+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,h_{1T}\bar{h}_{1s}^\perp-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1s}^\perp \\
& \qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar{h}_{1T}-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar{h}_{1T}-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar{h}_{1T}\Bigg] \\
& +\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\Bigg[\frac{M_A}{M_B}\,2x_{\scriptscriptstyle A}\,h_s\bar{h}_{1T}+2x_{\scriptscriptstyle B}\,g_{1s}\bar{g}_T'-\frac{m}{M_B}\,2\,g_{1s}\bar{h}_{1T} \\
& \qquad +\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1T}-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,h_{1s}^\perp\bar{h}_{1T}+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,2x_{\scriptscriptstyle B}\,h_{1s}^\perp\bar{h}_T^\perp \\
& \qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1T}-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar{h}_{1T}+\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar{h}_T^\perp\Bigg]\Bigg\}, \qquad (6.4)
\end{aligned}
$$

where the quark distribution functions in hadron A depend on $x_{\scriptscriptstyle A}$ and $\boldsymbol{k}_{aT}^2$, $f_1(x_{\scriptscriptstyle A},\boldsymbol{k}_{aT}^2)$ etc., while the antiquark distribution functions in hadron B depend on $x_{\scriptscriptstyle B}$ and $\boldsymbol{k}_{bT}^2$, $\bar{f}_1(x_{\scriptscriptstyle B},\boldsymbol{k}_{bT}^2)$ etc., and we use the shorthand notations

$$
\begin{aligned}
g_{1s} &= \lambda_A\,g_{1L}+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,g_{1T}, \\
\bar{g}_{1s} &= \lambda_B\,\bar{g}_{1L}+\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,\bar{g}_{1T}.
\end{aligned}
$$

Table 6.1: Contractions of the lepton tensor $L_{\mu\nu}^{(DY)}$ with tensor structures appearing in the hadron tensor.

$w^{\mu\nu}$	$L_{\mu\nu}w^{\mu\nu}$
$-g_\perp^{\mu\nu}$	$4Q^2\ \left(\frac{1}{2}-y+y^2\right)$
$a_\perp^{\{\mu}b_\perp^{\nu\}}-(a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu}$	$-4Q^2\,y\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\cos(\phi_a+\phi_b)$
$\frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho}+b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right)$	$4Q^2\,y\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\sin(\phi_a+\phi_b)$
$\hat{z}^{\{\mu}a_\perp^{\nu\}}$	$-4Q^2\,(1-2y)\sqrt{y(1-y)}\,\vert\boldsymbol{a}_\perp\vert\,\cos\phi_a$
$\hat{z}^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}$	$4Q^2\,(1-2y)\sqrt{y(1-y)}\,\vert\boldsymbol{a}_\perp\vert\,\sin\phi_a$

In many cases it is convenient to express the tensors with respect to measured directions, i.e. $\hat{x}$ and $\hat{y}$, $\hat{y}^\mu = \epsilon_\perp^{\mu\rho}\hat{x}_\rho$. One can use

$$\begin{aligned}
a_\perp^\mu &= (\hat{\boldsymbol{x}}\cdot\boldsymbol{a}_\perp)\,\hat{x}^\mu + (\epsilon_\perp^{\rho\sigma}x_\rho a_{\perp\sigma})\,\hat{y}^\mu \\
&= (\hat{\boldsymbol{x}}\cdot\boldsymbol{a}_\perp)\,\hat{x}^\mu + (\hat{\boldsymbol{x}}\wedge\boldsymbol{a}_\perp)\,\hat{y}^\mu \\
&= (\hat{\boldsymbol{x}}\cdot\boldsymbol{a}_\perp)\,\hat{x}^\mu + (\hat{\boldsymbol{y}}\cdot\boldsymbol{a}_\perp)\,\hat{y}^\mu \\
&= a^x\,\hat{x}^\mu + a^y\,\hat{y}^\mu, \qquad (6.5)
\end{aligned}$$

where $\boldsymbol{a}_\perp\wedge\boldsymbol{b}_\perp \equiv \epsilon_\perp^{\rho\sigma}a_{\perp\rho}b_{\perp\sigma}$. Other combinations give

$$\epsilon_\perp^{\mu\rho}a_{\perp\rho} = -a^y\,\hat{x}^\mu + a^x\,\hat{y}^\mu, \qquad (6.6)$$

$$\begin{aligned}
a_\perp^{\{\mu}b_\perp^{\nu\}} - (a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu} &= (a^xb^x - a^yb^y)\,\left(2\,\hat{x}^\mu\hat{x}^\nu + g_\perp^{\mu\nu}\right) \\
&\quad + (a^xb^y + a^yb^x)\,\hat{x}^{\{\mu}\hat{y}^{\nu\}}, \qquad (6.7)
\end{aligned}$$

$$a_\perp^{[\mu}b_\perp^{\nu]} = (a^xb^y - a^yb^x)\,\hat{x}^{[\mu}\hat{y}^{\nu]}, \qquad (6.8)$$

$$\begin{aligned}
\frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho}+b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right) &= -\,(a^xb^y + a^yb^x)\,\left(2\,\hat{x}^\mu\hat{x}^\nu + g_\perp^{\mu\nu}\right) \\
&\quad + (a^xb^x - a^yb^y)\,\hat{x}^{\{\mu}\hat{y}^{\nu\}}. \qquad (6.9)
\end{aligned}$$

Because the transverse direction is fixed by $\boldsymbol{q}_T = Q_T\,\hat{\boldsymbol{x}}$, one has specifically

$$k_{a\perp}^\mu + k_{b\perp}^\mu = Q_T\,\hat{x}^\mu, \qquad (6.10)$$

$$k_{a\perp}^\mu - k_{b\perp}^\mu = \frac{\boldsymbol{k}_{a\perp}^2 - \boldsymbol{k}_{b\perp}^2}{Q_T}\,\hat{x}^\mu + 2\,\frac{\epsilon_\perp^{\rho\sigma}k_{a\perp}^\rho k_{b\perp}^\sigma}{Q_T}\,\hat{y}^\mu. \qquad (6.11)$$

This allows one to pull the tensor structure outside the integration over transverse momenta.

The contractions with the lepton tensor, given in Table 6.1 use azimuthal angles defined with respect to the orthogonal momentum $\hat{\ell}$,

$$\hat{\ell}\cdot a_\perp = -|\boldsymbol{a}_\perp|\,\cos(\phi_a), \qquad (6.12)$$

$$\hat{\ell}_\mu\epsilon_\perp^{\mu\nu}a_{\perp\nu} = |\boldsymbol{a}_\perp|\,\sin(\phi_a). \qquad (6.13)$$

6.2 Drell-Yan after integration over transverse momenta

The hadronic tensor simplifies to

$$\begin{aligned}
\mathcal{W}^{\mu\nu} \;=\; & \frac{1}{3}\Bigg\{-g_\perp^{\mu\nu}\Big[f_1(x_A)\bar{f}_1(x_B) + \lambda_A\,\lambda_B\,g_1(x_A)\bar{g}_1(x_B)\Big] \\
& - \Big(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}} - S_{A\perp}\cdot S_{B\perp}\,g_\perp^{\mu\nu}\Big)\;h_1(x_A)\bar{h}_1(x_B) \\
& -\lambda_B\,\frac{M_A\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\,x_A\;(g_T(x_A) + \tilde{g}_T(x_A))\;\bar{g}_1(x_B) \\
& -\lambda_B\,\frac{M_B\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\,x_B\,h_1(x_A)\;\Big(\bar{h}_L(x_B) + \bar{\tilde{h}}_L(x_B)\Big) \\
& +\lambda_A\,\frac{M_A\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_A\;\Big(h_L(x_A) + \tilde{h}_L(x_A)\Big)\;\bar{h}_1(x_B) \\
& +\lambda_A\,\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_B\,g_1(x_A)\;\big(\bar{g}_T(x_B) + \bar{\tilde{g}}_T(x_B)\big)\Bigg\},
\end{aligned} \tag{6.14}$$

where the twist three functions are the ones given in chapter 2,

$$\tilde{g}_T(x) \;=\; g_T(x) - \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M^2}\,\frac{g_{1T}(x,\boldsymbol{k}_T^2)}{x} - \frac{m}{M}\,\frac{h_1(x)}{x}, \tag{6.15}$$

$$\tilde{h}_L(x) \;=\; h_L(x) + \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{M^2}\,\frac{h_{1L}^\perp(x,\boldsymbol{k}_T^2)}{x} - \frac{m}{M}\,\frac{g_1(x)}{x}. \tag{6.16}$$

We will consider the various possibilities for unpolarized (O) and longitudinally (L) or transversely polarized (T) target hadrons.

6.2.1 Drell-Yan cross sections for unpolarized hadrons

Integrated over the transverse momenta of the produced mu pair one has for unpolarized hadrons,

$$\mathcal{W}^{\mu\nu} = \frac{1}{3}\;(-g_\perp^{\mu\nu})\;f_1(x_A)\bar{f}_1(x_B), \tag{6.17}$$

and the cross section up to $\mathcal{O}(1/Q)$ is given by

$$\frac{d\sigma_{OO}(AB\to\mu^+\mu^-X)}{dx_A\,dx_B\,dy} = \frac{4\pi\,\alpha^2}{3\,Q^2}\;\left(\frac{1}{2} - y + y^2\right)\;f_1(x_A)\,\overline{f}_1(x_B), \tag{6.18}$$

and integrated over the muon angular distribution,

$$\frac{d\sigma_{OO}(AB\to\mu^+\mu^-X)}{dx_A\,dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\,f_1(x_A)\,\overline{f}_1(x_B). \tag{6.19}$$

6.2.2 Longitudinal double spin asymmetry in Drell-Yan scattering

Integrated over transverse momenta, there are no single spin asymmetries. For the case that both hadrons are longitudinally polarized, the hadronic tensor is

$$\mathcal{W}^{\mu\nu} = \frac{1}{3}\,\lambda_A\lambda_B\;(-g_\perp^{\mu\nu})\;g_1(x_A)\bar{g}_1(x_B), \tag{6.20}$$

and the cross section up to $\mathcal{O}(1/Q)$ is given by

$$\frac{d\sigma_{LL}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_A\,dx_B\,dy} = \frac{4\pi\,\alpha^2}{3\,Q^2}\;\left(\frac{1}{2} - y + y^2\right)\;\lambda_A\,\lambda_B\,g_1(x_A)\,\overline{g}_1(x_B), \tag{6.21}$$

and integrated over the muon angular distribution,

$$\frac{d\sigma_{LL}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_A\,dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\,\lambda_A\lambda_B\,g_1(x_A)\,\overline{g}_1(x_B). \tag{6.22}$$

6.2.3 Transverse double spin asymmetry in Drell-Yan scattering

For the case that both hadrons are transversely polarized, the hadronic tensor is

$$\mathscr{W}^{\mu\nu} = -\frac{1}{3}\left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}} - S_{A\perp}\cdot S_{B\perp}\,g_\perp^{\mu\nu}\right)\,h_1(x_A)\bar{h}_1(x_B) \tag{6.23}$$

and the cross section up to $\mathscr{O}(1/Q)$ is given by

$$\frac{d\sigma_{TT}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_A\,dx_B\,dy} = \frac{4\pi\,\alpha^2}{3\,Q^2}\,|\boldsymbol{S}_{A\perp}|\,|\boldsymbol{S}_{B\perp}|\,y(1-y)\,\cos(\phi_s^A+\phi_s^B)\,h_1(x_A)\,\bar{h}_1(x_B), \tag{6.24}$$

and integrated over the muon angular distribution,

$$\frac{d\sigma_{TT}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_A\,dx_B} = \frac{2\pi\,\alpha^2}{9\,Q^2}\,|\boldsymbol{S}_{A\perp}|\,|\boldsymbol{S}_{B\perp}|\,\cos(\phi_s^A+\phi_s^B)\,h_1(x_A)\,\bar{h}_1(x_B). \tag{6.25}$$

6.2.4 Longitudinal-transverse double spin asymmetry in Drell-Yan scattering

For the case that one hadron is longitudinally polarized and the second transverse, the hadronic tensor is

$$\begin{aligned}\mathscr{W}^{\mu\nu} &= \frac{1}{3}\,\lambda_A\left\{\frac{M_A\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_A\left(h_L(x_A)+\tilde{h}_L(x_A)\right)\bar{h}_1(x_B)\right.\\ &\qquad\left.+\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\,x_B\,g_1(x_A)\left(\bar{g}_T(x_B)+\bar{\tilde{g}}_T(x_B)\right)\right\},\end{aligned} \tag{6.26}$$

and the cross section up to $\mathscr{O}(1/Q)$ is given by

$$\begin{aligned}\frac{d\sigma_{LT}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_A\,dx_B\,dy} &= -\frac{4\pi\,\alpha^2}{3\,Q^2}\,\lambda_A\,|\boldsymbol{S}_{B\perp}|\,(1-2y)\sqrt{y(1-y)}\,\cos(\phi_s^B)\\ &\times\left[\frac{M_Ax_A}{Q}\left(h_L(x_A)+\tilde{h}_L(x_A)\right)\bar{h}_1(x_B)+\frac{M_Bx_B}{Q}\,g_1(x_A)\left(\bar{g}_T(x_B)+\bar{\tilde{g}}_T(x_B)\right)\right].\end{aligned} \tag{6.27}$$

which vanishes upon integration over the muon angle.

6.3 Azimuthal asymmetries in Drell-Yan scattering

We will consider separately the various possibilities involving unpolarized (O), longitudinally polarized (L) and transversely polarized (T) hadrons.

6.3.1 Azimuthal asymmetries in unpolarized Drell-Yan scattering

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$\begin{aligned}\mathscr{W}^{\mu\nu} &= \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\,\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\,\times\left\{-g_\perp^{\mu\nu}\left[f_1\bar{f}_1\right]\right.\\ &\qquad\left.+\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[-f_1\bar{f}_1+2x_A\,f^\perp\bar{f}_1\right]+\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[f_1\bar{f}_1-2x_B\,f_1\bar{f}^\perp\right]\right\},\\ &= -g_\perp^{\mu\nu}\,I[f_1\bar{f}_1]+z^{\{\mu}\hat{x}^{\nu\}}\left(\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_A(f^\perp+\tilde{f}^\perp)\bar{f}_1\right]\right.\\ &\qquad\left.-\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_B\,f_1\,(\bar{f}^\perp+\bar{\tilde{f}}^\perp)\right]\right),\end{aligned} \tag{6.28}$$

where the last expression involves integrals of the type

$$I\left[k_a^x\, f^\perp \bar f_1\right](x_A, x_B, \boldsymbol{q}_T) = \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\; \delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\, k_{aT}^x\, f^\perp(x_A,\boldsymbol{k}_{aT})\bar f_1(x_B,\boldsymbol{k}_{bT}). \qquad (6.29)$$

Note that a contribution proportional to $\hat z^{\{\mu}\epsilon_\perp^{\nu\}\rho}\hat x_\rho$ appears, but it is multiplied with integrals of the type $I[k_a^y\, f^\perp(x_A,|\boldsymbol{k}_{a\perp}|)\bar f_1(x_B,|Q_T\,\hat{\boldsymbol{x}}-\boldsymbol{k}_{a\perp}|)]$, which vanish.

The cross section is given by

$$\frac{d\sigma_{OO}(AB\to\mu^+\mu^- X)}{dx_A\, dx_B\, dy\, d^2\boldsymbol{q}_T} = \frac{4\pi\,\alpha^2}{Q^2}\Bigg\{\left(\frac{1}{2}-y+y^2\right) I\left[f_1\bar f_1\right] -(1-2y)\sqrt{y(1-y)}\,\cos\phi\Bigg(\frac{M_A}{Q}\, I\left[\frac{k_a^x}{M_A}\,x_A(f^\perp+\tilde f^\perp)\,\bar f_1\right] -\frac{M_B}{Q}\, I\left[\frac{k_b^x}{M_B}\,x_B f_1(\bar f^\perp+\bar{\tilde f}^\perp)\right]\Bigg)\Bigg\}. \qquad (6.30)$$

6.3.2 Azimuthal asymmetries in singly polarized Drell-Yan scattering

There is at tree level up to order $1/Q$ no azimuthal asymmetry in single spin asymmetries.

6.3.3 Azimuthal asymmetries in doubly polarized LL-asymmetries

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$\begin{aligned}
\mathcal{W}^{\mu\nu} &= \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\; \delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\\
&\quad\times\lambda_A\lambda_B\Bigg\{-g_\perp^{\mu\nu}\Big[g_{1L}\bar g_{1L}\Big] - \frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}} + (\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\, g_\perp^{\mu\nu}}{M_A M_B}\, h_{1L}^\perp \bar h_{1L}^\perp\\
&\quad+\frac{\hat z^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[g_{1L}\bar g_{1L} - 2x_A\, g_L^\perp \bar g_{1L} - \frac{M_B}{M_A}\,2x_B\, h_{1L}^\perp \bar h_L + \frac{m}{M_A}\,2\, h_{1L}^\perp \bar g_{1L} - \frac{\boldsymbol{k}_{b\perp}^2}{M_A M_B}\, h_{1L}^\perp \bar h_{1L}^\perp\right]\\
&\quad+\frac{\hat z^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[-g_{1L}\bar g_{1L} + 2x_B\, g_{1L}\bar g_L^\perp + \frac{M_A}{M_B}\,2x_A\, h_L \bar h_{1L}^\perp - \frac{m}{M_B}\,2\, g_{1L}\bar h_{1L}^\perp + \frac{\boldsymbol{k}_{a\perp}^2}{M_A M_B}\, h_{1L}^\perp \bar h_{1L}^\perp\right]\Bigg\},\\
&= \lambda_A\lambda_B\Bigg\{-g_\perp^{\mu\nu} I\left[g_{1L}\bar g_{1L}\right] - \Big(2\,\hat x^\mu \hat x^\nu + g_\perp^{\mu\nu}\Big)\; I\left[\frac{k_a^x k_b^x - k_a^y k_b^y}{M_A M_B}\, h_{1L}^\perp \bar h_{1L}^\perp\right]\\
&\quad+\hat z^{\{\mu}\hat x^{\nu\}}\Bigg(-\frac{M_A}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_A\,(g_L^\perp+\tilde g_L^\perp)\,\bar g_{1L}\right] - \frac{M_B}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_B\, h_{1L}^\perp(\bar h_L + \bar{\tilde h}_L)\right]\\
&\quad+\frac{M_B}{Q}\, I\left[\frac{k_b^x}{M_B}\, x_B\, g_{1L}(\bar g_L^\perp + \bar{\tilde g}_L^\perp)\right] + \frac{M_A}{Q}\, I\left[\frac{k_b^x}{M_B}\, x_A\,(h_L+\tilde h_L)\,\bar h_{1L}^\perp\right]\Bigg)\Bigg\},
\end{aligned} \qquad (6.31)$$

The cross section is given by

$$\frac{d\sigma_{LL}(\vec{A}\vec{B}\to\mu^+\mu^- X)}{dx_{\scriptscriptstyle A}\,dx_{\scriptscriptstyle B}\,dy\,d^2\boldsymbol{q}_T} = \frac{4\pi\,\alpha^2}{Q^2}\,\lambda_A\,\lambda_B\left\{\left(\frac{1}{2}-y+y^2\right)\,I\left[g_{1L}\bar{g}_{1L}\right]\right.$$
$$+y(1-y)\,\cos 2\phi\;I\left[\frac{k_a^x k_b^x - k_a^y k_b^y}{M_A M_B}\,h_{1L}^\perp \bar{h}_{1L}^\perp\right]$$
$$+(1-2y)\sqrt{y(1-y)}\,\cos\phi\left(\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1L}\right]\right.$$
$$+\frac{M_B}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp(\bar{h}_L+\tilde{\bar{h}}_L)\right]$$
$$-\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle B}\,g_{1L}(\bar{g}_L^\perp+\tilde{\bar{g}}_L^\perp)\right]$$
$$\left.\left.-\frac{M_A}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{\scriptscriptstyle A}\,(h_L+\tilde{h}_L)\,\bar{h}_{1L}^\perp\right]\right)\right\}. \quad (6.32)$$

6.3.4 Azimuthal asymmetries in doubly polarized LT-asymmetries

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$\mathscr{W}^{\mu\nu} = \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\;\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\;\lambda_A\left\{-g_\perp^{\mu\nu}\left[\frac{\boldsymbol{k}_{bT}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1L}\,\bar{g}_{1T}\right]\right.$$
$$-\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_A M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar{h}_{1T}^\perp - \frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,h_{1L}^\perp\bar{h}_{1T}$$
$$+\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1L}\,\bar{g}_{1T} - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,g_L^\perp\,\bar{g}_{1T} - \frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1L}^\perp\,\bar{h}_T\right.$$
$$\left.+\frac{m}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,h_{1L}^\perp\,\bar{g}_{1T} - \frac{\boldsymbol{k}_{b\perp}^2}{M_A M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar{h}_{1T}^\perp - \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1L}^\perp\bar{h}_{1T}\right]$$
$$+\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\left[-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1L}\,\bar{g}_{1T} + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,g_{1L}\,\bar{g}_T^\perp\right.$$
$$+\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_L\,\bar{h}_{1T}^\perp - \frac{m}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,g_{1L}\,\bar{h}_{1T}^\perp$$
$$\left.+\frac{\boldsymbol{k}_{a\perp}^2}{M_A M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar{h}_{1T}^\perp + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1L}^\perp\bar{h}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1L}^\perp\bar{h}_T^\perp\right]$$
$$+\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\left[\frac{M_A}{M_B}\,2x_{\scriptscriptstyle A}\,h_L\bar{h}_{1T} + 2x_{\scriptscriptstyle B}\,g_{1L}\bar{g}_T' - \frac{m}{M_B}\,2\,g_{1L}\bar{h}_{1T}\right.$$
$$\left.\left.+\frac{\boldsymbol{k}_{a\perp}^2}{M_A M_B}\,h_{1L}^\perp\bar{h}_{1T} - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_A M_B}\,h_{1L}^\perp\bar{h}_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_A M_B}\,2x_{\scriptscriptstyle B}\,h_{1L}^\perp\bar{h}_T^\perp\right]\right\}. \quad (6.33)$$

This can be rewritten as

$$
\begin{aligned}
\mathscr{W}^{\mu\nu} &= \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\, \delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\, \lambda_A \Bigg\{ -g_\perp^{\mu\nu}\left[g_{1L}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,\bar{g}_{1T}\right] \\
&\quad -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_A M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1L}^\perp\,\bar{h}_{1T}^\perp - \frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,h_{1L}^\perp\bar{h}_{1T} \\
&\quad -\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\left[\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}+\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar{h}_T+\bar{\tilde{h}}_T)\right] \\
&\quad +\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\Bigg[\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle B}\,g_{1L}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp)+\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,x_{\scriptscriptstyle A}\,(h_L+\tilde{h}_L)\,\bar{h}_{1T}^\perp \\
&\qquad\qquad -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp)\Bigg] \\
&\quad +\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\left[x_{\scriptscriptstyle B}\,g_{1L}\,(\bar{g}_T'+\bar{\tilde{g}}_T')+\frac{M_A}{M_B}\,x_{\scriptscriptstyle A}\,(h_L+\tilde{h}_L)\,\bar{h}_{1T}+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_A M_B}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp)\right]\Bigg\},
\end{aligned}
$$

or

$$
\begin{aligned}
\mathscr{W}^{\mu\nu} &= \lambda_A\Bigg\{-g_\perp^{\mu\nu}\,S_B^x\,I\left[\frac{k_b^x}{M_B}\,g_{1L}\,\bar{g}_{1T}\right]-\Big(\hat{x}^{\{\mu}S_{B\perp}^{\nu\}}+S_B^x\,g_\perp^{\mu\nu}\Big)\;I\left[\frac{k_a^x}{M_A}\,h_{1L}^\perp\bar{h}_1\right] \\
&\quad -\Bigg(\Big(2\hat{x}^\mu\hat{x}^\nu+g_\perp^{\mu\nu}\Big)\;S_B^x-\hat{x}^{\{\mu}\hat{y}^{\nu\}}\,S_B^y\Bigg) \\
&\qquad\qquad \times I\left[\frac{4\,k_a^x(k_b^x)^2-2\,k_b^x\,(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})-k_a^x\,\boldsymbol{k}_{b\perp}^2}{2\,M_A\,M_B^2}\,h_{1L}^\perp\bar{h}_{1T}^\perp\right] \\
&\quad +\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}\Bigg(\frac{M_B}{Q}\,x_{\scriptscriptstyle B}\,I\left[g_{1L}(\bar{g}_T+\bar{\tilde{g}}_T)\right]+\frac{M_A}{Q}\,x_{\scriptscriptstyle A}\,I\left[(h_L+\tilde{h}_L)\,\bar{h}_1\right] \\
&\qquad -\frac{M_B}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_A M_B}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar{h}_T-\bar{h}_T^\perp+\bar{\tilde{h}}_T-\bar{\tilde{h}}_T^\perp)\right] \\
&\qquad -\frac{M_A}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_A M_B}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg) \\
&\quad +\Big(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\,S_B^x-\hat{z}^{\{\mu}\hat{y}^{\nu\}}\,S_B^y\Big)\;\Bigg(\frac{M_B}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_{\scriptscriptstyle B}\,g_{1L}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp)\right] \\
&\qquad +\frac{M_A}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_{\scriptscriptstyle A}\,(h_L+\tilde{h}_L)\,\bar{h}_{1T}^\perp\right] \\
&\qquad -\frac{M_B}{Q}\,I\left[\frac{2\,k_a^x k_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_A M_B}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar{h}_T+\bar{h}_T^\perp+\bar{\tilde{h}}_T+\bar{\tilde{h}}_T^\perp)\right] \\
&\qquad -\frac{M_A}{Q}\,I\left[\frac{2\,k_a^x k_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_A M_B}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg)\Bigg\}. \qquad (6.34)
\end{aligned}
$$

The cross section is given by

$$
\begin{aligned}
\frac{d\sigma_{LT}(\vec{A}\vec{B}\to\mu^+\mu^-X)}{dx_{\scriptscriptstyle A}\,dx_{\scriptscriptstyle B}\,dy\,d^2\boldsymbol{q}_T} = \frac{4\pi\,\alpha^2}{Q^2}\,\lambda_A\,|\boldsymbol{S}_{B\perp}|\Bigg\{ & \left(\frac{1}{2}-y+y^2\right)\,\cos(\phi-\phi_s^{\scriptscriptstyle B})\,I\left[\frac{k_b^x}{M_B}\,g_{1L}\,\bar{g}_{1T}\right] \\
& +y(1-y)\,\cos(\phi+\phi_s^{\scriptscriptstyle B})\,I\left[\frac{k_a^x}{M_A}\,h_{1L}^\perp\bar{h}_1\right] \\
& +y(1-y)\,\cos(3\phi-\phi_s^{\scriptscriptstyle B})\,I\left[\frac{4\,k_a^x(k_b^x)^2-2\,k_b^x\,(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})-k_a^x\,\boldsymbol{k}_{b\perp}^2}{2\,M_A\,M_B^2}\,h_{1L}^\perp\bar{h}_{1T}^\perp\right] \\
& -(1-2y)\sqrt{y(1-y)}\,\cos\phi_s^{\scriptscriptstyle B}\Bigg(\frac{M_B}{Q}\,x_{\scriptscriptstyle B}\,I\left[g_{1L}(\bar{g}_T+\bar{\tilde{g}}_T)\right]+\frac{M_A}{Q}\,x_{\scriptscriptstyle A}\,I\left[(h_L+\tilde{h}_L)\,\bar{h}_1\right] \\
& \qquad -\frac{M_B}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar{h}_T-\bar{h}_T^\perp+\bar{\tilde{h}}_T-\bar{\tilde{h}}_T^\perp)\right] \\
& \qquad -\frac{M_A}{Q}\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg) \\
& -(1-2y)\sqrt{y(1-y)}\,\cos(2\phi-\phi_s^{\scriptscriptstyle B})\,\Bigg(\frac{M_B}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_{\scriptscriptstyle B}\,g_{1L}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp)\right] \\
& \qquad +\frac{M_A}{Q}\,I\left[\frac{2\,(k_b^x)^2-\boldsymbol{k}_{b\perp}^2}{2\,M_B^2}\,x_{\scriptscriptstyle A}\,(h_L+\tilde{h}_L)\,\bar{h}_{1T}^\perp\right] \\
& \qquad -\frac{M_B}{Q}\,I\left[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_{\scriptscriptstyle B}\,h_{1L}^\perp\,(\bar{h}_T+\bar{h}_T^\perp+\bar{\tilde{h}}_T+\bar{\tilde{h}}_T^\perp)\right] \\
& \qquad -\frac{M_A}{Q}\,I\left[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_AM_B}\,x_{\scriptscriptstyle A}\,(g_L^\perp+\tilde{g}_L^\perp)\,\bar{g}_{1T}\right]\Bigg)\Bigg\}. \qquad (6.35)
\end{aligned}
$$

6.3.5 Azimuthal asymmetries in doubly polarized TT-asymmetries

The relevant result expressed in terms of the twist two and twist three distribution functions and perpendicular tensors and vectors is

$$
\begin{aligned}
\mathscr{W}^{\mu\nu} \;=\; & \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\, \delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\;\times\Bigg\{-g_\perp^{\mu\nu}\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1T}\,\bar g_{1T}\right] \\
& -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}^\perp\,\bar h_{1T}^\perp \\
& -\frac{k_{b\perp}^{\{\mu}S_{A\perp}^{\nu\}}+(\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp})\,g_\perp^{\mu\nu}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar h_{1T}^\perp \\
& -\frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar h_{1T}-\left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}\right)\,h_{1T}\bar h_{1T} \\
& +\frac{\hat z^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\Bigg[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1T}\,\bar g_{1T}-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,g_T^\perp\,\bar g_{1T} \\
& \quad -\frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}^\perp\,\bar h_T+\frac{m}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,h_{1T}^\perp\,\bar g_{1T} \\
& \quad -\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}^\perp\,\bar h_{1T}^\perp-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar h_{1T}^\perp \\
& \quad -\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar h_{1T}+\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_T^\perp\,\bar h_{1T}^\perp \\
& \quad -\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar h_{1T}+\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar h_{1T}\Bigg] \\
& +\frac{\hat z^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\Bigg[-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,g_{1T}\,\bar g_{1T}+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,g_{1T}\,\bar g_T^\perp \\
& \quad +\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_T\,\bar h_{1T}^\perp-\frac{m}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,g_{1T}\,\bar h_{1T}^\perp \\
& \quad +\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}^\perp\,\bar h_{1T}^\perp+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar h_{1T}^\perp \\
& \quad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar h_{1T}-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1T}^\perp\bar h_T^\perp \\
& \quad +\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,h_{1T}\bar h_{1T}-\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar h_T^\perp\Bigg] \\
& +\frac{M_A\,\hat z^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\Bigg[-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,g_T'\,\bar g_{1T}-\frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}\,\bar h_T \\
& \quad +\frac{m}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2\,h_{1T}\,\bar g_{1T}-\frac{\boldsymbol{k}_{b\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar h_{1T}^\perp \\
& \quad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,h_{1T}\,\bar h_{1T}^\perp-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\,2x_{\scriptscriptstyle A}\,h_T^\perp\,\bar h_{1T}^\perp \\
& \quad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar h_{1T}-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,h_{1T}\bar h_{1T}-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,2x_{\scriptscriptstyle A}\,h_T^\perp\bar h_{1T}\Bigg] \\
& +\frac{M_B\,\hat z^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\Bigg[\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle A}\,h_T\,\bar h_{1T}+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,g_{1T}\bar g_T' \\
& \quad -\frac{m}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2\,g_{1T}\bar h_{1T}+\frac{\boldsymbol{k}_{a\perp}^2}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar h_{1T} \\
& \quad -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,h_{1T}^\perp\bar h_{1T}+\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_AM_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,2x_{\scriptscriptstyle B}\,h_{1T}^\perp\bar h_T^\perp \\
& \quad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar h_{1T}-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,h_{1T}\bar h_{1T}+\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,2x_{\scriptscriptstyle B}\,h_{1T}\bar h_T^\perp\Bigg]\Bigg\},
\end{aligned}
\tag{6.36}
$$

which can be rewritten as

$$
\begin{aligned}
\mathscr{W}^{\mu\nu} \quad = \quad & \frac{1}{3}\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\; \delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\ \times \Bigg\{ -g_\perp^{\mu\nu}\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, g_{1T}\,\bar{g}_{1T}\right] \\
& -\frac{k_{a\perp}^{\{\mu}k_{b\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})\,g_\perp^{\mu\nu}}{M_A M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, h_{1T}^\perp\,\bar{h}_{1T}^\perp \\
& -\frac{k_{b\perp}^{\{\mu}S_{A\perp}^{\nu\}}+(\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp})\,g_\perp^{\mu\nu}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, h_{1T}\,\bar{h}_{1T}^\perp \\
& -\frac{k_{a\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\, h_{1T}^\perp\bar{h}_{1T} - \left(S_{A\perp}^{\{\mu}S_{B\perp}^{\nu\}}+(\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,g_\perp^{\mu\nu}\right)\, h_{1T}\bar{h}_{1T} \\
& +\frac{\hat{z}^{\{\mu}k_{a\perp}^{\nu\}}}{Q}\Bigg[-\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_A\,(g_T^\perp+\tilde{g}_T^\perp)\,\bar{g}_{1T} \\
& \qquad -\frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_B\, h_{1T}^\perp\,(\bar{h}_T+\bar{\tilde{h}}_T) \\
& \qquad +\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_A\,(h_T^\perp+\tilde{h}_T^\perp)\,\bar{h}_{1T}^\perp + (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,x_A\,(h_T^\perp+\tilde{h}_T^\perp)\,\bar{h}_{1T}\Bigg] \\
& +\frac{\hat{z}^{\{\mu}k_{b\perp}^{\nu\}}}{Q}\Bigg[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_B\, g_{1T}\,(\bar{g}_T^\perp+\bar{\tilde{g}}_T^\perp) \\
& \qquad +\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_A\,(h_T+\tilde{h}_T)\,\bar{h}_{1T}^\perp \\
& \qquad -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\, x_B\, h_{1T}^\perp(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp) - (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\,x_B\, h_{1T}\,(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp)\Bigg] \\
& +\frac{M_A\,\hat{z}^{\{\mu}S_{A\perp}^{\nu\}}}{Q}\Bigg[-\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_A\,(g_T'+\tilde{g}_T')\,\bar{g}_{1T} - \frac{M_B}{M_A}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_B\, h_{1T}\,(\bar{h}_T+\bar{\tilde{h}}_T) \\
& \qquad -\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_A M_B}\,\frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{B\perp}}{M_B}\, x_A\,(h_T^\perp+\tilde{h}_T^\perp)\,\bar{h}_{1T}^\perp - \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{B\perp}}{M_A}\, x_A\,(h_T^\perp+\tilde{h}_T^\perp)\bar{h}_{1T}\Bigg] \\
& +\frac{M_B\,\hat{z}^{\{\mu}S_{B\perp}^{\nu\}}}{Q}\Bigg[\frac{M_A}{M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\, x_A\,(h_T+\tilde{h}_T)\,\bar{h}_{1T} + \frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\, x_B\, g_{1T}(\bar{g}_T'+\bar{\tilde{g}}_T') \\
& \qquad +\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{M_A M_B}\,\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{S}_{A\perp}}{M_A}\, x_B\, h_{1T}^\perp\,(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp) + \frac{\boldsymbol{k}_{b\perp}\cdot\boldsymbol{S}_{A\perp}}{M_B}\, x_B\, h_{1T}\,(\bar{h}_T^\perp+\bar{\tilde{h}}_T^\perp)\Bigg]\Bigg\}
\end{aligned}
$$

or

$$
\begin{aligned}
\mathscr{W}^{\mu\nu} \;=\;& -g_\perp^{\mu\nu}\left((\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\, I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_A M_B}\, g_{1T}\,\bar{g}_{1T}\right] + \left(S_A^x S_B^x - S_A^y S_B^y\right)\, I\left[\frac{k_a^x k_b^x - k_a^y k_b^y}{2\,M_A M_B}\, g_{1T}\,\bar{g}_{1T}\right]\right)\\
& - \left(S_{A\perp}^{\{\mu} S_{B\perp}^{\nu\}} + (\boldsymbol{S}_{A\perp}\cdot\boldsymbol{S}_{B\perp})\, g_\perp^{\mu\nu}\right)\left(I\left[h_1\bar{h}_1\right] - I\left[\frac{\boldsymbol{k}_{a\perp}^2\boldsymbol{k}_{b\perp}^2}{4\,M_A^2 M_B^2}\, h_{1T}^\perp\,\bar{h}_{1T}^\perp\right]\right)\\
& -\left(\left(\hat{x}^{\{\mu} S_{A\perp}^{\nu\}} + S_A^x\, g_\perp^{\mu\nu}\right) S_B^x + \left(\hat{y}^{\{\mu} S_{A\perp}^{\nu\}} + S_A^y\, g_\perp^{\mu\nu}\right) S_B^y\right) I\left[\frac{(k_b^x)^2-(k_b^y)^2}{2\,M_B^2}\, h_1\,\bar{h}_{1T}^\perp\right]\\
& -\left(\left(\hat{x}^{\{\mu} S_{B\perp}^{\nu\}} + S_B^x\, g_\perp^{\mu\nu}\right) S_A^x + \left(\hat{y}^{\{\mu} S_{B\perp}^{\nu\}} + S_B^y\, g_\perp^{\mu\nu}\right) S_A^y\right) I\left[\frac{(k_a^x)^2-(k_a^y)^2}{2\,M_A^2}\, h_{1T}^\perp\,\bar{h}_1\right]\\
& -\left(\left(2\hat{x}^\mu\hat{x}^\nu + g_\perp^{\mu\nu}\right)\left(S_A^x S_B^x - S_A^y S_B^y\right) - \hat{x}^{\{\mu}\hat{y}^{\nu\}}\,(S_A^x S_B^y + S_A^y S_B^x)\right)\\
&\qquad\qquad \times I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\,[(k_b^x)^2-(k_b^y)^2] - 4k_a^x k_a^y k_b^x k_b^y}{4\,M_A^2 M_B^2}\, h_{1T}^\perp\,\bar{h}_{1T}^\perp\right]\\
& -\hat{z}^{\{\mu} S_{A\perp}^{\nu\}}\, S_B^x\left(\frac{M_A}{Q}\, I\left[\frac{k_b^x}{M_B}\, x_A\,(g_T+\tilde{g}_T)\,\bar{g}_{1T}\right] + \frac{M_B}{Q}\, I\left[\frac{k_b^x}{M_B}\, x_B\, h_1\,(\bar{h}_T+\bar{\tilde{h}}_T)\right]\right)\\
& +\hat{z}^{\{\mu}\epsilon_\perp^{\nu\}\rho} S_{A\perp\rho}\, S_B^y\, \frac{M_B}{Q}\, I\left[\frac{k_b^x}{M_B}\, x_B\, h_1\,(\bar{h}_T^\perp + \bar{\tilde{h}}_T^\perp)\right]\\
& - \left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\, S_A^x S_B^x - \hat{z}^{\{\mu}\hat{y}^{\nu\}}\, S_A^y S_B^x\right)\left(\frac{M_A}{Q}\, I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2 M_B}\, x_A\,(g_T^\perp + \tilde{g}_T^\perp)\,\bar{g}_{1T}\right]\right.\\
&\qquad\qquad + \frac{M_B}{Q}\, I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2 M_B}\, x_B\, h_{1T}^\perp\,(\bar{h}_T + \bar{\tilde{h}}_T)\right]\\
&\qquad\qquad \left. - \frac{M_B}{Q}\, I\left[\frac{k_a^x k_a^y k_b^y}{M_A^2 M_B}\, x_B\, h_{1T}^\perp\,(\bar{h}_T^\perp + \bar{\tilde{h}}_T^\perp)\right]\right)\\
& - \left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\, S_A^y S_B^y + \hat{z}^{\{\mu}\hat{y}^{\nu\}}\, S_A^x S_B^y\right)\left(\frac{M_A}{Q}\, I\left[\frac{k_a^x k_a^y k_b^y}{M_A^2 M_B}\, x_A\,(g_T^\perp + \tilde{g}_T^\perp)\,\bar{g}_{1T}\right]\right.\\
&\qquad\qquad + \frac{M_B}{Q}\, I\left[\frac{k_a^x k_a^y k_b^y}{M_A^2 M_B}\, x_B\, h_{1T}^\perp\,(\bar{h}_T + \bar{\tilde{h}}_T)\right]\\
&\qquad\qquad \left. - \frac{M_B}{Q}\, I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\, k_b^x}{2\,M_A^2 M_B}\, x_B\, h_{1T}^\perp\,(\bar{h}_T^\perp + \bar{\tilde{h}}_T^\perp)\right]\right)\\
& +\hat{z}^{\{\mu} S_{B\perp}^{\nu\}}\, S_A^x\left(\frac{M_A}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_A\,(h_T+\tilde{h}_T)\,\bar{h}_1\right] + \frac{M_B}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_B\, g_{1T}\,(\bar{g}_T+\bar{\tilde{g}}_T)\right]\right)\\
& -\hat{z}^{\{\mu}\epsilon_\perp^{\nu\}\rho} S_{B\perp\rho}\, S_A^y\, \frac{M_A}{Q}\, I\left[\frac{k_a^x}{M_A}\, x_A\,(h_T^\perp + \tilde{h}_T^\perp)\bar{h}_1\right]\\
& + \left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\, S_A^x S_B^x - \hat{z}^{\{\mu}\hat{y}^{\nu\}}\, S_A^x S_B^y\right)\left(\frac{M_B}{Q}\, I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_A M_B^2}\, x_B\, g_{1T}\,(\bar{g}_T^\perp + \bar{\tilde{g}}_T^\perp)\right]\right.\\
&\qquad\qquad + \frac{M_A}{Q}\, I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_A M_B^2}\, x_A\,(h_T+\tilde{h}_T)\,\bar{h}_{1T}^\perp\right]\\
&\qquad\qquad \left. - \frac{M_A}{Q}\, I\left[\frac{k_a^y k_b^x k_b^y}{M_A M_B^2}\, x_A\,(h_T^\perp + \tilde{h}_T^\perp)\bar{h}_{1T}^\perp\right]\right)\\
& + \left(\hat{z}^{\{\mu}\hat{x}^{\nu\}}\, S_A^y S_B^y + \hat{z}^{\{\mu}\hat{y}^{\nu\}}\, S_A^y S_B^x\right)\left(\frac{M_B}{Q}\, I\left[\frac{k_a^y k_b^x k_b^y}{M_A M_B^2}\, x_A\, g_{1T}\,(\bar{g}_T^\perp + \bar{\tilde{g}}_T^\perp)\right]\right.\\
&\qquad\qquad + \frac{M_A}{Q}\, I\left[\frac{k_a^y k_b^x k_b^y}{M_A M_B^2}\, x_A\,(h_T+\tilde{h}_T)\,\bar{h}_{1T}^\perp\right]\\
&\qquad\qquad \left. - \frac{M_A}{Q}\, I\left[\frac{k_a^x[(k_b^x)^2-(k_b^y)^2]}{2\,M_A M_B^2}\, x_A\,(h_T^\perp + \tilde{h}_T^\perp)\bar{h}_{1T}^\perp\right]\right) \qquad (6.37)
\end{aligned}
$$

The cross section is given by

$$\begin{aligned}
\frac{d\sigma_{TT}(\vec{A}\vec{B}\to\mu^+\mu^- X)}{dx_{A}\,dx_{B}\,dy\,d^2\boldsymbol{q}_T} = \frac{4\pi\,\alpha^2}{Q^2}\,|\boldsymbol{S}_{A\perp}|\,|\boldsymbol{S}_{B\perp}| \Bigg\{ &\left(\frac{1}{2}-y+y^2\right)\,\cos(\phi_s^{A}-\phi_s^{B})\,I\left[\frac{\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp}}{2\,M_A M_B}\,g_{1T}\,\bar{g}_{1T}\right]\\
&+\left(\frac{1}{2}-y+y^2\right)\,\cos(\phi_s^{A}+\phi_s^{B}-2\phi) I\left[\frac{k_a^x k_b^x - k_a^y k_b^y}{2\,M_A M_B}\,g_{1T}\,\bar{g}_{1T}\right]\\
&+y(1-y)\,\cos(\phi_s^{A}+\phi_s^{B})\left(I\left[h_1\bar{h}_1\right]-I\left[\frac{\boldsymbol{k}^2_{a\perp}\boldsymbol{k}^2_{b\perp}}{4\,M_A^2 M_B^2}\,h_{1T}^\perp\,\bar{h}_{1T}^\perp\right]\right)\\
&+y(1-y)\,\cos(2\phi+\phi_s^{A}-\phi_s^{B})\,I\left[\frac{(k_b^x)^2-(k_b^y)^2}{2\,M_B^2}\,h_1\,\bar{h}_{1T}^\perp\right]\\
&+y(1-y)\,\cos(2\phi-\phi_s^{A}+\phi_s^{B})\,I\left[\frac{(k_a^x)^2-(k_a^y)^2}{2\,M_A^2}\,h_{1T}^\perp\,\bar{h}_1\right]\\
&+\ldots
\end{aligned}$$

$$
\begin{aligned}
&\ldots \\
&-y(1-y)\,\cos(4\phi-\phi_s^{A}-\phi_s^{B})\,I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\,[(k_b^x)^2-(k_b^y)^2]-4k_a^xk_a^yk_b^xk_b^y}{4\,M_A^2M_B^2}\,h_{1T}^{\perp}\,\bar h_{1T}^{\perp}\right] \\
&+(1-2y)\sqrt{y(1-y)}\,\cos\phi_s^{A}\,\cos(\phi_s^{B}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{A}\,(g_T+\tilde g_T)\,\bar g_{1T}\right]\right. \\
&\qquad\qquad\left.+\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{B}\,h_1\,(\bar h_T+\bar{\tilde h}_T)\right]\right) \\
&+(1-2y)\sqrt{y(1-y)}\,\sin\phi_s^{A}\,\sin(\phi_s^{B}-\phi)\,\frac{M_B}{Q}\,I\left[\frac{k_b^x}{M_B}\,x_{B}\,h_1\,(\bar h_T^{\perp}+\bar{\tilde h}_T^{\perp})\right] \\
&+(1-2y)\sqrt{y(1-y)}\,\cos(2\phi-\phi_s^{A})\,\cos(\phi_s^{B}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2M_B}\,x_{A}\,(g_T^{\perp}+\tilde g_T^{\perp})\,\bar g_{1T}\right]\right. \\
&\qquad\qquad+\frac{M_B}{Q}\,I\left[\frac{k_b^x\,[(k_a^x)^2-(k_a^y)^2]}{2\,M_A^2M_B}\,x_{B}\,h_{1T}^{\perp}\,(\bar h_T+\bar{\tilde h}_T)\right] \\
&\qquad\qquad\left.-\frac{M_B}{Q}\,I\left[\frac{k_a^xk_a^yk_b^y}{M_A^2M_B}\,x_{B}\,h_{1T}^{\perp}\,(\bar h_T^{\perp}+\bar{\tilde h}_T^{\perp})\right]\right) \\
&+(1-2y)\sqrt{y(1-y)}\,\sin(2\phi-\phi_s^{A})\,\sin(\phi_s^{B}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_a^xk_a^yk_b^y}{M_A^2M_B}\,x_{A}\,(g_T^{\perp}+\tilde g_T^{\perp})\,\bar g_{1T}\right]\right. \\
&\qquad\qquad+\frac{M_B}{Q}\,I\left[\frac{k_a^xk_a^yk_b^y}{M_A^2M_B}\,x_{B}\,h_{1T}^{\perp}\,(\bar h_T+\bar{\tilde h}_T)\right] \\
&\qquad\qquad\left.-\frac{M_B}{Q}\,I\left[\frac{[(k_a^x)^2-(k_a^y)^2]\,k_b^x}{2\,M_A^2M_B}\,x_{B}\,h_{1T}^{\perp}\,(\bar h_T^{\perp}+\bar{\tilde h}_T^{\perp})\right]\right) \\
&-(1-2y)\sqrt{y(1-y)}\,\cos\phi_s^{B}\,\cos(\phi_s^{A}-\phi)\,\left(\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{A}\,(h_T+\tilde h_T)\,\bar h_1\right]\right. \\
&\qquad\qquad\left.+\frac{M_B}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{B}\,g_{1T}\,(\bar g_T+\bar{\tilde g}_T)\right]\right) \\
&-(1-2y)\sqrt{y(1-y)}\,\sin\phi_s^{B}\,\sin(\phi_s^{A}-\phi)\,\frac{M_A}{Q}\,I\left[\frac{k_a^x}{M_A}\,x_{A}\,(h_T^{\perp}+\tilde h_T^{\perp})\bar h_1\right] \\
&-(1-2y)\sqrt{y(1-y)}\,\cos(2\phi-\phi_s^{B})\,\cos(\phi_s^{A}-\phi)\,\left(\frac{M_B}{Q}\,I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_AM_B^2}\,x_{B}\,g_{1T}\,(\bar g_T^{\perp}+\bar{\tilde g}_T^{\perp})\right]\right. \\
&\qquad\qquad+\frac{M_A}{Q}\,I\left[\frac{k_a^x\,[(k_b^x)^2-(k_b^y)^2]}{2\,M_AM_B^2}\,x_{A}\,(h_T+\tilde h_T)\,\bar h_{1T}^{\perp}\right] \\
&\qquad\qquad\left.-\frac{M_A}{Q}\,I\left[\frac{k_a^yk_b^xk_b^y}{M_AM_B^2}\,x_{A}\,(h_T^{\perp}+\tilde h_T^{\perp})\bar h_{1T}^{\perp}\right]\right) \\
&-(1-2y)\sqrt{y(1-y)}\,\sin(2\phi-\phi_s^{B})\,\sin(\phi_s^{A}-\phi)\,\left(\frac{M_B}{Q}\,I\left[\frac{k_a^yk_b^xk_b^y}{M_AM_B^2}\,x_{A}\,g_{1T}\,(\bar g_T^{\perp}+\bar{\tilde g}_T^{\perp})\right]\right. \\
&\qquad\qquad+\frac{M_A}{Q}\,I\left[\frac{k_a^yk_b^xk_b^y}{M_AM_B^2}\,x_{A}\,(h_T+\tilde h_T)\,\bar h_{1T}^{\perp}\right] \\
&\qquad\qquad\left.\left.-\frac{M_A}{Q}\,I\left[\frac{k_a^x[(k_b^x)^2-(k_b^y)^2]}{2\,M_AM_B^2}\,x_{A}\,(h_T^{\perp}+\tilde h_T^{\perp})\bar h_{1T}^{\perp}\right]\right)\right\}. \qquad (6.38)
\end{aligned}
$$

6.4 Convolutions and gaussian distributions

In order to study the behavior of the convolutions of distribution, it is useful to consider gaussian distributions,

$$f(x_{\scriptscriptstyle A},\boldsymbol{k}_{aT}) = f(x_{\scriptscriptstyle A},\boldsymbol{0}_T)\,\exp(-R_a^2\boldsymbol{k}_{aT}^2) = f(x_{\scriptscriptstyle A})\,\frac{R_a^2}{\pi}\,\exp(-R_a^2\boldsymbol{k}_{aT}^2), \tag{6.39}$$

$$\bar f(x_{\scriptscriptstyle B},\boldsymbol{k}_{bT}) = \bar f(x_{\scriptscriptstyle B},\boldsymbol{0}_T)\,\exp(-R_b^2\boldsymbol{k}_{bT}^2) = f(x_{\scriptscriptstyle B})\,\underbrace{\frac{R_b^2}{\pi}\,\exp(-R_b^2\boldsymbol{k}_{bT}^2)}_{\mathscr{P}(\boldsymbol{k}_{bT},R_b)}, \tag{6.40}$$

In that case the convolution becomes

$$\begin{aligned} I[f\,\bar f] &= \int d^2\boldsymbol{k}_{aT}\,d^2\boldsymbol{k}_{bT}\;\delta^2(\boldsymbol{k}_{aT}+\boldsymbol{k}_{bT}-\boldsymbol{q}_T)\,f(x_{\scriptscriptstyle A},\boldsymbol{k}_{aT})\,\bar f(x_{\scriptscriptstyle B},\boldsymbol{k}_{bT}) \\ &= \frac{\pi}{R_a^2+R_b^2}\,\exp\left(-\frac{Q_T^2\,R_a^2R_b^2}{R_a^2+R_b^2}\right)\,f(x_{\scriptscriptstyle A},\boldsymbol{0}_T)\,\bar f(x_{\scriptscriptstyle B},\boldsymbol{0}_T) \\ &= f(x_{\scriptscriptstyle A})\,\bar f(x_{\scriptscriptstyle B})\,\mathscr{P}(\boldsymbol{q}_{\scriptscriptstyle T};R) \end{aligned} \tag{6.41}$$

with $R^2 = R_a^2R_b^2/(R_a^2+R_b^2)$. The other convolutions that appear in the cross sections are of the form

$$I\Big[\frac{k_a^x}{M_a}\,f\,\bar f\Big] = \frac{R^2}{R_a^2}\,\frac{Q_T}{M_a}\,I[f\,\bar f], \tag{6.42}$$

$$I\Big[\frac{\boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{M_aM_b}\,f\,\bar f\Big] = \frac{R^2}{M_aM_bR_a^2R_b^2}\,\big(Q_T^2\,R^2-1\big)\;I[f\,\bar f], \tag{6.43}$$

$$I\Big[\frac{2(k_a^x)^2-\boldsymbol{k}_{aT}^2}{2M_a^2}\,f\,\bar f\Big] = I\Big[\frac{(k_a^x)^2-(k_a^y)^2}{2M_a^2}\,f\,\bar f\Big] = \frac{R^4}{R_a^4}\,\frac{Q_T^2}{2M_a^2}\,I[f\,\bar f], \tag{6.44}$$

$$I\Big[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{2M_aM_b}\,f\,\bar f\Big] = I\Big[\frac{k_a^xk_b^x-k_a^yk_b^y}{2M_aM_b}\,f\,\bar f\Big] = \frac{R^4}{R_a^2\,R_b^2}\,\frac{Q_T^2}{2M_aM_b}\,I[f\,\bar f], \tag{6.45}$$

$$I\Big[\frac{k_a^x[(k_b^x)^2-(k_b^y)^2]}{2M_aM_b^2}\,f\,\bar f\Big] = \frac{R^4}{R_a^2\,R_b^4}\,\frac{Q_T}{2M_aM_b^2}\,\big(Q_{\scriptscriptstyle T}^2R^2-1\big)\;I[f\,\bar f], \tag{6.46}$$

$$I\Big[\frac{k_b^xk_a^yk_b^y}{M_aM_b^2}\,f\,\bar f\Big] = -\frac{R^4}{R_a^2\,R_b^4}\,\frac{Q_T}{2M_aM_b^2}\,I[f\,\bar f], \tag{6.47}$$

$$\begin{aligned} &I\Big[\frac{k_a^x[(k_b^x)^2-(k_b^y)^2]-2k_b^xk_a^yk_b^y}{2M_aM_b^2}\,f\,\bar f\Big] \\ &\qquad = I\Big[\frac{4\,k_a^x(k_b^x)^2-2\,k_b^x\,(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})-k_a^x\,\boldsymbol{k}_{b\perp}^2}{2M_aM_b^2}\,f\,\bar f\Big] = \frac{R^6}{R_a^2\,R_b^4}\,\frac{Q_T^3}{2M_aM_b^2}\,I[f\,\bar f], \end{aligned} \tag{6.48}$$

$$I\Big[\frac{[(k_a^x)^2-(k_a^y)^2]\,[(k_b^x)^2-(k_b^y)^2]-4k_a^xk_a^yk_b^xk_b^y}{4M_a^2M_b^2}\,f\,\bar f\Big] = \frac{R^8}{R_a^4\,R_b^4}\,\frac{Q_T^4}{4M_a^2M_b^2}\,I[f\,\bar f]. \tag{6.49}$$

Another way to deconvolute the results are using the following weighted quantities,

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}}{M_a}\, I\Big[\frac{k_a^x}{M_a}\,f\,\bar f\Big] = 2\,f^{(1)}(x_{\scriptscriptstyle A})\,\bar f(x_{\scriptscriptstyle B}), \tag{6.50}$$

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}^2}{M_aM_b}\, I\Big[\frac{\boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{M_aM_b}\,f\,\bar f\Big] = 4\,f^{(1)}(x_{\scriptscriptstyle A})\,\bar f^{(1)}(x_{\scriptscriptstyle B}), \tag{6.51}$$

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}^2}{M_a^2}\, I\Big[\frac{2(k_a^x)^2-\boldsymbol{k}_{aT}^2}{2M_a^2}\,f\,\bar f\Big] = 2\,f^{(2)}(x_{\scriptscriptstyle A})\,\bar f(x_{\scriptscriptstyle B}), \tag{6.52}$$

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}^2}{M_aM_b}\, I\Big[\frac{2\,k_a^xk_b^x-\boldsymbol{k}_{aT}\cdot\boldsymbol{k}_{bT}}{2M_aM_b}\,f\,\bar f\Big] = 4\,f^{(1)}(x_{\scriptscriptstyle A})\,\bar f^{(1)}(x_{\scriptscriptstyle B}), \tag{6.53}$$

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}^3}{M_aM_b^2}\, I\Big[\frac{k_a^x[(k_b^x)^2-(k_b^y)^2]}{2M_aM_b^2}\,f\,\bar f\Big] = 8\,f^{(1)}(x_{\scriptscriptstyle A})\,\bar f^{(2)}(x_{\scriptscriptstyle B}), \tag{6.54}$$

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}^3}{M_aM_b^2}\, I\Big[\frac{k_b^xk_a^yk_b^y}{M_aM_b^2}\,f\,\bar f\Big] = -4\,f^{(1)}(x_{\scriptscriptstyle A})\,\bar f^{(2)}(x_{\scriptscriptstyle B}), \tag{6.55}$$

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}^3}{M_aM_b^2}\, I\Big[\frac{4\,k_a^x(k_b^x)^2-2\,k_b^x\,(\boldsymbol{k}_{a\perp}\cdot\boldsymbol{k}_{b\perp})-k_a^x\,\boldsymbol{k}_{b\perp}^2}{2M_aM_b^2}\,f\,\bar f\Big] = 12\,f^{(1)}(x_{\scriptscriptstyle A})\,\bar f^{(2)}(x_{\scriptscriptstyle B}), \tag{6.56}$$

$$\int d^2\boldsymbol{q}_{\scriptscriptstyle T}\ \frac{Q_{\scriptscriptstyle T}^4}{M_a^2M_b^2}\, I\Big[\frac{[(k_a^x)^2-(k_a^y)^2]\,[(k_b^x)^2-(k_b^y)^2]-4k_a^xk_a^yk_b^xk_b^y}{4M_a^2M_b^2}\,f\,\bar f\Big] = 24\,f^{(2)}(x_{\scriptscriptstyle A})\,\bar f^{(2)}(x_{\scriptscriptstyle B}), \tag{6.57}$$

where $f^{(n)}(x_{\scriptscriptstyle A})$ indicate the $(\boldsymbol{k}_{aT}^2/2M_a^2)^n$-moments of $f(x_{\scriptscriptstyle A},\boldsymbol{k}_{aT}^2)$,

$$f^{(n)}(x_{\scriptscriptstyle A}) = \int d^2\boldsymbol{k}_{aT}\ \left(\frac{\boldsymbol{k}_{aT}^2}{2M_a^2}\right)^n\ f(x_{\scriptscriptstyle A},\boldsymbol{k}_{aT}^2)\ =\ \frac{n!}{(2M_a^2R_a^2)^n}\,f(x_{\scriptscriptstyle A}). \tag{6.58}$$

Chapter 7

Lepton-hadron up to $\mathscr{O}(1/Q)$

7.1 Inclusion of gluon contributions

We will consider in this section the inclusion of diagrams with gluons connecting the soft and hard part. The additional contribution is given by four diagrams. Two of them have gluons connected to the lower soft part (the hadron → quark part), the others gluons connected to the upper soft part (the quark → hadron part). Including the contribution of the handbag one has

$$
\begin{aligned}
2M\,\mathscr{W}_{\mu\nu} \;=\; & \int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\left[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right] \\
& - \int d^4p\,d^4k\,d^4p_1\,\delta^4(p+q-k)\Bigg\{\mathrm{Tr}\left[\gamma_\alpha\,\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2+i\epsilon}\,\gamma_\nu\Phi_A^\alpha(p,p-p_1)\gamma_\mu\Delta(k)\right] \\
& \qquad + \mathrm{Tr}\left[\gamma_\mu\,\frac{(\not k-\not p_1+m)}{(k-p_1)^2-m^2-i\epsilon}\,\gamma_\alpha\Delta(k)\gamma_\nu\Phi_A^\alpha(p-p_1,p)\right]\Bigg\} \\
& - \int d^4p\,d^4k\,d^4k_1\,\delta^4(p+q-k)\Bigg\{\mathrm{Tr}\left[\gamma_\nu\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2+i\epsilon}\,\gamma_\alpha\Phi(p)\gamma_\mu\Delta_A^\alpha(k-k_1,k)\right] \\
& \qquad + \mathrm{Tr}\left[\gamma_\alpha\,\frac{(\not p-\not k_1+m)}{(p-k_1)^2-m^2-i\epsilon}\,\gamma_\mu\Delta_A^\alpha(k,k-k_1)\gamma_\nu\Phi(p)\right]\Bigg\}
\end{aligned} \tag{7.1}
$$

[Note that we have for a quark-quark-gluon blob used momentum p_1 (or k_1) for the gluon and $p-p_1$ (or $k-k_1$) for the quark. This is easier to extend when we consider multiple gluon correlation functions.] The momenta p_1 and k_1 connected to the soft hadronic parts are parametrized according to

$$p_1 = \left[p_1^-, \frac{x_1\,Q}{\sqrt{2}}, \boldsymbol{p}_{1T}\right], \tag{7.2}$$

$$k_1 = \left[\frac{z_1\,Q}{\sqrt{2}}, k_1^+, \boldsymbol{k}_{1T}\right], \tag{7.3}$$

The momentum appearing in the extra fermion propagator is $p-p_1+q = k-p_1$ with $(k-p_1)^2 = -x_1\,Q^2$, or $k-k_1-q = p-k_1$ with $(p-k_1)^2 = -z_1\,Q^2$. Thus one has in leading order in $1/Q$,

$$\frac{\not k-\not p_1+m}{(k-p_1)^2-m^2+i\epsilon} = \frac{\gamma^-}{Q\sqrt{2}} - \frac{\gamma^+}{(x_1-i\epsilon)\,Q\sqrt{2}} + \frac{\boldsymbol{\gamma}_T\cdot(\boldsymbol{k}_T-\boldsymbol{p}_{1T})-m}{(x_1-i\epsilon)\,Q^2}, \tag{7.4}$$

$$\frac{\not p-\not k_1+m}{(p-k_1)^2-m^2+i\epsilon} = \frac{\gamma^+}{Q\sqrt{2}} - \frac{\gamma^-}{(z_1-i\epsilon)\,Q\sqrt{2}} + \frac{\boldsymbol{\gamma}_T\cdot(\boldsymbol{p}_T-\boldsymbol{k}_{1T})-m}{(z_1-i\epsilon)\,Q^2}. \tag{7.5}$$

This can be used to consider separately the contributions of transverse (A_T^α) and longitudinal (A^+) gluons. For the transverse gluons, the trace of the first gluonic contribution becomes

$$\begin{aligned}
&-\int d^4p\, d^4k\, d^4p_1\, \delta^4(p+q-k)\,\mathrm{Tr}\left[\gamma_\alpha\,\frac{\not k-\not p_1+m}{(k-p_1)^2-m^2+i\epsilon}\,\gamma_\nu\Phi_A^\alpha(p,p-p_1)\gamma_\mu\Delta(k)\right]\\
=&\int d^4p\, d^4k\,\delta^4(p+q-k)\int\frac{d^4p_1}{(2\pi)^4}\int d^4x\int d^4y\; e^{i\,(p-p_1)\cdot x+i\,p_1\cdot y}\\
&\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA_T^\alpha(y)\,\psi(x)|P,S\rangle,
\end{aligned}$$

which starts off at order $1/Q$ and at this order requires leading parts from Φ_A^α (proportional to $P_+\Phi_A^\alpha P_-$) and leading parts from Δ (proportional to $P_-\Delta P_+$). As $\{\gamma^-,\gamma_T^\alpha\}=0$ and $\gamma^- P_+ = P_-\gamma^- = 0$ only the $\gamma^- = P_+\gamma^- P_-$ part in Eq. 7.4 contributes. This term is independent of any of the components of p_1, and we thus can immediately consider the distributions $\int d^4p_1\,\Phi_A^\alpha(p,p-p_1)$, or explicitly

$$\int d^4p\, d^4k\,\delta^4(p+q-k)\int d^4x\; e^{i\,p\cdot x}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA_T^\alpha(x)\,\psi(x)|P,S\rangle. \tag{7.6}$$

This contribution will be studied in the next section. Note that it can be written in terms of the covariant derivative as

$$\begin{aligned}
&\int d^4p\, d^4k\,\delta^4(p+q-k)\int d^4x\; e^{i\,p\cdot x}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, i\,D_T^\alpha(x)\,\psi(x)|P,S\rangle\\
&-\int d^4p\, d^4k\,\delta^4(p+q-k)\,p_T^\alpha\int d^4x\; e^{i\,p\cdot x}\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\frac{\gamma_\alpha\,\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\,\psi(x)|P,S\rangle.
\end{aligned}\tag{7.7}$$

In this section we consider next the contributions of longitudinal gluons (A^+). They lead to traces of the form

$$-\int d^4p\, d^4k\, d^4p_1\,\delta^4(p+q-k)\,\mathrm{Tr}\left[\gamma^-\,\frac{\not k-\not p_1+m}{(k-p_1)^2-m^2+i\epsilon}\,\gamma_\nu\Phi_A^+(p,p-p_1)\gamma_\mu\Delta(k)\right]$$

The first term in Eq. 7.4 does not contribute. The second term contributes at $\mathscr{O}(1)$ as the dominant contribution in Φ_A^+ is the part projected out by $\int dp_1^-\;P_+\Phi_A^+P_-$ which is of $\mathscr{O}(Q)$. Explicitly, we get for the first correction in Eq. 7.1

$$\begin{aligned}
&-\int d^4p\, d^4k\,\delta^4(p+q-k)\int\frac{d^4p_1}{(2\pi)^4}\int d^4x\int d^4y\; e^{i\,(p-p_1)\cdot x+i\,p_1\cdot y}\\
&\qquad\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^-\;\frac{\gamma^+}{(x_1-i\epsilon)\,Q\sqrt{2}}\,\gamma_\nu\, gA^+(y)\,\psi(x)|P,S\rangle\\
=&\int d^4p\, d^4k\,\delta^4(p+q-k)\int\frac{dx_1}{2\pi}\int d^4x\int dy^-\;\frac{e^{-i\,x_1\,p^+(x^--y^-)}}{x_1-i\epsilon}\;e^{i\,p\cdot x}\\
&\qquad\times\;\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\;\frac{\gamma^-\gamma^+}{2}\,\gamma_\nu\, gA^+(y)\,\psi(x)|P,S\rangle\Big|_{y^+=x^+,\;y_T=x_T}\\
=&\int d^4p\, d^4k\,\delta^4(p+q-k)\int d^4x\int dy^-\;\theta(y^--x^-)\,e^{i\,p\cdot x}\\
&\qquad\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\,P_+\,\gamma_\nu\, igA^+(y^-)\,\psi(x)|P,S\rangle\\
=&-\int d^4p\, d^4k\,\delta^4(p+q-k)\int d^4x\; e^{i\,p\cdot x}\\
&\qquad\times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\,P_+\,\gamma_\nu\, ig\int_\infty^{x^-} dy^-\;A^+(y^-)\,\psi(x)|P,S\rangle.
\end{aligned}\tag{7.8}$$

The second term in Eq. 7.1 gives

$$
\begin{aligned}
&- \int d^4p\, d^4k\, d^4p_1\, \delta^4(p+q-k)\, \mathrm{Tr}\left[\gamma_\mu \frac{(\not k - \not p_1 + m)}{(k-p_1)^2 - m^2 - i\epsilon}\, \gamma^- \Delta(k)\gamma_\nu \Phi_A^+(p-p_1,p)\right] \\
= \quad & \int d^4p\, d^4k\, \delta^4(p+q-k) \int d^4x\; e^{i\,p\cdot x} \\
& \times\langle P,S|\overline{\psi}(0)\, ig \int_\infty^0 dy^-\; A^+(y^-)\, \gamma_\mu\, P_-\, \Delta(k)\, \gamma_\nu\, \psi(x)|P,S\rangle
\end{aligned} \tag{7.9}
$$

The last two terms in Eq. 7.1 give

$$
\begin{aligned}
&- \int d^4p\, d^4k\, d^4k_1\, \delta^4(p+q-k) \Bigg\{ \mathrm{Tr}\left[\gamma_\nu \frac{(\not p - \not k_1 + m)}{(p-k_1)^2 - m^2 + i\epsilon}\, \gamma^+ \Phi(p)\gamma_\mu \Delta_A^-(k_1,k)\right] \\
&\qquad + \mathrm{Tr}\left[\gamma^+ \frac{(\not p - \not k_1 + m)}{(p-k_1)^2 - m^2 - i\epsilon}\, \gamma_\mu \Delta_A^-(k,k-k_1)\gamma_\nu \Phi(p)\right]\Bigg\} \\
= \quad & \int d^4p\, d^4k\, \delta^4(p+q-k) \int d^4x\; e^{i\,k\cdot x} \\
& \times\Bigg\{ \mathrm{Tr}\,\langle 0|\psi(x)\, a_h^\dagger a_h\, ig \int_{-\infty}^0 dy^+\; A^-(y^+)\, \overline{\psi}(0)\, \gamma_\nu\, P_+\, \Phi(p)\, \gamma_\mu|0\rangle \\
&\quad - \mathrm{Tr}\,\langle 0|\psi(x)\, ig \int_{-\infty}^{x^+} dy^+\; A^-(y^+)\, a_h^\dagger a_h\, \overline{\psi}(0)\, \gamma_\nu\, \Phi(p)\, P_-\, \gamma_\mu|0\rangle \Bigg\}.
\end{aligned}
$$

The result of multiple A^+- or A^--gluons together with the tree-level result gives in leading order in $1/Q$ (when the projectors P_+ and P_- don't matter) the exponentiated path-ordered result

$$
2M\, \mathscr{W}_{\mu\nu} = \int d^4p\, d^4k\, \delta^4(p+q-k)\; \mathrm{Tr}\left[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right] \tag{7.10}
$$

with

$$
\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4} \int d^4x\; e^{i\,p\cdot x} \langle P,S|\overline{\psi}_j(0)\mathscr{G}(0,\infty;0_T)\,\mathscr{G}(\infty,x^-;x_T)\psi_i(x)|P,S\rangle, \tag{7.11}
$$

$$
\Delta_{ij}(k,P_h,S_h) = \frac{1}{(2\pi)^4} \int d^4x\; e^{ik\cdot x}\,\langle 0|\mathscr{G}(-\infty,x^+;x_T)\psi_i(x)\, a_h^\dagger a_h\, \overline{\psi}_j(0)\mathscr{G}(0,-\infty;0_T)|0\rangle. \tag{7.12}
$$

Provided we assume that matrix elements containing bilocal operators $\overline{\psi}(0)\, A_T(y^\pm = \mp\infty, y_T)\, \psi(x)$ vanish for physical states, the above links can be connected resulting in a color gauge-invariant matrix element that must be used in the definition of the correlation functions.

Before considering the transverse gluons let us check the case of two A^+ gluons. For instance considering a gauge choice $A^- = 0$, one needs only to consider the absorption of the A^+ gluons in the 'distribution' part. Dressing the diagram leading to the first of the four terms above with another 'parallel' gluon one obtains a contribution

$$
\begin{aligned}
&\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{d^4p_1}{(2\pi)^4}\, \frac{d^4p_2}{(2\pi)^4} \int d^4x \int d^4y_1\, d^4y_2\; e^{i\,(p-p_1-p_2)\cdot x + i\,p_1\cdot y_1 + i\,p_2\cdot y_2} \\
&\quad \times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)\gamma^- \frac{\gamma^+}{(x_2 - i\epsilon)\, Q\sqrt{2}}\, \gamma^- \frac{\gamma^+}{(x_1+x_2-i\epsilon)\, Q\sqrt{2}}\, \gamma_\nu\, gA^+(y_2)\, gA^+(y_1)\, \psi(x)|P,S\rangle \\
= \quad & \int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{dx_1}{2\pi}\, \frac{dx_2}{2\pi} \int d^4x \int dy_1^-\, dy_2^-\; \frac{e^{-i\,(x_1+x_2)p^+(x^- - y_1^-)}\; e^{-i\,x_2 p^+(y_1^- - y_2^-)}}{(x_1+x_2-i\epsilon)\,(x_2-i\epsilon)}\; e^{i\,p\cdot x} \\
&\quad \times\langle P,S|\overline{\psi}(0)\gamma_\mu\Delta(k)P_+^2\, \gamma_\nu\, gA^+(y_2)\, gA^+(y_1)\, \psi(x)|P,S\rangle.
\end{aligned} \tag{7.13}
$$

The integration over x_1 and x_2 gives

$$
i\theta(y_1^- - x^-)\, i\theta(y_2^- - y_1^-), \tag{7.14}
$$

leading to the path ordering.

Dressing a diagram with an A_T^α-gluon with a longitudinal gluon leads to one contribution of the form

$$\begin{aligned}
&\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4} \int d^4x \int d^4y_1\, d^4y_2\; e^{i\,(p-p_1-p_2)\cdot x + i\,p_1\cdot y_1 + i\,p_2\cdot y_2} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\gamma^- \,\frac{\gamma^+}{(x_2-i\epsilon)\,Q\sqrt{2}}\,\gamma_\alpha\,\frac{\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA^+(y_2)\, gA_T^\alpha(y_1)\,\psi(x)|P,S\rangle \\
= &\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{dx_2}{2\pi} \int d^4x \int dy_2^-\; \frac{e^{-i\,x_2 p^+(x^- - y_2^-)}}{x_2 - i\epsilon}\, e^{i\,p\cdot x} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k) P_+\,\frac{\gamma_\alpha\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA^+(y_2)\, gA_T^\alpha(x)\,\psi(x)|P,S\rangle. \qquad (7.15)
\end{aligned}$$

The x_2 integration gives $\theta(y_2^- - x^-)$, the first term of the link. Note that the $A^+(y_1)A_T^\alpha(y_2)$ contribution vanishes at $\mathscr{O}(1/Q)$ because of the nonmatching Dirac structure.

As a final note of this section, we look for the contribution that combines with Eq. 7.15 into a covariant derivative. Since the term we consider is $\mathscr{O}(g^2)$ and $\mathscr{O}(1/Q)$ we expect the $i\partial_T^\alpha$ to be in the $\mathscr{O}(g)$ contribution also at $\mathscr{O}(1/Q)$. Again considering the $A^- = 0$ gauge, the only part at $\mathscr{O}(1/Q)$ that we sofar neglected is coming from the γ_T part of the fermion propagator instead of the γ^+ in Eq. 7.8,

$$\begin{aligned}
&-\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{d^4p_1}{(2\pi)^4} \int d^4x \int d^4y\; e^{i\,(p-p_1)\cdot x + i\,p_1\cdot y} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\gamma^- \,\frac{\gamma_\alpha(k^\alpha - p_1^\alpha)}{(x_1-i\epsilon)\,Q^2}\,\gamma_\nu\, gA^+(y)\,\psi(x)|P,S\rangle \\
= &\int d^4p\, d^4k\, \delta^4(p+q-k) \int \frac{dx_1}{2\pi} \int d^4x \int dy^-\; \frac{e^{-i\,x_1\,p^+(x^- - y^-)}}{x_1 - i\epsilon}\, e^{i\,p\cdot x} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\,\frac{\gamma_\alpha\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA^+(y)\, i\partial_T^\alpha\psi(x)|P,S\rangle \\
&+\int d^4p\, d^4k\, \delta^4(p+q-k)\,(k_T^\alpha - p_T^\alpha) \int \frac{dx_1}{2\pi} \int d^4x \int dy^-\; \frac{e^{-i\,x_1\,p^+(x^- - y^-)}}{x_1 - i\epsilon}\, e^{i\,p\cdot x} \\
&\qquad \times \langle P,S|\overline{\psi}(0)\gamma_\mu \Delta(k)\,\frac{\gamma_\alpha\gamma^-}{Q\sqrt{2}}\,\gamma_\nu\, gA^+(y)\,\psi(x)|P,S\rangle. \qquad (7.16)
\end{aligned}$$

We correctly obtain the first term in the link for the covariant derivative and the p_T^α-term in Eq. 7.7 (and for as far as we now performed the calculation, namely in $A^- = 0$ gauge the link contribution for the p_T-term arising from $g\,A_T = iD_T - i\partial_T$ in the fragmentation part).

7.2 The $\mathscr{O}(1/Q)$ contribution from transverse gluons

Up to $\mathscr{O}(1/Q)$ one needs to include the contributions of the handbag diagram, now calculated up to this order with in addition irreducible diagrams with one gluon coupling either to the soft part involving hadron H or the soft part involving hadron h. The expressions thus involve the quark-gluon correlation functions.

The full expression for the symmetric and antisymmetric parts of the hadronic tensor are,

$$
\begin{aligned}
2M\,\mathscr{W}_S^{\mu\nu} \;=\;& 2z_h \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\, \delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T) \\
&\times\Bigg\{ -g_\perp^{\mu\nu}\left[f_1 D_1 + g_{1s}G_{1s} + \frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}\, f_1 D_{1T}^\perp \right] \\
&- \frac{k_\perp^{\{\mu}p_\perp^{\nu\}} + (\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)\, g_\perp^{\mu\nu}}{M M_h}\, h_{1s}^\perp H_{1s}^\perp - \frac{k_\perp^{\{\mu}S_\perp^{\nu\}} + (\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp)\, g_\perp^{\mu\nu}}{M_h}\, h_{1T} H_{1s}^\perp \\
&- \frac{p_\perp^{\{\mu}S_{h\perp}^{\nu\}} + (\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu}}{M}\, h_{1s}^\perp H_{1T} - \left(S_\perp^{\{\mu}S_{h\perp}^{\nu\}} + (\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu}\right)\; h_{1T}H_{1T} \\
&- \frac{\left(k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho} + p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}\right)}{2MM_h}\, h_{1s}^\perp H_1^\perp - \frac{\left(k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho} + S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}\right)}{2M_h}\, h_{1T}H_1^\perp \\
&+ \frac{2\,\hat{t}^{\,\{\mu}k_\perp^{\nu\}}}{Q}\Bigg[-f_1 D_1 + f_1\frac{D^\perp}{z_h} - g_{1s}G_{1s} + g_{1s}\frac{G_s^\perp}{z_h} \\
&\qquad + \frac{M}{M_h}\, x_{\scriptscriptstyle B}\, h_s H_{1s}^\perp - \frac{m}{M_h}\, g_{1s}H_{1s}^\perp - \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1s}^\perp H_{1T} \\
&\qquad + \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1s}^\perp \frac{H_T^\perp}{z_h} - \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\, h_{1T}H_{1T} + \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\, h_{1T}\frac{H_T^\perp}{z_h}\Bigg] \\
&+ \frac{2\,\hat{t}^{\,\{\mu}p_\perp^{\nu\}}}{Q}\Bigg[x_{\scriptscriptstyle B}\, f^\perp D_1 + x_{\scriptscriptstyle B}\, g_s^\perp G_{1s} + \frac{M_h}{M}\, h_{1s}^\perp \frac{H_s}{z_h} - \frac{m}{M}\, h_{1s}^\perp G_{1s} + \frac{\boldsymbol{k}_\perp^2}{MM_h}\, h_{1s}^\perp H_{1s}^\perp \\
&\qquad + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1s}^\perp H_{1T} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, x_{\scriptscriptstyle B}\, h_T^\perp H_{1s}^\perp + \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\, x_{\scriptscriptstyle B}\, h_T^\perp H_{1T}\Bigg] \\
&+ \frac{2M\,\hat{t}^{\,\{\mu}S_\perp^{\nu\}}}{Q}\Bigg[x_{\scriptscriptstyle B}\, g_T' G_{1s} + \frac{M_h}{M}\, h_{1T}\frac{H_s}{z_h} - \frac{m}{M}\, h_{1T}G_{1s} + \frac{\boldsymbol{k}_\perp^2}{MM_h}\, h_{1T}H_{1s}^\perp \\
&\qquad - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\, x_{\scriptscriptstyle B}\, h_T^\perp H_{1s}^\perp + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, h_{1T}H_{1T} - \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M}\, x_{\scriptscriptstyle B}\, h_T^\perp H_{1T}\Bigg] \\
&+ \frac{2M_h\,\hat{t}^{\,\{\mu}S_{h\perp}^{\nu\}}}{Q}\Bigg[\frac{M}{M_h}\, x_{\scriptscriptstyle B}\, h_s H_{1T} + g_{1s}\frac{G_T'}{z_h} - \frac{m}{M_h}\, g_{1s}H_{1T} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\, h_{1s}^\perp H_{1T} \\
&\qquad - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\, h_{1s}^\perp\frac{H_T^\perp}{z_h} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, h_{1T}H_{1T} - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, h_{1T}\frac{H_T^\perp}{z_h}\Bigg] \\
&+ \frac{2\,\hat{t}^{\,\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{Q}\Bigg[\lambda_h f_1\frac{D_L^\perp}{z_h} + \frac{M}{M_h}\, x_{\scriptscriptstyle B}\, h_s H_1^\perp - \frac{m}{M_h}\, g_{1s}H_1^\perp \\
&\qquad - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\, f_1 D_{1T}^\perp + \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\, x_{\scriptscriptstyle B}\, f^\perp D_{1T}^\perp\Bigg] \\
&+ \frac{2\,\hat{t}^{\,\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho}}{Q}\Bigg[\frac{M_h}{M}\, h_{1s}^\perp\frac{H}{z_h} + \frac{\boldsymbol{k}_\perp^2}{MM_h}\, h_{1s}^\perp H_1^\perp + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, x_{\scriptscriptstyle B}\, h_T^\perp H_1^\perp\Bigg] \\
&+ \frac{2M\,\hat{t}^{\,\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho}}{Q}\Bigg[\frac{M_h}{M}\, h_{1T}\frac{H}{z_h} + \frac{\boldsymbol{k}_\perp^2}{MM_h}\, h_{1T}H_1^\perp - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\, x_{\scriptscriptstyle B}\, h_T^\perp H_1^\perp\Bigg] \\
&+ \frac{2M_h\,\hat{t}^{\,\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho}}{Q}\Bigg[f_1\frac{D_T}{z_h} + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\, f_1 D_{1T}^\perp - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M_h^2}\, x_{\scriptscriptstyle B}\, f^\perp D_{1T}^\perp\Bigg]\Bigg\} \qquad (7.17)
\end{aligned}
$$

and

$$
\begin{aligned}
2M\,\mathscr{W}_A^{\mu\nu} &= 2z_h \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \\
&\quad \times \Bigg\{ i\,\epsilon_\perp^{\mu\nu} \left[f_1 G_{1s} + g_{1s} D_1 \right] + i\frac{k_\perp^{[\mu} S_{h\perp}^{\nu]}}{M_h}\, g_{1s} D_{1T}^\perp \\
&\quad + i\frac{2\,\hat{t}^{[\mu} k_\perp^{\nu]}}{Q} \Bigg[-\frac{M}{M_h}\, x_B\, eH_1^\perp + \frac{m}{M_h}\, f_1 H_1^\perp - \lambda_h\, g_{1s} \frac{D_L^\perp}{z_h} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\, g_{1s} D_{1T}^\perp \\
&\qquad + \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\frac{m}{M}\, h_{1s}^\perp D_{1T}^\perp - \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\, x_B\, g_s^\perp D_{1T}^\perp \\
&\qquad - \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\frac{M}{M_h}\, x_B\, g_T' D_{1T}^\perp + \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp}\frac{m}{M_h}\, h_{1T} D_{1T}^\perp \Bigg] \\
&\quad + i\frac{2\,\hat{t}^{[\mu} p_\perp^{\nu]}}{Q}\frac{M_h}{M}\, h_{1s}^\perp \frac{E_s}{z_h} + i\frac{2M\,\hat{t}^{[\mu} S_\perp^{\nu]}}{Q}\frac{M_h}{M}\, h_{1T}\frac{E_s}{z_h} \\
&\quad + i\frac{2M_h\,\hat{t}^{[\mu} S_{h\perp}^{\nu]}}{Q} \Bigg[-g_{1s}\frac{D_T}{z_h} - \frac{\boldsymbol{k}_\perp^2}{M_h^2}\, g_{1s} D_{1T}^\perp - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M M_h}\frac{m}{M_h}\, h_{1s}^\perp D_{1T}^\perp \\
&\qquad + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{M_h^2}\, x_B\, g_s^\perp D_{1T}^\perp + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\frac{M}{M_h}\, x_B\, g_T' D_{1T}^\perp - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\frac{m}{M_h}\, h_{1T} D_{1T}^\perp \Bigg] \\
&\quad + i\frac{2\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} k_{\perp\rho}}{Q} \left[f_1\frac{G_s^\perp}{z_h} - f_1 G_{1s} + g_{1s}\frac{D^\perp}{z_h} - g_{1s} D_1 + \frac{M}{M_h}\, x_B\, eH_{1s}^\perp - \frac{m}{M_h}\, f_1 H_{1s}^\perp \right] \\
&\quad + i\frac{2\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} p_{\perp\rho}}{Q} \left[x_B\, f^\perp G_{1s} + x_B\, g_s^\perp D_1 + \frac{M_h}{M}\, h_{1s}^\perp\frac{E}{z_h} - \frac{m}{M}\, h_{1s}^\perp D_1 \right] \\
&\quad + i\frac{2M\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} S_{\perp\rho}}{Q} \left[x_B\, g_T' D_1 + \frac{M_h}{M}\, h_{1T}\frac{E}{z_h} - \frac{m}{M}\, h_{1T} D_1 \right] \\
&\quad + i\frac{2M_h\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} S_{h\perp\rho}}{Q} \left[f_1\frac{G_T'}{z_h} + \frac{M}{M_h}\, x_B\, eH_{1T} - \frac{m}{M_h}\, f_1 H_{1T} \right] \Bigg\}. \qquad (7.18)
\end{aligned}
$$

Note that one can make the replacement $\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} a_{\perp\rho} = \epsilon^{\mu\nu\rho\sigma} a_{\perp\rho}\hat{q}_\sigma$. The quark distribution functions in hadron H depend on x_B and $\boldsymbol{p}_T^2$, $f_1(x_B, \boldsymbol{p}_T^2)$ etc., while the quark fragmentation functions into hadron h depend on z_h and $-z_h\boldsymbol{k}_T$ (the perpendicular momentum of hadron h with respect to the quark), $D_1(z_h, -z_h\boldsymbol{k}_T)$ etc., and we use the shorthand notations

$$
\begin{aligned}
g_{1s} &= \lambda\, g_{1L} + \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\, g_{1T}, \\
G_{1s} &= \lambda_h\, G_{1L} + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_{h\perp}}{M_h}\, G_{1T}.
\end{aligned}
$$

The contractions with the lepton tensor, given in Table 7.1 use azimuthal angles defined with respect to the scattering plane and the (spacelike) virtual photon momentum,

$$\hat{\ell}\cdot a_\perp \;=\; -\hat{\ell}\cdot\boldsymbol{a}_\perp \;=\; -|\boldsymbol{a}_\perp|\,\cos(\phi_a), \qquad (7.19)$$

$$\hat{\ell}_\mu\epsilon_\perp^{\mu\nu} a_{\perp\nu} \;\equiv\; \hat{\ell}\wedge\boldsymbol{a}_\perp \;=\; |\boldsymbol{a}_\perp|\,\sin(\phi_a), \qquad (7.20)$$

where we have used $\boldsymbol{a}_\perp\wedge\boldsymbol{b}_\perp \equiv \epsilon_\perp^{\rho\sigma} a_{\perp\rho} b_{\perp\sigma}$. In many cases it is convenient to express the tensors in $\mathscr{W}^{\mu\nu}$

Table 7.1: Contractions of the lepton tensor $L_{\mu\nu}^{(\ell H)}$ with tensor structures appearing in the hadron tensor.

$w^{\mu\nu}$	$L_{\mu\nu} w^{\mu\nu}$
$-g_\perp^{\mu\nu}$	$\frac{4Q^2}{y^2}\left(1-y+\frac{1}{2}y^2\right)$
$a_\perp^{\{\mu}b_\perp^{\nu\}} - (a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu}$	$\frac{4Q^2}{y^2}\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\cos(\phi_a+\phi_b)$
$\frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} + b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right)$ $= a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} - (\epsilon_\perp^{\rho\sigma}a_{\perp\rho}b_{\perp\sigma})\,g_\perp^{\mu\nu}$ $= a_\perp^\mu\epsilon_\perp^{\nu\rho}b_{\perp\rho} + b_\perp^\mu\epsilon_\perp^{\nu\rho}a_{\perp\rho} + (a_\perp\cdot b_\perp)\,\epsilon_\perp^{\mu\nu}$	$-\frac{4Q^2}{y^2}\,(1-y)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\sin(\phi_a+\phi_b)$
$\hat{t}^{\,\{\mu}a_\perp^{\nu\}}$	$-\frac{4Q^2}{y^2}(2-y)\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\cos\phi_a$
$\hat{t}^{\,\{\mu}\,\epsilon_\perp^{\nu\}\rho}\,a_{\perp\rho}$	$\frac{4Q^2}{y^2}(2-y)\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\sin\phi_a$
$i\,\epsilon_\perp^{\mu\nu}$	$\lambda_e\,\frac{4Q^2}{y^2}\,y\left(1-\frac{y}{2}\right)$
$i\,a_\perp^{[\mu}b_\perp^{\nu]}$	$\lambda_e\,\frac{4Q^2}{y^2}\,y\left(1-\frac{y}{2}\right)\,\vert\boldsymbol{a}_\perp\vert\,\vert\boldsymbol{b}_\perp\vert\,\sin(\phi_b-\phi_a)$
$i\,\hat{t}^{\,[\mu}a_\perp^{\nu]}$	$-\lambda_e\,\frac{4Q^2}{y^2}\,y\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\sin\phi_a$
$i\,\hat{t}^{\,[\mu}\,\epsilon_\perp^{\nu]\rho}a_{\perp\rho}$	$-\lambda_e\,\frac{4Q^2}{y^2}\,y\sqrt{1-y}\,\vert\boldsymbol{a}_\perp\vert\,\cos\phi_a$

with respect to measured directions, i.e. $\hat{x} = -\hat{h}$ and $\hat{y}^\mu = \epsilon_\perp^{\mu\nu}\,\hat{x}_\nu$,

$$k_\perp^\mu - p_\perp^\mu \;=\; -P_{h\perp}/z \;=\; -Q_T\,\hat{h}^\mu, \tag{7.21}$$

$$a_\perp^\mu \;=\; (\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)\,\hat{h}^\mu + (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)\,\epsilon_\perp^{\mu\rho}\hat{h}_\rho, \tag{7.22}$$

$$\epsilon_\perp^{\mu\rho}\,a_{\perp\rho} \;=\; -(\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)\,\hat{h}^\mu + (\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)\,\epsilon_\perp^{\mu\rho}\,\hat{h}_\rho, \tag{7.23}$$

$$\begin{aligned} a_\perp^{\{\mu}b_\perp^{\nu\}} - (a_\perp\cdot b_\perp)\,g_\perp^{\mu\nu} \;=\; & \Big[2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp) - (\boldsymbol{a}_\perp\cdot\boldsymbol{b}_\perp)\Big]\;\Big(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\Big) \\ & + \Big[(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\wedge\boldsymbol{b}_\perp) + (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp)\Big]\;\hat{h}^{\{\mu}\epsilon_\perp^{\nu\}\rho}\hat{h}_\rho, \end{aligned} \tag{7.24}$$

$$\begin{aligned} \frac{1}{2}\left(a_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}b_{\perp\rho} + b_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}a_{\perp\rho}\right) \;=\; & -\Big[(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\wedge\boldsymbol{b}_\perp) + (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp)\Big]\;\Big(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\Big) \\ & + \Big[2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp) - (\boldsymbol{a}_\perp\cdot\boldsymbol{b}_\perp)\Big]\;\hat{h}^{\{\mu}\epsilon_\perp^{\nu\}\rho}\hat{h}_\rho, \end{aligned} \tag{7.25}$$

$$a_\perp^{[\mu}b_\perp^{\nu]} \;=\; \Big[(\hat{\boldsymbol{h}}\cdot\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\wedge\boldsymbol{b}_\perp) - (\hat{\boldsymbol{h}}\wedge\boldsymbol{a}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{b}_\perp)\Big]\;\hat{h}^{[\mu}\epsilon_\perp^{\nu]\rho}\hat{h}_\rho = (\boldsymbol{a}_\perp\wedge\boldsymbol{b}_\perp)\,\epsilon_\perp^{\mu\nu}. \tag{7.26}$$

This allows one to pull the tensor structure outside the integration over transverse momenta. A useful relation is $\hat{\boldsymbol{h}}\cdot(\boldsymbol{p}_\perp + \boldsymbol{k}_\perp) = (\boldsymbol{p}_\perp^2 - \boldsymbol{k}_\perp^2)/Q_T$.

In the next sections we will consider the cases of production of unpolarized and polarized leptoproduction separately, that means either one sums over all final state configurations of, say, a produced particle, or one determines from the final state configurations the spin vector S_h (characterized by λ_h and S_{hT}) that determines the production matrix. For each of the above cases we will consider separately the case of unpolarized (O) and longitudinally polarized (L) leptons and of unpolarized (O), longitudinally polarized (L) or transversely polarized initial hadron state (T).

7.3 Lepto-production integrated over transverse momenta

The hadronic tensor simplifies to

$$
\begin{aligned}
2M \int d^2P_{h\perp}\, \mathcal{W}^{\mu\nu} \;=\;& 2z_h \Bigg\{ -g_\perp^{\mu\nu} \Big[f_1 D_1 + \lambda\lambda_h\, g_1 G_1\Big] - \left(S_\perp^{\{\mu} S_{h\perp}^{\nu\}} + (\boldsymbol{S}_\perp \cdot \boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu}\right)\, h_1 H_1 \\
& + i\,\epsilon_\perp^{\mu\nu} \Big[f_1 \lambda_h\, G_1 + \lambda\, g_1 D_1 \Big] \\
& + \lambda_h\, \frac{2M\, \hat t^{\,\{\mu} S_\perp^{\nu\}}}{Q} \left[x_{\scriptscriptstyle B}\, g_T G_1 + \frac{M_h}{M}\, h_1 \left(\frac{H_L}{z_h} + H_{1L}^{\perp(1)} - \frac{m}{M_h}\, G_1 \right) \right] \\
& + \lambda\, \frac{2M_h\, \hat t^{\,\{\mu} S_{h\perp}^{\nu\}}}{Q} \left[\frac{M}{M_h}\, x_{\scriptscriptstyle B}\, h_L H_1 + g_1 \left(\frac{G_T}{z_h} - G_{1T}^{(1)} - \frac{m}{M_h}\, H_1 \right) \right] \\
& + \frac{2M_h\, \hat t^{\,\{\mu} \epsilon_\perp^{\nu\}\rho} S_{\perp\rho}}{Q}\, h_1 \left(\frac{H}{z_h} + H_1^{\perp(1)} \right) + \frac{2M_h\, \hat t^{\,\{\mu} \epsilon_\perp^{\nu\}\rho}\, S_{h\perp\rho}}{Q}\, f_1 \left(\frac{D_T}{z_h} + D_{1T}^{\perp(1)} \right) \\
& + i\lambda_h\, \frac{2M \hat t^{\,[\mu} S_\perp^{\nu]}}{Q}\, \frac{M_h}{M}\, h_1 \frac{E_L}{z_h} - i\lambda\, \frac{2M_h\, \hat t^{\,[\mu} S_{h\perp}^{\nu]}}{Q}\, g_1 \left(\frac{D_T}{z_h} + D_{1T}^{\perp(1)} \right) \\
& + i \frac{2M_h\, \hat t^{\,[\mu} \epsilon_\perp^{\nu]\rho} S_{h\perp\rho}}{Q} \left[\frac{M}{M_h}\, 2x_{\scriptscriptstyle B}\, e H_1 + f_1 \left(\frac{G_T}{z_h} - G_{1T}^{(1)} - \frac{m}{M_h}\, H_1 \right) \right] \\
& + i \frac{2M\, \hat t^{\,[\mu} \epsilon_\perp^{\nu]\rho} S_{\perp\rho}}{Q} \left[x_{\scriptscriptstyle B}\, g_T D_1 + \frac{M_h}{M}\, h_1 \left(\frac{E}{z_h} - \frac{m}{M_h}\, D_1 \right) \right] \Bigg\}, \qquad (7.27)
\end{aligned}
$$

where the quark distribution functions in hadron H depend on $x_{\scriptscriptstyle B}$, while the quark fragmentation functions into hadron h depend on z_h.

7.3.1 Unpolarized leptons and hadrons

The relevant part of the hadronic tensor is

$$
2M \int d^2P_{h\perp}\, \mathcal{W}^{\mu\nu} = -g_\perp^{\mu\nu}\, 2z\, f_1(x_{\scriptscriptstyle B}) D_1(z) + \frac{2M_h\, \hat t^{\,\{\mu} \epsilon_\perp^{\nu\}\rho}\, S_{h\perp\rho}}{Q}\, 2z\, f_1(x_{\scriptscriptstyle B}) \left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z) \right). \qquad (7.28)
$$

The semi-inclusive cross section is given by

$$
\begin{aligned}
\frac{d\sigma_{OO}(\ell H \to \ell' \vec h X)}{dx_{\scriptscriptstyle B}\, dy\, dz} =& \frac{4\pi\alpha^2\, s}{Q^4} \Bigg\{ \left(\frac{y^2}{2} + 1 - y \right) x_{\scriptscriptstyle B} f_1(x_{\scriptscriptstyle B})\, D_1(z) \\
& + 2|\boldsymbol{S}_{h\perp}|\, (2-y)\sqrt{1-y}\, \sin(\phi_s^h)\, \frac{M_h}{Q}\, x_{\scriptscriptstyle B}\, f_1(x_{\scriptscriptstyle B}) \left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z) \right) \Bigg\}. \qquad (7.29)
\end{aligned}
$$

We remind that $D_{1T}^{\perp(1)}$ is related to D_T via

$$
D_{1T}^{\perp(1)}(z) = -z \int_z^1 dy\, \frac{D_T(y)}{y^3}, \qquad (7.30)
$$

or equivalently

$$
\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z) = z \frac{d}{dz}\, D_{1T}^{\perp(1)}. \qquad (7.31)
$$

7.3.2 Unpolarized leptons and longitudinally polarized hadrons

The relevant part of the hadronic tensor is

$$
\begin{aligned}
2M \int d^2P_{h\perp}\, \mathcal{W}^{\mu\nu} =& -g_\perp^{\mu\nu}\, \lambda\, \lambda_h\, 2z\, g_1(x_{\scriptscriptstyle B}) G_1(z) \\
& + \frac{2M_h\, \hat t^{\,\{\mu} S_{h\perp}^{\nu\}}}{Q}\, \lambda \left[\frac{M}{M_h}\, 2x_{\scriptscriptstyle B} z\, h_L(x_{\scriptscriptstyle B}) H_1(z) + 2z\, g_1(x_{\scriptscriptstyle B}) \frac{\tilde G_T(z)}{z} \right], \qquad (7.32)
\end{aligned}
$$

where

$$\frac{\tilde{G}_T(z)}{z} = \frac{G_T(z)}{z} - G_{1T}^{(1)}(z) - \frac{m}{M_h}\,H_1(z) \tag{7.33}$$

is a pure quark-quark-gluon (twist-three) matrix element. The semi-inclusive cross section is given by

$$\begin{aligned}\frac{d\Delta\sigma_{OL}(\ell\vec{H}\to\ell'\vec{h}X)}{dx_B\,dy\,dz} &= \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\left\{\lambda_h\,\left(\frac{y^2}{2}+1-y\right)\,x_B\,g_1(x_B)\,G_1(z)\right.\\ &\left. -2|\boldsymbol{S}_{h\perp}|\,(2-y)\sqrt{1-y}\,\cos(\phi_s^h)\left[\frac{M}{Q}\,x_B^2\,h_L(x_B)H_1(z)+\frac{M_h}{Q}\,x_B\,g_1(x_B)\frac{\tilde{G}_T(z)}{z}\right]\right\}\end{aligned} \tag{7.34}$$

where $\Delta\sigma_{OL}$ indicates that only the part involving polarization is given, which can be extracted as a cross section difference, e.g. between λ = 1 and λ = -1.

7.3.3 Unpolarized leptons and transversely polarized hadrons

The relevant part of the hadronic tensor is

$$\begin{aligned}2M\int d^2P_{h\perp}\,\mathscr{W}^{\mu\nu} &= -\left(S_\perp^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})\,g_\perp^{\mu\nu}\right)\,2z\,h_1(x_B)H_1(z)\\ &+\frac{2M\,\hat{t}^{\{\mu}S_\perp^{\nu\}}}{Q}\,\lambda_h\left[2x_Bz\,g_T(x_B)G_1(z)+\frac{M_h}{M}\,4z\,h_1(x_B)\frac{\tilde{H}_L(z)}{z}\right]\\ &+\frac{2M_h\,\hat{t}^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho}}{Q}\,2z\,h_1(x_B)\frac{\tilde{H}(z)}{z},\end{aligned} \tag{7.35}$$

where

$$\frac{\tilde{H}_L(z)}{z} = \frac{H_L(z)}{z} + \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{2M_h^2}\,H_{1L}^\perp(z,-z\boldsymbol{k}_T) - \frac{m}{M_h}\,G_1(z) \tag{7.36}$$

$$\frac{\tilde{H}(z)}{z} = \frac{H(z)}{z} + \int d^2\boldsymbol{k}_T\,\frac{\boldsymbol{k}_T^2}{M_h^2}H_1^\perp(z,-z\boldsymbol{k}_T) \tag{7.37}$$

are pure quark-quark-gluon matrix elements. The semi-inclusive cross section is given by

$$\begin{aligned}\frac{d\Delta\sigma_{OT}(\ell\vec{H}\to\ell'\vec{h}X)}{dx_B\,dy\,dz} &= \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\left\{2\,(2-y)\sqrt{1-y}\,\sin(\phi_s)\,\frac{M_h}{Q}\,x_B\,h_1(x_B)\,\frac{\tilde{H}(z)}{z}\right.\\ &-2\lambda_h\,(2-y)\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T(x_B)G_1(z)+\frac{M_h}{Q}\,x_B\,h_1(x_B)\frac{\tilde{H}_L(z)}{z}\right]\\ &\left.-|\boldsymbol{S}_{h\perp}|\,(1-y)\,\cos(\phi_s+\phi_s^h)\,x_B\,h_1(x_B)H_1(z)\right\}.\end{aligned} \tag{7.38}$$

7.3.4 Polarized leptons and unpolarized hadrons

The relevant part of the hadronic tensor is

$$\begin{aligned}2M\int d^2P_{h\perp}\,\mathscr{W}^{\mu\nu} &= i\,\epsilon_\perp^{\mu\nu}\,\lambda_h\,2z\,f_1(x_B)G_1(z)\\ &+i\frac{2M_h\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho}S_{h\perp\rho}}{Q}\left[\frac{M}{M_h}\,2x_Bz\,e(x_B)H_1(z)+2z\,f_1(x_B)\frac{\tilde{G}_T(z)}{z}\right].\end{aligned} \tag{7.39}$$

The semi-inclusive cross section is given by

$$\frac{d\Delta\sigma_{LO}(\vec{\ell}H \to \ell'\vec{h}X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e \Bigg\{ \lambda_h\,y\,\left(1-\frac{y}{2}\right)\,x_B\,f_1(x_B)\,G_1(z)$$
$$-2\,|\boldsymbol{S}_{h\perp}|\,y\sqrt{1-y}\,\cos(\phi_s^h)\left[\frac{M}{Q}\,x_B^2\,e(x_B)H_1(z) + \frac{M_h}{Q}\,x_B\,f_1(x_B)\frac{\tilde{G}_T(z)}{z}\right]\Bigg\}. \quad (7.40)$$

7.3.5 Polarized leptons and longitudinally polarized hadrons

The relevant part of the hadronic tensor is

$$2M\int d^2P_{h\perp}\,\mathscr{W}^{\mu\nu} = i\,\epsilon_\perp^{\mu\nu}\,\lambda\,2z\,g_1(x_B)D_1 - i\frac{2M_h\,\hat{t}^{\,[\mu}S_{h\perp}^{\nu]}}{Q}\,\lambda\,2z\,g_1(x_B)\,\left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z)\right). \quad (7.41)$$

The semi-inclusive cross section is given by

$$\frac{d\Delta\sigma_{LL}(\vec{\ell}\vec{H} \to \ell'\vec{h}X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,\lambda\Bigg\{ y\,\left(1-\frac{y}{2}\right)\,x_B g_1(x_B)\,D_1(z)$$
$$+2\,|\boldsymbol{S}_{h\perp}|\,y\sqrt{1-y}\,\sin(\phi_s^h)\,\frac{M_h}{Q}\,x_B\,g_1(x_B)\,\left(\frac{D_T}{z_h}(z) + D_{1T}^{\perp(1)}(z)\right)\Bigg\}. \quad (7.42)$$

7.3.6 Polarized leptons and transversely polarized hadrons

The relevant part of the hadronic tensor is

$$2M\int d^2P_{h\perp}\,\mathscr{W}^{\mu\nu} = i\frac{2M_h\hat{t}^{\,[\mu}S_\perp^{\nu]}}{Q}\,\lambda_h\,2z\,h_1(x_B)\frac{E_L(z)}{z}$$
$$+i\frac{2M\,\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}}{Q}\left[2x_B z\,g_T(x_B)D_1(z) + \frac{M_h}{M}\,2z\,h_1(x_B)\frac{\tilde{E}(z)}{z}\right], \quad (7.43)$$

where

$$\frac{\tilde{E}(z)}{z} = \frac{E(z)}{z} - \frac{m}{M_h}\,D_1(z). \quad (7.44)$$

The semi-inclusive cross section is given by

$$\frac{d\Delta\sigma_{LT}(\vec{\ell}\vec{H} \to \ell'\vec{h}X)}{dx_B\,dy\,dz} =$$
$$\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,|\boldsymbol{S}_\perp|\Bigg\{-2\,y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T(x_B)D_1(z) + \frac{M_h}{Q}\,x_B\,h_1(x_B)\frac{\tilde{E}(z)}{z}\right]$$
$$-2\,\lambda_h\,y\sqrt{1-y}\,\sin(\phi_s)\,\frac{M_h}{Q}\,x_B\,h_1(x_B)\,\frac{\tilde{E}_L(z)}{z}\Bigg\}. \quad (7.45)$$

7.4 Transverse momenta in lepto-production without polarization in the final state

In the case that no polarization in the final state is observed or for the case that a spin 0 particle is produced (e.g. semi-inclusive leptopionproduction, the hadronic tensor simplifies to:

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} = & \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T \times \Bigg\{ -g_\perp^{\mu\nu}\, 2z\, f_1 D_1 + i\,\epsilon_\perp^{\mu\nu}\, 2z\, g_{1s} D_1 \\
& + \frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}p_\perp^{\nu\}} + p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2MM_h} 2z h_{1s}^\perp H_1^\perp + \frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}S_\perp^{\nu\}} + S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2M_h} 2z h_{1T} H_1^\perp \\
& + \frac{\hat{t}^{\{\mu}k_\perp^{\nu\}}}{Q}\left[-4z\, f_1 D_1 + 4\, f_1 D^\perp\right] + \frac{\hat{t}^{\{\mu}p_\perp^{\nu\}}}{Q}\, 4xz\, f^\perp D_1 \\
& + \frac{k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[-\frac{M}{M_h}\, 4xz\, h_s H_1^\perp + \frac{m}{M_h}\, 4z\, g_{1s} H_1^\perp\right] \\
& + \frac{p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[-\frac{M_h}{M}\, 4\, h_{1s}^\perp H - \frac{\boldsymbol{k}_\perp^2}{MM_h}\, 4z\, h_{1s}^\perp H_1^\perp - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, 4xz\, h_T^\perp H_1^\perp\right] \\
& + \frac{M\, S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[-\frac{M_h}{M}\, 4\, h_{1T} H - \frac{\boldsymbol{k}_\perp^2}{MM_h}\, 4z\, h_{1T} H_1^\perp + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\, 4xz\, h_T^\perp H_1^\perp\right] \\
& + i\frac{\hat{t}^{[\mu}k_\perp^{\nu]}}{Q}\left[-\frac{M}{M_h}\, 4xz\, e H_1^\perp + \frac{m}{M_h}\, 4z\, f_1 H_1^\perp\right] \\
& + i\frac{k_{\perp\rho}\epsilon_\perp^{\rho[\mu}\hat{t}^{\nu]}}{Q}\left[4\, g_{1s} D^\perp - 4z\, g_{1s} D_1\right] \\
& + i\frac{p_{\perp\rho}\epsilon_\perp^{\rho[\mu}\hat{t}^{\nu]}}{Q}\left[4xz\, g_s^\perp D_1 + \frac{M_h}{M}\, 4\, h_{1s}^\perp E - \frac{m}{M}\, 4z\, h_{1s}^\perp D_1\right] \\
& + i\frac{M\, S_{\perp\rho}\epsilon_\perp^{\rho[\mu}\hat{t}^{\nu]}}{Q}\left[4xz\, g_T' D_1 + \frac{M_h}{M}\, 4\, h_{1T} E - \frac{m}{M}\, 4z\, h_{1T} D_1\right]\Bigg\} \qquad (7.46)
\end{aligned}
$$

We will again consider all situations separately for different polarizations of the lepton and target hadrons.

7.4.1 $\boxed{\ell H \to \ell' h X}$ (unpolarized hadrons)

The hadronic tensor is given by

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} = & \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \times \Bigg\{ -g_\perp^{\mu\nu}\, 2z\, f_1 D_1 \\
& + \frac{\hat{t}^{\{\mu}k_\perp^{\nu\}}}{Q}\left[-4z\, f_1 D_1 + 4\, f_1 D^\perp\right] + \frac{\hat{t}^{\{\mu}p_\perp^{\nu\}}}{Q}\, 4xz\, f^\perp D_1 \Bigg\} \\
= & -g_\perp^{\mu\nu}\, 2z\, I[f_1 D_1] \\
& + \frac{\hat{t}^{\{\mu}\hat{h}^{\nu\}}}{Q}\left(4z\, I\left[(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)\, f_1\, \left(\frac{D^\perp}{z} - D_1\right)\right] + 4xz\, I\left[(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)\, f^\perp D_1\right]\right), \qquad (7.47)
\end{aligned}
$$

where the last expression involves integrals of the type

$$
I\left[(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)\, f^\perp D_1\right](x_B, z) = \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\, (\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\, f^\perp(x_B, \boldsymbol{p}_T) D_1(z, -z\boldsymbol{k}_T), \qquad (7.48)
$$

with $\boldsymbol{q}_T = -\boldsymbol{P}_{h\perp}/z = Q_T\,\hat{\boldsymbol{h}}$. Note that a contribution proportional to $\hat{h}_\rho\epsilon^{\rho\{\mu}\hat{t}^{\nu\}}$ appears, but it is multiplied with integrals of the type $I[(\hat{\boldsymbol{h}}\wedge\boldsymbol{p}_\perp) f_1(x_B, |\boldsymbol{p}_\perp|) D(z, z|Q_T\,\hat{\boldsymbol{h}} - \boldsymbol{p}_\perp|)]$, which vanish.

The most explicit differential cross section is immediately obtained from the first expression for the hadronic tensor and is given by

$$\frac{d\sigma_{OO}(\ell + H \to \ell' + h + \text{jet} + X)}{dx_B\, dy\, dz\, d^2\boldsymbol{P}_{h\perp}\, d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\, s}{Q^4} \left\{ \left(\frac{y^2}{2} + 1 - y \right) x_B f_1\, D_1 + 2(2-y)\sqrt{1-y}\, \frac{|\boldsymbol{P}_{h\perp}|}{zQ}\, \cos(\phi_h)\; x_B f_1\, \left(\frac{D^\perp}{z} - D_1 \right) - 2(2-y)\sqrt{1-y}\, \frac{|\boldsymbol{p}_\perp|}{Q}\, \cos(\phi_j)\, \left(x_B^2 f^\perp D_1 - x_B f_1 D_1 + x_B f_1 \frac{D^\perp}{z} \right) \right\}, \tag{7.49}$$

where the arguments of distribution and fragmentation functions are $f_1(x_B, \boldsymbol{p}_\perp)$, $D_1(z, \boldsymbol{P}_{h\perp} - z\boldsymbol{p}_\perp)$, etc.

The semi-inclusive cross section where the jet direction is not determined can be found also from the (first) most general expression above, but this is cumbersome, since one must be aware that integrating over $d^2\boldsymbol{p}_\perp$, the argument of the fragmentation functions depend on $\boldsymbol{P}_{h\perp} - z\boldsymbol{p}_\perp$. It is easier to start with the second expression for the hadronic tensor and obtain

$$\frac{d\sigma_{OO}(\ell + H \to \ell' + h + X)}{dx_B\, dy\, dz\, d^2\boldsymbol{P}_{h\perp}} = \frac{4\pi\alpha^2\, s}{Q^4} \left\{ \left(\frac{y^2}{2} + 1 - y \right) I[x_B f_1\, D_1] - 2(2-y)\sqrt{1-y}\, \cos(\phi_h) \left(\frac{M_h}{Q}\, I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\, x_B f_1\, \left(\frac{D^\perp}{z} - D_1 \right) \Big] + \frac{M}{Q}\, I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\, x_B^2 f^\perp D_1 \Big] \right) \right\}, \tag{7.50}$$

which involves the above defined convolutions over distribution and fragmentation functions.

Returning to the previous cross section, one can integrate over the transverse momenta of the produced hadrons in the jet and find

$$\frac{d\sigma_{OO}(\ell + H \to \ell' + h + \text{jet} + X)}{dx_B\, dy\, dz\, d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\, s}{Q^4} \left\{ \left(\frac{y^2}{2} + 1 - y \right) x_B f_1(x_B, \boldsymbol{p}_\perp)\, D_1(z) - 2(2-y)\sqrt{1-y}\, \frac{|\boldsymbol{p}_\perp|}{Q}\, \cos(\phi_j)\; x_B^2 f^\perp(x_B, \boldsymbol{p}_\perp) D_1(z) \right\}, \tag{7.51}$$

where the arguments of distribution and fragmentation functions are $f_1(x_B, \boldsymbol{p}_\perp)$, $D_1(z)$, etc. Integrating also over the transverse momentum of the jet we obtain the result discussed earlier,

$$\frac{d\sigma_{OO}(\ell + H \to \ell' + X)}{dx_B\, dy\, dz} = \frac{4\pi\alpha^2\, s}{Q^4} \left(\frac{y^2}{2} + 1 - y \right) x_B f_1(x_B) D_1(z).$$

7.4.2 $\boxed{\ell\vec{H} \to \ell' h X}$ (longitudinally polarized target)

The hadronic tensor is given by

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} &= \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \\
&\quad \times\lambda \Bigg\{ \frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}p_\perp^{\nu\}} + p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}k_\perp^{\nu\}}\right)}{2MM_h}\, 2z h_{1L}^\perp H_1^\perp \\
&\quad + \frac{k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[-\frac{M}{M_h}\,4xz\,h_L H_1^\perp + \frac{m}{M_h}\,4z\,g_{1L}H_1^\perp\right] \\
&\quad + \frac{p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[-\frac{M_h}{M}\,4\,h_{1L}^\perp H - \frac{\boldsymbol{k}_\perp^2}{MM_h}\,4z\,h_{1L}^\perp H_1^\perp\right]\Bigg\}. \\
&= \lambda\Bigg\{\hat{h}_\rho\epsilon_\perp^{\rho\{\mu}\hat{h}^{\nu\}}\,2z\,I\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp) - \boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\,h_{1L}^\perp H_1^\perp\right] \\
&\quad - \hat{h}_\rho\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}\Bigg(\frac{M}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,\left(x\,h_L - \frac{m}{M}\,g_{1L}\right)\,H_1^\perp\Big] \\
&\quad + \frac{M_h}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,h_{1L}^\perp\,\left(\frac{H}{z} + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\Big]\Bigg)\Bigg\}. \qquad (7.52)
\end{aligned}$$

The most explicit differential cross section is given by

$$\begin{aligned}
&\frac{d\Delta\sigma_{OL}(\ell + \vec{H} \to \ell' + h + \text{jet} + X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\Bigg\{-(1-y)\,\frac{|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{p}_\perp|}{zMM_h}\,\sin(\phi_h + \phi_j)\,x_B\,h_{1L}^\perp H_1^\perp \\
&\quad + (1-y)\,\frac{\boldsymbol{p}_\perp^2}{MM_h}\,\sin(2\phi_j)\,x_B\,h_{1L}^\perp H_1^\perp \\
&\quad - 2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{P}_{h\perp}|}{zQ}\,\sin\phi_h\,\frac{M}{M_h}\,x_B\,\left(x_B h_L - \frac{m}{M}\,g_{1L}\right)\,H_1^\perp \\
&\quad + 2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\sin\phi_j\,\Bigg[\frac{M}{M_h}\,x_B\,\left(x_B h_L - \frac{m}{M}\,g_{1L}\right)\,H_1^\perp \\
&\quad + \frac{M_h}{M}\,x_B h_{1L}^\perp\,\left(\frac{H}{z} + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\Bigg]\Bigg\}, \qquad (7.53)
\end{aligned}$$

where the arguments of distribution and fragmentation functions are $h_{1L}^\perp(x_B, \boldsymbol{p}_\perp)$, etc. and $H_1^\perp(z, \boldsymbol{P}_{h\perp} - z\boldsymbol{p}_\perp)$, etc. The cross section averaged over the jet angle is found from the second form for $\mathcal{W}^{\mu\nu}$,

$$\begin{aligned}
&\frac{d\Delta\sigma_{OL}(\ell + \vec{H} \to \ell' + h + X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\Bigg\{(1-y)\,\sin(2\phi_h)\,I\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp) - \boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\,x_B h_{1L}^\perp H_1^\perp\right] \\
&\quad + 2(2-y)\sqrt{1-y}\,\sin\phi_h\,\Bigg(\frac{M}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_B\,\left(x_B h_L - \frac{m}{M}\,g_{1L}\right)\,H_1^\perp\Big] \\
&\quad + \frac{M_h}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x_B h_{1L}^\perp\,\left(\frac{H}{z} + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\,H_1^\perp\right)\Big]\Bigg)\Bigg\}. \qquad (7.54)
\end{aligned}$$

Integrating over the transverse momenta of the produced hadrons in the jet one obtains

$$\frac{d\Delta\sigma_{OL}(\ell+\vec{H}\rightarrow \ell'+h+\mathrm{jet}+X)}{dx_{B}\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda\left\{+2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\sin\phi_j\,\frac{M_h}{M}\,x_{B}h_{1L}^{\perp}(x_{B},\boldsymbol{p}_\perp)\frac{\tilde{H}(z)}{z}\right\}, \qquad (7.55)$$

where $h_{1L}^{\perp}(x_{B},\boldsymbol{p}_\perp)$ is a 'leading twist' distribution and the fragmentation function $\tilde{H}(z)$,

$$\frac{\tilde{H}(z)}{z} = \int d^2\boldsymbol{k}_{T}\,\frac{\tilde{H}(z,\boldsymbol{k}_{T})}{z} = \int d^2\boldsymbol{k}_{T}\,\left(\frac{H}{z}+\frac{\boldsymbol{k}_{T}^2}{M_h^2}\,H_1^{\perp}\right), \qquad (7.56)$$

is a 'twist three' quark-quark-gluon matrix element.

7.4.3 $\boxed{\ell\vec{H} \to \ell' h X}$ (transversely polarized target)

The hadronic tensor is given by

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} &= \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \\
&\quad \times \Bigg\{ \frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu} p_\perp^{\nu\}} + p_{\perp\rho}\epsilon_\perp^{\rho\{\mu} k_\perp^{\nu\}}\right)}{2MM_h}\, \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\, 2z h_{1T}^\perp H_1^\perp \\
&\quad + \frac{\left(k_{\perp\rho}\epsilon_\perp^{\rho\{\mu} S_\perp^{\nu\}} + S_{\perp\rho}\epsilon_\perp^{\rho\{\mu} k_\perp^{\nu\}}\right)}{2M_h}\, 2z h_{1T} H_1^\perp \\
&\quad - \frac{k_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\, \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, 4z\, \left(x\, h_T - \frac{m}{M}\, g_{1T}\right)\, H_1^\perp \\
&\quad - \frac{p_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[\frac{M_h}{M}\, \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\, 4\, h_{1T}^\perp\, \left(H + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\, zH_1^\perp\right) + \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h}\, 4xz\, h_T^\perp H_1^\perp\right] \\
&\quad - \frac{M\, S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}}{Q}\left[\frac{M_h}{M}\, 4\, h_{1T}\, \left(H + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\, zH_1^\perp\right) - \frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}\, 4xz\, h_T^\perp H_1^\perp\right]\Bigg\} \\
&= \frac{1}{2}\left(\hat{h}_\rho\epsilon_\perp^{\rho\{\mu} S_\perp^{\nu\}} + S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{h}_\perp^{\nu\}}\right)\, I\Big[\frac{(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{M_h}\, 2z\, h_1 H_1^\perp\Big] \\
&\quad + \hat{h}_\rho\epsilon_\perp^{\rho\{\mu}\hat{h}^{\nu\}}\, (\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\, I\Big[\frac{4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp) - \boldsymbol{p}_\perp^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp) - 2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)}{2\,M^2\,M_h}\, 2z\, h_{1T}^\perp H_1^\perp\Big] \\
&\quad + \left(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\right)\, (\hat{\boldsymbol{h}}\wedge\boldsymbol{S}_\perp) \\
&\qquad \times I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp) + \boldsymbol{p}_\perp^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp) - 4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{2\,M^2\,M_h}\, 2z\, h_{1T}^\perp H_1^\perp\Big] \\
&\quad - S_{\perp\rho}\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}\Bigg(\frac{M}{Q}\, I\left[\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{2MM_h}\, 4z\, \left(x_B h_T - \frac{m}{M}\, g_{1T} - x_B h_T^\perp\right)\, H_1^\perp\right] \\
&\qquad\qquad + \frac{M_h}{Q}\, I\left[4z\, h_1\, \left(\frac{H}{z} + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\, H_1^\perp\right)\right]\Bigg) \\
&\quad - h_\rho\epsilon_\perp^{\rho\{\mu}\hat{t}^{\nu\}}\, (\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\, \Bigg(\frac{M_h}{Q}\, I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2 - \boldsymbol{p}_\perp^2}{2M^2}\, 4z\, h_{1T}^\perp\, \left(\frac{H}{z} + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\, H_1^\perp\right)\Big] \\
&\qquad\qquad + \frac{M}{Q}\, I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp) - \boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{2MM_h}\, 4z\, \left(x_B h_T - \frac{m}{M}\, g_{1T} + x_B h_T^\perp\right)\, H_1^\perp\Big]\Bigg) \\
&\quad - \hat{h}^{\{\mu}\hat{t}^{\nu\}}\, (\hat{\boldsymbol{h}}\wedge\boldsymbol{S}_\perp)\, \Bigg(\frac{M_h}{Q}\, I\Big[\frac{\boldsymbol{p}_\perp^2 - 2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2}{2M^2}\, 4z\, h_{1T}^\perp\, \left(\frac{H}{z} + \frac{\boldsymbol{k}_\perp^2}{M_h^2}\, H_1^\perp\right)\Big] \\
&\qquad\qquad + \frac{M}{Q}\, I\Big[\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp - 2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)}{2MM_h}\, 4z\, \left(x_B h_T - \frac{m}{M}\, g_{1T} + x_B h_T^\perp\right)\, H_1^\perp\Big]\Bigg)
\end{aligned}
\tag{7.57}
$$

The most explicit differential cross section is given by

$$\begin{aligned}
\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp| \Bigg\{ &(1-y)\frac{|\boldsymbol{p}_\perp|}{M_h}\,\sin(\phi_j+\phi_s)\,x_B h_1 H_1^\perp \\
&+(1-y)\frac{|\boldsymbol{p}_\perp|^3}{2\,M^2\,M_h}\,\sin(3\phi_j-\phi_s)\,x_B h_{1T}^\perp H_1^\perp \\
&-(1-y)\frac{|\boldsymbol{P}_{h\perp}|}{zM_h}\,\sin(\phi_h+\phi_s)\,x_B h_1 H_1^\perp \\
&-(1-y)\frac{\boldsymbol{p}_\perp^2\,|\boldsymbol{P}_{h\perp}|}{2\,M^2\,M_h}\,\sin(2\phi_j+\phi_h-\phi_s)\,x_B h_{1T}^\perp H_1^\perp \\
&+2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\sin(\phi_s)\;x_B h_1\,\frac{\tilde{H}}{z} \\
&+2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\frac{\boldsymbol{p}_\perp^2}{2\,M^2}\,\sin(2\phi_j-\phi_s)\,x_B h_{1T}^\perp\,\frac{\tilde{H}}{z} \\
&+2(2-y)\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2M_h\,Q}\,\sin(2\phi_j-\phi_s)\;x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}-x_B h_T^\perp\right)\,H_1^\perp \\
&+2(2-y)\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2M_h\,Q}\,\sin(\phi_s)\;x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}+x_B h_T^\perp\right)\,H_1^\perp \\
&-2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|\,|\boldsymbol{P}_{h\perp}|}{2z\,M_h\,Q}\,\sin(\phi_h+\phi_j-\phi_s)\;x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}-x_B h_T^\perp\right)\,H_1^\perp \\
&-2(2-y)\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|\,|\boldsymbol{P}_{h\perp}|}{2z\,M_h\,Q}\,\sin(\phi_h-\phi_j+\phi_s)\;x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}+x_B h_T^\perp\right)\,H_1^\perp \Bigg\},
\end{aligned} \tag{7.58}$$

where the distribution functions depend on x_B and $\boldsymbol{p}_\perp$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_\perp^2 = (\boldsymbol{P}_{h\perp}-z\boldsymbol{p}_\perp)^2$. We have used or can use

$$\begin{aligned}
&h_1(x_B,\boldsymbol{p}_T) = h_{1T}(x_B,\boldsymbol{p}_T)+\frac{\boldsymbol{p}_T^2}{2\,M^2}\,h_{1T}^\perp(x_B,\boldsymbol{p}_T),\\
&x_B h_T(x_B,\boldsymbol{p}_T)-\frac{m}{M}\,g_{1T}(x_B,\boldsymbol{p}_T)-x_B h_T^\perp(x_B,\boldsymbol{p}_T) = -2\,h_1(x_B,\boldsymbol{p}_T)+x_B\tilde{h}_{1L}^\perp(x_B,\boldsymbol{p}_T),\\
&x_B h_T(x_B,\boldsymbol{p}_T)-\frac{m}{M}\,g_{1T}(x_B,\boldsymbol{p}_T)+x_B h_T^\perp(x_B,\boldsymbol{p}_T) = \\
&\qquad\qquad -\frac{\boldsymbol{p}_T^2}{M^2}\,h_{1T}^\perp(x_B,\boldsymbol{p}_T)+x_B\left(\tilde{h}_T(x_B,\boldsymbol{p}_T)+\tilde{h}_T^\perp(x_B,\boldsymbol{p}_T)\right).
\end{aligned} \tag{7.59}$$

Integrating only over the jet directions one obtains

$$\begin{aligned}
\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp| \Bigg\{ &(1-y)\,\sin(\phi_h+\phi_s)\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_B h_1 H_1^\perp\Big] \\
&+(1-y)\,\sin(3\phi_h-\phi_s)\,I\Big[\frac{4(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)-\boldsymbol{p}_\perp^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{2\,M^2\,M_h}\,x_B h_{1T}^\perp H_1^\perp\Big] \\
&+2(2-y)\sqrt{1-y}\,\sin(\phi_s)\,\Bigg(\frac{M_h}{Q}\,I\Big[x_B h_1\,\frac{\tilde{H}}{z}\Big] \\
&\qquad+\frac{M}{Q}\,I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{2MM_h}\,x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}+x_B h_T^\perp\right)\,H_1^\perp\Big]\Bigg) \\
&+2(2-y)\sqrt{1-y}\,\sin(2\phi_h-\phi_s)\,\Bigg(\frac{M_h}{Q}\,I\Big[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2\,M^2}\,x_B h_{1T}^\perp\,\frac{\tilde{H}}{z}\Big] \\
&\qquad+\frac{M}{Q}\,I\Big[\frac{\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp-2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)}{2MM_h}\,x_B\,\left(x_B h_T-\frac{m}{M}\,g_{1T}-x_B h_T^\perp\right)\,H_1^\perp\Big]\Bigg)\Bigg\}.
\end{aligned} \tag{7.60}$$

The cross section integrated over the transverse momenta of hadrons h within the jet becomes

$$\begin{aligned}\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+\mathrm{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \\ \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\Bigg\{2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\sin\phi_s\;x_B h_1(x_B,\boldsymbol{p}_\perp)\frac{\tilde{H}(z)}{z} \\ +2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\frac{\boldsymbol{p}_\perp^2}{2\,M^2}\,\sin(2\phi_j-\phi_s)\,x_B h_{1T}^\perp(x_B,\boldsymbol{p}_\perp)\frac{\tilde{H}(z)}{z}\Bigg\},\end{aligned} \tag{7.61}$$

where the fragmentation function is integrated over transverse momenta. Integrating further over the jet angle one finds

$$\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\,2(2-y)\sqrt{1-y}\,\frac{M_h}{Q}\,\sin\phi_s\;x_B\,h_1(x_B)\,\frac{\tilde{H}(z)}{z}. \tag{7.62}$$

7.4.4 $\boxed{\vec{\ell}H\to\ell' hX}$ (unpolarized hadrons)

The hadronic tensor is given by

$$\begin{aligned}2M\,\mathcal{W}^{\mu\nu} &= \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T) \\ &\quad\times\left\{i\frac{\hat{t}^{\,[\mu}k_\perp^{\nu]}}{Q}\left[-\frac{M}{M_h}\,4xz\,eH_1^\perp+\frac{m}{M_h}\,4z\,f_1H_1^\perp\right]\right\} \\ &= -i\,\hat{t}^{\,[\mu}\hat{h}^{\nu]}\,\frac{M}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,\Big(x\,e-\frac{m}{M}\,f_1\Big)\,H_1^\perp\Big]\end{aligned} \tag{7.63}$$

The most explicit differential cross section is given by

$$\begin{aligned}\frac{d\Delta\sigma_{LO}(\vec{\ell}+H\to\ell'+h+\mathrm{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \\ \frac{4\pi\alpha^2\,s}{Q^4}\Bigg\{2\lambda_e y\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\sin\phi_j\,\frac{M}{M_h}\,x_B\,\Big(x_Be-\frac{m}{M}\,f_1\Big)\,H_1^\perp \\ -2\lambda_e y\sqrt{1-y}\,\frac{|\boldsymbol{P}_{h\perp}|}{zQ}\,\sin\phi_h\,\frac{M}{M_h}\,x_B\,\Big(x_Be-\frac{m}{M}\,f_1\Big)\,H_1^\perp\Bigg\},\end{aligned} \tag{7.64}$$

where the arguments of distribution and fragmentation functions are $\tilde{e}(x_B,\boldsymbol{p}_\perp)$, etc. and $H_1^\perp(z,\boldsymbol{P}_{h\perp}-z\boldsymbol{p}_\perp)$. Integrating over the transverse momenta of the produced hadrons in the jet one obtains zero. Integrating over the jet angle the result is

$$\begin{aligned}\frac{d\Delta\sigma_{LO}(\vec{\ell}+H\to\ell'+h+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \\ \frac{4\pi\alpha^2\,s}{Q^4}\,2\lambda_e y\sqrt{1-y}\,\sin\phi_h\,\frac{M}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_B\,\Big(x_Be-\frac{m}{M}\,f_1\Big)\,H_1^\perp\Big].\end{aligned} \tag{7.65}$$

7.4.5 $\boxed{\vec{\ell}\vec{H} \to \ell' h X}$ (longitudinally polarized target)

The hadronic tensor is given by

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} \;=\; & \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\; \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\;\times\;\lambda\Bigg\{ i\,\epsilon_\perp^{\mu\nu}\,2z\,g_{1L} D_1 \\
& + i\frac{k_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\Big[4\,g_{1L} D^\perp - 4z\,g_{1L} D_1\Big] \\
& + i\frac{p_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\Big[4xz\,g_L^\perp D_1 + \frac{M_h}{M}\,4\,h_{1L}^\perp E - \frac{m}{M}\,4z\,h_{1L}^\perp D_1\Big]\Bigg\}. \\
& \lambda\Bigg\{ i\,\epsilon_\perp^{\mu\nu}\,2z\,I\Big[g_{1L} D_1\Big] + i\,\hat{h}_{\perp\rho}\epsilon_\perp^{[\mu}\hat{t}^{\nu]}\Bigg(\frac{M_h}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,g_{1L}\,\Big(\frac{D^\perp}{z} - D_1\Big)\Big] \\
& + \frac{M}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x g_L^\perp D_1\Big] + \frac{M_h}{Q}\,4z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,h_{1L}^\perp\,\Big(\frac{E}{z} - \frac{m}{M}D_1\Big)\Big]\Bigg)\Bigg\}. \qquad (7.66)
\end{aligned}
$$

The most explicit differential cross section is given by

$$
\begin{aligned}
& \frac{d\Delta\sigma_{LL}(\vec{\ell} + \vec{H} \to \ell' + h + \text{jet} + X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \\
& \quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\lambda\Bigg\{ y\left(1 - \frac{y}{2}\right) x_B\,g_{1L}\,D_1 \\
& \quad - 2y\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\cos(\phi_j)\Bigg[x_B^2\,g_L^\perp\,D_1 + x_B\,g_{1L}\,\left(\frac{D^\perp}{z} - D_1\right) \\
& \qquad + \frac{M_h}{M}\,x_B\,h_{1L}^\perp\,\left(\frac{E}{z} - \frac{m}{M_h}\,D_1\right)\Bigg] \\
& \quad + 2y\sqrt{1-y}\,\frac{|\boldsymbol{P}_{h\perp}|}{z\,Q}\,\cos(\phi_h)\,x_B\,g_{1L}\,\left(\frac{D^\perp}{z} - D_1\right)\Bigg\}, \qquad (7.67)
\end{aligned}
$$

where the distribution functions depend on x_B and $\boldsymbol{p}_\perp$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_\perp^2 = (\boldsymbol{P}_{h\perp} - z\boldsymbol{p}_\perp)^2$. Integrating only over the jet angle one finds

$$
\begin{aligned}
& \frac{d\Delta\sigma_{LL}(\vec{\ell} + \vec{H} \to \ell' + h + X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \\
& \quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\lambda\Bigg\{ y\left(1 - \frac{y}{2}\right)\,I\left[x_B\,g_{1L}\,D_1\right] \\
& \quad - 2y\sqrt{1-y}\,\cos(\phi_h)\Bigg(\frac{M_h}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp}{M_h}\,x_B\,g_{1L}\,\left(\frac{D^\perp}{z} - D_1\right)\Big] \\
& \qquad + \frac{M}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x_B^2 g_L^\perp D_1\Big] \\
& \qquad + \frac{M_h}{Q}\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,x_B h_{1L}^\perp\,\left(\frac{E}{z} - \frac{m}{M}D_1\right)\Big]\Bigg)\Bigg\}. \qquad (7.68)
\end{aligned}
$$

Integrating over the transverse momenta of the produced hadrons we obtain

$$\frac{d\Delta\sigma_{LL}(\vec{\ell}+\vec{H}\to\ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\lambda\Bigg\{y\left(1-\frac{y}{2}\right)x_B\,g_{1L}\,D_1 -2y\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|}{Q}\,\cos(\phi_j)\left[x_B^2\,g_L^\perp\,D_1+\frac{M_h}{M}\,x_B\,h_{1L}^\perp\,\left(\frac{E}{z}-\frac{m}{M_h}\,D_1\right)\right]\Bigg\} \tag{7.69}$$

with the fragmentation function integrated over $\boldsymbol{k}_T$, and thus only depending on z. Integrating also over the transverse jet momentum, one is left with the leading result for longitudinally polarized targets,

$$\frac{d\sigma_{LL}(\vec{\ell}+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\lambda\,y\left(1-\frac{y}{2}\right)x_B\,g_1(x_B)\,D_1(z). \tag{7.70}$$

7.4.6 $\boxed{\vec{\ell}\vec{H}\to\ell' hX}$ (transversely polarized target)

The hadronic tensor is given by

$$\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} &= \int d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\;\times\Bigg\{i\,\epsilon_\perp^{\mu\nu}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\,2z\,g_{1T}D_1 \\
&\quad +i\frac{k_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\left[4\,g_{1T}D^\perp-4z\,g_{1T}D_1\right] \\
&\quad +i\frac{p_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\left[4xz\,g_T^\perp D_1+\frac{M_h}{M}\,4\,h_{1T}^\perp E-\frac{m}{M}\,4z\,h_{1T}^\perp D_1\right] \\
&\quad +i\frac{M\,S_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}}{Q}\left[4xz\,g_T' D_1+\frac{M_h}{M}\,4\,h_{1T}E-\frac{m}{M}\,4z\,h_{1T}D_1\right]\Bigg\}. \\
&= i\,\epsilon_\perp^{\mu\nu}\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\,2z\,I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,g_{1T}D_1\Big] \\
&\quad +i\,\hat{h}_\rho\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{S}_\perp)\Bigg(\frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,g_{1T}\,\left(\frac{D^\perp}{z}-D_1\right)\Big] \\
&\qquad +\frac{M}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,xg_T^\perp D_1\Big] \\
&\qquad +\frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}h_{1T}^\perp\,\left(\frac{E}{z}-\frac{m}{M}\,D_1\right)\Big]\Bigg) \\
&\quad +i\,\hat{t}^{[\mu}\hat{h}^{\nu]}\,(\hat{\boldsymbol{h}}\wedge\boldsymbol{S}_\perp)\Bigg(\frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,g_{1T}\,\left(\frac{D^\perp}{z}-D_1\right)\Big] \\
&\qquad +\frac{M}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,xg_T^\perp D_1\Big] \\
&\qquad +\frac{M_h}{Q}\,4z\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}h_{1T}^\perp\,\left(\frac{E}{z}-\frac{m}{M}\,D_1\right)\Big]\Bigg) \\
&\quad +iS_{\perp\rho}\epsilon_\perp^{\rho\,[\mu}\hat{t}^{\nu]}\,\Bigg(\frac{M}{Q}\,4z\,I\Big[xg_TD_1\Big]+\frac{M_h}{Q}\,4z\,I\Big[h_1\,\left(\frac{E}{z}-\frac{m}{M}\,D_1\right)\Big] \\
&\qquad +\frac{M_h}{Q}\,4z\,I\Big[\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,g_{1T}\,\left(\frac{D^\perp}{z}-D_1\right)\Big]\Bigg).
\end{aligned} \tag{7.71}$$

The most explicit differential cross section is given by

$$\begin{aligned}
&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\rightarrow \ell'+h+\text{jet}+X)}{dx_{B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_{\perp}} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\Bigg\{ y\left(1-\frac{y}{2}\right)\frac{|\boldsymbol{p}_\perp|}{M}\,\cos(\phi_j-\phi_s)\;x_{B}\,g_{1T}\,D_1 \\
&\qquad -2y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_{B}^2\,g_T\,D_1+\frac{M_h}{Q}\,x_{B}\,h_1\,\frac{\tilde{E}}{z}\right] \\
&\qquad -2y\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2MQ}\,\cos(2\phi_j-\phi_s)\left[x_{B}^2\,g_T^\perp\,D_1+\frac{M_h}{M}\,x_{B}\,h_{1T}^\perp\,\frac{\tilde{E}}{z}\right] \\
&\qquad -2y\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2MQ}\,\cos(\phi_j)\,\cos(\phi_j-\phi_s)\;x_{B}\,g_{1T}\,\frac{\tilde{D}^\perp}{z} \\
&\qquad +2y\sqrt{1-y}\,\frac{|\boldsymbol{p}_\perp|\,|\boldsymbol{P}_{h\perp}|}{2zMQ}\,\cos(\phi_h)\,\cos(\phi_j-\phi_s)\;x_{B}\,g_{1T}\,\frac{\tilde{D}^\perp}{z}\Bigg\}, \qquad (7.72)
\end{aligned}$$

where the distribution functions depend on x_B and $\boldsymbol{p}_\perp$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_\perp^2 = (\boldsymbol{P}_{h\perp} - z\boldsymbol{p}_\perp)^2$. Integrating only over the jet angle one obtains

$$\begin{aligned}
&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\rightarrow \ell'+h+X)}{dx_{B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\Bigg\{ y\left(1-\frac{y}{2}\right)\cos(\phi_h-\phi_s)\;I\Big[\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp}{M}\,g_{1T}D_1\Big] \\
&\qquad -2y\sqrt{1-y}\,\cos(\phi_s)\Bigg(\frac{M}{Q}\,I\Big[x^2 g_T D_1\Big]+\frac{M_h}{Q}\,I\Big[xh_1\,\frac{\tilde{E}}{z}\Big] \\
&\qquad\qquad +\frac{M_h}{Q}\,I\Big[\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,xg_{1T}\,\frac{\tilde{D}^\perp}{z}\Big]\Bigg) \\
&\qquad -2y\sqrt{1-y}\,\cos(2\phi_h-\phi_s)\Bigg(\frac{M}{Q}\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,x^2 g_T^\perp D_1\Big] \\
&\qquad\qquad +\frac{M_h}{Q}\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)^2-\boldsymbol{p}_\perp^2}{2M^2}\,xh_{1T}^\perp\,\frac{\tilde{E}}{z}\Big] \\
&\qquad\qquad +\frac{M_h}{Q}\,I\Big[\frac{2(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_\perp)(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_\perp)-\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp}{2MM_h}\,xg_{1T}\,\frac{\tilde{D}^\perp}{z}\Big]\Bigg)\Bigg\}. \qquad (7.73)
\end{aligned}$$

Integrating over the transverse momenta of the produced hadrons we obtain

$$\begin{aligned}
&\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\rightarrow \ell'+h+\text{jet}+X)}{dx_{B}\,dy\,dz\,d^2\boldsymbol{p}_{\perp}} = \\
&\quad \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\Bigg\{ y\left(1-\frac{y}{2}\right)\frac{|\boldsymbol{p}_\perp|}{M}\,\cos(\phi_j-\phi_s)\;x_{B}\,g_{1T}\,D_1 \\
&\qquad -2y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_{B}^2\,g_T\,D_1+\frac{M_h}{Q}\,x_{B}\,h_1\,\frac{\tilde{E}}{z}\right] \\
&\qquad +2y\sqrt{1-y}\,\frac{\boldsymbol{p}_\perp^2}{2MQ}\,\cos(2\phi_j+\phi_s)\left[x_{B}^2\,g_T^\perp\,D_1+\frac{M_h}{M}\,x_{B}\,h_{1T}^\perp\,\frac{\tilde{E}}{z}\right]\Bigg\}, \qquad (7.74)
\end{aligned}$$

with the fragmentation function integrated over $\boldsymbol{k}_T$, and thus only depending on z. Integrating also over the transverse jet momentum, one is left with

$$\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\to\ell'+h+X)}{dx_B\,dy\,dz} = -\frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e|\boldsymbol{S}_\perp|\,2y\sqrt{1-y}\,\cos(\phi_s)\left[\frac{M}{Q}\,x_B^2\,g_T(x_B)\,D_1(z)+\frac{M_h}{Q}\,x_B\,h_1(x_B)\frac{\tilde{E}(z)}{z}\right]. \quad (7.75)$$

7.5 Transverse momenta in lepto-production with polarization in the final state

For the case that polarization in the final state is measured, we will restrict ourselves to the leading order (parton model) results. The hadronic tensor, omitting the unpolarized contribution discussed in the previous section becomes

$$\begin{aligned}2M\,\mathscr{W}^{\mu\nu} = &\int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\\ &\times\Bigg\{-g_\perp^{\mu\nu}\left[2z\,f_1D_1+2z\,g_{1s}G_{1s}+\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}\,2z\,f_1D_{1T}^\perp\right]\\ &-\frac{k_\perp^{\{\mu}p_\perp^{\nu\}}+(\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp)\,g_\perp^{\mu\nu}}{MM_h}\,2z\,h_{1s}^\perp H_{1s}^\perp-\frac{k_\perp^{\{\mu}S_\perp^{\nu\}}+(\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp)\,g_\perp^{\mu\nu}}{M_h}\,2z\,h_{1T}H_{1s}^\perp\\ &-\frac{p_\perp^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp})\,g_\perp^{\mu\nu}}{M}\,2z\,h_{1s}^\perp H_{1T}-\left(S_\perp^{\{\mu}S_{h\perp}^{\nu\}}+(\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})\,g_\perp^{\mu\nu}\right)\,2z\,h_{1T}H_{1T}\\ &+i\,\epsilon_\perp^{\mu\nu}\,2z\,f_1G_{1s}+i\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\,2z\,g_{1s}D_{1T}^\perp\Bigg\}. \quad (7.76)\end{aligned}$$

As before we will consider the various possibilities separately.

7.5.1 $\ell H\to\ell'\vec{h}X$

The hadronic tensor relevant for this situation is

$$2M\,\mathscr{W}^{\mu\nu} = \int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\,(-g_\perp^{\mu\nu})\,\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}\,2z\,f_1D_{1T}^\perp. \quad (7.77)$$

The cross section is given by

$$\begin{aligned}&\frac{d\Delta\sigma_{OO}(\ell+H\to\ell'+\vec{h}+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} =\\ &\quad\frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_{h\perp}|\Bigg\{\left(1-y+\frac{1}{2}\,y^2\right)\,\frac{|\boldsymbol{P}_{h\perp}|}{zM_h}\,\sin(\phi_h-\phi_s^h)\,x_Bf_1D_{1T}^\perp\\ &\qquad-\left(1-y+\frac{1}{2}\,y^2\right)\,\frac{|\boldsymbol{p}_\perp|}{M_h}\,\sin(\phi_j-\phi_s^h)\,x_Bf_1D_{1T}^\perp\Bigg\}. \quad (7.78)\end{aligned}$$

Averaged over the transverse momenta of the hadrons h in the jet this gives zero.

7.5.2 $\boxed{\ell \vec{H} \to \ell' \vec{h} X}$ (longitudinally polarized target)

The symmetric hadronic tensor relevant for this situation is

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} \quad = \quad & \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\; \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\, \lambda \left\{ -g_\perp^{\mu\nu}\, 2z\, g_{1L} G_{1s} \right. \\
& \qquad - \frac{k_\perp^{\{\mu} p_\perp^{\nu\}} + (\boldsymbol{k}_\perp \cdot \boldsymbol{p}_\perp)\, g_\perp^{\mu\nu}}{M M_h}\, 2z\, h_{1L}^\perp H_{1s}^\perp \\
& \qquad \left. - \frac{p_\perp^{\{\mu} S_{h\perp}^{\nu\}} + (\boldsymbol{p}_\perp \cdot \boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu}}{M}\, 2z\, h_{1L}^\perp H_{1T} \right\}. \qquad (7.79)
\end{aligned}
$$

We will give only the cross section averaged over the transverse momenta of the hadrons in the jet. This is

$$
\boxed{
\begin{aligned}
& \frac{d\Delta\sigma_{OL}(\ell + \vec{H} \to \ell' + \vec{h} + \text{jet} + X)}{dx_{\scriptscriptstyle B}\, dy\, dz\, d^2\boldsymbol{p}_\perp} = \\
& \quad \frac{4\pi\alpha^2\, s}{Q^4}\, \lambda \left\{ \lambda_h \left(1 - y - \frac{y^2}{2}\right) x_{\scriptscriptstyle B}\, g_{1L}(x_{\scriptscriptstyle B}, \boldsymbol{p}_\perp)\, G_1(z) \right. \\
& \qquad \left. - |\boldsymbol{S}_{h\perp}|\, (1-y)\, \frac{|\boldsymbol{p}_\perp|}{M}\, \cos(\phi_j + \phi_s^h)\; x_{\scriptscriptstyle B}\, h_{1L}^\perp(x_{\scriptscriptstyle B}, \boldsymbol{p}_\perp)\, H_1(z) \right\}. \qquad (7.80)
\end{aligned}
}
$$

Integrating over the jet directions one obtains

$$
\frac{d\Delta\sigma_{OL}(\ell + \vec{H} \to \ell' + \vec{h} + X)}{dx_{\scriptscriptstyle B}\, dy\, dz} = \frac{4\pi\alpha^2\, s}{Q^4}\, \lambda\, \lambda_h \left(1 - y - \frac{y^2}{2}\right) x_{\scriptscriptstyle B}\, g_1(x_{\scriptscriptstyle B})\, G_1(z). \qquad (7.81)
$$

7.5.3 $\boxed{\ell \vec{H} \to \ell' \vec{h} X}$ (transversely polarized target)

The symmetric tensor relevant in this case is

$$
\begin{aligned}
2M\,\mathcal{W}^{\mu\nu} \quad = \quad & \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\; \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T) \left\{ -g_\perp^{\mu\nu}\, \frac{\boldsymbol{p}_\perp \cdot \boldsymbol{S}_\perp}{M}\, 2z\, g_{1T} G_{1s} \right. \\
& \qquad - \frac{k_\perp^{\{\mu} p_\perp^{\nu\}} + (\boldsymbol{k}_\perp \cdot \boldsymbol{p}_\perp)\, g_\perp^{\mu\nu}}{M M_h}\, \frac{\boldsymbol{p}_\perp \cdot \boldsymbol{S}_\perp}{M}\, 2z\, h_{1T}^\perp H_{1s}^\perp \\
& \qquad - \frac{k_\perp^{\{\mu} S_\perp^{\nu\}} + (\boldsymbol{k}_\perp \cdot \boldsymbol{S}_\perp)\, g_\perp^{\mu\nu}}{M_h}\, 2z\, h_{1T} H_{1s}^\perp \\
& \qquad - \frac{p_\perp^{\{\mu} S_{h\perp}^{\nu\}} + (\boldsymbol{p}_\perp \cdot \boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu}}{M}\, \frac{\boldsymbol{p}_\perp \cdot \boldsymbol{S}_\perp}{M}\, 2z\, h_{1T}^\perp H_{1T} \\
& \qquad \left. - \left(S_\perp^{\{\mu} S_{h\perp}^{\nu\}} + (\boldsymbol{S}_\perp \cdot \boldsymbol{S}_{h\perp})\, g_\perp^{\mu\nu} \right) 2z\, h_{1T} H_{1T} \right\}. \qquad (7.82)
\end{aligned}
$$

We will give only the cross section averaged over the transverse momenta of the hadrons in the jet. This is

$$\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\rightarrow \ell'+h+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\left\{\lambda_h\left(1-y-\frac{y^2}{2}\right)\frac{|\boldsymbol{p}_\perp|}{M}\cos(\phi_j-\phi_s)\;x_B\,g_{1T}(x_B,\boldsymbol{p}_\perp)\,G_1(z)\right.$$
$$-|\boldsymbol{S}_{h\perp}|\left[(1-y)\cos(\phi_s+\phi_s^h)\;x_B\,h_{1T}(x_B,\boldsymbol{p}_\perp)\,H_1(z)\right.$$
$$\left.\left.-(1-y)\,\frac{\boldsymbol{p}_\perp^2}{M^2}\cos(\phi_j+\phi_s^h)\,\cos(\phi_j-\phi_s)\;x_B\,h_{1T}^\perp(x_B,\boldsymbol{p}_\perp)\,H_1(z)\right]\right\}. \tag{7.83}$$

Integrating over the jet directions one obtains

$$\frac{d\Delta\sigma_{OT}(\ell+\vec{H}\rightarrow \ell'+\vec{h}+X)}{dx_B\,dy\,dz} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_\perp|\,|\boldsymbol{S}_{h\perp}|\,(1-y)\,\cos(\phi_s+\phi_s^h)\;x_B\,h_1(x_B)\,H_1(z) \tag{7.84}$$

7.5.4 $\vec{\ell}H\rightarrow \ell'\vec{h}X$

The hadronic tensor relevant for this situation is the antisymmetric tensor

$$2M\,\mathscr{W}^{\mu\nu} = \int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\;(i\,\epsilon_\perp^{\mu\nu})\;2z\,f_1 G_{1s}. \tag{7.85}$$

The cross section is given by

$$\frac{d\Delta\sigma_{LO}(\vec{\ell}+H\rightarrow \ell'+\vec{h}+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\left\{\lambda_h\,y\left(1-\frac{1}{2}\,y\right)\;x_B\,f_1 G_{1L}\right.$$
$$+|\boldsymbol{S}_{h\perp}|\left[y\left(1-\frac{1}{2}\,y\right)\frac{|\boldsymbol{p}_\perp|}{M_h}\cos(\phi_j+\phi_s^h)\,x_B\,f_1 G_{1T}\right.$$
$$\left.\left.-y\left(1-\frac{1}{2}\,y\right)\frac{|\boldsymbol{P}_{h\perp}|}{zM_h}\cos(\phi_h+\phi_s^h)\,x_B\,f_1 G_{1T}\right]\right\}, \tag{7.86}$$

where the distributions depend on x_B and $\boldsymbol{p}_\perp$ and the fragmentation functions depend on z and $z^2\boldsymbol{k}_\perp^2$. Averaged over the hadrons in the jet, one obtains

$$\frac{d\Delta\sigma_{LO}(\vec{\ell}+H\rightarrow \ell'+\vec{h}+\text{jet}+X)}{dx_B\,dy\,dz\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,\lambda_h\,y\left(1-\frac{1}{2}\,y\right)\;x_B\,f_1(x_B,\boldsymbol{p}_\perp)G_1(z), \tag{7.87}$$

where $G_1(z)$ is obtained by integrating G_{1L} over the transverse momenta. A similar expression discussed earlier is obtained after integrating over the transverse momentum of the jet.

7.5.5 $\vec{\ell}\vec{H}\rightarrow \ell'\vec{h}X$ (longitudinally polarized target)

The antisymmetric hadronic tensor relevant for this situation is

$$2M\,\mathscr{W}^{\mu\nu} = \int d^2\boldsymbol{k}_T\,d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\left(i\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\right)\lambda\,2z\,g_{1L}D_{1T}^\perp. \tag{7.88}$$

An asymmetry is obtained in

$$\frac{d\Delta\sigma_{LL}(\vec{\ell}+\vec{H}\to\ell'+\vec{h}+\text{jet}+X)}{dx_{\scriptscriptstyle B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,\lambda\left\{|\boldsymbol{S}_{h\perp}|\left[-y\,\left(1-\frac{y}{2}\right)\,\frac{|\boldsymbol{p}_\perp|}{M_h}\,\sin(\phi_j-\phi_s^h)\;x_{\scriptscriptstyle B}\,g_{1L}\,D_{1T}^\perp + y\,\left(1-\frac{y}{2}\right)\,\frac{|\boldsymbol{P}_{h\perp}|}{zM_h}\,\sin(\phi_h-\phi_s^h)\;x_{\scriptscriptstyle B}\,g_{1L}\,D_{1T}^\perp\right]\right\}, \qquad (7.89)$$

which vanishes upon integration over the transverse momenta of the produced hadrons.

7.5.6 $\boxed{\vec{\ell}\vec{H}\to\ell'\vec{h}X}$ (transversely polarized target)

The antisymmetric tensor relevant in this case is

$$2M\,\mathscr{W}^{\mu\nu} = \int d^2\boldsymbol{k}_{\scriptscriptstyle T}\,d^2\boldsymbol{p}_{\scriptscriptstyle T}\;\delta^2(\boldsymbol{p}_{\scriptscriptstyle T}+\boldsymbol{q}_{\scriptscriptstyle T}-\boldsymbol{k}_{\scriptscriptstyle T})\,\left(i\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\right)\,\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{S}_\perp}{M}\,2z\,g_{1T}D_{1T}^\perp. \qquad (7.90)$$

An asymmetry is obtained in

$$\frac{d\Delta\sigma_{LT}(\vec{\ell}+\vec{H}\to\ell'+\vec{h}+\text{jet}+X)}{dx_{\scriptscriptstyle B}\,dy\,dz\,d^2\boldsymbol{P}_{h\perp}\,d^2\boldsymbol{p}_\perp} = \frac{4\pi\alpha^2\,s}{Q^4}\,\lambda_e\,|\boldsymbol{S}_\perp|\,|\boldsymbol{S}_{h\perp}|\left[-y\,\left(1-\frac{y}{2}\right)\,\frac{\boldsymbol{p}_\perp^2}{MM_h}\,\sin(\phi_j-\phi_s^h)\,\cos(\phi_j-\phi_s)\;x_{\scriptscriptstyle B}\,g_{1T}\,D_{1T}^\perp + y\,\left(1-\frac{y}{2}\right)\,\frac{|\boldsymbol{p}_\perp|\,|\boldsymbol{P}_{h\perp}|}{zMM_h}\,\sin(\phi_h-\phi_s^h)\,\cos(\phi_j-\phi_s)\;x_{\scriptscriptstyle B}\,g_{1T}\,D_{1T}^\perp\right], \qquad (7.91)$$

which again vanishes upon integration over the transverse momenta of the produced hadrons.

7.6 Convolutions and gaussian distributions

In order to study the behavior of the convolutions of distribution and fragmentation functions it is useful to consider gaussian distributions,

$$\begin{aligned} f(x,\boldsymbol{p}_{\scriptscriptstyle T}) &= f(x,\boldsymbol{0}_{\scriptscriptstyle T})\,\exp(-R_H^2\boldsymbol{p}_{\scriptscriptstyle T}^2) \\ &= f(x)\,\frac{R_H^2}{\pi}\,\exp(-R_H^2\boldsymbol{p}_{\scriptscriptstyle T}^2) \equiv f(x)\,\mathscr{P}(\boldsymbol{p}_{\scriptscriptstyle T};R_H), \qquad (7.92) \\ D(z,-z\boldsymbol{k}_{\scriptscriptstyle T}) &= D(z,\boldsymbol{0}_{\scriptscriptstyle T})\,\exp(-R_h^2\boldsymbol{k}_{\scriptscriptstyle T}^2). \\ &= D(z)\,\frac{R_h^2}{\pi\,z^2}\,\exp(-R_h^2\boldsymbol{k}_{\scriptscriptstyle T}^2) = \frac{D(z)}{z^2}\,\mathscr{P}(\boldsymbol{k}_{\scriptscriptstyle T};R_h) = D(z)\,\mathscr{P}\left(-z\boldsymbol{k}_{\scriptscriptstyle T};\frac{R_h}{z}\right), \qquad (7.93) \end{aligned}$$

In that case the convolution becomes

$$\begin{aligned} I[f\,D] &= \int d^2\boldsymbol{p}_{\scriptscriptstyle T}\,d^2\boldsymbol{k}_{\scriptscriptstyle T}\;\delta^2(\boldsymbol{p}_{\scriptscriptstyle T}+\boldsymbol{q}_{\scriptscriptstyle T}-\boldsymbol{k}_{\scriptscriptstyle T})\,f(x_{\scriptscriptstyle B},\boldsymbol{p}_{\scriptscriptstyle T})\,D(z,-z\boldsymbol{k}_{\scriptscriptstyle T}) \\ &= \frac{\pi}{R_H^2+R_h^2}\,\exp\left(-\frac{Q_{\scriptscriptstyle T}^2\,R_H^2R_h^2}{R_H^2+R_h^2}\right)\,f(x,\boldsymbol{0}_{\scriptscriptstyle T})\,D(z,\boldsymbol{0}_{\scriptscriptstyle T}) \\ &= f(x_{\scriptscriptstyle B})\,D(z)\,\frac{\mathscr{P}(\boldsymbol{q}_{\scriptscriptstyle T};R)}{z^2}, \qquad (7.94) \end{aligned}$$

where $R^2 = R_H^2\,R_h^2/(R_H^2+R_h^2)$. The other convolutions that appear in the cross sections are of the form

$$I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol p_T}{M}\,f\,D\Big] \;=\; \frac{R^2}{R_H^2}\,\frac{Q_T}{M}\,I[f\,D] = \frac{R^2}{R_H^2}\,\frac{Q_T}{M}\,f(x_B)\,D(z)\,\frac{\mathscr{P}(\boldsymbol q_T;R)}{z^2}, \tag{7.95}$$

$$I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol k_T}{M_h}\,f\,D\Big] \;=\; -\frac{R^2}{R_h^2}\,\frac{Q_T}{M_h}\,I[f\,D], \tag{7.96}$$

$$I\Big[\frac{\boldsymbol p_T\cdot\boldsymbol k_T}{MM_h}\,f\,D\Big] \;=\; \frac{R^2}{MM_hR_H^2R_h^2)}\left(1-Q_T^2\,R^2\right)\,I[f\,D], \tag{7.97}$$

$$I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2-\boldsymbol p_T^2}{2M^2}\,f\,D\Big] \;=\; \frac{R^4}{R_H^4}\,\frac{Q_T^2}{2M^2}\,I[f\,D], \tag{7.98}$$

$$I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)^2-\boldsymbol k_T^2}{2M_h^2}\,f\,D\Big] \;=\; \frac{R^4}{R_h^4}\,\frac{Q_T^2}{2M_h^2}\,I[f\,D], \tag{7.99}$$

$$I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-\boldsymbol p_T\cdot\boldsymbol k_T}{2MM_h}\,f\,D\Big] \;=\; -\frac{R^4}{R_H^2R_h^2}\,\frac{Q_T^2}{2MM_h}\,I[f\,D], \tag{7.100}$$

$$I\Big[\frac{4(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\boldsymbol p_T\cdot\boldsymbol k_T)-\boldsymbol p_T^2\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)}{2M^2M_h}\,f\,D\Big] \;=\; -\frac{R^6}{R_H^4R_h^2}\,\frac{Q_T^3}{2M^2M_h}\,I[f\,D]. \tag{7.101}$$

Another way to deconvolute the results are the following weighted quantities,

$$\int d^2\boldsymbol q_T\;\frac{Q_T}{M}\,I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol p_T}{M}\,f\,D\Big] = 2\,f^{(1)}(x)\,D(z), \tag{7.102}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T}{M_h}\,I\Big[\frac{\hat{\boldsymbol h}\cdot\boldsymbol k_T}{M_h}\,f\,D\Big] = 2\,f(x)\,D^{(1)}(z), \tag{7.103}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{MM_h}\,I\Big[\frac{\boldsymbol p_T\cdot\boldsymbol k_T}{MM_h}\,f\,D\Big] = -4\,f^{(1)}(x)\,D^{(1)}(z), \tag{7.104}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{M^2}\,I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2-\boldsymbol p_T^2}{2M^2}\,f\,D\Big] = 2\,f^{(2)}(x)\,D(z), \tag{7.105}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{M_h^2}\,I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)^2-\boldsymbol k_T^2}{2M_h^2}\,f\,D\Big] = 2\,f(x)\,D^{(2)}(z), \tag{7.106}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^2}{MM_h}\,I\Big[\frac{2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-\boldsymbol p_T\cdot\boldsymbol k_T}{2MM_h}\,f\,D\Big] = -4\,f^{(1)}(x)\,D^{(1)}(z), \tag{7.107}$$

$$\int d^2\boldsymbol q_T\;\frac{Q_T^3}{M^2M_h}\,I\Big[\frac{4(\hat{\boldsymbol h}\cdot\boldsymbol p_T)^2(\hat{\boldsymbol h}\cdot\boldsymbol k_T)-2(\hat{\boldsymbol h}\cdot\boldsymbol p_T)\,(\boldsymbol p_T\cdot\boldsymbol k_T)-\boldsymbol p_T^2\,(\hat{\boldsymbol h}\cdot\boldsymbol k_T)}{2M^2M_h}\,f\,D\Big] = -12\,f^{(2)}(x)\,D^{(1)}(z), \tag{7.108}$$

where $f^{(n)}(x)$ indicate the $(\boldsymbol k_T^2/2M^2)^n$-moments of $f(x,\boldsymbol k_T^2)$, and similarly $D^{(n)}(z)$ indicate the $(\boldsymbol k_T^2/2M_h^2)^n$-moments of $D(z,z^2\boldsymbol k_T^2)$, in terms of the Gaussian parametrizations given by

$$f^{(n)}(x) = \int d^2\boldsymbol p_T\;\left(\frac{\boldsymbol p_T^2}{2M^2}\right)^n\,f(x,\boldsymbol p_T^2) \;=\; \frac{n!}{(2M^2R_H^2)^n}\,f(x), \tag{7.109}$$

$$D^{(n)}(z) = z^2\int d^2\boldsymbol k_T\;\left(\frac{\boldsymbol k_T^2}{2M^2}\right)^n\,D(z,-z^2\boldsymbol k_T^2) \;=\; \frac{n!}{(2M_h^2R_h^2)^n}\,D(z). \tag{7.110}$$

7.7 Lepton-hadron semi-inclusive DIS including Z-exchange

[Rainer Jakob, October 9, 1997]

Differential cross section for semi-inclusive lepton hadron scattering $e+H\rightarrow e'+h+X$

$$\begin{aligned}\frac{d\sigma}{dx_B\,dy\,dz_h\,d^2\boldsymbol{q}_T} &= \frac{\pi\alpha_{em}^2}{2\,Q^4}\,y\,z_h\;L_{\mu\nu}^{(\gamma\gamma)}\mathscr{W}^{\mu\nu}_{(\gamma\gamma)} \\ &+ \frac{\pi\alpha_w\alpha_{em}}{2\,(Q^2-M_Z^2+i\Gamma_Z M_Z)\;Q^2}\,y\,z_h\;L_{\mu\nu}^{(Z\gamma)}\mathscr{W}^{\mu\nu}_{(Z\gamma)} \\ &+ \frac{\pi\alpha_w\alpha_{em}}{2\,(Q^2-M_Z^2-i\Gamma_Z M_Z)\;Q^2}\,y\,z_h\;L_{\mu\nu}^{(\gamma Z)}\mathscr{W}^{\mu\nu}_{(\gamma Z)} \\ &+ \frac{\pi\alpha_w^2}{2\,((Q^2-M_Z^2)^2+\Gamma_Z^2 M_Z^2)}\,y\,z_h\;L_{\mu\nu}^{(ZZ)}\mathscr{W}^{\mu\nu}_{(ZZ)}\end{aligned} \tag{7.111}$$

leptonic tensors for e^+e^-

$$L_{\mu\nu}^{(\gamma\gamma)} = \frac{1}{2}\mathrm{Tr}\left[\gamma_\mu\,\not{\ell}\,\gamma_\nu\,\not{\ell}'\,\frac{1+\lambda\gamma_5\,\not{s}}{2}\right] = 2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - \mathrm{g}_{\mu\nu}\mathrm{Q}^2 + 2\mathrm{i}\,\lambda\,\epsilon_{\mu\nu\rho\sigma}\mathrm{q}^\rho\ell^\sigma \tag{7.112}$$

$$L_{\mu\nu}^{(Z\gamma)} = \frac{1}{2}\mathrm{Tr}\left[\gamma_\mu\,\not{\ell}(\mathrm{g}_\mathrm{V}^\ell+\mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\nu\,\not{\ell}'\right] = \mathrm{g}_\mathrm{A}^\ell\left(2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - \mathrm{g}_{\mu\nu}\mathrm{Q}^2\right) + 2\mathrm{i}\mathrm{g}_\mathrm{V}^\ell\epsilon_{\mu\nu\rho\sigma}\ell^\rho\ell'^\sigma + \lambda\ldots \tag{7.113}$$

$$L_{\mu\nu}^{(\gamma Z)} = \frac{1}{2}\mathrm{Tr}\left[(\mathrm{g}_\mathrm{V}^\ell+\mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\mu\,\not{\ell}\,\gamma_\nu\,\not{\ell}'\right] = \mathrm{g}_\mathrm{A}^\ell\left(2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - \mathrm{g}_{\mu\nu}\mathrm{Q}^2\right) + 2\mathrm{i}\mathrm{g}_\mathrm{V}^\ell\epsilon_{\mu\nu\rho\sigma}\ell^\rho\ell'^\sigma + \lambda\ldots \tag{7.114}$$

$$\begin{aligned}L_{\mu\nu}^{(ZZ)} &= \frac{1}{2}\mathrm{Tr}\left[(\mathrm{g}_\mathrm{V}^\ell+\mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\mu\,\not{\ell}(\mathrm{g}_\mathrm{V}^\ell+\mathrm{g}_\mathrm{A}^\ell\gamma_5)\gamma_\nu\,\not{\ell}'\right] = \\ &\qquad (g_A^{\ell\,2}+g_V^{\ell\,2})\left(2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - g_{\mu\nu}Q^2\right) + (2g_A^\ell g_V^\ell)2i\epsilon_{\mu\nu\rho\sigma}\ell^\rho\ell'^\sigma + \lambda\ldots\end{aligned} \tag{7.115}$$

October 9, 1997

hadronic tensor for semi-inclusive lepton hadron scattering $e + H \to e' + h + X$

$$2M\,\mathscr{W}^{\mu\nu}_{(\gamma\gamma)} = \int dp^- dk^+ d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\ \mathrm{Tr}\Big[\Phi(\mathrm{p})\gamma^\mu\Delta(\mathrm{k})\gamma^\nu\Big] \tag{7.116}$$

$$\begin{aligned}2M\,\mathscr{W}^{\mu\nu}_{S(\gamma\gamma)} = 2z_h \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{ & \\ -\, g_\perp^{\mu\nu}\Bigg[f_1 D_1 + g_{1s}G_{1s} + \frac{\epsilon_\perp^{\rho\sigma} k_{\perp\rho} S_{h\perp\sigma}}{M_h} f_1 D_{1T}^\perp - \frac{\epsilon_\perp^{\rho\sigma} p_{\perp\rho} S_{\perp\sigma}}{M} f_{1T}^\perp D_1 & \\ - \frac{\boldsymbol{p}_\perp\cdot\boldsymbol{k}_\perp\ \boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp} - \boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}\ \boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{MM_h} f_{1T}^\perp D_{1T}^\perp\Bigg] & \\ -\,(S_\perp^{\{\mu}S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{S}_\perp\cdot\boldsymbol{S}_{h\perp})h_{1T}H_{1T} - \frac{k_\perp^{\{\mu}p_\perp^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp}{MM_h}(h_{1s}^\perp H_{1s}^\perp + h_1^\perp H_1^\perp) & \\ -\,\frac{p_\perp^{\{\mu}S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{p}_\perp\cdot\boldsymbol{S}_{h\perp}}{M} h_{1s}^\perp H_{1T} - \frac{k_\perp^{\{\mu}S_\perp^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\cdot\boldsymbol{S}_\perp}{M_h} h_{1T}H_{1s}^\perp & \\ -\,\frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho} + p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2MM_h}(h_{1s}^\perp H_1^\perp - h_1^\perp H_{1s}^\perp) & \\ +\,\frac{p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho} + S_{h\perp}^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho}}{2M} h_1^\perp H_{1T} - \frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho} + S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2M_h} h_{1T}H_1^\perp\Bigg\} &\end{aligned} \tag{7.117}$$

$$\begin{aligned}2M\,\mathscr{W}^{\mu\nu}_{A(\gamma\gamma)} = 2z_h \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{ & \\ i\,\epsilon_\perp^{\mu\nu}(f_1G_{1s} + g_{1s}D_1) - i\,\frac{p_\perp^{[\mu}S_\perp^{\nu]}}{M} f_{1T}^\perp G_{1s} + i\,\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h} g_{1s}D_{1T}^\perp\Bigg\} &\end{aligned} \tag{7.118}$$

$$2M\,\mathscr{W}^{\mu\nu}_{(Z\gamma)} = \int dp^- dk^+ d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\,\mathrm{Tr}\Big[\Phi(\mathrm{p})\,(\mathrm{g_V}+\mathrm{g_A}\gamma_5)\,\gamma^\mu\Delta(\mathrm{k})\gamma^\nu\Big] \tag{7.119}$$

$$\begin{aligned}
2M\,\mathscr{W}^{\mu\nu}_{S(Z\gamma)} = 2z_h\int d^2\boldsymbol{k}_T\;d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\Bigg\{ &\\
-\,g_\perp^{\mu\nu}\Bigg[g_V\bigg(f_1D_1+g_{1s}G_{1s}+\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}f_1D_{1T}^\perp-\frac{\epsilon_\perp^{\rho\sigma}p_{\perp\rho}S_{\perp\sigma}}{M}f_{1T}^\perp D_1\bigg)&\\
-\,g_A\bigg(f_1G_{1s}+g_{1s}D_1+\frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}g_{1s}D_{1T}^\perp-\frac{\epsilon_\perp^{\rho\sigma}p_{\perp\rho}S_{\perp\sigma}}{M}f_{1T}^\perp G_{1s}\bigg)&\\
-\,g_V\,\frac{\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{k}_\perp\;\boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}\;-\;\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}\;\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{MM_h}f_{1T}^\perp D_{1T}^\perp\Bigg]&\\
-\,(S_\perp^{\{\mu}S_{h\perp}^{\nu\}}+g_\perp^{\mu\nu}\;\boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}\,)g_Vh_{1T}H_{1T}&\\
-\,\frac{k_\perp^{\{\mu}p_\perp^{\nu\}}+g_\perp^{\mu\nu}\;\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{p}_\perp}{MM_h}\;\big[g_V(h_{1s}^\perp H_{1s}^\perp+h_1^\perp H_1^\perp)-i\;g_A(h_{1s}^\perp H_1^\perp-h_1^\perp H_{1s}^\perp)\big]&\\
-\,\frac{p_\perp^{\{\mu}S_{h\perp}^{\nu\}}+g_\perp^{\mu\nu}\;\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}}{M}(g_Vh_{1s}^\perp H_{1T}+i\;g_Ah_1^\perp H_{1T})&\\
-\,\frac{k_\perp^{\{\mu}S_\perp^{\nu\}}+g_\perp^{\mu\nu}\;\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{M_h}(g_Vh_{1T}H_{1s}^\perp-i\;g_Ah_{1T}H_1^\perp)&\\
-\,\frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho}+p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2MM_h}\Big[g_V\,\big(h_{1s}^\perp H_1^\perp-h_1^\perp H_{1s}^\perp\big)+i\;g_A\,\big(h_{1s}^\perp H_{1s}^\perp+h_1^\perp H_1^\perp\big)\Big]&\\
+\,\frac{p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho}+S_{h\perp}^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho}}{2M}\;\big(g_Vh_1^\perp H_{1T}-i\;g_Ah_{1s}^\perp H_{1T}\big)&\\
-\,\frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho}+S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2M_h}\;\big(g_Vh_{1T}H_1^\perp+i\;g_Ah_{1T}H_{1s}^\perp\big)&\\
-\,i\;g_A\frac{S_{h\perp}^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho}+S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho}}{2}h_{1T}H_{1T}\Bigg\}&
\end{aligned} \tag{7.120}$$

$$\begin{aligned}
2M\,\mathscr{W}^{\mu\nu}_{A(Z\gamma)} = 2z_h\int d^2\boldsymbol{k}_T\;d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\Bigg\{ &\\
+\,i\;\epsilon_\perp^{\mu\nu}\Bigg[+g_V(f_1G_{1s}+g_{1s}D_1)-g_A(f_1D_1+g_{1s}G_{1s})&\\
+\,g_A\,\frac{\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{k}_\perp\;\boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}\;-\;\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{\boldsymbol{h}\perp}\;\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{MM_h}f_{1T}^\perp D_{1T}^\perp\Bigg]&\\
-\,i\;\frac{p_\perp^{[\mu}S_\perp^{\nu]}}{M}(g_Vf_{1T}^\perp G_{1s}-g_Af_{1T}^\perp D_1)+i\;\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}(g_Vg_{1s}D_{1T}^\perp-g_Af_1D_{1T}^\perp)\Bigg\}&
\end{aligned} \tag{7.121}$$

note: for the interference of γ and Z exchange there are the simple relations for the symmetric (S) and anti-symmetric (A) parts

$$W^{\mu\nu}_{S(\gamma Z)} = \Big(W^{\mu\nu}_{S(Z\gamma)}\Big)^* \qquad \text{and} \qquad W^{\mu\nu}_{A(\gamma Z)} = -\Big(W^{\mu\nu}_{A(Z\gamma)}\Big)^* \tag{7.122}$$

$$2M\,\mathscr{W}^{\mu\nu}_{(ZZ)} = \int dp^- dk^+ d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T\,\delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T) \times \mathrm{Tr}\Big[\Phi(\mathrm{p})\,(\mathrm{g_V} + \mathrm{g_A}\gamma_5)\,\gamma^\mu \Delta(\mathrm{k})\,(\mathrm{g_V} + \mathrm{g_A}\gamma_5)\,\gamma^\nu\Big] \quad (7.123)$$

$$\begin{aligned}
2M\,\mathscr{W}^{\mu\nu}_{S(ZZ)} = 2z_h \int d^2\boldsymbol{k}_T\; d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{ \\
-\,g_\perp^{\mu\nu}\Bigg[(g_V^2 + g_A^2)\Bigg(f_1 D_1 + g_{1s}G_{1s} + \frac{\epsilon_\perp^{\rho\sigma}p_{\perp\rho}S_{\perp\sigma}}{M}f_{1T}^\perp D_1 - \frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}f_1 D_{1T}^\perp \\
+\,\frac{\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{k}_\perp\;\boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}\;-\;\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}\;\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{MM_h}f_{1T}^\perp D_{1T}^\perp\Bigg) \\
+\,2g_V g_A\Bigg(f_1 G_{1s} + g_{1s}D_1 + \frac{\epsilon_\perp^{\rho\sigma}k_{\perp\rho}S_{h\perp\sigma}}{M_h}g_{1s}D_{1T}^\perp - \frac{\epsilon_\perp^{\rho\sigma}p_{\perp\rho}S_{\perp\sigma}}{M}f_{1T}^\perp G_{1s}\Bigg)\Bigg] \\
+\,(g_V^2 - g_A^2)\Bigg[-\,(S_\perp^{\{\mu}S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{h\perp})h_{1T}H_{1T} - \frac{k_\perp^{\{\mu}p_\perp^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{p}_\perp}{MM_h}\,\big(h_{1s}^\perp H_{1s}^\perp + h_1^\perp H_1^\perp\big) \\
-\,\frac{p_\perp^{\{\mu}S_{h\perp}^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}}{M}h_{1s}^\perp H_{1T} - \frac{k_\perp^{\{\mu}S_\perp^{\nu\}} + g_\perp^{\mu\nu}\,\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{M_h}h_{1T}H_{1s}^\perp \\
-\,\frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho} + p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2MM_h}\,\big(h_{1s}^\perp H_1^\perp - h_1^\perp H_{1s}^\perp\big) \\
+\,\frac{p_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{h\perp\rho} + S_{h\perp}^{\{\mu}\epsilon_\perp^{\nu\}\rho}p_{\perp\rho}}{2M}h_1^\perp H_{1T} - \frac{k_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}S_{\perp\rho} + S_\perp^{\{\mu}\epsilon_\perp^{\nu\}\rho}k_{\perp\rho}}{2M_h}h_{1T}H_1^\perp\Bigg]\Bigg\}
\end{aligned} \quad (7.124)$$

$$\begin{aligned}
2M\,\mathscr{W}^{\mu\nu}_{A(ZZ)} = 2z_h \int d^2\boldsymbol{k}_T\; d^2\boldsymbol{p}_T\;\delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\Bigg\{ \\
i\,\epsilon_\perp^{\mu\nu}\Bigg[-\,2g_V g_A\Bigg(f_1 D_1 + g_{1s}G_{1s} - \frac{\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{k}_\perp\;\boldsymbol{S}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}\;-\;\boldsymbol{p}_\perp\!\cdot\!\boldsymbol{S}_{h\perp}\;\boldsymbol{k}_\perp\!\cdot\!\boldsymbol{S}_\perp}{MM_h}f_{1T}^\perp D_{1T}^\perp\Bigg) \\
+\,(g_V^2 + g_A^2)\,(f_1 G_{1s} + g_{1s}D_1)\Bigg] \\
+\,i\,\frac{p_\perp^{[\mu}S_\perp^{\nu]}}{M}\Big[+\,2g_V g_A f_{1T}^\perp D_1 - (g_V^2 + g_A^2)f_{1T}^\perp G_{1s}\Big] \\
-\,i\,\frac{k_\perp^{[\mu}S_{h\perp}^{\nu]}}{M_h}\Big[+\,2g_V g_A f_1 D_{1T}^\perp - (g_V^2 + g_A^2)g_{1s}D_{1T}^\perp\Big]\Bigg\}
\end{aligned} \quad (7.125)$$

Chapter 8

Calculations of distribution and fragmentation functions

8.1 Quark distribution functions in the bag model

The starting point is the calculation of the lightcone correlation function for a target at rest $P = (M, \mathbf{0})$, expressed as a spatial integral,

$$\begin{aligned}
\Phi^{[\Gamma]}(x, \boldsymbol{k}_T) &= \frac{1}{2(2\pi)^3} \int da^-\, d^2\boldsymbol{a}_T\ \exp\left(-ixMa^-/\sqrt{2} + i\boldsymbol{k}_T\cdot\boldsymbol{a}_T\right)\ \langle P|\overline{\psi}(a)\Gamma\psi(0)|P\rangle\big|_{a^+=0} \\
&= \frac{1}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(ixMa_z + i\boldsymbol{k}_T\cdot\boldsymbol{a}_T\right)\ \langle P|\overline{\psi}(-a_z, \boldsymbol{a})\Gamma\psi(0)|P\rangle \\
&= \frac{1}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(ixMa_z + i\boldsymbol{k}_T\cdot\boldsymbol{a}_T\right)\ \langle P|\overline{\psi}(-a_z, \boldsymbol{a})\Gamma\psi(0)|P\rangle. \qquad (8.1)
\end{aligned}$$

The $\boldsymbol{k}_T$-integrated distribution functions become

$$\Phi^{[\Gamma]}(x) = \frac{1}{2\pi\sqrt{2}} \int da\ \exp\left(ixMa\right)\ \langle P|\overline{\psi}(-a, 0, 0, a)\Gamma\psi(0)|P\rangle. \qquad (8.2)$$

In the bag model one can consider the bag as a wave packet, i.e. a superposition of plane waves centered around $\boldsymbol{P} = \mathbf{0}$. The forward scattering off a bag gives the distribution function,

$$\begin{aligned}
\Phi^{[\Gamma]}_{\text{bag}}(x, \boldsymbol{k}_T) &= \frac{2M}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(ixMa_z + i\boldsymbol{k}_T\cdot\boldsymbol{a}_T\right) \\
&\qquad \int_{\text{bag}} d^3r\ \langle \text{bag}|\overline{\psi}(-a_z, \boldsymbol{r}+\boldsymbol{a})\Gamma\psi(\boldsymbol{r})|\text{bag}\rangle. \qquad (8.3)
\end{aligned}$$

Assuming that all quarks in the bag are in the lowest eigenmode, i.e. $\psi(x) = \sum_n \psi_n(x)\, a_n$ is restricted to one mode,

$$\psi_0(t, \boldsymbol{r}) = \psi_0(\boldsymbol{r})\, e^{-i\omega t/R}, \qquad (8.4)$$

with for massless quarks ω being the solution of $j_0(\omega) = j_1(\omega)$, i.e. $\omega \approx 2.043$, one obtains

$$\Phi^{[\Gamma]}_{\text{bag}}(x, \boldsymbol{k}_T) = \frac{2M}{(2\pi)^3\sqrt{2}} \int d^3a\ \exp\left(i\boldsymbol{k}\cdot\boldsymbol{a}\right) \int_{\text{bag}} d^3r\ \overline{\psi}_0(\boldsymbol{r}+\boldsymbol{a})\Gamma\psi_0(\boldsymbol{r}), \qquad (8.5)$$

where $\boldsymbol{k} \equiv (\boldsymbol{k}_T, (xMR-\omega)/R)$. This can be rewritten as

$$\Phi^{[\Gamma]}_{\text{bag}}(x, \boldsymbol{k}_T) = \frac{2M}{(2\pi)^3} \overline{\psi}_0(\boldsymbol{k}) \frac{\Gamma}{\sqrt{2}} \psi_0(\boldsymbol{k})\bigg|_{k_z=(xMR-\omega)/R}, \qquad (8.6)$$

where

$$\psi_0(\boldsymbol{k}) = \int_{\text{bag}} d^3r\ \exp\left(-i\boldsymbol{k}\cdot\boldsymbol{r}\right)\ \psi_0(\boldsymbol{r}). \qquad (8.7)$$

8.2 Quark distribution functions in the spectator model

To illustrate the calculation we will consider a simple nucleon-quark-diquark vertex connecting a nucleon with momentum P and mass M, a quark with momentum k and mass m and a scalar diquark. The essence of the model is that the diquark will be considered as an on-mass-shell spectator, i.e. $(P-k)^2 = M^2_{\rm diquark}$. The quark-quark correlation function is then obtained as the product of the tree graph for $N \to q + {\rm diquark}$.

The matrix element of a quark field interpolating between the nucleon state and a diquark state is

$$\langle X|\psi_i(0)|P,S\rangle = \frac{i}{k^2+m^2+i\epsilon}\,(\not k+m)_{ik}\,\Upsilon_{kl}\,U_l(P,S). \tag{8.8}$$

For a scalar diquark (with mass M_s), the vertex will be taken

$$\Upsilon^s_{ij}(P,k,P-k) = (1)_{ij}\,g_s(k^2) \tag{8.9}$$

(to be discussed later). The quark-quark correlation function for a scalar diquark becomes

$$\begin{aligned}
\Phi_{s\,ij}(k) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-ik\cdot x}\langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle \\
&= \frac{1}{(2\pi)^4}\,\frac{(\not k+m)_{ik}}{k^2-m^2}\,\Upsilon_{kk'}U_{k'}(P,S)\,(2\pi)\,\delta\left((P-k)^2-M_s^2\right)\,\overline{U}_{l'}(P,S)(\gamma_0\Upsilon^\dagger\gamma_0)_{l'l}\,\frac{(\not k+m)_{lj}}{k^2-m^2} \\
&= \left[(\not k+m)(\not P+M)\frac{(1+\gamma_5\not S)}{2}(\not k+m)\right]_{ij}\frac{1}{(2\pi)^3}\,\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((P-k)^2-M_s^2\right) \\
&\overset{S=0}{=} \left[(\not k+m)(\not P+M)(\not k+m)\right]_{ij}\,\frac{1}{2(2\pi)^3}\,\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((P-k)^2-M_s^2\right),
\end{aligned} \tag{8.10}$$

which gives for an unpolarized nucleon

$$\begin{aligned}
\Phi_s(k) &= \frac{1}{2(2\pi)^3}\,\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((P-k)^2-M_s^2\right) \\
&\quad\times\Bigg\{\left(mM^2+m^2M-mM_s^2+(M+m)k^2\right) \\
&\qquad -\left(k^2-m^2\right)\not P+\left((M+m)^2-M_s^2+k^2-m^2\right)\not k\Bigg\}.
\end{aligned} \tag{8.11}$$

The result for $\Phi_s(k)$ can be used to calculate the projections

$$\Phi_s^{[\Gamma]}(x,\boldsymbol{k}_T) = \frac{1}{2}\int dk^-\;{\rm Tr}(\Gamma\Phi_s)\Big|_{k^+=xP^+}, \tag{8.12}$$

which depend on $x = k^+/P^+$ and $\boldsymbol{k}_T^2$. The integration over k^- can be rewritten in covariant form as

$$\begin{aligned}
\Phi_s^{[\Gamma]}(x,\boldsymbol{k}_T) &= \int d(2k\cdot P)\,dk^2\;\delta\left(2x\,k\cdot P - x^2M^2-\boldsymbol{k}_T^2-k^2\right)\,\frac{{\rm Tr}(\Gamma\Phi_s)}{4\,P^+} \\
&= \int dP_s^2\,dk^2\;\delta\left(x(1-x)M^2-\boldsymbol{k}_T^2-(1-x)\,k^2-x\,P_s^2\right)\,\frac{{\rm Tr}(\Gamma\Phi_s)}{4\,P^+},
\end{aligned} \tag{8.13}$$

where $P_s = P-k$ and the latter form is suitable in the spectator model where one has the delta function $\delta(P_s^2-M_s^2)$ in the integrand. For the $\boldsymbol{k}_T$-integrated functions,

$$\Phi_s^{[\Gamma]}(x) = \frac{\pi}{2}\int dk^-\,d\boldsymbol{k}_T^2\;{\rm Tr}(\Gamma\Phi_s)\Big|_{k^+=xP^+}, \tag{8.14}$$

one has

$$\begin{aligned}
\Phi_s^{[\Gamma]}(x) &= \pi\int dP_s^2\,dk^2\;\theta\left(x(1-x)M^2-(1-x)\,k^2-x\,P_s^2\right)\,\frac{{\rm Tr}(\Gamma\Phi_s)}{4\,P^+} \\
&= \pi\int_{P_s^2(min)}^{\infty} dP_s^2\int_{-\infty}^{xM^2-\frac{x}{1-x}P_s^2} dk^2\;\frac{{\rm Tr}(\Gamma\Phi_s)}{4\,P^+},
\end{aligned} \tag{8.15}$$

where $P_s^2(min)$ is the minimum mass contributing in the antiquark-nucleon spectral function, which is larger than $(M-m)^2$ in order to render the nucleon stable.

Writing

$$\Phi_s(k) = \tilde{\Phi}_s(k)\,\delta(P_s^2 - M_s^2), \tag{8.16}$$

we get the 'spectator model' results

$$\begin{aligned}\Phi_s^{[\Gamma]}(x,\boldsymbol{k}_T) &= \int dk^2\;\delta\left(x(1-x)M^2 - \boldsymbol{k}_T^2 - (1-x)\,k^2 - x\,M_s^2\right)\;\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,P^+}\\ &= \left.\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,(1-x)\,P^+}\right|_{k^2\,=\,k^2(x,\boldsymbol{k}_T^2)},\end{aligned} \tag{8.17}$$

with

$$-k^2(x,\boldsymbol{k}_T^2) = \frac{\boldsymbol{k}_T^2}{1-x} + \frac{x}{1-x}\,M_s^2 - x\,M^2. \tag{8.18}$$

The $\boldsymbol{k}_T$-integrated result is

$$\begin{aligned}\Phi_s^{[\Gamma]}(x) &= \pi\int_0^\infty d\boldsymbol{k}_T^2\;\left.\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,(1-x)\,P^+}\right|_{k^2\,=\,k^2(x,\boldsymbol{k}_T^2)}\\ &= \pi\int_{\frac{x}{1-x}\,M_s^2 - xM^2}^\infty d(-k^2)\;\frac{\mathrm{Tr}(\Gamma\tilde{\Phi}_s)}{4\,P^+}.\end{aligned} \tag{8.19}$$

For the practical calculations it is convenient to introduce the quantities

$$\mu_s^2(x) = m^2(1-x) + xM_s^2 - x(1-x)M^2, \tag{8.20}$$

$$\lambda_s^2(x) = \Lambda^2(1-x) + xM_s^2 - x(1-x)M^2, \tag{8.21}$$

such that we have for the often appearing denominators,

$$m^2 - k^2 = \frac{\boldsymbol{k}_T^2 + \mu^2(x)}{1-x}, \tag{8.22}$$

$$\Lambda^2 - k^2 = \frac{\boldsymbol{k}_T^2 + \lambda^2(x)}{1-x}. \tag{8.23}$$

The function $\mu_s^2(x)$ has endpoints $\mu^2(0) = m^2$ and $\mu^2(1) = M_s^2$, acquiring a minimum at the point $x_0 = (M^2 + m^2 - M_s^2)/2M^2$ with the value

$$\mu^2(x_0) = \frac{\left((m+M_s)^2 - M^2\right)\,\left(M^2 - (m-M_s)^2\right)}{4\,M^2}, \tag{8.24}$$

which should be positive to avoid problems, i.e. $|M-m| \le M_s \le M+m$ and $|M-M_s| \le m \le M+M_s$, at least if $0 \le x_0 \le 1$, implying $|M_s^2 - m^2| \le M^2$. The condition for a valid application of the spectator approach for distribution functions becomes

$$M_s \ge M - m, \tag{8.25}$$

$$M_s \ge M - \Lambda. \tag{8.26}$$

The condition on the quark mass m is only relevant if the 'quark propagator' pole is kept. In many cases it is convenient to cancel it by the choice of the vertex function $g(k^2)$.

8.3 Quark-hadron vertices

The simplest way to construct valid quark-hadron vertices is to start with nonrelativistic (two-component) spinors and replace them by a (rest frame) Dirac spinor multiplied with the appropriate projection operator. Furthermore, to get the correct charge conjugation behavior, it is safest to start with the charge conjugation operator acting on positive energy spinors. Thirdly, one starts writing down everything in the hadron rest frame. Technically, the ingredients are for the (renormalized) spinors (in the rest frame),

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{8.27}$$

and for the charge conjugated u-spinors with $C = i\gamma^2\gamma^0$,

$$C\,u^{(1)} = v^{(2)}, \qquad C\,u^{(2)} = -v^{(1)}. \tag{8.28}$$

Projection operators involving these spinors are in the hadron rest frame given by

$$\sum_{\alpha=1}^{2} u^{(\alpha)}\overline{u}^{(\alpha)} = \frac{1+\gamma^0}{2} = \frac{\not P + M}{2M}, \tag{8.29}$$

$$\sum_{\alpha=1}^{2} v^{(\alpha)}\overline{v}^{(\alpha)} = \frac{\gamma^0 - 1}{2} = \frac{\not P - M}{2M}, \tag{8.30}$$

$$\sum_{\alpha=1}^{2} u^{(\alpha)}\overline{v}^{(\alpha)} = -\frac{(1+\gamma^0)\gamma_5}{2} = -\frac{(\not P + M)\gamma_5}{2M}, \tag{8.31}$$

$$\sum_{\alpha=1}^{2} v^{(\alpha)}\overline{u}^{(\alpha)} = \frac{(1-\gamma^0)\gamma_5}{2} = -\frac{(\not P - M)\gamma_5}{2M}. \tag{8.32}$$

For spin 1, the rest-system spin states are $\epsilon_+ = -(\epsilon_x + i\epsilon_y)/\sqrt{2}$, $\epsilon_0 = \epsilon_z$ and $\epsilon_- = (\epsilon_x - i\epsilon_y)/\sqrt{2}$, or explicitly

$$\epsilon_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}, \qquad \epsilon_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \epsilon_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \tag{8.33}$$

and the summation over states is

$$\sum_\alpha \epsilon_\mu^{(\alpha)}\epsilon_\nu^{(\alpha)*} = -g_{\mu\nu} + \frac{P_\mu P_\nu}{M^2}. \tag{8.34}$$

The *pion* → quark-antiquark spin-space vertex is e.g. obtained by constructing a spin zero quark-antiquark combination,

$$\begin{aligned} \Upsilon(\pi)_{ij} &= \frac{1}{\sqrt{2}}\left(u_i^{(1)}(C\,u^{(2)})_j^T - u_i^{(2)}(C\,u^{(1)})_j^T\right) \\ &= \frac{1}{\sqrt{2}}\left(u_i^{(1)}\overline{v}_j^{(1)} + u_i^{(2)}\overline{v}_j^{(2)}\right), \end{aligned} \tag{8.35}$$

i.e.

$$\Upsilon(\pi) \propto -\frac{1}{\sqrt{2}}\frac{(\not P_\pi + M_\pi)}{2M_\pi}\gamma_5 = \frac{1}{\sqrt{2}}\gamma_5\frac{(\not P_\pi - M_\pi)}{2M_\pi}, \tag{8.36}$$

$$\gamma_0\Upsilon^\dagger(\pi)\gamma_0 \propto \frac{1}{\sqrt{2}}\gamma_5\frac{(\not P_\pi + M_\pi)}{2M_\pi} = -\frac{1}{\sqrt{2}}\frac{(\not P_\pi - M_\pi)}{2M_\pi}\gamma_5. \tag{8.37}$$

The expression for $\Phi(k)$ is similar as for the nucleon with scalar diquark case with the expression between square brackets,

$$\begin{aligned} -\frac{1}{2}&\left[(\not k + m)\,\frac{(\not P + M)}{2M}\,\gamma_5\,(\not P - M)\,\gamma_5\,\frac{(\not P + M)}{2M}\,(\not k + m)\right] \\ &= \frac{1}{2}\left[(\not k + m)\,(\not P + M)\,(\not k + m)\right]. \end{aligned} \tag{8.38}$$

In this expression the 'spectator-antiquark' sum is taken to be $\not P - M$.

Combining the spinors to spin 1 one obtains the $\rho \rightarrow$ quark-antiquark spin-space vertex, e.g. for the spin 0 component one has

$$\begin{aligned}
\Upsilon_{ij}(\rho; S_z = 0) &= \frac{1}{\sqrt{2}} \left(u_i^{(1)} (C\, u^{(2)})_j^T + u_i^{(2)} (C\, u^{(1)})_j^T \right) \\
&= \frac{1}{\sqrt{2}} \left(u_i^{(1)} \overline{v}_j^{(1)} - u_i^{(2)} \overline{v}_j^{(2)} \right) \\
&= \frac{1}{\sqrt{2}} \left(\left(\sigma_z u^{(1)} \right)_i \overline{v}_j^{(1)} + \left(\sigma_z u^{(2)} \right)_i \overline{v}_j^{(2)} \right), \qquad (8.39)
\end{aligned}$$

which using that $\sigma_z u = \gamma_z \gamma_5 ((1+\gamma_0)/2) u$ gives the vertex for a vector meson,

$$\Upsilon^\mu(\rho) \propto \frac{1}{\sqrt{2}} \gamma^\mu \frac{(\not{P}_\rho - M_\rho)}{2M_\rho}, \qquad (8.40)$$

$$\gamma_0 \Upsilon^{\mu\dagger}(\rho) \gamma_0 \propto \frac{1}{\sqrt{2}} \frac{(\not{P}_\rho - M_\rho)}{2M_\rho} \gamma^\mu. \qquad (8.41)$$

The 'spin' part in $\Phi(k)$ becomes (for an unpolarized ρ-meson)

$$\begin{aligned}
&\frac{1}{2} \left[(\not{k} + m)\, \gamma_\mu \frac{(\not{P} - M)}{2M} (\not{P} - M) \frac{(\not{P} + M)}{2M} \gamma_\nu (\not{k} + m) \right] \left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{M^2} \right) \\
&\quad = \frac{1}{2} \left[(\not{k} + m)\, (\not{P} + M)\, (\not{k} + m) \right]. \qquad (8.42)
\end{aligned}$$

In this expression the 'spectator-antiquark' sum is taken to be $-\not{P} + M$.

In order to find the baryon $\rightarrow$ quark-diquark vertex it is useful to first build a nucleon spinor either from a quark and a diquark with spin zero,

$$U_i^{(1)} = u_i^{(1)}, \qquad (8.43)$$

or to build a nucleon spinor from a quark and a diquark with spin one, e.g.

$$\begin{aligned}
U_i^{(1)} &= -\sqrt{\frac{1}{3}}\, \epsilon_0\, u_i^{(1)} + \sqrt{\frac{2}{3}}\, \epsilon_+\, u_i^{(2)} \\
&= -\sqrt{\frac{1}{3}}\, \epsilon_0 \left(\sigma_z u^{(1)} \right)_i + \sqrt{\frac{1}{6}}\, \epsilon_+ \left(\sigma_- u^{(1)} \right)_i \\
&= -\sqrt{\frac{1}{3}} \left(\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}\, u^{(1)} \right)_i, \qquad (8.44)
\end{aligned}$$

from which one obtains the vertices

$$\Upsilon^s(N) \propto 1, \qquad (8.45)$$

$$\begin{aligned}
\Upsilon^{a\,\mu}(N) &\propto \frac{1}{\sqrt{3}} \left(\gamma_\nu \gamma_5 \frac{(\not{P} + M)}{2M} \right) \left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{M^2} \right) \\
&\propto \frac{1}{\sqrt{3}} \gamma_5 \left(\gamma^\mu + \frac{P^\mu}{M} \right), \qquad (8.46)
\end{aligned}$$

$$\gamma_0 \Upsilon^{a\,\mu\dagger}(N) \gamma_0 \propto -\frac{1}{\sqrt{3}} \left(\gamma^\mu + \frac{P^\mu}{M} \right) \gamma_5. \qquad (8.47)$$

For obtaining the short expressions note that a projection operator $\not{P} + M$ on the 'nucleon side', i.e. left side, becomes irrelevant. From the symmetric SU(6) wave function one deduces that one (with the above factor in Υ^a) need both vertices Υ^s and Υ^a with equal relative strength. For the axial-vector diquark contribution to $\Phi(k)$ the part between brackets in the scalar diquark case is replaced by

$$\begin{aligned}
&-\frac{1}{3} \left[(\not{k} + m)\, \gamma_\mu\, \gamma_5 \frac{(\not{P} + M)}{2M} (\not{P} + M) \frac{1 + \gamma_5 \not{S}}{2} \frac{(\not{P} + M)}{2M} \gamma_5\, \gamma_\nu\, (\not{k} + m) \right] \left(-g^{\mu\nu} + \frac{P^\mu P^\nu}{M^2} \right) \\
&\quad = \left[(\not{k} + m)\, (\not{P} + M) \left(\frac{1}{2} - \frac{1}{6} \gamma_5 \not{S} \right) (\not{k} + m) \right]. \qquad (8.48)
\end{aligned}$$

In this expression the spin-sum for the axial vector diquark is taken to be $-g^{\mu\nu} + P^\mu P^\nu / M^2$.

8.4 Quark fragmentation functions in the spectator model

To illustrate this calculation we will also consider the simple nucleon-quark-diquark vertex connecting a nucleon with momentum P and mass M, a quark with momentum k and mass m and a scalar diquark. The essence again is that the diquark will be considered as an on-mass-shell spectator, i.e. $(k-P_h)^2 = M^2_{\rm diquark}$. The quark-quark correlation function is then obtained as the product of the tree graph for $q \to N + $ diquark.

For a scalar diquark (with mass M_s), the vertex is as for the distribution functions,

$$\Upsilon^s_{ij}(P,k,P-k) = (1)_{ij}\, g_s(k^2), \tag{8.49}$$

The correlation function that is the starting point for the calculation of the fragmentation functions is

$$\begin{aligned}
\Delta_{s\,ij}(k) &= \frac{1}{(2\pi)^4}\int d^4x\; e^{ik\cdot x}\langle 0|\psi_i(x)a_h^\dagger a_h\overline{\psi}_j(0)|0\rangle \\
&= \frac{1}{(2\pi)^4}\,\frac{(\not k+m)_{ik}}{k^2-m^2}\,\Upsilon^\dagger_{kk'}\,U_{k'}(P_h,S_h)\,(2\pi)\,\delta\left((k-P_h)^2-M_s^2\right)\,\overline{U}_{l'}(P_h,S_h)\,\gamma_0\Upsilon_{l'l}\gamma_0\,\frac{(\not k+m)_{lj}}{k^2-m^2} \\
&= \left[(\not k+m)(\not P_h+M_h)\frac{(1+\gamma_5\not S_h)}{2}(\not k+m)\right]_{ij}\frac{1}{(2\pi)^3}\,\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((k-P_h)^2-M_s^2\right) \\
&\overset{S_h=0}{=} \left[(\not k+m)(\not P_h+M_h)(\not k+m)\right]_{ij}\,\frac{1}{2(2\pi)^3}\,\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((k-P_h)^2-M_s^2\right),
\end{aligned} \tag{8.50}$$

which is the same expression as for the distribution function and gives for the production of an unpolarized nucleon

$$\begin{aligned}
\Delta_s(k) &= \frac{1}{2(2\pi)^3}\,\frac{g^2(k^2)}{(k^2-m^2)^2}\,\delta\left((k-P_h)^2-M_s^2\right) \\
&\quad\times\Big\{\left(mM_h^2+m^2M_h-mM_s^2+(M_h+m)k^2\right) \\
&\qquad -\left(k^2-m^2\right)\not P_h+\left((M_h+m)^2-M_s^2+k^2-m^2\right)\not k\Big\}.
\end{aligned} \tag{8.51}$$

The result for $\Delta_s(k)$ can be used to calculate the projections

$$\Delta_s^{[\Gamma]}(z,\boldsymbol{P}_{h\perp}) = \frac{1}{4z}\int dk^+\,\mathrm{Tr}(\Gamma\Delta_s)\Big|_{k^-=P_h^-/z;\;\boldsymbol{k}_T=-\boldsymbol{P}_{h\perp}/z}, \tag{8.52}$$

which depend on $z = P_h^-/k^-$ and $\boldsymbol{P}_{h\perp}^2$ or $\boldsymbol{k}_T^2$. The integration over k^+ can be rewritten in covariant form as

$$\begin{aligned}
\Delta_s^{[\Gamma]}(z,\boldsymbol{P}_{h\perp}) &= \frac{1}{2z}\int d(2k\cdot P_h)\,dk^2\;\delta\left(\frac{2\,k\cdot P_h}{z}-\frac{M_h^2}{z^2}-\boldsymbol{k}_T^2-k^2\right)\,\frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\,P_h^-}\Bigg|_{\boldsymbol{k}_T=-\boldsymbol{P}_{h\perp}/z} \\
&= \frac{1}{2z}\int dP_s^2\,dk^2\;\delta\left(\frac{(1-z)\,k^2}{z}-\frac{(1-z)\,M_h^2}{z^2}-\boldsymbol{k}_T^2-\frac{P_s^2}{z}\right)\,\frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\,P_h^-}\Bigg|_{\boldsymbol{k}_T=-\boldsymbol{P}_{h\perp}/z} \\
&= \frac{1}{2z}\,\Phi_s^{[\Gamma]}(1/z,-\boldsymbol{P}_{h\perp}/z),
\end{aligned} \tag{8.53}$$

where $P_s = k - P_h$ and the second form is suitable in the spectator model where one has the delta function $\delta(P_s^2 - M_s^2)$ in the integrand.

For the $\boldsymbol{P}_{h\perp}$-integrated functions,

$$\Delta_s^{[\Gamma]}(z) = \frac{\pi\,z^2}{2}\int dk^+\,d\boldsymbol{k}_T^2\,\mathrm{Tr}(\Gamma\Delta_s)\Big|_{k^-=P_h^-/z}, \tag{8.54}$$

one has

$$\begin{aligned}
\Delta_s^{[\Gamma]}(z) &= \frac{\pi\,z}{2}\int dP_s^2\,dk^2\;\theta\left(\frac{(1-z)\,k^2}{z}-\frac{(1-z)\,M_h^2}{z^2}-\frac{P_s^2}{z}\right)\,\frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\,P_h^-} \\
&= \frac{\pi\,z}{2}\int_{P_s^2(min)}^\infty dP_s^2\int^\infty_{\frac{M_h^2}{z}+\frac{1}{1-z}P_s^2} dk^2\;\frac{\mathrm{Tr}(\Gamma\Delta_s)}{4\,P_h^-} = \frac{1}{2}\,z\,\Phi_s^{[\Gamma]}(1/z),
\end{aligned} \tag{8.55}$$

where $P_s^2(min)$ is the minimum mass contributing in the antiquark-nucleon spectral function, which is larger than $(M_h - m)^2$ in order to render the hadron h stable.

Writing

$$\Delta_s(k) = \tilde{\Delta}_s(k)\, \delta(P_s^2 - M_s^2), \tag{8.56}$$

we get the 'spectator model' results

$$\begin{aligned} \Delta_s^{[\Gamma]}(z, \boldsymbol{P}_{h\perp}) &= \frac{1}{2z} \int dk^2\ \delta\left(\frac{1-z}{z}\, k^2 - \frac{(1-z)\, M_h^2}{z^2} - \boldsymbol{k}_T^2 - \frac{M_s^2}{z}\right) \left.\frac{\text{Tr}(\Gamma\tilde{\Delta}_s)}{4\, P_h^-}\right|_{\boldsymbol{k}_T = -\boldsymbol{P}_{h\perp}/z} \\ &= \frac{1}{2(1-z)} \left.\frac{\text{Tr}(\Gamma\tilde{\Delta}_s)}{4\, P_h^-}\right|_{k^2 = k^2(z, \boldsymbol{k}_T^2)}, \end{aligned} \tag{8.57}$$

with

$$k^2(z, \boldsymbol{k}_T^2) = \frac{z}{1-z}\, \boldsymbol{k}_T^2 + \frac{M_s^2}{1-z} + \frac{M_h^2}{z}. \tag{8.58}$$

The $\boldsymbol{P}_{h\perp}$-integrated result is

$$\begin{aligned} \Delta_s^{[\Gamma]}(z) &= \frac{\pi\, z^2}{2(1-z)} \int_0^\infty d\boldsymbol{k}_T^2\ \left.\frac{\text{Tr}(\Gamma\tilde{\Delta}_s)}{4\, P_h^-}\right|_{k^2 = k^2(z, \boldsymbol{k}_T^2)} \\ &= \frac{\pi\, z}{2} \int_{\frac{1}{1-z}\, M_s^2 + \frac{1}{z}\, M_h^2}^{\infty} dk^2\ \frac{\text{Tr}(\Gamma\tilde{\Delta}_s)}{4\, P_h^-}. \end{aligned} \tag{8.59}$$

For the practical calculations it is convenient to use the functions also encountered for the distribution functions,

$$\mu_s^2(1/z) = \frac{1-z}{z^2}\, M_h^2 + \frac{M_s^2}{z} - \frac{1-z}{z}\, m^2, \tag{8.60}$$

$$\lambda_s^2(1/z) = \frac{1-z}{z^2}\, M_h^2 + \frac{M_s^2}{z} - \frac{1-z}{z}\, \Lambda^2, \tag{8.61}$$

such that we have for the often appearing denominators,

$$k^2 - m^2 = \frac{z}{1-z}\ \left(\boldsymbol{k}_T^2 + \mu^2(1/z)\right), \tag{8.62}$$

$$k^2 - \Lambda^2 = \frac{z}{1-z}\ \left(\boldsymbol{k}_T^2 + \lambda^2(1/z)\right). \tag{8.63}$$

With the argument $1/z$ and $0 \le z \le 1$, the function $\mu_s^2(1/z)$ has endpoints $\mu^2(1/z) \to M_h^2/z$ for $z \to 0$ and $\mu^2(1) = M_s^2$, acquiring a minimum at the point $z_0 = 2M_h^2/(M_h^2 + m^2 - M_s^2)$ with the value

$$\mu^2(1/z_0) = \frac{\left((m+M_s)^2 - M_h^2\right)\ \left(M_h^2 - (m-M_s)^2\right)}{4\, M_h^2}, \tag{8.64}$$

which should be positive to avoid problems, i.e $|M_h - m| \le M_s \le M_h + m$ and $|M_h - M_s| \le m \le M_h + M_s$, at least if $0 \le z_0 \le 1$, implying $m^2 - M_s^2 \ge M_h^2$. This implies the following condition for employment of the spectator model,

$$M_s \ge m - M_h, \tag{8.65}$$

$$M_s \ge \Lambda - M_h. \tag{8.66}$$

The first condition on the quark mass is not relevant if the pole in the quark propagator is cancelled by a special choice of $g(k^2)$.

Chapter 9

Perturbative corrections

9.1 Inclusive leptoproduction

In order to illustrate the inclusion of perturbative QCD corrections, we start with inclusive lepton-hadron scattering, for which the tree level result, corresponding to $\gamma^*(q) + q(p) \to q(k)$ with $k = p + q$ in leading order in $1/Q$ is given by

$$\begin{aligned}
2M\,W^{\mu\nu}(P,q) &= \int dp^-\,dp^+ d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,(\not p + \not q + m)\,\gamma^\nu\right)\,\delta\left((p+q)^2 - m^2\right) \\
&\approx \int dp^-\,d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,\frac{\not q}{2q^-}\,\gamma^\nu\right) \\
&\approx \frac{1}{4}\,f_1(x_B)\ \mathrm{Tr}\left(\gamma^-\,\gamma^\mu\,\gamma^+\,\gamma^\nu\right) \;=\; -g_\perp^{\mu\nu}\,f_1(x_B)
\end{aligned} \tag{9.1}$$

(Note that $g_T^{\mu\nu} = g_\perp^{\mu\nu}$ in this case as $q_T = 0$).

Perturbative corrections to this result for the nonsinglet structure functions come from the process $\gamma^*(q)+q(p) \to q(k)+G(l)$. This leads (omitting mass and vertex corrections) to the following contributions at leading order in $1/Q$,

$$\begin{aligned}
2M\,W^{\mu\nu}(P,q) &= \int dp^-\,d^2\boldsymbol{p}_\perp\,\theta(\mu^2 - \boldsymbol{p}_\perp^2)\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,\frac{\not q}{2q^-}\,\gamma^\nu\right) \\
&+ \frac{g^2\,C_F}{(2\pi)^3}\int d^4p\,d^4k\,d^4l\,\delta(l^2)\,\delta(k^2)\,\delta^4(p+q-k-l)\,d_{\alpha\beta}(l) \\
&\qquad \times\Big\{\theta\left(\boldsymbol{p}_\perp'^2 - \mu^2\right)\ \mathrm{Tr}[\Phi(p)\gamma^\alpha \not k_t \gamma^\mu \not k \gamma^\nu \not k_t \gamma^\beta]/\hat t^2 \\
&\qquad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not k_s \gamma^\alpha \not k \gamma^\nu \not k_t \gamma^\beta]/\hat s\hat t \\
&\qquad + \mathrm{Tr}[\Phi(p)\gamma^\alpha \not k_t \gamma^\mu \not k \gamma^\beta \not k_s \gamma^\nu]/\hat s\hat t \\
&\qquad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not k_s \gamma^\alpha \not k \gamma^\beta \not k_s \gamma^\nu]/\hat s^2\Big\},
\end{aligned} \tag{9.2}$$

where $\boldsymbol{p}_\perp' = \boldsymbol{p}_\perp - \boldsymbol{l}_\perp$ and where $d_{\alpha\beta}(l)$ is the gluon summation in the final state. This depends on the choice of gauge. For this a convenient choice is the axial gauge $\hat q \cdot A = 0$, in which case one has

$$d_{\alpha\beta}(l) = -g_{\alpha\beta} + \frac{l_\alpha q_\beta + q_\alpha l_\beta}{l\cdot q} - \frac{q^2\,l_\alpha\,l_\beta}{(l\cdot q)^2}. \tag{9.3}$$

For any gluon field linked to a matrix element or constituting a final state (which means essentially on-mass-shell compared to Q^2), the above gauge choice implies a polarization summation that is equivalent to the gauge choice $n_l \cdot A = 0$, where n_l is the lightlike vector constructed from l and q,

$$n_l = \frac{x_p\sqrt{2}}{Q}\left(l + \frac{2\,l\cdot q}{Q^2}\,q\right). \tag{9.4}$$

As we have seen that gauge choice is important for a parton inpretation of the correlation functions. The theta functions cutting off or taking into account the first rung of the ladder contribution avoids double counting. Transverse momenta larger than μ^2 are not included in the soft part.

Implementing the energy-momentum conservation to eliminate integration over p^+, k^- and $\boldsymbol{l}_\perp$, using $\delta(l^2)$ to eliminate the integration over l^+, introducing

$$f_1(x,\mu^2) = \int d^2\boldsymbol{p}_\perp\, \theta(\mu^2 - \boldsymbol{p}_\perp^2)\, f_1(x,\boldsymbol{p}_\perp^2) \tag{9.5}$$

and going to components (assuming P and q to have no perpendicular components) one finds

$$\begin{aligned} 2M\, W^{\mu\nu}(P,q) &= -g_\perp^{\mu\nu}\, f_1(x_B,\mu^2) \\ &+ \frac{g^2\, C_F}{(2\pi)^3} \int dp^- dk^+ d^2\boldsymbol{p}_\perp\, d^2\boldsymbol{k}_\perp\, \frac{dl^-}{2l^-}\, \delta(k^2)\, d_{\alpha\beta}(l) \\ &\qquad \times \Big\{ \theta\left(\boldsymbol{k}_\perp^2 - \mu^2\right)\ \mathrm{Tr}[\Phi(p)\gamma^\alpha \not k_t \gamma^\mu \not k \gamma^\nu \not k_t \gamma^\beta]/\hat t^2 \\ &\qquad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not k_s \gamma^\alpha \not k \gamma^\nu \not k_t \gamma^\beta]/\hat s \hat t \\ &\qquad + \mathrm{Tr}[\Phi(p)\gamma^\alpha \not k_t \gamma^\mu \not k \gamma^\beta \not k_s \gamma^\nu]/\hat s \hat t \\ &\qquad + \mathrm{Tr}[\Phi(p)\gamma^\mu \not k_s \gamma^\alpha \not k \gamma^\beta \not k_s \gamma^\nu]/\hat s^2 \Big\}. \end{aligned} \tag{9.6}$$

After this step we can parametrize the momenta

$$k = \left[z_k\, \frac{Q}{\sqrt{2}}, \frac{(1-z_k)(1-x_p)}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{k}_\perp \right], \tag{9.7}$$

$$l = \left[(1-z_k)\, \frac{Q}{\sqrt{2}}, \frac{z_k(1-x_p)}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{p}_\perp - \boldsymbol{k}_\perp \right], \tag{9.8}$$

$$q = \left[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \boldsymbol{0}_\perp \right], \tag{9.9}$$

$$p = \left[p^-, \frac{Q}{x_p\sqrt{2}}, \boldsymbol{p}_\perp \right], \tag{9.10}$$

and rewrite the integration for the $\mathscr{O}(g^2)$ correction as

$$\frac{g^2\, C_F}{(2\pi)^3}\, \frac{Q}{2\sqrt{2}} \int \frac{dx_p}{x_p^2} \int dz_k \int d^2\boldsymbol{k}_\perp\, \delta\left(\boldsymbol{k}_\perp^2 - \frac{z_k(1-z_k)(1-x_p)}{x_p}\, Q^2 \right) \int dp^-\, d^2\boldsymbol{p}_\perp\, \ldots\,. \tag{9.11}$$

The transverse momentum of the outgoing quark thus satisfies

$$\frac{\boldsymbol{k}_\perp^2}{Q^2} = \frac{z_k(1-z_k)(1-x_p)}{x_p}. \tag{9.12}$$

Positivity of $\boldsymbol{k}_\perp^2$ restricts the domain for z_k and x_p to the regions $0 \le z_k \le 1$ and $0 \le x_p \le 1$, while the theta function provides (in the first term) a regularization near the endpoints, for x_p not too close to unity

$$\frac{x_p}{1-x_p}\, \frac{\mu^2}{Q^2} \le z_k \le 1 - \frac{x_p}{1-x_p}\, \frac{\mu^2}{Q^2}. \tag{9.13}$$

It is useful to have explicit expressions for the vectors

$$k_s = p + q = k + l = \left[\frac{Q}{\sqrt{2}}, \frac{1-x_p}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{p}_\perp \right], \tag{9.14}$$

$$k_t = p - l = k - q = \left[-(1-z_k)\, \frac{Q}{\sqrt{2}}, \frac{1-z_k+x_p z_k}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{k}_\perp \right], \tag{9.15}$$

$$k_u = k - p = q - l = \left[z_k\, \frac{Q}{\sqrt{2}}, -\frac{x_p + z_k - x_p z_k}{x_p}\, \frac{Q}{\sqrt{2}}, \boldsymbol{k}_\perp - \boldsymbol{p}_\perp \right]. \tag{9.16}$$

The Mandelstam variables for the subprocess are

$$\hat s = k_s^2 = \frac{1-x_p}{x_p}\, Q^2, \tag{9.17}$$

$$\hat t = k_t^2 = -\frac{1-z_k}{x_p}\, Q^2, \tag{9.18}$$

$$\hat u = k_u^2 = -\frac{z_k}{x_p}\, Q^2, \tag{9.19}$$

satisfying $\hat{s}+\hat{t}+\hat{u}+Q^2=0$ and the inverse relations are:

$$x_p = \frac{Q^2}{\hat{s}+Q^2}, \tag{9.20}$$

$$z_k = -\frac{\hat{u}}{\hat{s}+Q^2}. \tag{9.21}$$

Useful inner products are:

$$\begin{aligned} 2\,p\cdot q &= \hat{s}+Q^2 &=& \frac{1}{x_p}\,Q^2 = -(\hat{u}+\hat{t}),\\ 2\,k\cdot q &= -(\hat{t}+Q^2) &=& -\frac{x_p+z_k-1}{x_p}\,Q^2 = \hat{s}+\hat{u},\\ 2\,l\cdot q &= -(\hat{u}+Q^2) &=& \frac{z_k-x_p}{x_p}\,Q^2 = \hat{s}+\hat{t}, \end{aligned} \tag{9.22}$$

The vectors $\hat{p}\propto x_p\,p\sqrt{2}/Q$, $\hat{k}\propto x_p\,k\sqrt{2}/Q$ and $\hat{l}\propto x_p\,l\sqrt{2}/Q$ are lightlike vectors, e.g. $\hat{p}\;\propto\;n_+$.

For the transverse and longitudinal structure function in

$$2M\,W^{\mu\nu}(P,q) = -2\,F_T(x_{\scriptscriptstyle B},Q^2)\,g_\perp^{\mu\nu} + 2\,F_L(x_{\scriptscriptstyle B},Q^2)\,\frac{\tilde{P}^\mu\tilde{P}^\nu}{\tilde{P}^2} \tag{9.23}$$

(Note $F_T = F_1 = M\,W_T = M\,W_1$ and $2\,F_L = F_2/x_{\scriptscriptstyle B} - 2\,F_1$) we then obtain the results

$$\begin{aligned} 2\,F_T(x_{\scriptscriptstyle B},Q^2) &= f_1(x_{\scriptscriptstyle B},\mu^2)\\ &+\frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1\frac{dx_p}{x_p}\int_0^1 dz_k\int d^2\boldsymbol{k}_\perp\,\delta\left(\boldsymbol{k}_\perp^2-\frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2\right)\,\frac{1}{(x_p-z_k)^2}\,f_1\left(\frac{x_{\scriptscriptstyle B}}{x_p}\right)\\ &\times\left\{\frac{(1-x_p)\,(x_p^2-2x_pz_k-2x_p^2z_k+3z_k^2+6x_pz_k^2+2x_p^2z_k^2-4z_k^3-4x_pz_k^3+2z_k^4)}{1-z_k}\,\theta\left(\boldsymbol{k}_\perp^2-\mu^2\right)\right.\\ &\qquad\left.-\,2\,z_k\,(x_p+z_k+x_pz_k-z_k^2)+\frac{(1-z_k)\,(x_p^2+z_k^2)}{1-x_p}\right\}\\ &= f_1(x_{\scriptscriptstyle B},\mu^2)\\ &+\frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1\frac{dx_p}{x_p}\int_0^1 dz_k\int d^2\boldsymbol{k}_\perp\,\delta\left(\boldsymbol{k}_\perp^2-\frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2\right)\,f_1\left(\frac{x_{\scriptscriptstyle B}}{x_p}\right)\,\theta\left(\boldsymbol{k}_\perp^2-\mu^2\right)\\ &\qquad\times\frac{2-2x_p+x_p^2-2z_k+4x_pz_k-2x_p^2z_k+z_k^2-2x_pz_k^2+2x_p^2z_k^2}{(1-x_p)(1-z_k)}\\ &= f_1(x_{\scriptscriptstyle B},\mu^2)\\ &+\frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1\frac{dx_p}{x_p}\int_0^1 dz_k\,\theta\left(\frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2-\mu^2\right)\,f_1\left(\frac{x_{\scriptscriptstyle B}}{x_p}\right)\\ &\qquad\times\left\{\frac{1+x_p^2}{(1-x_p)\,(1-z_k)}+\frac{1-z_k-2x_p+2z_kx_p-2z_kx_p^2}{1-x_p}\right\} \end{aligned} \tag{9.24}$$

$$2\,F_L(x_{\scriptscriptstyle B},Q^2) = \frac{g^2\,C_F}{(2\pi)^3}\int_{x_B}^1\frac{dx_p}{x_p}\int_0^1 dz_k d^2\boldsymbol{k}_\perp\,\delta\left(\boldsymbol{k}_\perp^2-\frac{z_k(1-z_k)(1-x_p)}{x_p}\,Q^2\right)\,4x_pz_k\,f_1\left(\frac{x_{\scriptscriptstyle B}}{x_p}\right). \tag{9.25}$$

The lower limit in the x_p integration comes from the support property of $\Phi(p)$, namely $P^+-p^+\geq 0$ or $p^+/P^+ = x_{\scriptscriptstyle B}/x_p \leq 1$. We note a singular part $\propto 1/(1-z_k)$ in F_T coming from the first term in the calculation. This is a collinear singularity ($\boldsymbol{k}_\perp^2\to 0$) which is regulated by the theta function.

In order to perform the integration over z_k one notes that

$$\begin{aligned} \int_0^{1-\delta} dz_k\;\frac{f(z_k)}{1-z_k} &= \int_0^1 dz_k\;\frac{f(z_k)-f(1)}{1-z_k}-f(1)\ln(\delta)\\ &= \int_0^1 dz_k\;\frac{f(z_k)}{(1-z_k)_+}-f(1)\ln(\delta), \end{aligned} \tag{9.26}$$

or in a functional sense

$$\frac{1}{1-z_k} = \frac{1}{(1-z_k)_+} - \delta(1-z_k)\,\ln\delta. \tag{9.27}$$

One obtains

$$\begin{aligned} 2\,F_T(x_B,Q^2) &= f_1(x_B;\mu^2) + \frac{g^2\,C_F}{8\pi^2}\int_{x_B}^1 \frac{dx_p}{x_p}\,f_1\left(\frac{x_B}{x_p}\right)\left\{\frac{1+x_p^2}{1-x_p}\,\ln\left(\frac{Q^2}{\mu^2}\right)\right. \\ &\qquad \left. + \frac{1+x_p^2}{1-x_p}\,\ln\left(\frac{1-x_p}{x_p}\right) + \frac{1-2x_p-2x_p^2}{2(1-x_p)}\right\}, \qquad (9.28) \\ 2\,F_L(x_B,Q^2) &= \frac{g^2\,C_F}{8\pi^2}\int_{x_B}^1 dx_p\;2\,x_p\,f_1\left(\frac{x_B}{x_p}\right) = \frac{\alpha_s}{2\pi}\int_{x_B}^1 \frac{dy}{y}\;2\,C_F\,\frac{x_B}{y}\,f_1(y). \qquad (9.29) \end{aligned}$$

The first expressions gives the scale dependence of $f_1(x;\mu^2)$, the second the perturbative result for the longitudinal structure function. The scale dependence of the structure function is determined by the singular term and requires an appropriate treatment of the singularities near $x_p = 1$.

$$f_1(x_B;Q^2) = f_1(x_B;\mu^2) + \frac{\alpha_s}{2\pi}\int_{x_B}^1 \frac{dy}{y}\,P_{qq}\left(\frac{x_B}{y}\right)\,f_1(y)\ln\left(\frac{Q^2}{\mu^2}\right), \tag{9.30}$$

where

$$P_{qq}(z) = C_F\,\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(1-z)\right]. \tag{9.31}$$

Since the singular piece only comes from the first of the four terms in the calculation, it is possible to obtain the evolution without considering the full process and only consider the gluon ladder graph contribution in the quark-quark correlation function. Implicitly here is of course the use in a hard scattering process with a large momentum scale, which defines the lightlike direction n_-. Requiring that $k^+ = xP^+$ and implementing momentum conservation, the momenta for $q(p) \to q(k) + G(l)$ can be parametrized as

$$\begin{aligned} k &= \left[p^- - \frac{x_p}{1-x_p}\,\frac{(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp)^2}{2x\,P^+},\; xP^+,\; \boldsymbol{k}_\perp\right], \\ l &= \left[\frac{x_p}{1-x_p}\,\frac{(\boldsymbol{k}_\perp - \boldsymbol{p}_\perp)^2}{2x\,P^+},\; \frac{1-x_p}{x_p}\,xP^+,\; \boldsymbol{p}_\perp - \boldsymbol{k}_\perp\right], \\ p &= \left[p^-,\; \frac{x}{x_p}\,P^+,\; \boldsymbol{p}_\perp\right]. \end{aligned} \tag{9.32}$$

The quantities k^2, $2k\cdot P$ and $(P-k)^2$ then can be expressed in $\boldsymbol{k}_T^2$ and x_p,

$$\tau = k^2 = 2k^+p^- - \frac{x_p}{1-x_p}\,(\boldsymbol{p}_\perp^2 - 2\boldsymbol{k}_\perp\cdot\boldsymbol{p}_\perp) - \frac{\boldsymbol{k}_\perp^2}{1-x_p} \approx -\frac{\boldsymbol{k}_\perp^2}{1-x_p}. \tag{9.33}$$

$$\sigma = 2k\cdot P = -\frac{x_p}{1-x_p}\,\frac{\boldsymbol{k}_T^2}{x} + x\,M^2. \tag{9.34}$$

$$M_R^2 = (P-k)^2 \approx \left(\frac{x_p}{x} - 1\right)\,\frac{\boldsymbol{k}_T^2}{x}. \tag{9.35}$$

In Fig. 9.1 we show the region limited by $x \le x_p \le 1$ and $\boldsymbol{k}_T^2 \le \mu^2$, which is described in the perturbative calculation. Using the 'large' ($Q^2 \gg \boldsymbol{k}_\perp^2$) vector

$$q = \left[\frac{Q^2}{2x\,P^+},\; -x\,P^+,\; \boldsymbol{0}_\perp\right] \tag{9.36}$$

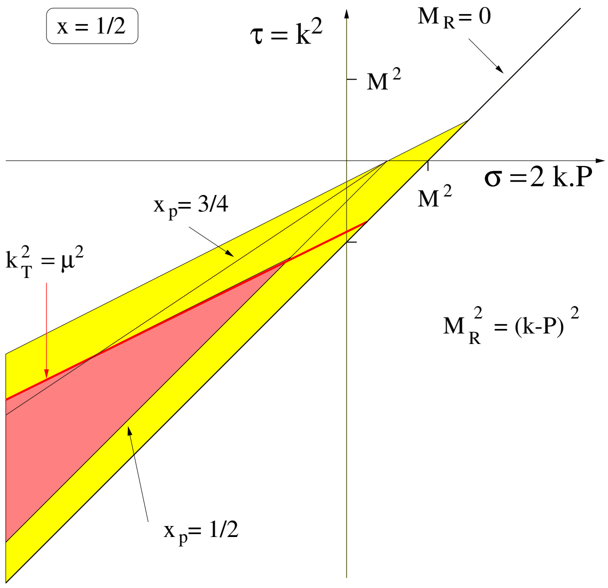


Figure 9.1: The (dark-shaded) region $\boldsymbol{k}_T^2 \geq \mu^2$ and $x \leq x_p \leq 1$ for $x = 1/2$ which is described by the ladder diagram.

for the choice of gauge, one obtains

$$\begin{aligned}
\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp) &= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp) \\
&\quad + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int dk^- \int dl^- dl^+ d^2\boldsymbol{l}_\perp\,\delta(l^2)\,d_{\alpha\beta}(l)\,\frac{\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}\gamma^+\not{k}\gamma^\beta]}{(k^2)^2} \\
&= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp) \\
&\quad + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int \frac{dl^+}{2l^+}\,d_{\alpha\beta}(l)\,\frac{\int dp^- d^2\boldsymbol{p}_\perp\,\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}\gamma^+\not{k}\gamma^\beta]}{(k^2)^2} \\
&= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,\Phi^{[\gamma^+]}(x,\boldsymbol{k}_\perp) \\
&\quad + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{4}\int \frac{dx_p}{x_p^2}\,\frac{x_p}{1-x_p}\,d_{\alpha\beta}(l)\,\frac{\int dp^- d^2\boldsymbol{p}_\perp\,\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}\gamma^+\not{k}\gamma^\beta]}{(k^2)^2}
\end{aligned}$$

or

$$\begin{aligned}
f_1(x,\boldsymbol{k}_\perp^2) &= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,f_1(x,\boldsymbol{k}_\perp^2) + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{\alpha_s\,C_F}{2\pi}\,\frac{1}{\pi\,\boldsymbol{k}_\perp^2}\int dx_p\,\frac{1-x_p}{x_p}\,f_1\left(\frac{x}{x_p}\right) \\
&\qquad \times \frac{Q^4\,(1+x_p^2)\,(1-x_p)^2 + 2Q^2\,\boldsymbol{k}_\perp^2\,x_p^3\,(1-x_p) + \boldsymbol{k}_\perp^4\,x_p^4}{\left(Q^2\,(1-x_p)^2 - \boldsymbol{k}_\perp^2\,x_p^2\right)^2} + \ldots \\
&= \theta(\mu^2 - \boldsymbol{k}_\perp^2)\,f_1(x,\boldsymbol{k}_\perp^2) + \theta(\boldsymbol{k}_\perp^2 - \mu^2)\,\frac{\alpha_s(\mu^2)}{2\pi}\,\frac{1}{\pi\,\boldsymbol{k}_\perp^2}\int_x^1 \frac{dy}{y}\,P_{qq}\left(\frac{x}{y}\right)\,f_1(y,\mu^2) + \ldots
\end{aligned} \tag{9.37}$$

where the last step includes the iteration of ladder graphs. The splitting function $P_{qq}(z)$ is as given above. The appropriate coefficient of the δ function $\delta(1-z)$ is easily found by using the result for $f_1(x,\boldsymbol{k}_\perp)$ for a free quark. This requires that $\int dz\,P_{qq}(z) = 0$.

9.2 Semi-inclusive leptoproduction

For 1-particle inclusive lepton-hadron scattering the tree-level result is given by

$$\begin{aligned} 2M\,\mathscr{W}^{\mu\nu}(q,P,P_h) &= \int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,\Delta(k)\,\gamma^\nu\right)\;\theta\left(\mu^2-\boldsymbol{p}_T^2\right)\,\theta\left(\mu^2-\boldsymbol{k}_T^2\right) \\ &= \int d^2\boldsymbol{p}_T\,\theta\left(\mu^2-\boldsymbol{p}_T^2\right)\int d^2\boldsymbol{k}_T\,\theta\left(\mu^2-\boldsymbol{k}_T^2\right)\,\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T) \\ &\qquad \int dp^-\int dk^+\;\mathrm{Tr}\left[\Phi(p)\,\gamma^\mu\,\Delta(k)\,\gamma^\nu\right]. \end{aligned} \tag{9.38}$$

The natural frame to work in is the one in which the two hadrons do not have transverse components (frame II). The momenta of the photon and the hadrons can be parametrized

$$P_h = \left[\bar{z}_h\,\frac{\tilde{Q}}{\sqrt{2}},\,\frac{M_h^2}{\bar{z}_h\tilde{Q}\sqrt{2}},\,\boldsymbol{0}_T\right], \tag{9.39}$$

$$q = \left[\frac{\tilde{Q}}{\sqrt{2}},\,-\frac{\tilde{Q}}{\sqrt{2}},\,\boldsymbol{q}_T\right], \tag{9.40}$$

$$P = \left[\frac{\bar{x}_B M^2}{\tilde{Q}\sqrt{2}},\,\frac{1}{\bar{x}_B}\,\frac{\tilde{Q}}{\sqrt{2}},\,\boldsymbol{0}_T\right]. \tag{9.41}$$

We note that $Q^2 = \tilde{Q}^2 + Q_T^2$, where $Q_T^2 = \boldsymbol{q}_T^2 = g_T^{\mu\nu}q_\mu q_\nu$. We do not assume that Q_T^2 is small at this point. Up to $\mathscr{O}(1/Q^2)$ corrections one has

$$\bar{x}_B = -\frac{q^+}{P^+} = -\frac{P_h\cdot q}{P_h\cdot P}, \tag{9.42}$$

$$\bar{z}_h = \frac{P_h^-}{q^-} = \frac{P\cdot P_h}{P\cdot q} \tag{9.43}$$

while the usual scaling variables become

$$x_B = \frac{Q^2}{2P\cdot q} = \bar{x}_B\,\frac{Q^2}{\tilde{Q}^2} = \bar{x}_B\left(1-\frac{Q_T^2}{Q^2}\right)^{-1}, \tag{9.44}$$

$$z_h = -\frac{2P_h\cdot q}{Q^2} = \bar{z}_h\,\frac{\tilde{Q}^2}{Q^2} = \bar{z}_h\left(1-\frac{Q_T^2}{Q^2}\right). \tag{9.45}$$

The quark momenta are parametrized as

$$k = \left[z_k\,\frac{\tilde{Q}}{\sqrt{2}},\,k^+,\,\boldsymbol{k}_T\right], \tag{9.46}$$

$$p = \left[p^-,\,\frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\,\boldsymbol{p}_T\right], \tag{9.47}$$

with for the tree-level calculation after implementing energy-momentum conservation $z_k = x_p = 1$. This condition must be dropped when one considers perturbative corrections, in which case one obtains the additional contribution

$$\begin{aligned} 2M\,\mathscr{W}^{\mu\nu}(q,P,P_h) &= \ldots \\ &+ \frac{g^2\,C_F}{(2\pi)^3}\int d^4p\,d^4k\,d^4l\;\delta^4(p+q-k-l)\,\delta(l^2)\,d_{\alpha\beta}(l) \\ &\qquad \times\Big\{\theta(\boldsymbol{p}_T'^2-\mu^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{t}^2 \\ &\qquad\quad +\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{s}\hat{t} \\ &\qquad\quad +\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\mu^2-\boldsymbol{k}_T^2)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}\hat{t} \\ &\qquad\quad +\theta(\mu^2-\boldsymbol{p}_T^2)\,\theta(\boldsymbol{k}_T'^2-\mu^2)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}^2\Big\}, \end{aligned} \tag{9.48}$$

where $\boldsymbol{p}'_T = \boldsymbol{p}_T - \boldsymbol{l}_T$ and $\boldsymbol{k}'_T = \boldsymbol{k}_T + \boldsymbol{l}_T$. Implementing energy-momentum conservation we get

$$\begin{aligned} 2M\,\mathcal{W}^{\mu\nu}(q,P,P_h) &= \ldots \\ &+ \frac{g^2\,C_F}{(2\pi)^3}\int dp^-\,dk^+\,d^2\boldsymbol{p}_T\,d^2\boldsymbol{k}_T\,dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T-\boldsymbol{l}_T)\,\delta(l^2)\,d_{\alpha\beta}(l) \\ &\qquad \times \big\{\theta(\boldsymbol{p}'^2_T-\mu^2)\,\theta(\mu^2-\boldsymbol{k}^2_T)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha \not{k}_t\gamma^\mu\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{t}^2 \\ &\qquad + \theta(\mu^2-\boldsymbol{p}^2_T)\,\theta(\mu^2-\boldsymbol{k}^2_T)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{s}\hat{t} \\ &\qquad + \theta(\mu^2-\boldsymbol{p}^2_T)\,\theta(\mu^2-\boldsymbol{k}^2_T)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}\hat{t} \\ &\qquad + \theta(\mu^2-\boldsymbol{p}^2_T)\,\theta(\boldsymbol{k}'^2_T-\mu^2)\,\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}^2\big\}, \end{aligned} \tag{9.49}$$

and the appropriate parametrization for the gluon momentum l is

$$l = \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\,\frac{1-x_p}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\boldsymbol{l}_T\right], \tag{9.50}$$

First we consider the Mandelstam variables starting with

$$k_s = p+q = k+l = \left[\frac{\tilde{Q}}{\sqrt{2}},\,\frac{1-x_p}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\boldsymbol{k}'_T\right], \tag{9.51}$$

$$k_t = p-l = k-q = \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\,\frac{\tilde{Q}}{\sqrt{2}},\boldsymbol{p}'_T\right], \tag{9.52}$$

$$k_u = p-k = l-q = \left[-z_k\,\frac{\tilde{Q}}{\sqrt{2}},\,\frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\boldsymbol{p}_T-\boldsymbol{k}_T\right]. \tag{9.53}$$

The Mandelstam variables for the subprocess are

$$\hat{s} = k_s^2 = \frac{1-x_p}{x_p}\,\tilde{Q}^2 - 2\,\boldsymbol{k}_T\cdot\boldsymbol{l}_T = \frac{1-x_p}{x_p}\,Q^2 - \frac{Q_T^2}{x_p} - 2\,\boldsymbol{p}_T\cdot\boldsymbol{q}_T, \tag{9.54}$$

$$\hat{t} = k_t^2 = -\frac{1-z_k}{x_p}\,\tilde{Q}^2 + 2\,\boldsymbol{p}_T\cdot\boldsymbol{l}_T = -(1-z_k)\,Q^2 - z_k\,Q_T^2 + 2\,\boldsymbol{k}_T\cdot\boldsymbol{q}_T, \tag{9.55}$$

$$\hat{u} = k_u^2 = -\frac{z_k}{x_p}\,\tilde{Q}^2 = -\frac{z_k}{x_p}\,(Q^2-Q_T^2), \tag{9.56}$$

satisfying $\hat{s}+\hat{t}+\hat{u}+Q^2=0$ and the inverse relations are:

$$x_p = \frac{\tilde{Q}^2}{\hat{s}+Q^2+2\,\boldsymbol{p}_T\cdot\boldsymbol{q}_T}, \tag{9.57}$$

$$z_k = -\frac{\hat{t}+Q^2-2\,\boldsymbol{k}_T\cdot\boldsymbol{q}_T}{\tilde{Q}^2}. \tag{9.58}$$

The implementation of the l-integration and $\delta(l^2)$ depends on which of the four terms one considers. For the first term,

$$\text{in term 1:}\qquad \int dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta(l^2) = \int\frac{dl^+}{2l^+}\int d^2\boldsymbol{p}'_T = \frac{1}{2}\int_{x_B}^1\frac{dx_p}{x_p}\,\frac{1}{1-x_p}\int d^2\boldsymbol{p}'_T, \tag{9.59}$$

(the range in x_p coming from the support of $\Phi(p)$) leading to the contribution

$$\begin{aligned} \ldots\ + \int d^2\boldsymbol{p}'_T\,\theta(\boldsymbol{p}'^2_T-\mu^2)\,d^2\boldsymbol{k}_T\,\theta(\mu^2-\boldsymbol{k}^2_T)\;\delta^2(\boldsymbol{p}'_T+\boldsymbol{q}_T-\boldsymbol{k}_T)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int_{x_B}^1\frac{dx_p}{x_p}\,\frac{1}{1-x_p} \\ \times\int dp^-\,d^2\boldsymbol{p}_T\int dk^+\;d_{\alpha\beta}(l)\,\mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{t}^2 \end{aligned} \tag{9.60}$$

with

$$\text{in term 1:}\qquad 1-z_k = \frac{x_p}{1-x_p}\,\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2} = \frac{x_p}{1-x_p}\,\frac{(\boldsymbol{p}'_T-\boldsymbol{p}_T)^2}{\tilde{Q}^2}. \tag{9.61}$$

For the fourth term we rewrite

$$\text{in term 4:} \qquad dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta(l^2) = \int \frac{dl^-}{2l^-}\,d^2\boldsymbol{k}'_T = \frac{1}{2}\int_{\bar{z}_h}^1 dz_k\,\frac{1}{1-z_k}\int d^2\boldsymbol{k}'_T, \tag{9.62}$$

(the support in z_k coming from the support of $\Delta(k)$) and obtain

$$\begin{aligned}\ldots\quad &+\int d^2\boldsymbol{p}_T\,\theta(\mu^2-\boldsymbol{p}_T^2)\,d^2\boldsymbol{k}'_T\,\theta(\boldsymbol{k}'^2_T-\mu^2)\;\delta^2(\boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}'_T)\,\frac{g^2\,C_F}{(2\pi)^3}\,\frac{1}{2}\int_{\bar{z}_h}^1 dz_k\,\frac{1}{1-z_k}\\ &\times\int dp^-\int dk^+\,d^2\boldsymbol{k}_T\;d_{\alpha\beta}(l)\,\mathrm{Tr}[\Phi(p)\gamma^\mu \not{k}_s\gamma^\alpha\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}^2\end{aligned} \tag{9.63}$$

with

$$\text{in term 4:} \qquad \frac{1-x_p}{x_p} = \frac{1}{x_p}-1 = \frac{1}{1-z_k}\,\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2} = \frac{1}{1-z_k}\,\frac{(\boldsymbol{k}'_T-\boldsymbol{k}_T)^2}{\tilde{Q}^2}. \tag{9.64}$$

For the other two terms we use

$$\text{in term 2 \& 3:} \qquad \int dl^-\,dl^+\,d^2\boldsymbol{l}_T\;\delta(l^2) = \frac{1}{2}\int_{\bar{z}_h}^1 dz_k\int_{\bar{x_B}}^1 \frac{dx_p}{x_p^2}\int d^2\boldsymbol{l}_T\;\delta\left(\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2}-\frac{(1-z_k)(1-x_p)}{x_p}\right), \tag{9.65}$$

leading to the contribution

$$\begin{aligned}\ldots\quad &+\frac{g^2\,C_F}{(2\pi)^3}\int d^2\boldsymbol{p}_T\,\theta(\mu^2-\boldsymbol{p}_T^2)\,d^2\boldsymbol{k}_T\,\theta(\mu^2-\boldsymbol{k}_T^2)\\ &\times\frac{1}{2}\int_{z_h}^1 dz_k\int_{\bar{x_B}}^1\frac{dx_p}{x_p}\,\frac{1}{1-x_p}\int d^2\boldsymbol{l}_T\;\delta\left(\frac{\boldsymbol{l}_T^2}{\tilde{Q}^2}-\frac{(1-z_k)(1-x_p)}{x_p}\right),\\ &\times\int dp^-\int dk^+\;d_{\alpha\beta}(l)\Big\{\mathrm{Tr}[\Phi(p)\gamma^\mu\not{k}_s\gamma^\alpha\Delta(k)\gamma^\nu\not{k}_t\gamma^\beta]/\hat{s}\hat{t}\\ &\qquad + \mathrm{Tr}[\Phi(p)\gamma^\alpha\not{k}_t\gamma^\mu\Delta(k)\gamma^\beta\not{k}_s\gamma^\nu]/\hat{s}\hat{t}\Big\},\end{aligned} \tag{9.66}$$

in which to use $\boldsymbol{l}_T = \boldsymbol{p}_T+\boldsymbol{q}_T-\boldsymbol{k}_T$ and consequently $\boldsymbol{p}'_T = \boldsymbol{k}_T-\boldsymbol{q}_T$ and $\boldsymbol{k}'_T = \boldsymbol{p}_T+\boldsymbol{q}_T$.

9.3 Ordering the transverse momentum dependence

We make an angular expansion of a quark or gluon correlator in field space, only making the angular structure of p_T explicit. We write

$$\Phi(x, p_T) = \Phi(x, p_T^2) + p_T^i\, \Phi_i(x, p_T^2) + p_T^{ij}\, \Phi_{ij}(x, p_T^2) + \ldots \tag{9.67}$$

We make a table with in the columns all possible TMD functions of the type $\Phi(x, p_T)$, $p_T^i\, \Phi(x, p_T)$, etc. We use abbreviations $\Phi^{(n)}(x, p_T^2) \equiv (-p_T^2/2)^n\, \Phi(x, p_T^2)$.

Symmetry in p_T			
TMDs	rank 0	rank 1	rank 2
$\Phi(x, p_T^2)$	Φ	–	–
$p_T^\alpha\, \Phi$	–	$p_T^\alpha\, \Phi$	–
$p_T^{\alpha\beta}\, \Phi$	–	–	$p_T^{\alpha\beta}\, \Phi$
$p_T^i\, \Phi_i(x, p_T^2)$	–	$p_T^i\, \Phi_i$	–
$p_T^\alpha p_T^i\, \Phi_i$	$\Phi^{(1)\alpha}$	–	$p_T^{\alpha i}\Phi_i$
$p_T^{\alpha\beta} p_T^i\, \Phi_i$	–	$-\frac{1}{2}\, p_T^{\{\alpha}\Phi^{(1)\beta\}}$ $+\frac{1}{2}\, g_T^{\alpha\beta}\, p_T^i\, \Phi_i^{(1)}$	–
$p_T^{ij}\, \Phi_{ij}(x, p_T^2)$	–	–	$p_T^{ij}\, \Phi_{ij}$
$p_T^\alpha p_T^{ij}\, \Phi_{ij}$	–	$-p_T^i \Phi_i^{(1)\alpha}$	–
$p_T^{\alpha\beta} p_T^{ij}\, \Phi_{ij}$	$\Phi^{(2)\alpha\beta}$	–	$-\frac{2}{3}\, p_T^{i\{\alpha}\Phi_i^{(1)\beta\}}$ $+\frac{2}{3}\, g_T^{\alpha\beta}\, p_T^{ij}\, \Phi_{ij}^{(1)}$
$p_T^{ijk}\, \Phi_{ijk}(x, p_T^2)$	–	–	–
$p_T^\alpha p_T^{ijk}\, \Phi_{ijk}$	–	–	$-p_T^{ij}\, \Phi_{ij}^{(1)\alpha}$
$p_T^{\alpha\beta} p_T^{ijk}\, \Phi_{ijk}$	–	$\ldots$	–

Symmetry in ∂_T			
TMDs	rank 0	rank 1	rank 2
$\Phi(x, p_T)$	Φ	–	–
$p_T^\alpha\, \Phi$	–	$\tilde{\Phi}_\partial^\alpha$	–
$p_T^{\alpha\beta}\, \Phi$	$\pi^2\, C_{GG}\, \Phi_{GG}^{\alpha\beta}$	–	$\tilde{\Phi}_{\partial\partial}^{\alpha\beta}$
$\Phi(x, p_T)$	–	–	–
$p_T^\alpha\, \Phi$	$\pi\, C_G\, \Phi_G^\alpha$	–	–
$p_T^{\alpha\beta}\, \Phi$	–	$\frac{\pi}{2}\, C_G \left(\tilde{\Phi}_{\partial G}^{\{\alpha\beta\}} + \tilde{\Phi}_{G\partial}^{\{\alpha\beta\}}\right)$ $-\frac{\pi}{2}\, C_G\, g_T^{\alpha\beta} \left(\tilde{\Phi}_{G\partial}^{ii} + \tilde{\Phi}_{\partial G}^{ii}\right)$	–

9.4 Evolution and transverse momentum dependence

In order to investigate generally the high $\boldsymbol{p}_T$-tails of the distribution and fragmentation functions, we separate the transverse momentum dependence and Lorentz structure, relevant to determine the leading or subleading character of the correlation functions from the Dirac and target spin structure. Using p_T^α and

$$p_T^{\alpha\beta} = p_T^\alpha p_T^\beta - \frac{1}{2}\, p_T^2\, g_T^{\alpha\beta}, \tag{9.68}$$

in order to make an angular expansion in the transverse momenta, we write (we use $\gamma^+ = \not{n}$ and $\gamma^- = \not{P}$ with a dimensionful n-vector, satisfying $P \cdot n = 1$)

$$\begin{aligned}\Phi(x, p_T) \quad = \quad & \left[\Phi_2(x, p_T^2) + \left(\frac{p_{T\,\alpha}}{M}\right)\, \Phi_2^\alpha(x, p_T^2) + \left(\frac{p_{T\,\alpha\beta}}{M^2}\right)\, \Phi_2^{\alpha\beta}(x, p_T^2) + \ldots\right] \not{P} \\ & + M\left[\Phi_3(x, p_T^2) + \left(\frac{p_{T\,\alpha}}{M}\right)\, \Phi_3^\alpha(x, p_T^2) + \ldots\right] \\ & + M^2\left[\Phi_4(x, p_T^2) + \left(\frac{p_{T\,\alpha}}{M}\right)\, \Phi_4^\alpha(x, p_T^2) + \ldots\right] \not{n} \\ & + \ldots \end{aligned} \tag{9.69}$$

where the quantities $\Phi_t^{\alpha\cdots}(x, p_T^2)$ are symmetric and traceless in the transverse indices $(\alpha\ldots)$ and depend on the arguments x and p_T^2 and get the (transverse) Lorentz structure from the Dirac gamma-matrices and the spin vectors. The integrated and p_T-weighted quantities are given by

$$\Phi(x) = \int d^2p_T\; \Phi(x, p_T) = \Phi_2(x)\, \not{P} + M\, \Phi_3(x) + M^2\, \Phi_4(x)\, \not{n} \tag{9.70}$$

$$\frac{1}{M}\, \Phi_\partial^\alpha(x) = \int d^2p_T\; \left(\frac{p_T^\alpha}{M}\right)\, \Phi(x, p_T) = \Phi_2^{\alpha(1)}(x)\, \not{P} + M\, \Phi_3^{\alpha(1)}(x) + \ldots \tag{9.71}$$

$$\frac{1}{M^2}\, \Phi_{\partial\partial}^{\alpha\beta}(x) = \int d^2p_T\; \left(\frac{p_{T\,\alpha\beta}}{M^2}\right)\, \Phi(x, p_T^2) = \Phi_2^{\alpha\beta(2)}(x)\, \not{P} + \ldots, \tag{9.72}$$

with the transverse moments defined as

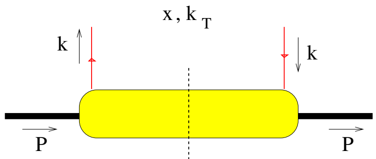
$$\Phi_t^{\alpha\ldots(n)}(x, p_T^2) \equiv \left(\frac{-p_T^2}{2M^2}\right)^n\; \Phi_t^{\alpha\cdots}(x, p_T^2). \tag{9.73}$$

Similarly we have the (integrated) quark-quark-gluon correlation functions

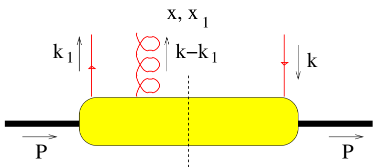
$$\Phi_A^\alpha(x, x_1) = M\, \Phi_{A;3}^\alpha(x, x_1)\, \not{P} + M^2\, \Phi_{A;4}^\alpha(x, x_1) + \ldots, \tag{9.74}$$

$$\Phi_{AA}^{\alpha\beta}(x, x_1, x_2) = M^2\, \Phi_{AA;4}^{\alpha\beta}(x, x_1, x_2)\, \not{P} + \ldots, \tag{9.75}$$

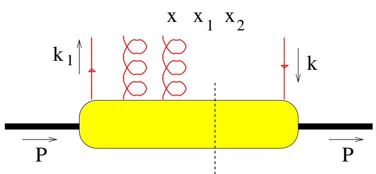
and similarly $\Phi_D^\alpha(x, x_1)$, etc. Of course, the non-integrated correlation functions can be expanded in the transverse momenta as done above. Some parts of $\Phi_A^\alpha(x, x_1)$ can after integration over one of the arguments be related to the (twist-three) quark-quark correlation functions $\Phi_3(x)$ via the QCD equations of motion. In the above parametrization, the functions on the right-hand side depending on x and p_T^2 have canonical dimension -2, while functions depending on x are dimensionless. Graphically represented we have



The quantity $\Phi(x, k_T) = \int dk \cdot P\; \Phi(k; P, S)$ with canonical dimension -1 or after further integration the quantities $\Phi(x)$ and $\frac{1}{M}\, \Phi_\partial^\alpha(x)$.



$\Phi_A^\alpha(x, x_1)$



$\Phi_{AA}^{\alpha\beta}(x, x_1, x_2)$

A hard part can be connected to the soft parts via an integration over the momenta of the soft part, $\int d^4k\,d^4k_1\,\ldots$ Taking the dimensions following from this integration into account we get for the simplest kernel, starting with two quarks with the relevant momentum components being y and k_T and leaving two quarks with non-integrated momentum components x and p_T

$$\begin{aligned}
&\int d^4k\,\ldots H(p;k) = \\
&\quad \frac{\alpha_s}{p_T^2}\int dy\,d^2k_T\,\ldots\Big\{\not n\,H_{2,2}(y;x)\,\not P + H_{3,3}(y;x) + \not P\,H_{4,4}(y;x)\,\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{1}{p_T^2}\int dy\,d^2k_T\,\ldots\not P\,H_{4,2}(y,x)\,\not P\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{p_{Tij}}{p_T^2}\int dy\,d^2k_T\,\ldots\Big\{\not n\,H^{ij}_{2,2}(y,x)\,\not P + H^{ij}_{3,3}(y,x) + \not P\,H^{ij}_{4,4}(y,x)\,\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{1}{p_T^2}\int dy\,d^2k_T\,\ldots\Big\{k_{Tij}\,\not n H'^{ij}_{2,2}(y,x)\,\not P + k_{Tij}\,H'^{ij}_{3,3}(y,x) + k_{Tij}\,\not P H'^{ij}_{4,4}(y,x)\,\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{p_{Ti}}{p_T^2}\int dy\,d^2k_T\,\ldots\Big\{k_{Tj}\,\not n H''^{j,i}_{2,2}(y,x)\,\not P + k_{Tj}\,H''^{j,i}_{3,3}(y,x) + k_{Tj}\,\not P H''^{j,i}_{4,4}(y,x)\,\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{p_{Ti}}{p_T^2}\int dy\,d^2k_T\,\ldots\Big\{H^{i}_{3,2}(y,x)\not P + \not P H^{i}_{4,3}(y,x)\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{1}{p_T^2}\int dy\,d^2k_T\,\ldots\Big\{k_{Ti}\,H^{i}_{3,2}(y,x)\,\not P + k_{Ti}\,\not P H^{i}_{4,3}(y,x)\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,p_{Ti}\int dy\,d^2k_T\,\ldots\Big\{\not n H^{i}_{2,3}(y,x) + H^{i}_{3,4}(y,x)\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\int dy\,d^2k_T\,\ldots\Big\{k_{Ti}\,\not n\,H'^{i}_{2,3}(y,x) + k_{Ti}\,H'^{i}_{3,4}(y,x)\not n\Big\}\\
&\quad + \ldots
\end{aligned} \tag{9.76}$$

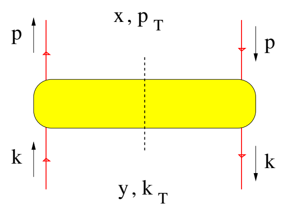
(the Dirac structure preceding H refers to the fermions with momentum k, the structure following H to the fermions with momentum p). The kernel connecting the soft quark-quark-gluon correlator to a quark-quark correlator can be analyzed to contain (we now make the k_T dependence implicit)

$$\begin{aligned}
&\int d^4k\,d^4k_1\,\ldots H^{\alpha}_{A}(p;k,k_1) = \\
&\quad \frac{\alpha_s}{p_T^2}\,\frac{1}{p_T^2}\int dy\,dy_1\,\ldots\Big\{H^{\alpha}_{A;4,2}(y,y_1;x)\not P + \not P H^{\alpha}_{A;5,3}(y,y_1;x)\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{p_{Ti}}{p_T^2}\int dy\,dy_1\,\ldots\Big\{\not n H^{\alpha;i}_{A;3,2}(y,y_1;x)\not P + H^{\alpha;i}_{A;4,3}(y,y_1;x) + \not P H^{\alpha;i}_{A;5,4}(y,y_1;x)\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{1}{p_T^2}\int dy\,dy_1\,\ldots\Big\{k_{Ti}\,\not n H'^{\alpha;i}_{A;3,2}(y,y_1;x)\not P + k_{Ti}\,H'^{\alpha;i}_{A;4,3}(y,y_1;x) + k_{Ti}\,\not P H'^{\alpha;i}_{A;5,4}(y,y_1;x)\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\int dy\,dy_1\,\ldots\Big\{\not n H^{\alpha}_{A;3,3}(y,y_1;x) + H^{\alpha}_{A;4,4}(y,y_1;x)\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{p_{Tij}}{p_T^2}\int dy\,dy_1\,\ldots\Big\{\not n H^{\alpha;ij}_{A;3,3}(y,y_1;x) + H^{\alpha;ij}_{A;4,4}(y,y_1;x)\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{1}{p_T^2}\int dy\,dy_1\,\ldots\Big\{k_{Tij}\,\not n H'^{\alpha;ij}_{A;3,3}(y,y_1;x) + k_{Tij}\,H'^{\alpha;ij}_{A;4,4}(y,y_1;x)\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,\frac{p_{Ti}}{p_T^2}\int dy\,dy_1\,\ldots\Big\{k_{Tj}\,\not n H''^{\alpha;j,i}_{A;3,3}(y,y_1;x) + k_{Tj}\,H''^{\alpha;j,i}_{A;4,4}(y,y_1;x)\not n\Big\}\\
&\quad + \frac{\alpha_s}{p_T^2}\,p_{Ti}\int dy\,dy_1\,\ldots\not n H^{\alpha;i}_{A;3,4}(y,y_1;x)\not n\\
&\quad + \frac{\alpha_s}{p_T^2}\int dy\,dy_1\,\ldots k_{Ti}\,\not n H'^{\alpha;i}_{A;3,4}(y,y_1;x)\not n\\
&\quad + \ldots
\end{aligned} \tag{9.77}$$

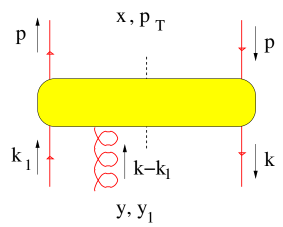
To include the qqGG soft contributions we need

$$
\begin{aligned}
&\int d^4k\, d^4k_1\, d^4k_2\, \ldots H^{\alpha\beta}_{AA}(p;k,k_1,k_2) = \\
&\quad \frac{\alpha_s}{p_T^2}\int dy\, dy_1\, dy_2\, \ldots \not{n} H^{\alpha\beta}_{AA;4,4}(y,y_1,y_2;x)\not{n} \\
&\quad + \frac{\alpha_s}{p_T^2}\frac{1}{p_T^2}\int dy\, dy_1\, dy_2\, \ldots \left\{\not{n} H^{\alpha\beta}_{AA;4,2}(y,y_1,y_2;x)\not{P} + H^{\alpha\beta}_{AA;5,3}(y,y_1,y_2;x) + \not{P} H^{\alpha\beta}_{AA;6,4}(y,y_1,y_2;x)\not{n}\right\} \\
&\quad + \frac{\alpha_s}{p_T^2}\frac{p_{Ti}}{p_T^2}\int dy\, dy_1\, dy_2\, \ldots \left\{\not{n} H^{\alpha\beta;i}_{AA;4,3}(y,y_1,y_2;x) + H^{\alpha\beta;i}_{AA;5,4}(y,y_1,y_2;x)\not{n}\right\} \\
&\quad + \frac{\alpha_s}{p_T^2}\frac{1}{p_T^2}\int dy\, dy_1\, dy_2\, \ldots \left\{k_{Ti}\,\not{n} H'^{\alpha\beta;i}_{AA;4,3}(y,y_1,y_2;x) + k_{Ti}\, H'^{\alpha\beta;i}_{AA;5,4}(y,y_1,y_2;x)\not{n}\right\} \\
&\quad + \frac{\alpha_s}{p_T^2}\frac{p_{Tij}}{p_T^2}\int dy\, dy_1\, dy_2\, \ldots \not{n} H^{\alpha\beta;ij}_{AA;4,4}(y,y_1,y_2;x)\not{n} \\
&\quad + \frac{\alpha_s}{p_T^2}\frac{1}{p_T^2}\int dy\, dy_1\, dy_2\, \ldots k_{Tij}\,\not{n} H'^{\alpha\beta;ij}_{AA;4,4}(y,y_1,y_2;x)\not{n} \\
&\quad + \frac{\alpha_s}{p_T^2}\frac{p_{Ti}}{p_T^2}\int dy\, dy_1\, dy_2\, \ldots k_{Tj}\,\not{n} H''^{\alpha\beta;j,i}_{AA;4,4}(y,y_1,y_2;x)\not{n} \\
&\quad + \ldots
\end{aligned}
\tag{9.78}
$$

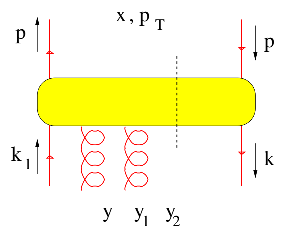
Graphically the kernels are represented by



$\int d^4k\, \ldots H(p;k)$



$\int d^4k\, d^4k_1\, \ldots H^{\alpha}_{A}(p;k,k_1)$



$\int d^4k\, d^4k_1\, d^4k_2\, \ldots H^{\alpha\beta}_{AA}(p;k,k_1,k_2)$

Combining the contributions we can extract the large-p_T-behavior For instance, for the corrections to Φ_2 we find leading contributions proportional to α_s/p_T^2. Collecting these and similarly for other quark-quark correlators, we get

$$\Delta\Phi_2(x,p_T^2) = \frac{\alpha_s}{p_T^2}\int dy\ \Phi_2(y)\not P\not n H_{2,2}(y;x) \tag{9.79}$$

$$\Delta\Phi_2^i(x,p_T^2) = \frac{\alpha_s}{p_T^2}\frac{M^2}{p_T^2}\Big[\int dy\ \Phi_{2\,j}^{(1)}(y)H''^{j,i}_{2,2}(y;x) + \int dy\ \Phi_3(y)H^i_{3,2}(y;x) + \int dy\,dy_1\ \Phi_{A\,\alpha;3}(y,y_1)\not P\not n H^{\alpha;i}_{A;3,2}(y,y_1;x)\Big] \tag{9.80}$$

$$\Delta\Phi_2^{ij}(x,p_T^2) = \frac{\alpha_s}{p_T^2}\frac{M^2}{p_T^2}\int dy\ \Phi_2(y)\not P\not n H^{ij}_{2,2}(y;x) \tag{9.81}$$

$$\Delta\Phi_3(x,p_T^2) = \frac{\alpha_s}{p_T^2}\Big[\int dy\ \Phi_3(y)H_{3,3}(y;x) + \int dy\ \Phi_{2\,i}^{(1)}(y)\not P\not n H'^{\,i}_{2,3}(y;x) + \int dy\,dy_1\ \Phi_{A\,\alpha;3}(y,y_1)\not P\not n H^{\alpha}_{A;3,3}(y,y_1;x)\Big] \tag{9.82}$$

$$\begin{aligned}\Delta\Phi_3^i(x,p_T^2) = {} & \frac{\alpha_s}{p_T^2}\int dy\ \Phi_2(y)\not P\not n H^i_{2,3}(y;x) \\ & + \frac{\alpha_s}{p_T^2}\frac{M^2}{p_T^2}\Big[\int dy\ \Phi_{3\,j}^{(1)}(y)H''^{;j,i}_{3,3}(y;x) + \int dy\ \Phi_4(y)\not n\not P H^i_{4,3}(y;x) \\ & + \int dy\,dy_1\ \Phi_{A\,\alpha;4}(y,y_1)H^{\alpha;i}_{A;4,3}(y,y_1;x) \\ & + \int dy\,dy_1\ \Phi^{(1)}_{A\,\alpha;3\,j}(y,y_1)\not P\not n H''^{\,\alpha;j,i}_{A;3,3}(y,y_1;x) \\ & + \int dy\,dy_1\,dy_2\ \Phi_{AA\,\alpha\beta;4}(y,y_1,y_2)\not P\not n H^{\alpha\beta;i}_{AA;4,3}(y,y_1,y_2;x)\Big]\Big]\end{aligned} \tag{9.83}$$

$$\Delta\Phi_3^{ij}(x,p_T^2) = \frac{\alpha_s}{p_T^2}\frac{M^2}{p_T^2}\Big[\int dy\ \Phi_3(y)H^{ij}_{3,3}(y;x) + \int dy\ \Phi_{A\,\alpha;3}(y,y_1)\not P\not n H^{\alpha;ij}_{A;3,3}(y,y_1;x)\Big] \tag{9.84}$$

$$\begin{aligned}\Delta\Phi_4(x,p_T^2) = {} & \frac{\alpha_s}{p_T^2}\Big[\int dy\ \Phi_4(y)\not n\not P H_{4,4}(y;x) + \int dy\ \Phi_{3\,i}^{(1)}(y)\not P\not n H'^{\,i}_{3,4}(y;x) \\ & + \int dy\,dy_1\ \Phi_{A\,\alpha;4}(y,y_1)H^{\alpha}_{A;4,4}(y,y_1;x) \\ & + \int dy\,dy_1\ \Phi^{(1)}_{A\,\alpha;3\,i}(y,y_1)\not P\not n H'^{\,\alpha;i}_{A;3,4}(y,y_1;x) \\ & + \int dy\,dy_1\,dy_2\ \Phi_{AA\,\alpha\beta;4}(y,y_1,y_2)\not P\not n H^{\alpha\beta}_{AA;4,4}(y,y_1,y_2;x)\Big]\end{aligned} \tag{9.85}$$

$$\Delta\Phi_4^i(x,p_T^2) = \frac{\alpha_s}{p_T^2}\Big[\int dy\ \Phi_3(y)H^i_{3,4}(y;x) + \int dy\,dy_1\ \Phi_{A\,\alpha;3}(y,y_1)\not P\not n H^{\alpha;i}_{A;3,4}(y,y_1;x)\Big] \tag{9.86}$$

$$\begin{aligned}\Delta\Phi_4^{ij}(x,p_T^2) = {} & \frac{\alpha_s}{p_T^2}\Big[\int dy\ \Phi_4(y)\not n\not P H^{ij}_{4,4}(y;x) \\ & + \int dy\,dy_1\ \Phi_{A\,\alpha;4}(y,y_1)H^{\alpha;ij}_{A;4,4}(y,y_1;x) \\ & + \int dy\,dy_1\,dy_2\ \Phi_{AA\,\alpha\beta;4}(y,y_1,y_2)\not P\not n H^{\alpha\beta;ij}_{AA;4,4}(y,y_1,y_2;x)\Big]\end{aligned} \tag{9.87}$$

9.5 Equations of motion for various parts

For the matrix elements we write

$$\begin{aligned}
\Phi &= \gamma^- \Phi_2 + \frac{M}{P^+}\Phi_3 + \left(\frac{M}{P^+}\right)^2 \gamma^+ \Phi_4 && (9.88)\\
&= \frac{1}{2}\gamma^-\gamma^+\left[\gamma^-\Phi_2 + \frac{M}{P^+}\Phi_3\right] + \frac{1}{2}\gamma^+\gamma^-\left[\frac{M}{P^+}\Phi_3 + \left(\frac{M}{P^+}\right)^2\gamma^+\Phi_4\right], && (9.89)\\
\gamma^+\Phi &= \frac{1}{2}\gamma^+\gamma^-\left[2\,\Phi_2 + \frac{M}{P^+}\gamma^+\Phi_3\right], && (9.90)\\
\gamma^-\Phi &= \frac{1}{2}\gamma^-\gamma^+\left[\frac{M}{P^+}\gamma^-\Phi_3 + 2\left(\frac{M}{P^+}\right)^2\Phi_4\right], && (9.91)\\
\frac{1}{M}\Phi_D^\alpha &= \gamma^-\Phi_{D,3}^\alpha + \frac{M}{P^+}\Phi_{D,4}^\alpha + \left(\frac{M}{P^+}\right)^2\gamma^+\Phi_{D,5}^\alpha && (9.92)\\
&= \frac{1}{2}\gamma^-\gamma^+\left[\gamma^-\Phi_{D,3}^\alpha + \frac{M}{P^+}\Phi_{D,4}^\alpha\right] + \frac{1}{2}\gamma^+\gamma^-\left[\frac{M}{P^+}\Phi_{D,4}^\alpha + \left(\frac{M}{P^+}\right)^2\gamma^+\Phi_{D,5}^\alpha\right]. && (9.93)
\end{aligned}$$

The equations of motion imply:

$$\gamma^+ iD^-\psi + \gamma^- iD^+\psi + \gamma^\alpha iD_\alpha\psi - m\,\psi = 0. \quad (9.94)$$

Multiplying with a good projector $\gamma^-\gamma^+/2$, we get for the matrix elements

$$\frac{1}{2}\gamma^-\gamma^+\left[\gamma^-\Phi_D^+ + \gamma_\alpha\,\Phi_D^\alpha - m\,\Phi\right] = \frac{1}{2}\gamma^-\gamma^+\left[xP^+\gamma^-\Phi + \gamma_\alpha\,\Phi_D^\alpha - m\,\Phi\right] = 0 \quad (9.95)$$

or for the leading (twist-3) part in the expansion

$$\gamma_\alpha\,\Phi_{D,3}^\alpha = x\,\Phi_3 - \frac{m}{M}\Phi_2. \quad (9.96)$$

9.6 The unpolarized case

The quark-quark correlation function depending on $k^+ = xP^+$ and $\boldsymbol{k}_T$ is for a spin 0 or unpolarized hadron, including only leading $(M/P^+)^0$ and subleading $(M/P^+)^1$ contributions, given by

$$\begin{aligned}
\Phi(x,\boldsymbol{k}_T) &= \frac{1}{2}\left\{f_1(x,\boldsymbol{k}_T) + i\,h_1^\perp(x,\boldsymbol{k}_T)\,\frac{\not{k}_T}{M}\right\}\gamma^-\\
&+ \frac{M}{2P^+}\left\{e(x,\boldsymbol{k}_T) + f^\perp(x,\boldsymbol{k}_T)\,\frac{\not{k}_T}{M} + i\,h(x,\boldsymbol{k}_T)\,\frac{[\gamma^-,\gamma^+]}{2}\right\} && (9.97)
\end{aligned}$$

The $\boldsymbol{k}_T$-integrated results are

$$\begin{aligned}
\Phi(x) &= \frac{1}{2}f_1(x)\,\gamma^-\\
&+ \frac{M}{2P^+}\left\{e(x) + i\,h(x)\,\frac{[\gamma^-,\gamma^+]}{2}\right\}, && (9.98)\\
\frac{1}{M}\Phi_\partial^\alpha(x) &= \frac{1}{2}\,i\,h_1^{\perp(1)}(x)\,\gamma_T^\alpha\,\gamma^-\\
&+ \frac{M}{2P^+}f^{\perp(1)}(x)\,\gamma_T^\alpha. && (9.99)
\end{aligned}$$

The relevant projections are (we do use the tensor $\sigma^{+-} = (i/2)[\gamma^+, \gamma^-]$ but we use $\gamma_T^\alpha \gamma^+$ instead of the tensor $\sigma^{\alpha+} = i\gamma_T^\alpha \gamma^+$),

$$\Phi^{[\gamma^+]}(x, \boldsymbol{k}_T) = f_1(x, \boldsymbol{k}_T), \tag{9.100}$$

$$\Phi^{[1]}(x, \boldsymbol{k}_T) = \frac{M}{P^+}\, e(x, \boldsymbol{k}_T), \tag{9.101}$$

$$\Phi^{[\sigma^{+-}]}(x, \boldsymbol{k}_T) = \frac{M}{P^+}\, i\, h(x, \boldsymbol{k}_T) \tag{9.102}$$

$$\Phi^{[\gamma_T^\alpha \gamma^+]}(x, \boldsymbol{k}_T) = -i\, k_T^\alpha\, h_1^\perp(x, \boldsymbol{k}_T), \quad \text{and} \quad \Phi_\partial^{\alpha[\gamma^\beta \gamma^+]}(x) = -i\, g_T^{\alpha\beta}\, h_1^{\perp(1)}(x), \tag{9.103}$$

$$\Phi^{[\gamma_T^\alpha]}(x, \boldsymbol{k}_T) = \frac{M}{P^+}\, k_T^\alpha\, f^\perp(x, \boldsymbol{k}_T). \quad \text{and} \quad \Phi_\partial^{\alpha[\gamma^\beta]}(x) = g_T^{\alpha\beta}\, \frac{M}{P^+}\, f^{\perp(1)}(x). \tag{9.104}$$

For the $\overline{\psi}(0)\, iD_T^\alpha \psi(x)$ correlation function (bilocal, i.e. integrated over d^4k_1) one obtains from the equations of motion

$$\frac{1}{M}\, \Phi_D^{\alpha[\gamma^+]}(x, \boldsymbol{k}_T) = \frac{k_T^\alpha}{M} \left(x\, f^\perp + i\, \frac{m}{M}\, h_1^\perp \right) \tag{9.105}$$

$$\frac{1}{M}\, \Phi_{D\alpha}^{[\gamma_T^\alpha \gamma^+]}(x, \boldsymbol{k}_T) = (x\, e - m\, f_1 - i\, x\, h) \tag{9.106}$$

Using the quark-quark correlations multiplied with k_T^α,

$$\frac{k_T^\alpha}{M}\, \Phi^{[\gamma^+]}(x, \boldsymbol{k}_T) = \frac{k_T^\alpha}{M}\, f_1(x, \boldsymbol{k}_T), \tag{9.107}$$

$$\frac{k_{T\alpha}}{M}\, \Phi^{[\gamma_T^\alpha \gamma^+]}(x, \boldsymbol{k}_T) = \frac{1}{M}\, \Phi_{\partial\alpha}^{[\gamma_T^\alpha \gamma^+]}(x, \boldsymbol{k}_T) = 2i\, h_1^{\perp(1)}(x, \boldsymbol{k}_T), \tag{9.108}$$

the difference $\Phi_A^\alpha = \Phi_D^\alpha - k_T^\alpha\, \Phi$ is given by

$$\frac{1}{M}\, \Phi_A^{\alpha[\gamma^+]}(x, \boldsymbol{k}_T) \quad = \quad \frac{k_T^\alpha}{M} \Bigg(\underbrace{x f^\perp - f_1}_{x\tilde{f}^\perp} + i\, \frac{m}{M}\, h_1^\perp \Bigg) \tag{9.109}$$

$$\frac{1}{M}\, \Phi_{A\alpha}^{[\sigma^{\alpha+}]}(x, \boldsymbol{k}_T) \quad = \quad \underbrace{x\, e - \frac{m}{M}\, f_1}_{x\tilde{e}}\, \underbrace{-i\, x\, h - 2i\, h_1^{\perp(1)}}_{-ix\tilde{h}} \tag{9.110}$$

The most general form of quark-quark-gluon correlation functions integrated over all transverse momenta but not integrated over x_1 are

$$\Phi_D^\alpha(x, x_1) = \frac{M}{2P^+}\, E_D(x, x_1) \gamma_T^\alpha\, \gamma^-, \tag{9.111}$$

from which one finds that

$$\frac{1}{M}\, \Phi_D^\alpha(x) = \int dx_1\; E_D(x, x_1)\, \gamma_T^\alpha\, \gamma^- \tag{9.112}$$

(similarly for $\Phi_A(x, x_1)$). The projections are

$$\Phi_{D\alpha}^{[\gamma_T^\alpha \gamma^+]}(x, x_1) = 2\, \frac{M}{P^+}\, E_D(x, x_1) \tag{9.113}$$

$$\frac{1}{M}\, \Phi_{D\alpha}^{[\gamma_T^\alpha \gamma^+]}(x) = 2 \int dx_1\; E_D(x, x_1). \tag{9.114}$$

9.7 Sample calculations

9.7.1 The fermion propagator

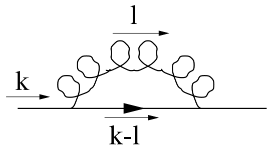
We will demonstrate perturbative calculations, in particular when one uses a lightcone gauge in detail for the example of the self-energy contribution to $\Gamma^{(2)}(p)$. Writing the (truncated) Green's function as

$$\Gamma^{(2)}(k) = -i\, (S_F)^{-1}(k) = -i[\not{k} - m_0 - \Sigma(k)], \tag{9.115}$$

one has the expansion

$$S_F(k) = \frac{1}{\not k - m_0} + \frac{1}{\not k - m_0}\,\Sigma(k)\,\frac{1}{\not k - m_0} + \ldots . \tag{9.116}$$

The contribution of the one-loop diagram,



to the self-energy in the lightcone gauge is given by

$$\Sigma(k) = -ig^2\,C_F \int \frac{d^4l}{(2\pi)^4}\,\frac{\gamma^\alpha(\not k - \not l + m_0)\gamma^\beta\,d_{\alpha\beta}(l)}{(l^2+i\epsilon)\,((k-l)^2 - m_0^2 + i\epsilon)}, \tag{9.117}$$

with

$$d^{\alpha\beta}(l) = g^{\alpha\beta} - \frac{l^\alpha n_-^\beta + n_-^\alpha l^\beta}{l^+}. \tag{9.118}$$

The numerator can be written as

$$\text{numerator} = -2(\not k - \not l) + 4m_0 + \gamma^+\,\frac{(\not k - \not l + m_0)\not l}{l^+} + \frac{\not l(\not k - \not l + m_0)}{l^+}\,\gamma^+ .$$

Because of the expansion around the zeroth order result, it is convenient to rewrite $\not l$ as $(\not k - m_0) - (\not k - \not l - m_0)$, leading for the numerator to

$$\begin{aligned}\text{numerator} \;=\;& -2(\not k - \not l) + 4m_0 \\ & -\gamma^+\,\frac{(\not k - \not l + m_0)(\not k - m_0)}{l^+} - \frac{(\not k - m_0)(\not k - \not l + m_0)}{l^+}\,\gamma^+ \\ & +\gamma^+\,\frac{(\not k - \not l + m_0)(\not k - \not l - m_0)}{l^+} + \frac{(\not k - \not l - m_0)(\not k - \not l + m_0)}{l^+}\,\gamma^+ .\end{aligned}$$

We consider the case $k_T \approx k^- \approx 0$ and $k^+ = x\,P^+$, while $l^- = \alpha\,\boldsymbol{l}_T^2/2P^+$ and $l^+ = y\,P^+$. We need the following integrals[1]

$$\begin{aligned}\Sigma_1(k) \;\equiv\;& -i\,\frac{g^2}{(2\pi)^4}\int d^4l\,\frac{1}{(l^2+i\epsilon)\,((k-l)^2 - m_0^2 + i\epsilon)} \\ =\;& \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy \int \frac{d\alpha}{2\pi\,i}\,\frac{1}{(\alpha y - 1 + i\epsilon)(\alpha(y-x) - 1 + i\epsilon)} \\ =\;& \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\Theta_{11}^0(y, y-x) \\ =\;& \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2} \\ =\;& \frac{\alpha_s}{4\pi}\,\ln \Lambda^2, \end{aligned} \tag{9.119}$$

$$\begin{aligned}\Sigma_2(k) \;\equiv\;& -i\,\frac{g^2}{(2\pi)^4}\int d^4l\,\frac{k^+}{(l^2+i\epsilon)\,((k-l)^2 - m_0^2 + i\epsilon)\,l^+} \\ =\;& \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{x}{y}\int \frac{d\alpha}{2\pi\,i}\,\frac{1}{(\alpha y - 1 + i\epsilon)(\alpha(y-x) - 1 + i\epsilon)} \\ =\;& \frac{\alpha_s}{4\pi^2}\int \frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{x}{y}\,\Theta_{11}^0(y, y-x) \\ =\;& \Sigma_1\int \frac{dy}{y}\;x\Theta_{11}^0(y, y-x). \end{aligned} \tag{9.120}$$

[1] reminder: $\Theta_{11}^0(y, y-x) = \frac{1}{x}\,[\theta(y)\,\theta(x-y) - \theta(-y)\,\theta(y-x)]$, thus one has for the y-integration (for positive x) $\int_0^x dy$

We also need the integral

$$
\begin{aligned}
-i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{l^+}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)\,k^+}
&= \frac{\alpha_s}{4\pi^2}\int\frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{y}{x}\int\frac{d\alpha}{2\pi\,i}\;\frac{1}{(\alpha y-1+i\epsilon)(\alpha(y-x)-1+i\epsilon)}\\
&= \frac{\alpha_s}{4\pi^2}\int\frac{d^2l_T}{\boldsymbol{l}_T^2}\int dy\;\frac{y}{x}\,\Theta^0_{11}(y,y-x)\\
&= \frac{1}{2}\,\Sigma_1.
\end{aligned}
\tag{9.121}
$$

It is useful to have the following integrals

$$
\begin{aligned}
\not L_1(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not l}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)}\\
&= \frac{1}{2}\,\not k\,\Sigma_1 &(9.122)\\
\not L_2(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not k-\not l-m_0}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)}\\
&= \frac{1}{2}\,\not k\,\Sigma_1 - m_0\,\Sigma_1, &(9.123)\\
\not R_1(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not k-\not l-m_0}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)\,l^+}\\
&= \frac{1}{k^+}\big[\not k\,\underbrace{(\Sigma_2-\Sigma_1)}_{\Sigma_3} - m_0\,\Sigma_2\big], &(9.124)\\
\not R_2(k) &\equiv -i\,\frac{g^2}{(2\pi)^4}\int d^4l\;\frac{\not k-\not l+m_0}{(l^2+i\epsilon)\,((k-l)^2-m_0^2+i\epsilon)\,l^+}\\
&= \frac{1}{k^+}\big[\not k\,\underbrace{(\Sigma_2-\Sigma_1)}_{\Sigma_3} + m_0\,\Sigma_2\big] = \not R_1(k) + \frac{2m_0}{k^+}\,\Sigma_2, &(9.125)
\end{aligned}
$$

where one should realize that the only nonvanishing contribution in $\not l$ is $\not l = l^+\gamma^- = \frac{l^+}{k^+}\,\not k$. Note that $\gamma^+\not R_2(k) = 2\Sigma_3 - \not R_1(k)\gamma^+$ and $\not R_2(k)\gamma^+ = 2\Sigma_3 - \gamma^+\not R_1(k)$. It is now straightforward to find the result

$$
\begin{aligned}
\Sigma(k) &= -\not k\Sigma_1 + 4m_0\,\Sigma_1 - (\not k-m_0)\,\not R_2(k)\gamma^+ - \gamma^+\not R_2(k)\,(\not k-m_0)\\
&= -(\not k-m_0)\,\Sigma_1 + 3m_0\,\Sigma_1 - 4\Sigma_3\,(\not k-m_0) + (\not k-m_0)\,\gamma^+\not R_1(k) + \not R_1(k)\gamma^+\,(\not k-m_0)\\
&.
\end{aligned}
\tag{9.126}
$$

Inserting in the expansion for $S_F(k)$ one obtains

$$
\begin{aligned}
S_F(k) &= \frac{(1-\Sigma_1)}{(\not k-m_0)} + \frac{1}{(\not k-m_0)}\,3m_0\,\Sigma_1\,\frac{1}{(\not k-m_0)}\\
&\quad -\frac{1}{(\not k-m_0)}\,\gamma^+\not R_2(k) - \not R_2(k)\gamma^+\,\frac{1}{(\not k-m_0)}\\
&= \underbrace{\big(1-\not R_2(k)\gamma^+\big)}_{U_2^{-1}(k)}\,\frac{(1-\Sigma_1)}{(\not k-m_0(1+3\Sigma_1))}\,\underbrace{\big(1-\gamma^+\not R_2(k)\big)}_{U_1(k)}\\
&= U_2^{-1}(k)\,\frac{(1-\Sigma_1)}{(\not k-m)}\,U_1(k)
\end{aligned}
\tag{9.127}
$$

with $m = m_0(1+3\Sigma_1)$ and up to the required (first order) precision

$$\begin{aligned} U_1(k) &= 1 - \gamma^+ \not{R}_2(k) = 1 - 2\Sigma_3 + \not{R}_1(k)\gamma^+ \\ &= 1 - \frac{1}{k^+}\,\Sigma_3(k)\,\gamma^+\,\not{k} - \frac{m}{k^+}\,\Sigma_2(k)\,\gamma^+ \end{aligned} \tag{9.128}$$

$$\begin{aligned} U_2(k) &= 1 + \not{R}_2(k)\gamma^+ = 1 + 2\Sigma_3 - \gamma^+ \not{R}_1(k) \\ &= 1 + \frac{1}{k^+}\,\Sigma_3(k)\,\not{k}\,\gamma^+ + \frac{m}{k^+}\,\Sigma_2(k)\,\gamma^+. \end{aligned} \tag{9.129}$$

9.8 Evolution of $f_1(x)$

The tree level contribution is given by

$$f_1^{[\alpha_s^0]}(x) = \frac{1}{4}\,f_1(x)\,\mathrm{Tr}\left[\gamma^-\gamma^+\right] \tag{9.130}$$

$$= f_1(x). \tag{9.131}$$

The ladder graph gives using

$$k = \left[\frac{\alpha}{2}\,\boldsymbol{k}_T^2, x, k_T\right], \tag{9.132}$$

$$l = \left[\frac{\alpha}{2}\,\boldsymbol{k}_T^2, x-y, k_T\right], \tag{9.133}$$

$$p = \left[0, y, 0_T\right], \tag{9.134}$$

and the trace

$$\begin{aligned} \mathrm{Tr}\left[\gamma^-\gamma^\alpha \not{k}\gamma^+\not{k}\gamma^\beta\right]\,d_{\alpha\beta}(l) &= -8\,\boldsymbol{k}_T^2 - 32\,\frac{(k^+)^2\,l^-}{l^+} + 16\,\frac{k^+\,\boldsymbol{k}_T\cdot\boldsymbol{l}_T}{l^+} \\ &= \boldsymbol{k}_T^2\left[-8 - 16\,\alpha\,\frac{x^2}{(x-y)} + 16\,\frac{x}{(x-y)}\right] \\ &= \boldsymbol{k}_T^2\,\frac{[8(x+y) - 16\,\alpha\,x^2]}{(x-y)} \end{aligned} \tag{9.135}$$

the result

$$\begin{aligned} f_1^{[\alpha_s^1,\mathrm{ladder}]}(x) &= -i\pi\,\frac{g^2\,C_F}{(2\pi)^4}\int \frac{dy\,d\alpha}{2}\,\frac{d^2k_T}{\boldsymbol{k}_T^2}\,\frac{[2(x+y) - 4\,x^2\,\alpha]}{(x-y)\,[\alpha(x-y)-1+i\epsilon]\,[\alpha\,x - 1 + i\epsilon]^2}\,f_1(y) \\ &= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int dy\,\left[\frac{(x+y)}{(x-y)}\,\Theta_{21}^0(x,x-y) - \frac{2\,x^2}{(x-y)}\,\Theta_{21}^1(x,x-y)\right] f_1(y) \\ &= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int dy\,\left[\frac{(y^2+x^2)}{y(y-x)}\,\Theta_{11}^0(x,x-y) + \frac{x}{y}\,\delta(x)\right] f_1(y) \\ &= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int dy\,\frac{(y^2+x^2)}{y(y-x)}\,\Theta_{11}^0(x,x-y) f_1(y) \\ &= \frac{\alpha_s}{2\pi^2}\,C_F\int\frac{d^2k_T}{\boldsymbol{k}_T^2}\int_x^\infty \frac{dy}{y}\,\frac{(y^2+x^2)}{y(y-x)}\,f_1(y) \end{aligned} \tag{9.136}$$

(note that the support of $f_1(y)$ is $-1 \le y \le 1$). For the self-energy graphs on the two fermion legs we need the traces

$$\begin{aligned} \mathrm{Tr}\left[\gamma^-\gamma^+\,\frac{1}{\not{k}}\,\Sigma(k)\right] &= \mathrm{Tr}\left[\gamma^-\gamma^+\,\frac{1}{\not{k}}\left(-\not{k}\Sigma_1 + \not{k}\not{R}(k)\gamma^+ + \gamma^+\not{R}(k)\not{k}\right)\right] \\ &= \mathrm{Tr}\left[\gamma^-\gamma^+\,\frac{1}{\not{k}}\left(-\not{k}\Sigma_1 - \not{k}\not{k}\,\frac{\gamma^+}{k^+}\,\Sigma_3 - \frac{\gamma^+}{k^+}\,\not{k}\not{k}\Sigma_3\right)\right] \\ &= -4\,\Sigma_1 - 16\,\Sigma_3 = 4(3\,\Sigma_1 - 4\,\Sigma_2), \end{aligned} \tag{9.137}$$

$$\mathrm{Tr}\left[\gamma^-\Sigma(k)\gamma^+\,\frac{1}{\not{k}}\right] = 4(3\,\Sigma_1 - 4\,\Sigma_2). \tag{9.138}$$

Then (taking into account a factor 1/2)

$$f_1^{[\alpha_s^1,\text{self}]}(x) = (-\Sigma_1 - 4\Sigma_3)\, f_1(x) = (3\,\Sigma_1 - 4\,\Sigma_2)\, f_1(x) \tag{9.139}$$

$$= \Sigma_1 \left(3 - 4 \int \frac{dy}{y}\; x\Theta^0_{11}(y, y-x)\right) f_1(x)$$

$$= \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \left\{\frac{3}{2}\, f_1(x) - 2\, f_1(x) \int dy\; \frac{x}{x-y}\; \Theta^0_{11}(y, y-x)\right\} \tag{9.140}$$

Using

$$\int dy\; \frac{x}{x-y}\left[\Theta^0_{11}(y, y-x) + \Theta^0_{11}(x, x-y)\right] = 0. \tag{9.141}$$

the total result becomes

$$f_1^{[\alpha_s^1]}(x) = \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \left\{\int dy \left[\frac{y^2+x^2}{y(y-x)}\, f_1(y) - \frac{2x}{y-x}\, f_1(x)\right] \Theta^0_{11}(x, x-y) + \frac{3}{2}\, f_1(x)\right\}$$

The integral runs from $x \le y \le \infty$, but for the first term it is limited to $x \le y \le 1$ using the support properties of f_1. In the next step we introduce $\beta = x/y$ and find that

$$x \le y \le 1 \quad\Longleftrightarrow\quad x \le \beta \le 1,$$
$$x \le y \le \infty \quad\Longleftrightarrow\quad 0 \le \beta \le 1,$$

Thus

$$f_1^{[\alpha_s^1]}(x) = \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \left\{\int_x^1 \frac{d\beta}{\beta}\; \frac{1+\beta^2}{(1-\beta)}\; f_1\left(\frac{x}{\beta}\right) - 2\, f_1(x) \int_0^1 d\beta\; \frac{1}{1-\beta} + \frac{3}{2}\, f_1(x)\right\}$$

$$= \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \int_x^1 \frac{d\beta}{\beta} \left[\frac{1+\beta^2}{(1-\beta)_+} + \frac{3}{2}\,\delta(1-\beta)\right] f_1\left(\frac{x}{\beta}\right) \tag{9.142}$$

$$= \frac{\alpha_s}{2\pi^2}\, C_F \int \frac{d^2k_T}{\boldsymbol{k}_T^2} \int_x^1 \frac{d\beta}{\beta}\; P^{[f]}(\beta)\; f_1\left(\frac{x}{\beta}\right).$$

9.9 Gribov-Lipatov reciprocity and evolution of D_1

For the fragmentation function D_1, the tree level contribution is given by

$$D_1^{[\alpha_s^0]}(z) = \frac{1}{4}\, D_1(z)\, \text{Tr}\left[\gamma^+\gamma^-\right] = D_1(z). \tag{9.143}$$

We note that the result can be obtained from the distribution functions by interchanging everywhere the + and − components and then later make replacements $x \to 1/z$ and $f_1(x) \to f_1(1/z) \to D_1(z)$.

The momenta for the ladder graph in the case of fragmentation can be written

$$k = \left[x, \frac{\alpha}{2}\, \boldsymbol{k}_T^2, k_T\right] \quad\longrightarrow\quad \left[\frac{1}{z}, \frac{\alpha}{2}\, \boldsymbol{k}_T^2, k_T\right], \tag{9.144}$$

$$l = \left[x-y, \frac{\alpha}{2}\, \boldsymbol{k}_T^2, k_T\right] \tag{9.145}$$

$$p = [y, 0, 0_T]\,, \tag{9.146}$$

involving also simply an interchange of the lightcone components.

We now look at the evolution equations of fragmentation functions (FF) by looking at those for distribution functions (DF) in the domain $x \ge 1$. Starting with the result for DF's obtained via $f_1^{[\alpha_s^1]}(x)$,

$$\frac{d}{d\tau}\, f(x,\tau) = \frac{\alpha_s}{2\pi}\, C_F \int_0^\infty dy \left[\frac{\mathscr{N}\left(\frac{x}{y}\right)}{1-\frac{x}{y}}\, f(y) - \frac{\left(\frac{x}{y}\right)\, \mathscr{N}(1)}{1-\frac{x}{y}}\, f(x)\right] \Theta_{11}(x, x-y), \tag{9.147}$$

allowing for different numerators in order to treat the various distribution functions (f_1, g_1 and h_1). We consider $\mathscr{N}(\beta) = 2\,\beta^p$, which will make it easy to get the results for $\mathscr{N}(\beta) = 1+\beta^2$ or $\mathscr{N}(\beta) = 2\beta$. In all these cases $\mathcal{N}(1) = 2$, the numerator result coming from the fermion self energy. We find (defining $\beta = x/y$) for the DF

$$\frac{d}{d\tau}\, f(x) \quad = \quad \frac{\alpha_s}{2\pi} \int_0^1 \frac{d\beta}{\beta} \Bigg[\underbrace{\frac{2\,\beta^p}{1-\beta}\, f\left(\frac{x}{\beta}\right)}_{x\le\beta\le 1} - \underbrace{\frac{2\,\beta}{1-\beta}\, f(x)}_{0\le\beta\le 1} \Bigg]. \tag{9.148}$$

For the FF we employ an continuation to the region $x \ge 1$ and then a substitution $x \to 1/z$. For a quark with $k^- = xP_h^-$ to produce a hadron with P_h^- the radiated gluon can have momentum $\ell^- = (x-y)P_h^-$ with $1 \le y \le x$. This is obtained via the generalized Θ-functions if one uses for the timelike fragmenting quark a propagator prescription $1/(p^2 - m^2 - i\epsilon)$. Thus one has

$$\begin{aligned}
\Theta^0_{11}(x_1,x_2) = \int \frac{d\alpha}{2\pi\, i}\, \frac{1}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)} &= \frac{[\theta(x_1)\,\theta(-x_2) - \theta(-x_1)\theta(x_2)]}{x_1 - x_2} \\
&= \frac{\theta(x_1) - \theta(x_2)}{x_1 - x_2} = \frac{\theta(-x_2) - \theta(-x_1)}{x_1 - x_2}, \\
\Theta^0_{\underline{1}1}(x_1,x_2) = \int \frac{d\alpha}{2\pi\, i}\, \frac{1}{(\alpha x_1 - 1 - i\epsilon)(\alpha x_2 - 1 + i\epsilon)} &= -\frac{[\theta(x_1)\,\theta(x_2) - \theta(-x_1)\theta(-x_2)]}{x_1 - x_2} \\
&= -\frac{\theta(x_1) - \theta(-x_2)}{x_1 - x_2} = -\frac{\theta(x_2) - \theta(-x_1)}{x_1 - x_2},
\end{aligned}$$

and for positive x-values

$$\begin{aligned}
\Theta^0_{11}(x,x-y) &\overset{x\ge 0}{=} \frac{\theta(y-x)}{y}, \\
\Theta^0_{\underline{1}1}(x,x-y) &\overset{x\ge 0}{=} -\frac{\theta(x-y)}{y}.
\end{aligned}$$

This modification happens only for the ladder graph, not for the self-energy. With the further replacements $z \to 1/x$ and $f(x) \to f(1/z) \to D(z)$ one obtains

$$\begin{aligned}
\frac{d}{d\tau}\, f(x) &= \frac{\alpha_s}{2\pi} \int_1^\infty \frac{d\beta}{\beta}\, \underbrace{\frac{\mathcal{N}(\beta)}{\beta - 1}\, f\left(\frac{x}{\beta}\right)}_{1\le\beta\le x} - \int_0^1 \frac{d\beta}{\beta}\, \frac{\beta\,\mathcal{N}(1)}{1-\beta}\, f(x), \\
\frac{d}{d\tau}\, D(z) &= \frac{\alpha_s}{2\pi} \int_1^\infty \frac{d\beta}{\beta}\, \underbrace{\frac{\mathcal{N}(\beta)}{(\beta-1)}\, D(\beta z)}_{1\le\beta\le 1/z} - \int_0^1 \frac{d\beta}{\beta}\, \underbrace{\frac{\beta\,\mathcal{N}(1)}{1-\beta}\, D(z)}_{0\le\beta<1} \\
&= \frac{\alpha_s}{2\pi} \int_0^1 \frac{d\beta}{\beta} \Bigg[\underbrace{\frac{\beta\,\mathcal{N}(1/\beta)}{(1-\beta)}\, D\left(\frac{z}{\beta}\right)}_{z\le\beta\le 1} - \underbrace{\frac{\beta\,\mathcal{N}(1)}{1-\beta}\, D(z)}_{0\le\beta<1} \Bigg] \\
&= \frac{\alpha_s}{2\pi} \int_z^1 \frac{d\beta}{\beta}\, \frac{\beta\,\mathcal{N}(1/\beta)}{(1-\beta)_+}\, D\left(\frac{z}{\beta}\right)
\end{aligned} \tag{9.149}$$

We find the splitting functions

$$P^{[f]}(\beta) = \frac{\mathcal{N}(\beta)}{(1-\beta)_+}, \tag{9.150}$$

$$P^{[D]}(\beta) = \frac{\beta\,\mathcal{N}(1/\beta)}{(1-\beta)_+}. \tag{9.151}$$

For a polynomial $\mathscr{N}(\beta) = 2\,\beta^p$ the moments become

$$
\begin{aligned}
P^{[f]}(\beta) &= \frac{2\beta^p}{(1-\beta)_+} \\
\longrightarrow \quad A_n^{[f]} &= 2\int_0^1 d\beta\ \frac{\beta^{n-1+p}-1}{1-\beta} = -2\sum_{j=1}^{n-1+p}\frac{1}{j} = -2\sum_{j=1}^{n}\frac{1}{j}\underbrace{-\frac{2}{n+1}\ldots-\frac{2}{n-1+p}}_{+\mathscr{A}(n)} \\
P^{[D]}(\beta) &= \frac{2\beta^{p-1}}{(\beta-1)_+} \\
\longrightarrow \quad A_n^{[D]} &= 2\int_1^\infty d\beta\ \frac{\beta^{p-n-2}-1}{\beta-1} \\
P^{[D]}(\beta) &= \frac{2\beta^{1-p}}{(1-\beta)_+} \\
\longrightarrow \quad A_n^{[D]} &= 2\int d\beta\ \frac{\beta^{n-p}-1}{1-\beta} = -2\sum_{j=1}^{n-p}\frac{1}{j} = -2\sum_{j=1}^{n}\frac{1}{j} + \frac{2}{n}\underbrace{+\frac{2}{n-1}\ldots+\frac{2}{n-p+1}}_{+\mathscr{A}(-n)}
\end{aligned}
$$

Appendix A

Lightcone coordinates

We use two different notations for four-vectors,

$$a^\mu = (a^0, a^1, a^2, a^3) = (a^0, \boldsymbol{a}), \tag{A.1}$$
$$a^\mu = [a^-, a^+, a^1, a^2] = [a^-, a^+, \boldsymbol{a}_T]. \tag{A.2}$$

The metric tensor is given by $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ or $g_{++} = g_{--} = 0$ and $g_{+-} = 1$ and defines the scalar product

$$a \cdot b = g_{\mu\nu} a^\mu b^\nu \equiv g^{ab}. \tag{A.3}$$

The first vector notation corresponds to the expansion of a vector in a Cartesian basis with orthogonal vectors $\hat{t}^\mu$, $\hat{x}^\mu$, $\hat{y}^\mu$, $\hat{z}^\mu$ with t timelike ($t^2 = 1$) and the others spacelike ($\hat{x}^2$ = -1, etc.), the second notation with an expansion in two lightlike vectors n_+^μ and n_-^μ ($n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$).

The antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ is fixed by

$$\epsilon^{0123} = \epsilon^{-+12} = 1. \tag{A.4}$$

We employ the notation

$$\epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma \equiv \epsilon^{abcd}. \tag{A.5}$$

A useful property is the following way of bringing a vector into the antisymmetric combination,

$$\epsilon^{\mu\nu\rho\sigma}\, g^{\alpha\beta} = \epsilon^{\alpha\nu\rho\sigma}\, g^{\mu\beta} + \epsilon^{\mu\alpha\rho\sigma}\, g^{\nu\beta} + \epsilon^{\mu\nu\alpha\sigma}\, g^{\rho\beta} + \epsilon^{\mu\nu\rho\alpha}\, g^{\sigma\beta}. \tag{A.6}$$

The following fourth rank tensor is also useful in many applications,

$$S^{\mu\nu\rho\sigma} = g^{\mu\nu}\, g^{\rho\sigma} - g^{\mu\rho}\, g^{\nu\sigma} + g^{\mu\sigma}\, g^{\nu\rho}. \tag{A.7}$$

In several cases we will also be dealing only with transverse vectors, projected out by

$$g_T^{\mu\nu} \equiv g^{\mu\nu} - n_+^\mu n_-^\nu - n_-^\mu n_+^\nu, \tag{A.8}$$

with as only nonvanishing elements $g_T^{11} = g_T^{22} = -1$ and the corresponding antisymmetric tensor with $\epsilon_T^{12} = 1$ given by

$$\epsilon_T^{\alpha\beta} = \epsilon^{-+\alpha\beta} \tag{A.9}$$

(or equivalently $g_\perp^{\mu\nu}$ and $\epsilon_\perp^{\mu\nu}$). Given a transverse vector a_T^μ, in the appropriate frame determined by two components (a^1, a^2) or determined by its length $|a_T| = \sqrt{-a_T^2}$ and an azimuthal angle ϕ_a, such that

$$a_T^\mu = (a^1, a^2) = |a_T|(\cos\phi_a, \sin\phi_a), \tag{A.10}$$

we can consider the vector

$$\bar{a}_T^\mu = \epsilon_T^{\mu\nu} a_\nu = (-a^2, a^1), \tag{A.11}$$

which has the same length, $|\boldsymbol{a}_T| = |\bar{\boldsymbol{a}}_T|$ but an azimuthal angle $\phi_{\bar{a}} = \phi_a + \pi/2$. We note that $\bar{\bar{a}}_T^\mu = -a_T^\mu$. Again we use the notation with vectors as indices, e.g.

$$g_T^{ab} \equiv g_T^{\mu\nu} a_\mu b_\nu = a_T \cdot b_T = -\boldsymbol{a}_T \cdot \boldsymbol{b}_T, \tag{A.12}$$
$$\epsilon_T^{ab} \equiv \epsilon_T^{\mu\nu} a_{T\mu} b_{T\nu} \equiv a_T \wedge b_T = a_T \cdot \bar{b}_T, \tag{A.13}$$

and we note that we can write $a_T^\mu = g_T^{a\mu}$ and $\bar{a}_T^\mu = g_T^{\bar{a}\mu} = \epsilon_T^{\mu a}$. The symmetric traceless tensor constructed from two (transverse) four vectors is denoted

$$S_T^{a\mu b\nu} = a_T^{\{\mu} b_T^{\nu\}} - (a_T \cdot b_T)\, g_T^{\mu\nu}. \tag{A.14}$$

In terms of the azimuthal angles,

$$g_T^{ab} = -|\boldsymbol{a}_T|\,|\boldsymbol{b}_T|\,\cos(\phi_b - \phi_a), \tag{A.15}$$
$$\epsilon_T^{ab} = |\boldsymbol{a}_T|\,|\boldsymbol{b}_T|\,\sin(\phi_b - \phi_a), \tag{A.16}$$
$$S_T^{abcd} = |\boldsymbol{a}_T|\,|\boldsymbol{b}_T|\,|\boldsymbol{c}_T|\,|\boldsymbol{d}_T|\,\cos(\phi_b + \phi_d - \phi_a - \phi_c). \tag{A.17}$$

For products of tensors we have the relations

$$g_T^{ij}\, g_T^{ab} = g_T^{ib}\, g_T^{ja} + \epsilon_T^{ia}\, \epsilon_T^{jb}, \tag{A.18}$$
$$\epsilon_T^{ij}\, \epsilon_T^{ab} = g_T^{ia}\, g_T^{jb} - g_T^{ib}\, g_T^{ja}, \tag{A.19}$$
$$\epsilon_T^{ab}\, g_T^{ij} = \epsilon_T^{ib}\, g_T^{aj} + \epsilon_T^{ai}\, g_T^{bj}. \tag{A.20}$$

Employing two unit-length transverse vectors $\hat{x}^\mu$ and $\hat{y}^\mu = \bar{\hat{x}}^\mu$, one easily derives the expansions

$$a_T^\mu = g_T^{a\mu} = -g_T^{ax}\, g_T^{x\mu} - g_T^{ay}\, g_T^{y\mu} = a^x\, \hat{x}^\mu + a^y\, \hat{y}^\mu, \tag{A.21}$$
$$\bar{a}_T^\mu = \epsilon_T^{\mu a} = g_T^{\bar{a}\mu} = g_T^{ay}\, g_T^{x\mu} - g_T^{ax}\, g_T^{y\mu} = -a^y\, \hat{x}^\mu + a^x\, \hat{y}^\mu. \tag{A.22}$$

where $a^x \equiv -(a_T \cdot \hat{x}) = \boldsymbol{a}_T \cdot \hat{\boldsymbol{x}}$ and $a^y \equiv -(a_T \cdot \hat{y}) = \hat{x} \wedge a_T$. For second rank tensors one obtains

$$a_T^{\{\mu} b_T^{\nu\}} - (a_T \cdot b_T)\, g_T^{\mu\nu} = S_T^{a\mu b\nu} = S_T^{axbx}\, S_T^{x\mu x\nu} + S_T^{axby}\, S_T^{x\mu y\nu}$$
$$= (a^x b^x - a^y b^y)\,(2\hat{x}^\mu \hat{x}^\nu + g_T^{\mu\nu}) + (a^x b^y + a^y b^x)\, \hat{x}^{\{\mu} \hat{y}^{\nu\}}, \tag{A.23}$$
$$a_T^{[\mu} b_T^{\nu]} = g_T^{a[\mu} g_T^{\nu]b} = \epsilon_T^{ab}\, \epsilon_T^{\mu\nu} = (a^x b^y - a^y b^x)\, \epsilon_T^{\mu\nu}, \tag{A.24}$$
$$\frac{1}{2}\left(a_T^{\{\mu} \bar{b}_T^{\nu\}} + b_T^{\{\mu} \bar{a}_T^{\nu\}}\right) = -S_T^{axby}\, S_T^{x\mu x\nu} + S_T^{axbx}\, S_T^{x\mu y\nu}$$
$$= -(a^x b^y + a^y b^x)\,(2\hat{x}^\mu \hat{x}^\nu + g_T^{\mu\nu}) + (a^x b^x - a^y b^y)\, \hat{x}^{\{\mu} \hat{y}^{\nu\}}, \tag{A.25}$$
$$\frac{1}{2}\left(a_T^{\{\mu} \bar{b}_T^{\nu\}} - b_T^{\{\mu} \bar{a}_T^{\nu\}}\right) = \epsilon_T^{ab}\, g_T^{\mu\nu} = (a^x b^y - a^y b^x)\, g_T^{\mu\nu}, \tag{A.26}$$

Finally we want to give the following useful way of transforming a tensor involving external momenta q_T into quark transverse momenta k_T and p_T. Having $p_T + k_T = q_T$ one sees that

$$S_T^{q\alpha q\beta} \quad = \quad S_T^{p\alpha p\beta} + S_T^{k\alpha k\beta} + 2\, S_T^{p\alpha k\beta}. \tag{A.27}$$

and elimination of the mixed combination in

$$2(p_T {\cdot} k_T)\, S_T^{p\alpha k\beta} = k_T^2\, S_T^{p\alpha p\beta} + p_T^2\, S_T^{k\alpha k\beta}. \tag{A.28}$$

The latter can be obtained by using $|k_T|\hat{x} = k_T$ and working out $S_T^{p\alpha p\beta}$ and $S_T^{p\alpha k\beta}$.

Spherical basis

Transverse tensors can also be labeled according to their rotational behavior in E(2). We can employ a spherical basis, starting with

$$a_T^{(1)} \equiv a^x + i\, a^y = |a_T|\, e^{i\varphi} \tag{A.29}$$

generalized to

$$a_T^{(m)} \equiv (a_x + i\, a^y)^m = |a_T|^m\, e^{im\varphi}. \tag{A.30}$$

These are labeled by integer m values. The transverse vector has rank $m = 1$, the symmetric rank-2 tensor $k_T^{\alpha\beta}$ has rank $m = 2$. The components of a tensor of rank m are not independent. Being traceless and symmetric they satisfy

$$a_T^{i_1 \ldots i_m} = \frac{1}{m!}\, a_T^{\{i_1 \ldots i_m\}} - \text{Traces}. \tag{A.31}$$

The two independent components can be recast in

$$a_T^{i_1\ldots i_m} \iff \frac{|a_T|^m}{2^{m-1}}\, e^{\pm i m\varphi}. \tag{A.32}$$

The components are actually proportional to $\cos(m\varphi)$ or $\sin(m\varphi)$. For example we have

$$\begin{aligned} a_T^{(3)} &= (a^x + i\, a^y)^3 && \text{(A.33)}\\ &= (a^x a^x a^x - 3\, a^x a^y a^y) + i(3\, a^x a^x a^y - a^y a^y a^y), && \text{(A.34)}\end{aligned}$$

and read off that

$$a_T^{xxx} = -a_T^{xyy} = \frac{1}{4}\,(a^x a^x a^x - 3\, a^x a^y a^y) = \frac{|a_T|^3}{4}\,\cos(3\varphi), \tag{A.35}$$

$$a_T^{xxy} = -a_T^{yyy} = \frac{1}{4}\,(3\, a^x a^x a^y - a^y a^y a^y) = \frac{|a_T|^3}{4}\,\sin(3\varphi). \tag{A.36}$$

In order to consistently include tensor built from different transverse vectors we define

$$T_{a_1\ldots a_m}^{i_1\ldots i_m} = \frac{1}{m!} a_1^{\{i_1} \ldots a_m^{i_m\}} - \text{Traces}. \tag{A.37}$$

Such tensors are simply the product of lower rank tensors, e.g.

$$T_{ab}^{(2)} = a_T^{(1)} b_T^{(1)} = (a^x + i a^y)(b^x + i b^y) = (a^x b^x - a^y b^y) + i(a^x b^y + a^y b^x) = |a_T|\,|b_T|\, e^{i(\varphi_a+\varphi_b)}. \tag{A.38}$$

More general

$$T_{ab}^{(m_a+m_b)} = a_T^{(m_a)} b_T^{(m_b)} = |a_T|^{m_a}\, |b_T|^{m_b}\, e^{i(m_a\varphi_a + m_b\varphi_b)}. \tag{A.39}$$

We can reduce the rank by constructing

$$T_{ab}^{(m_a-m_b)} = a_T^{(m_a)} b_T^{(m_b)*} = |a_T|^{m_a}\, |b_T|^{m_b}\, e^{i(m_a\varphi_a - m_b\varphi_b)}. \tag{A.40}$$

A wellknown reduction is the scalar constructed from two vectors,

$$\begin{aligned} T_{ab}^{(0)} &= a_T^{(1)} b_T^{(1)*} = |a_T|\,|b_T|\, e^{i(\phi_a-\phi_b)} && \text{(A.41)}\\ &= (a^x + i a^y)(b^x - i b^y) = (a^x b^x + a^y b^y) - i(a^x b^y - a^y b^x) && \text{(A.42)}\\ &= (\boldsymbol{a}_T \cdot \boldsymbol{b}_T) - i(\boldsymbol{a}_T \wedge \boldsymbol{b}_T). && \text{(A.43)}\end{aligned}$$

We define the explicit components for any irreducible tensor as

$$\Gamma_{A_1\ldots A_m}^{x\ldots x} \equiv \frac{|\Gamma_{A_1\ldots A_m}^{(m)}|}{2^{m-1}}\,\cos(\varphi_1 + \ldots + \varphi_m), \tag{A.44}$$

$$\frac{1}{m}\left(\Gamma_{A_1\ldots A_m}^{yx\ldots x} + \Gamma_{A_1\ldots A_m}^{xy\ldots x} + \ldots + \Gamma_{A_1\ldots A_m}^{xx\ldots y}\right) \equiv \frac{|\Gamma_{A_1\ldots A_m}^{(m)}|}{2^{m-1}}\,\sin(\varphi_1 + \ldots + \varphi_m), \tag{A.45}$$

where A_i can be any object, spin vectors, gamma matrices, etc., as long as they have a transverse vector character. The other components are now fixed. In terms of

$$\Gamma_{A_1\ldots A_m}^{(m)} \equiv |\Gamma_{A_1\ldots A_m}^{(m)}|\, e^{i(\varphi_1+\ldots+\varphi_m)}, \tag{A.46}$$

one sees that the Euclidean contraction of two tensors can be written as

$$\begin{aligned} a_T^{i_1\ldots i_m}\, \Gamma_{A_1\ldots A_m}^{i_1\ldots i_m} &= \frac{1}{2^m}\left(a_T^{(m)} \Gamma_{A_1\ldots A_m}^{(m)*} + a_T^{(m)*} \Gamma_{A_1\ldots A_m}^{(m)}\right)\\ &= \frac{|a_T|^m}{2^{m-1}}\, |\Gamma_{A_1\ldots A_m}^{(m)}|\, \cos(\varphi_1 + \ldots + \varphi_m - m\varphi). && \text{(A.47)}\end{aligned}$$

Furthermore, we have

$$\int \frac{d\varphi}{2\pi}\; a_T^{(m)}(\varphi)\, a_T^{(m')*}(\varphi) = |a_T|^{2m}\, \delta_{mm'}, \tag{A.48}$$

This immediately gives

$$\int \frac{d\varphi}{2\pi}\; a_T^{(m)}(\varphi)\, a_T^{i_1\ldots i_m}\, \Gamma_{A_1\ldots A_m}^{i_1\ldots i_m} = \left(\frac{|a_T|^2}{2}\right)^m\, \Gamma_{A_1\ldots A_m}^{(m)}, \tag{A.49}$$

and thus also

$$\int \frac{d\varphi}{2\pi}\; a_T^{\alpha_1\ldots\alpha_m}(\varphi)\, a_T^{i_1\ldots i_m}\, \Gamma_{A_1\ldots A_m}^{i_1\ldots i_m} = \left(\frac{|a_T|^2}{2}\right)^m\, \Gamma_{A_1\ldots A_m}^{\alpha_1\ldots\alpha_m}. \tag{A.50}$$

Integrating over transverse momenta

We can write down the following relations for integrating $\boldsymbol{k}_T^2$ dependent functions over $\boldsymbol{k}_T$,

$$\int d^2k_T\; k_T^\alpha k_T^i \ldots = -\frac{1}{2}\,g_T^{\alpha i}\int d^2k_T\; \boldsymbol{k}_T^2\; \ldots \tag{A.51}$$

$$\int d^2k_T\; k_T^\alpha k_T^\beta\, k_T^i k_T^j \ldots = \frac{1}{8}\,\left(g_T^{\alpha i}g_T^{\beta j}+g_T^{\alpha j}g_T^{\beta i}+g_T^{\alpha\beta}g_T^{ij}\right)\int d^2k_T\; \boldsymbol{k}_T^4 \ldots, \tag{A.52}$$

$$\int d^2k_T\; k_T^\alpha k_T^\beta\,\left(k_T^i k_T^j+\frac{1}{2}\,\boldsymbol{k}_T^2\,g_T^{ij}\right)\ldots = \frac{1}{8}\,\left(g_T^{\alpha i}g_T^{\beta j}+g_T^{\alpha j}g_T^{\beta i}-g_T^{\alpha\beta}g_T^{ij}\right)\int d^2k_T\; \boldsymbol{k}_T^4 \ldots, \tag{A.53}$$

$$\int d^2k_T\left(k_T^\alpha k_T^\beta+\frac{1}{2}\,\boldsymbol{k}_T^2\,g_T^{\alpha\beta}\right)\left(k_T^i k_T^j+\frac{1}{2}\,\boldsymbol{k}_T^2\,g_T^{ij}\right)\ldots = \frac{1}{8}\,\left(g_T^{\alpha i}g_T^{\beta j}+g_T^{\alpha j}g_T^{\beta i}-g_T^{\alpha\beta}g_T^{ij}\right)\int d^2k_T\; \boldsymbol{k}_T^4 \ldots, \tag{A.54}$$

or using $S_T^{k\alpha k\beta} = 2\,k_T^\alpha k_T^\beta - k_T^2\,g_T^{\alpha\beta} = 2\,k_T^\alpha k_T^\beta + \boldsymbol{k}_T^2\,g_T^{\alpha\beta}$, one has the relations

$$\int d^2k_T\; S_T^{k\alpha k i}\ldots = 0 \tag{A.55}$$

$$\int d^2k_T\; S_T^{k\alpha k\beta}\,S_T^{kikj}\ldots = \frac{1}{2}\,S_T^{\alpha i\beta j}\int d^2k_T\;(k_T^2)^2\ldots, \tag{A.56}$$

We also want to discuss the full products of irreducible tensors,

$$k_T^i, \tag{A.57}$$

$$k_T^{ij} = k_T^i k_T^j - \frac{1}{2}\,k_T^2\,g_T^{ij}, \tag{A.58}$$

$$k_T^{ijk} = k_T^i k_T^j k_T^k - \frac{1}{4}\,k_T^2\left(g_T^{ij}k_T^k+g_T^{ik}k_T^j+g_T^{jk}k_T^i\right), \tag{A.59}$$

$$\begin{aligned}
k_T^{ijkl} &= k_T^i k_T^j k_T^k k_T^l \\
&\quad -\frac{1}{6}\,k_T^2\left(g_T^{ij}k_T^k k_T^l+g_T^{ik}k_T^j k_T^l+g_T^{il}k_T^j k_T^k+g_T^{jk}k_T^i k_T^l+g_T^{jl}k_T^i k_T^k+g_T^{kl}k_T^i k_T^j\right)\\
&\quad +\frac{1}{24}\,(k_T^2)^2\left(g_T^{ij}\,g_T^{kl}+g_T^{ik}\,g_T^{jl}+g_T^{il}\,g_T^{jk}\right),\\
&= k_T^i k_T^j k_T^k k_T^l \\
&\quad -\frac{1}{6}\,k_T^2\left(g_T^{ij}k_T^{kl}+g_T^{ik}k_T^{jl}+g_T^{il}k_T^{jk}+g_T^{jk}k_T^{il}+g_T^{jl}k_T^{ik}+g_T^{kl}k_T^{ij}\right)\\
&\quad -\frac{1}{8}\,(k_T^2)^2\left(g_T^{ij}\,g_T^{kl}+g_T^{ik}\,g_T^{jl}+g_T^{il}\,g_T^{jk}\right).
\end{aligned} \tag{A.60}$$

These are all traceless symmetric tensors,

$$g_{T\,ij}\,k_T^{ij} = g_{T\,ij}\,k_T^{ijk} = g_{T\,ij}\,k_T^{ijkl} = 0, \tag{A.61}$$

Products of tensors can be decomposed into these irreducible ones, such as

$$k_T^i\,k_T^\alpha = k_T^{i\alpha} + \frac{1}{2}\,k_T^2\,g_T^{i\alpha}, \tag{A.62}$$

$$\begin{aligned}
k_T^i\,k_T^{\alpha\beta} &= k_T^i k_T^\alpha k_T^\beta - \frac{1}{2}\,k_T^2\,g_T^{\alpha\beta}k_T^i\\
&= k_T^{i\alpha\beta} + \frac{1}{4}\,k_T^2\left(g_T^{i\alpha}k_T^\beta+g_T^{i\beta}k_T^\alpha-g_T^{\alpha\beta}k_T^i\right)
\end{aligned} \tag{A.63}$$

$$\begin{aligned}
k_T^i\,k_T^{\alpha\beta\gamma} &= k_T^{i\alpha\beta\gamma}\\
&\quad +\frac{1}{12}\,k_T^2\left(2g_T^{i\alpha}\,k_T^{\beta\gamma}+2g_T^{i\beta}\,k_T^{\alpha\gamma}+2g_T^{i\gamma}\,k_T^{\alpha\beta}-g_T^{\beta\gamma}\,k_T^{i\alpha}-g_T^{\alpha\gamma}\,k_T^{i\beta}-g_T^{\alpha\beta}\,k_T^{i\gamma}\right)
\end{aligned} \tag{A.64}$$

$$\begin{aligned}
k_T^{ij}\,k_T^{\alpha\beta} &= k_T^{ij\alpha\beta}\\
&\quad +\frac{1}{6}\,k_T^2\left(g_T^{i\alpha}k_T^{j\beta}+g_T^{i\beta}k_T^{j\alpha}+g_T^{j\alpha}k_T^{i\beta}+g_T^{j\beta}k_T^{i\alpha}-2g_T^{ij}k_T^{\alpha\beta}-2g_T^{\alpha\beta}k_T^{ij}\right)\\
&\quad +\frac{1}{8}\,(k_T^2)^2\left(g_T^{i\alpha}g_T^{j\beta}+g_T^{i\beta}g_T^{j\alpha}-g_T^{ij}g_T^{\alpha\beta}\right),
\end{aligned} \tag{A.65}$$

Convolutions for transverse momentum integrations

We look at the convolutions

$$I[F] \equiv \int d^2p_T\, d^2k_T\, \delta^2(p_T + k_T - q_T)\, F(q_T^2, p_T^2, k_T^2), \tag{A.66}$$

and calculate weighted results. Linear weighting gives

$$\int d^2p_T\, d^2k_T\, \delta^2(p_T + k_T - q_T)\; p_T^\alpha\, F(p_T^2, k_T^2) = q_T^\alpha\, A(q_T^2), \tag{A.67}$$

with

$$q_T^2\, A(q_T^2) = \frac{1}{2}\, I\left[(q_T^2 + p_T^2 - k_T^2) F(p_T^2, k_T^2)\right].$$

Quadratic weighting gives

$$\int d^2p_T\, d^2k_T\, \delta^2(p_T + k_T - q_T)\; S_T^{p\alpha p\beta}\, F(p_T^2, k_T^2) = S_T^{q\alpha q\beta}\, B_1(q_T^2), \tag{A.68}$$

$$\int d^2p_T\, d^2k_T\, \delta^2(p_T + k_T - q_T)\; S_T^{p\alpha k\beta}\, F(p_T^2, k_T^2) = S_T^{q\alpha q\beta}\, B_2(q_T^2), \tag{A.69}$$

with

$$q_T^4\, B_1(q_T^2) = I\left[S_T^{pqpq}\, F(p_T^2, k_T^2)\right] = \frac{1}{2}\, I\left[\left(q_T^4 - 2q_T^2\, k_T^2 + (p_T^2 - k_T^2)^2\right) F(p_T^2, k_T^2)\right],$$
$$q_T^4\, B_2(q_T^2) = I\left[S_T^{pqkq}\, F(p_T^2, k_T^2)\right] = \frac{1}{2}\, I\left[\left(q_T^2(p_T^2 + k_T^2) - (p_T^2 - k_T^2)^2\right) F(p_T^2, k_T^2)\right].$$

For quartic weighting, we get

$$\int d^2p_T\, d^2k_T\, \delta^2(p_T + k_T - q_T)\; S_T^{p\alpha p\beta}\, S_T^{pipj}\, F(p_T^2, k_T^2) = S_T^{q\alpha q\beta}\, S_T^{qiqj}\, (2\, D_4 - D_1) + q_T^4\, S_T^{\alpha i\beta j}\, (D_1 - D_4), \tag{A.70}$$

$$\int d^2p_T\, d^2k_T\, \delta^2(p_T + k_T - q_T)\; S_T^{p\alpha p\beta}\, S_T^{kikj}\, F(p_T^2, k_T^2) = S_T^{q\alpha q\beta}\, S_T^{qiqj}\, (2\, D_5 - D_2) + q_T^4\, S_T^{\alpha i\beta j}\, (D_2 - D_5), \tag{A.71}$$

$$\int d^2p_T\, d^2k_T\, \delta^2(p_T + k_T - q_T)\; S_T^{p\alpha k\beta}\, S_T^{pikj}\, F(p_T^2, k_T^2) = S_T^{q\alpha q\beta}\, S_T^{qiqj}\, (2\, D_6 - D_3) + q_T^4\, S_T^{\alpha i\beta j}\, (D_3 - D_6), \tag{A.72}$$

with

$$q_T^4\, D_1(q_T^2) = I\left[S_T^{pppp}\, F(p_T^2, k_T^2)\right] = I\left[p_T^4\, F(p_T^2, k_T^2)\right], \tag{A.73}$$

$$q_T^4\, D_2(q_T^2) = I\left[S_T^{pkpk}\, F(p_T^2, k_T^2)\right] = \frac{1}{2}\, I\left[\left(q_T^4 - 2q_T^2\,(k_T^2 + p_T^2) + (p_T^2 - k_T^2)^2\right) F(p_T^2, k_T^2)\right], \tag{A.74}$$

$$q_T^4\, D_3(q_T^2) = I\left[S_T^{ppkk}\, F(p_T^2, k_T^2)\right] = I\left[S_T^{pkkp}\, F(p_T^2, k_T^2)\right] = I\left[p_T^2 k_T^2\, F(p_T^2, k_T^2)\right], \tag{A.75}$$

$$q_T^8\, D_4(q_T^2) = I\left[(S_T^{pqpq})^2\, F(p_T^2, k_T^2)\right], \tag{A.76}$$

$$q_T^8\, D_5(q_T^2) = I\left[S_T^{pqpq}\, S_T^{kqkq}\, F(p_T^2, k_T^2)\right], \tag{A.77}$$

$$q_T^8\, D_6(q_T^2) = I\left[(S_T^{pqkq})^2\, F(p_T^2, k_T^2)\right]. \tag{A.78}$$

Appendix B

Frames in leptoproduction

B.1 The inclusive case

Consider $\ell(k) + H(P) \to \ell'(k') + X$ in which the momentum transfer is $q = k - k'$. Using lightcone coordinates the momenta k, q and P satisfying $q^2 = -Q^2$, $2\,P\cdot q = Q^2/x$ and $P\cdot q = y\,P\cdot k$ are given by

$$k = \Big[\frac{1}{y}\,\frac{Q}{\sqrt{2}}, \frac{y\,\boldsymbol{\ell}_T^2}{Q\sqrt{2}}, \underbrace{\frac{1}{y}\,\boldsymbol{q}_T + Q\,\frac{\sqrt{1-y}}{y}\,\hat{\ell}_T}_{\boldsymbol{\ell}_T}\Big], \tag{B.1}$$

$$q = \Big[\frac{Q}{\sqrt{2}}, \frac{Q_T^2 - Q^2}{Q\sqrt{2}}, \boldsymbol{q}_T\Big], \tag{B.2}$$

$$P = \Big[\left(\frac{xM}{Q\sqrt{2}}\right), \frac{Q}{x\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.3}$$

where $P^- = M^2/2P^+$ will be neglected in the rest. Here we have allowed for an arbitrary transverse momentum in q being $\boldsymbol{q}_T = Q_T\,\hat{q}_T$. We note that the parton momenta satisfying $p^2 = (p+q)^2 = 0$ (up to mass effects and with $|\boldsymbol{p}_T| \ll Q$) and the final state remnant momenta are given by

$$p = xP + p_T = \Big[0, \frac{Q}{\sqrt{2}}, (\boldsymbol{p}_T)\Big], \tag{B.4}$$

$$p' = p + q = \Big[\frac{Q}{\sqrt{2}}, \frac{Q_T^2}{Q\sqrt{2}}, \boldsymbol{q}_T\Big], \tag{B.5}$$

$$P' = P - p = \Big[0, \frac{(1-x)}{x}\,\frac{Q}{\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.6}$$

$$P' + p' = \Big[\frac{Q}{\sqrt{2}}, \frac{xQ_T^2 + (1-x)Q^2}{xQ\sqrt{2}}, \boldsymbol{q}_T\Big]. \tag{B.7}$$

Note that the hadronic mass is given by

$$W^2 = \frac{(1-x)}{x}\,Q^2 \qquad \text{or} \qquad x = \frac{Q^2}{W^2 + Q^2}. \tag{B.8}$$

Specific frames are:
(1) $Q_T = 0$.

In this frame one has for the momenta

$$k = \Big[\frac{1}{y}\,\frac{Q}{\sqrt{2}}, \frac{(1-y)}{y}\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{1-y}}{y}\,\hat{\ell}_T\Big], \tag{B.9}$$

$$q = \Big[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.10}$$

$$P = \Big[0, \frac{Q}{x\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.11}$$

$$k' = k - q = \Big[\frac{(1-y)}{y}\,\frac{Q}{\sqrt{2}}, \frac{1}{y}\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{(1-y)}}{y}\,\hat{\ell}_T\Big], \tag{B.12}$$

$$p' = xP + q = \Big[\frac{Q}{\sqrt{2}}, 0, \mathbf{0}_T\Big]. \tag{B.13}$$

(2) $\boldsymbol{q}_T = -Q\sqrt{1-y}\,\hat{\ell}_T$ (HERA frame).
In this frame the electron and target hadron are parallel and one has

$$k = \Big[\frac{1}{y}\,\frac{Q}{\sqrt{2}}, 0, \mathbf{0}_T\Big], \tag{B.14}$$

$$q = \Big[\frac{Q}{\sqrt{2}}, -y\,\frac{Q}{\sqrt{2}}, -Q\sqrt{1-y}\,\hat{\ell}_T\Big], \tag{B.15}$$

$$P = \Big[0, \frac{Q}{x\sqrt{2}}, \mathbf{0}_T\Big], \tag{B.16}$$

$$k' = k - q = \Big[\frac{(1-y)}{y}\,\frac{Q}{\sqrt{2}}, y\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{(1-y)}}{y}\,\hat{\ell}_T\Big], \tag{B.17}$$

$$p' = xP + q = \Big[\frac{Q}{\sqrt{2}}, (1-y)\,\frac{Q}{\sqrt{2}}, Q\,\frac{\sqrt{(1-y)}}{y}\,\hat{\ell}_T\Big]. \tag{B.18}$$

(3) $Q_T = Q$.
In this frame one has in essence that q is transverse, implying for the hadronic momenta

$$q = \Big[\frac{Q}{\sqrt{2}}, 0, Q\,\hat{q}_T\Big], \tag{B.19}$$

$$P = \Big[0, \frac{Q}{x\sqrt{2}}, \mathbf{0}_T\Big]. \tag{B.20}$$

(4) $Q_T \downarrow Q$ (Bjorken frame).
First boosting the original momenta by multiplying the plus components with a factor $\sqrt{Q_T^2 - Q^2}/Q$ and the minus components with the inverse factor one obtains

$$q = \Big[\sqrt{\frac{Q_T^2 - Q^2}{2}}, \sqrt{\frac{Q_T^2 - Q^2}{2}}, Q\,\hat{q}_T\Big], \tag{B.21}$$

$$P = \Big[0, \frac{Q^2}{x\sqrt{2(Q_T^2 - Q^2)}}, \mathbf{0}_T\Big]. \tag{B.22}$$

In the limit $Q_T \downarrow Q$ this frame has a purely transverse momentum exchange and an infinite momentum for the target hadron.

B.2 Diffractive scattering

Diffractive deep inelastic scattering is considered as the situation in which a proton momentum P and photon momentum q in the initial state, lead to a colorless proton remnant P' with invariant mass squared $M_Y^2 \sim M^2 = P^2$. We consider specifically the regime, where $(P+q)^2 = W^2 \gg Q^2 = -q^2 \gg M^2$. This implies that

$$x_B = Q^2/2P{\cdot}q = \frac{Q^2}{W^2+Q^2+M^2} \approx \frac{Q^2}{W^2+Q^2} \approx \frac{Q^2}{W^2} \ll 1. \tag{B.23}$$

The momenta are written in lightcone components

$$P = \left[\frac{M^2}{2P^+}, P^+, \mathbf{0}_T\right] \approx \left[0, P^+, \mathbf{0}_T\right], \tag{B.24}$$

$$q = \left[q^-, \frac{-Q^2}{2q^-}, \mathbf{0}_T\right], \tag{B.25}$$

with $2P^+q^- \approx W^2 + Q^2$ and they exchange a momentum p, leaving a proton remnant momentum P',

$$p = \left[\frac{t+\boldsymbol{p}_T^2}{2x_P P^+}, x_P P^+, \boldsymbol{p}_T\right], \tag{B.26}$$

$$P' = \left[\frac{M_Y^2+\boldsymbol{p}_T^2}{2(1-x_P)P^+}, (1-x_P)P^+, -\boldsymbol{p}_T\right]. \tag{B.27}$$

The exchanged momentum squared $-t = p^2$ is smaller or of the same order as Q^2. Two often used frames to look at this problem are the brick-wall frame (A) or the photon-proton CM frame (B). We'll add a third one and refer to it as the diffractive frame (C). We have

$$(A) \quad q^- = Q/\sqrt{2} \quad \text{and} \quad P^+ \approx Q/x_B\sqrt{2}, \tag{B.28}$$

$$(B) \quad q^- = \sqrt{(W^2+Q^2)/2} \approx W/\sqrt{2} \quad \text{and} \quad P^+ \approx \sqrt{(W^2+Q^2)/2} \approx W/\sqrt{2}, \tag{B.29}$$

$$(C) \quad q^- = M_X/\sqrt{2} \quad \text{and} \quad P^+ = \frac{W^2+Q^2}{/} M_X\sqrt{2}. \tag{B.30}$$

In the brick wall frame the photon only has a space-like component and it naturally becomes the photon-parton CM system. The photon-proton CM frame or the diffractive frame are more natural for diffractive events. We see that the quantities t and $\boldsymbol{p}_T^2$ appearing in the exchanged momentum satisfy

$$t + \frac{\boldsymbol{p}_T^2}{1-x_P} = -x_P\left(\frac{M_Y^2}{1-x_P} - M^2\right) \quad \text{or} \quad t + \boldsymbol{p}_T^2 \approx -x_P(M_Y^2 - M^2). \tag{B.31}$$

The exchanged momentum will be added to the momentum q producing a final state spanning a rapidity range with invariant mass squared M_X^2. Taking it to consist of two (massless) momenta

$$q_1 = \left[\alpha q^-, \frac{Q_T^2}{2\alpha q^-}, \boldsymbol{q}_T\right], \tag{B.32}$$

$$q_2 = \left[(1-\alpha)q^-, \frac{Q_T^2}{2(1-\alpha)q^-}, -\boldsymbol{q}_T\right], \tag{B.33}$$

$$P_X = q_1 + q_2 = \left[q^-, \frac{Q_T^2}{2\alpha(1-\alpha)q^-}, \mathbf{0}_T\right], \tag{B.34}$$

where we have introduced $Q_T^2 = \boldsymbol{q}_T^2$ and we find that the invariant mass squared M_X^2 is

$$M_X^2 = \frac{Q_T^2}{\alpha(1-\alpha)}, \quad \text{thus} \quad \alpha \approx \frac{Q_T^2}{M_X^2+Q_T^2} \approx \frac{Q_T^2}{M_X^2} \text{ and } 1-\alpha \approx \frac{M_X^2}{M_X^2+Q_T^2} \approx 1, \tag{B.35}$$

where the approximation has already assumed $\alpha \ll 1$. The system X spans a rapidity range from $\eta_1 = \frac{1}{2}\ln(q_1^-/q_1^+) = \ln(\alpha q^-\sqrt{2}/Q_T)$ to $\eta_2 = \frac{1}{2}\ln(q_2^-/q_2^+) = \ln((1-\alpha)q^-\sqrt{2}/Q_T)$, leading to

$$\Delta\eta = \ln\left(\frac{1-\alpha}{\alpha}\right) \approx -\ln\alpha - \alpha \approx -\ln\alpha. \tag{B.36}$$

A sizable rapidity range requires α to be small (and $1-\alpha$ close to one), which makes M_X^2 substantially larger than $\boldsymbol{q}_T^2$. Including the transferred transverse momentum component p_T in p the final state momentum P_X doesn't change the validity of the various approximations above.

Matching the plus components in $p + q = P_X$,

$$\frac{-Q^2}{2q^-} + x_P P^+ = \frac{Q_T^2}{2\alpha(1-\alpha)q^-} \approx \frac{M_X^2}{2q^-}, \tag{B.37}$$

gives

$$x_P \approx \frac{M_X^2 + Q^2}{W^2 + Q^2} \quad \text{and} \quad \beta \equiv \frac{x_B}{x_P} = \frac{Q^2}{M_X^2 + Q^2}. \tag{B.38}$$

Looking now at the (exchanged) momentum

$$k = q_2 - q = p - q_1 = \Big[-\alpha q^-, x_P P^+ - \frac{Q_T^2}{2\alpha q^-}, \boldsymbol{p}_T - \boldsymbol{q}_T\Big] \approx \Big[-\alpha q^-, \beta x_P P^+, \boldsymbol{p}_T - \boldsymbol{q}_T\Big] \tag{B.39}$$

it in essence carries a fraction α of q and a fraction x_B of P, and has a low (negative) invariant mass squared of the order of $2\alpha x_B P^+ q^- - \ldots Q_T^2 \approx -\alpha Q^2 - \ldots Q_T^2 \approx -Q_T^2\,(Q^2 + \ldots M_X^2)/M_X^2$ Explicitly we get in the parton-photon CM frame (A) and the nucleon-photon CM frame (B) the momentum

$$k \overset{(A)}{=} \Big[\frac{-Q_T^2}{M_X^2}\frac{Q}{\sqrt{2}}, \frac{Q}{\sqrt{2}}, \boldsymbol{p}_T - \boldsymbol{q}_T\Big] \tag{B.40}$$

$$k \overset{(B)}{=} \Big[-\alpha\frac{\sqrt{W^2+Q^2}}{\sqrt{2}}, \frac{Q^2}{\sqrt{W^2+Q^2}\sqrt{2}}, \boldsymbol{p}_T - \boldsymbol{q}_T\Big], \tag{B.41}$$

which acquire natural meaning as 'parton' with momentum $\beta p = x_B P$ taken from the momentum $p = x_P P$ (frame A, Pomeron picture) or as 'parton' with momentum αq in the 'photon' interacting with the target P via a colorless exchange (frame B, dipole picture). Finally the gap between the final state stuff X and the target remnant at $\eta(P') = \frac{1}{2}\ln((M_Y^2 + \boldsymbol{p}_T^2)/(P^+)^2)$ is given by

$$\Delta\eta_{\text{gap}} = \eta(P') - \eta_1 = \frac{1}{2}\ln\left(\frac{(W^2+Q^2)^2}{4M_X^2(M_Y^2+\boldsymbol{p}_T^2)}\right) \approx \ln\left(\frac{M_W^2+Q^2}{2M_X M_Y}\right) \tag{B.42}$$

$$= \ln\left(\frac{M_X^2+Q^2}{2M_X M_Y}\right) - \ln(x_P) = \ln\left(\frac{Q^2}{2M_X M_Y}\right) - \ln(x_B). \tag{B.43}$$

We add to the Pomeron and dipole pictures a third (also equivalent) picture in terms of TMDs, which is natural in the limit $W \to \infty$ or $x_P \to 0$ (and thus also $x_B \to 0$). For $x_P = 0$ the momentum transfer squared $t = -\boldsymbol{p}_T^2$. The natural frame is frame B, where now a diffractive TMD $f_{\text{diff}}^p(\boldsymbol{p}_T)$ is combined with photon TMD PDFs $f^{q/\gamma}(\alpha, \boldsymbol{q}_T)$ and $f^{\bar{q}/\gamma}(\alpha, \boldsymbol{q}_T)$. The transverse momentum q_T (linked to M_X) belongs to the photon PDF and is conjugate to the 'dipole size', while the transverse momentum p_T (experimentally linked to t) belongs to the proton TMD and relates to the 'transverse distance' in the Wilson loop in the matrix element in the definition of the 'diffractive proton TMD'.

B.2.1 Double diffractive scattering

In the situation of hadron-hadron scattering, we can for each of the hadrons (P_1 and P_2) and remnants (P_1' and P_2') a similar situation leading to a soft transverse-like exchange to momenta

$$p_1 = \Big[\frac{t + \boldsymbol{p}_{1T}^2}{2\xi_1 P_1^+}, \xi_1\, P_1^+, \boldsymbol{p}_{1T}\Big], \tag{B.44}$$

$$p_2 = \Big[\xi_2\, P_2^-, \frac{t_2 + \boldsymbol{p}_{2T}^2}{2\xi_2 P_2^-}, \boldsymbol{p}_{2T}\Big], \tag{B.45}$$

with like x_P in the previous section $t_i + \boldsymbol{p}_{iT}^2 \propto \xi_i \times (\text{hadronic scale})^2$ and $\xi_i \ll 1$. We now also consider production of M_X^2 in the intermediate region as originating from two massless momenta q_1 and q_2 with transverse momenta $\pm\boldsymbol{q}_T$ (for which $\boldsymbol{q}_T^2 = Q_T^2$), in the appropriate frame where $P_1^+ = M_X/x_1\sqrt{2}$ and $P_2^- = M_X/x_2\sqrt{2}$ (thus $s = x_1\,x_2\,M_X^2$), given by

$$q_1 = \Big[\alpha\frac{M_X}{\sqrt{2}}, (1-\alpha)\frac{M_X}{\sqrt{2}}, -\boldsymbol{q}_T\Big], \tag{B.46}$$

$$q_2 = \Big[(1-\alpha)\frac{M_X}{\sqrt{2}}, \alpha\frac{M_X}{\sqrt{2}}, \boldsymbol{q}_T\Big], \tag{B.47}$$

$$q = \Big[\frac{M_X}{\sqrt{2}}, \frac{M_X}{\sqrt{2}}, \boldsymbol{0}_T\Big]. \tag{B.48}$$

The (for our purposes again small) fraction α is determined by

$$\alpha(1-\alpha) = \frac{Q_T^2}{M_X^2} \quad \text{or} \quad \alpha \approx \frac{Q_T^2}{M_X^2 + Q_T^2} \approx \frac{Q_T^2}{M_X^2} \text{ and } 1-\alpha \approx \frac{M_X^2}{M_X^2 + Q_T^2}. \tag{B.49}$$

In this case the simple picture of a produced 'dipole' implies

$$\begin{aligned} k &= p_1 - q_1 = \Big[\Big(\frac{x_1}{\xi_1}\,\frac{t_1 + \boldsymbol{p}_{1T}^2}{M_X^2} - \alpha\Big)\frac{M_X}{\sqrt{2}}, \Big(\frac{\xi_1}{x_1} + \alpha - 1\Big)\frac{M_X}{\sqrt{2}}, \boldsymbol{p}_{1T} - \boldsymbol{q}_T\Big] & \text{(B.50)} \\ &= q_2 - p_2 == \Big[\Big(1 - \alpha - \frac{\xi_2}{x_2}\Big)\frac{M_X}{\sqrt{2}}, \Big(\alpha - \frac{x_2}{\xi_2}\,\frac{t_2 + \boldsymbol{p}_{2T}^2}{M_X^2}\Big)\frac{M_X}{\sqrt{2}}, -\boldsymbol{p}_{2T-}\boldsymbol{q}_T\Big]. & \text{(B.51)} \end{aligned}$$

The solution implies

$$\beta_1 = \frac{x_1}{\xi_1} = 1 + \frac{t_2 + \boldsymbol{p}_{2T}^2}{M_X^2} \approx 1 \quad \text{and} \quad \beta_2 = \frac{x_2}{\xi_2} = 1 - \frac{t_1 + \boldsymbol{p}_{1T}^2}{M_X^2} \approx 1. \tag{B.52}$$

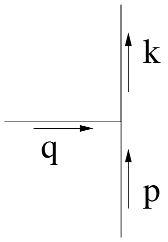
Thus

$$k \approx \Big[-\alpha\,\frac{M_X}{\sqrt{2}}, \alpha\,\frac{M_X}{\sqrt{2}}, -\boldsymbol{q}_T\Big]. \tag{B.53}$$

Appendix C

Kinematics

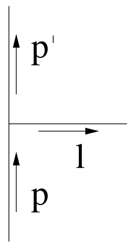
C.1 Single parton case: absorption of hard momentum



$p^2 = k^2 = 0$

$$p \approx \left[0,\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$
$$q \approx \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad q^2 = -Q^2$$
$$k \approx \left[\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

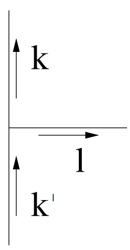
C.2 Single-parton distribution: branching



$p^2 = l^2 = 0$

$$p \approx \left[0,\ \frac{1}{x_p}\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$
$$l \approx \left[(1-z_k)\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\frac{\tilde{Q}}{\sqrt{2}},\ -\boldsymbol{p}_T\right] \qquad \boldsymbol{p}_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\tilde{Q}^2$$
$$p' \approx \left[-(1-z_k)\frac{\tilde{Q}}{\sqrt{2}},\ \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{p}_T\right] \qquad p'^2 = -\frac{1}{(1-x_p)}\boldsymbol{p}_T^2$$

C.3 Single-parton fragmentation: branching



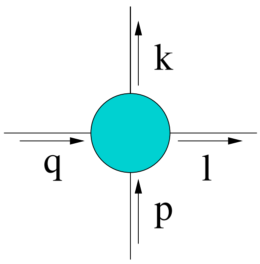
$k^2 = l^2 = 0$

$$k \approx \left[z_k\frac{\tilde{Q}}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$
$$l \approx \left[(1-z_k)\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{k}_T\right] \qquad \boldsymbol{k}_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\tilde{Q}^2$$
$$k' \approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{k}_T\right] \qquad k'^2 = \frac{z_k}{(1-z_k)}\boldsymbol{k}_T^2$$

C.4 Single-parton case: absorption and branching

In this case we choose a parametrization satisfying

$$x_p = \frac{\tilde{Q}^2}{2p\cdot q} = -\frac{q\cdot k}{p\cdot k} \quad \text{and} \quad z_k = \frac{p\cdot k}{p\cdot q} = -\frac{k\cdot q}{\tilde{Q}^2}$$



$p^2 = k^2 = l^2 = 0$

$$p \approx \left[0,\ \frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ -\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \qquad q^2 = -Q^2 = -(\tilde{Q}^2 + Q_T^2)$$

$$k \approx \left[z_k\,\frac{\tilde{Q}}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right] \qquad \left(\boldsymbol{q}_T^2 = Q_T^2\right)$$

$$l \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}}, \boldsymbol{q}_T\right] \qquad Q_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\,\tilde{Q}^2 = \frac{(1-z_k)(1-x_p)}{1-z_k+x_p z_k}\,Q^2$$

$$\begin{aligned} p_s &= p+q = k+l \\ &\approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \\ p_t &= k-p = q+l \\ &\approx \left[z_k\,\frac{\tilde{Q}}{\sqrt{2}},\ -\frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right] \\ p_u &= p-l = k-q \\ &\approx \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \end{aligned}$$

$$\hat{s} = p_s^2 = \frac{z_k(1-x_p)}{x_p}\,\tilde{Q}^2 = \frac{z_k}{(1-z_k)}\,Q_T^2$$

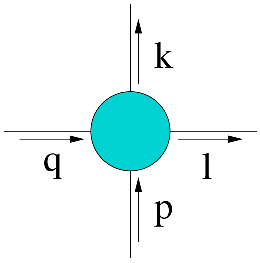
$$\hat{t} = p_t^2 = -\frac{z_k}{x_p}\,\tilde{Q}^2 = -\frac{z_k}{(1-x_p)(1-z_k)}\,Q_T^2$$

$$\hat{u} = p_u^2 = -\frac{(1-z_k)}{x_p}\,\tilde{Q}^2 = -\frac{1}{(1-x_p)}\,Q_T^2$$

C.5 Single-parton case (again): absorption and branching

In this case we choose a slightly different parametrization corresponding with

$$x_p = \frac{Q^2}{2p\cdot q} = -\frac{q\cdot k}{p\cdot k}\,\frac{Q^2}{\tilde{Q}^2} \quad \text{and} \quad z_k = \frac{p\cdot k}{p\cdot q} = -\frac{k\cdot q}{\tilde{Q}^2}$$



$p^2 = k^2 = l^2 = 0$

$$p \approx \left[0,\ \frac{1}{x_p}\,\frac{Q^2}{\tilde{Q}\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ -\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \qquad q^2 = -Q^2 = -(\tilde{Q}^2 + Q_T^2)$$

$$k \approx \left[z_k\,\frac{\tilde{Q}}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right] \qquad (\boldsymbol{q}_T^2 = Q_T^2)$$

$$l \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \left(\frac{Q^2}{x_p\tilde{Q}^2} - 1\right)\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \qquad Q_T^2 = \frac{(1-z_k)(1-x_p)}{x_p z_k}\,Q^2 = \frac{(1-z_k)(1-x_p)}{x_p + z_k - 1}\,\tilde{Q}^2$$

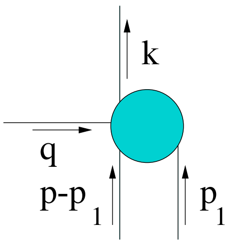
$$\begin{aligned} p_s &= p + q = k + l \\ &\approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ \left(\frac{Q^2}{x_p\tilde{Q}^2} - 1\right)\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \\ p_t &= k - p = q + l \\ &\approx \left[z_k\,\frac{\tilde{Q}}{\sqrt{2}},\ -\frac{1}{x_p}\,\frac{Q^2}{\tilde{Q}\sqrt{2}},\ \mathbf{0}_T\right] \\ p_u &= p - l = k - q \\ &\approx \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{q}_T\right] \end{aligned}$$

$$\hat{s} = p_s^2 = \frac{(1-x_p)}{x_p}\,Q^2 = \frac{z_k}{(1-z_k)}\,Q_T^2$$

$$\hat{t} = p_t^2 = -\frac{z_k}{x_p}\,Q^2 = -\frac{z_k^2}{(1-x_p)(1-z_k)}\,Q_T^2$$

$$\hat{u} = p_u^2 = -\frac{(1-z_k)}{x_p}\,Q^2 = -\frac{z_k}{(1-x_p)}\,Q_T^2$$

C.6 Multi-parton distribution: absorption



$p_1^2 = p^2 = p \cdot p_1 = k^2 = 0$

$$p \approx \left[0,\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p_1 \approx \left[0,\ x_1\,\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad q^2 = -Q^2$$

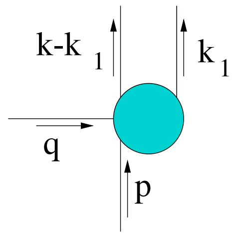
$$k \approx \left[\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

In general ($p = \sum p_i$):

$$k - p_i \approx \left[\frac{Q}{\sqrt{2}},\ -x_i\,\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (k-p_i)^2 = -x_i\,Q^2$$

$$q + p_i \approx \left[\frac{Q}{\sqrt{2}},\ -(1-x_i)\,\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (q+p_i)^2 = -(1-x_i)\,Q^2$$

C.7 Multi-parton fragmentation: absorption



$$p \approx \left[0,\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$q \approx \left[\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad q^2 = -Q^2$$

$$k_1 \approx \left[z_1\,\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

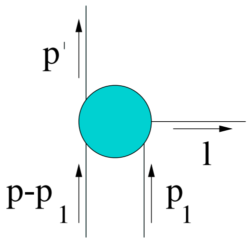
$$k_1^2 = k^2 = k\cdot k_1 = p^2 = 0 \qquad k \approx \left[\frac{Q}{\sqrt{2}},\ 0,\ \mathbf{0}_T\right]$$

In general ($k = \sum k_i$):

$$k_i - p \approx \left[z_i\,\frac{Q}{\sqrt{2}},\ -\frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (k_i-p)^2 = -z_i\,Q^2$$

$$k_i - q \approx \left[-(1-z_i)\,\frac{Q}{\sqrt{2}},\ \frac{Q}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad (q-k_i)^2 = -(1-z_i)\,Q^2$$

C.8 Multi-parton distribution: branching



$$p \approx \left[0,\ \frac{1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p_1 \approx \left[0,\ \frac{x_1}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$l \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ -\boldsymbol{p}_T\right] \qquad \boldsymbol{p}_T^2 = \frac{(1-z_k)(1-x_p)}{x_p}\,\tilde{Q}^2$$

$$p_1^2 = p^2 = p\cdot p_1 = l^2 = 0 \qquad p' \approx \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{p}_T\right] \qquad p'^2 = -\frac{1}{1-x_p}\,\boldsymbol{p}_T^2$$

In general ($p = \sum p_i$):

$$l - p_i \approx \left[(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{((1-x_i)-x_p)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ -\boldsymbol{p}_T\right] \qquad (l-p_i)^2 = -\frac{x_i}{(1-x_p)}\,\boldsymbol{p}_T^2$$

$$p' - p_i \approx \left[-(1-z_k)\,\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(x_p-x_i)}{x_p}\,\frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{p}_T\right] \qquad (p'-p_i)^2 = -\frac{(1-x_i)}{(1-x_p)}\,\boldsymbol{p}_T^2$$

Equivalently one can also use the following parametrization (where l defines the minus direction)

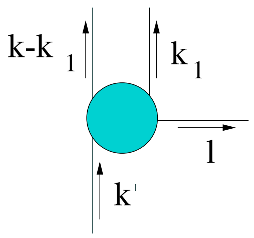
$$p \approx \left[0,\ \frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p_1 \approx \left[0,\ x_1\,\frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right] \qquad l - p_i \approx \left[\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ -x_i\,\frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ -\mathbf{0}_T\right]$$

$$l \approx \left[\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ 0,\ \mathbf{0}_T\right] \qquad p' - p_i \approx \left[-\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ (1-x_i)\,\frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right]$$

$$p' \approx \left[-\frac{|\boldsymbol{p}_T|}{(1-x_p)\sqrt{2}},\ \frac{|\boldsymbol{p}_T|}{\sqrt{2}},\ \mathbf{0}_T\right]$$

C.9 Multi-parton fragmentation: branching



$$k_1^2 = k^2 = k \cdot k_1 = l^2 = 0$$

$$k \approx \left[z_k \, \frac{\tilde{Q}}{\sqrt{2}},\ 0,\ \mathbf{0}_T \right]$$

$$k_1 \approx \left[z_1 \, z_k \, \frac{\tilde{Q}}{\sqrt{2}},\ 0,\ \mathbf{0}_T \right]$$

$$l \approx \left[(1 - z_k) \, \frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1 - x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{k}_T \right] \qquad \boldsymbol{k}_T^2 = \frac{(1 - z_k)(1 - x_p)}{x_p} \, \tilde{Q}^2$$

$$k' \approx \left[\frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1 - x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{k}_T \right] \qquad k'^2 = \frac{z_k}{(1 - z_k)} \, \boldsymbol{k}_T^2$$

In general ($k = \sum k_i$):

$$k_i + l \approx \left[(1 - (1 - z_i) z_k) \, \frac{\tilde{Q}}{\sqrt{2}},\ \frac{(1 - x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}},\ \boldsymbol{k}_T \right] \qquad (l + k_i)^2 = -\frac{z_i \, z_k}{(1 - z_k)} \, \boldsymbol{k}_T^2$$

$$k_i - k' \approx \left[-(1 - z_i \, z_k) \, \frac{\tilde{Q}}{\sqrt{2}},\ -\frac{(1 - x_p)}{x_p} \, \frac{\tilde{Q}}{\sqrt{2}},\ -\boldsymbol{k}_T \right] \qquad (k_i - k')^2 = -\frac{(1 - z_i) \, z_k}{(1 - z_k)} \, \boldsymbol{k}_T^2$$

Appendix D

Useful formulae

D.1 Combining denominators

Feynman trick:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{(xA + (1-x)B)^2}. \tag{D.1}$$

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \dots dx_n \; \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + \dots + x_n A_n)^n}. \tag{D.2}$$

D.2 Some indefinite integrals

$$(\mathrm{D} = 2) \qquad \int dp \, \sqrt{p^2 \pm m^2} = \frac{1}{2} p \sqrt{p^2 \pm M^2} + \frac{1}{2} M^2 \, \ln\left(p + \sqrt{p^2 \pm M^2}\right) \tag{D.3}$$

$$(\mathrm{D} = 0) \qquad \int dp \, \frac{1}{\sqrt{p^2 \pm M^2}} = \ln\left(p + \sqrt{p^2 \pm M^2}\right) \tag{D.4}$$

$$(\mathrm{D} = 0) \qquad \frac{\pi}{2} \int dp \, \frac{p}{p^2 \pm M^2} = \pi \, \ln\left(p^2 \pm M^2\right) \tag{D.5}$$

$$(\mathrm{D} = \text{-}1) \qquad \int dp \, \frac{1}{p^2 + M^2} = \frac{1}{M} \, \tan^{-1}\left(\frac{p}{M}\right), \tag{D.6}$$

$$(\mathrm{D} = \text{-}1) \qquad \int dp \, \frac{1}{p^2 - M^2} = \frac{1}{M} \, \ln\left|\frac{p - M}{p + M}\right|, \tag{D.7}$$

D.3 Special functions

Beta function

$$\begin{aligned} B(\mu, \nu) &= \int_0^1 dx \; x^{\mu-1} (1-x)^{\nu-1} && \text{(D.8)} \\ &= \int_0^\infty dy \; y^{\nu-1} (1+y)^{-\mu-\nu} && \text{(D.9)} \\ &= 2 \int_0^{\pi/2} d\theta \; \sin^{2\mu-1}\theta \, \cos^{2\nu-1}\theta && \text{(D.10)} \\ &= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}. && \text{(D.11)} \end{aligned}$$

The Gamma function is defined for $\mathrm{Re}(z) > 0$ as

$$\Gamma(z) = \int_0^\infty dx \; x^{z-1} \, \exp(-x) \tag{D.12}$$

and satisfies $\Gamma(n) = (n-1)!$, $\Gamma(\frac{1}{2}) = \pi^{1/2}$ and the relation

$$z\,\Gamma(z) = \Gamma(z+1). \tag{D.13}$$

The function has simple poles with residue $(-)^n(1/n!)$ at the points $z = -n$. Near negative integers the expansion can be obtained from

$$\begin{aligned}\Gamma(1+\epsilon) &= 1 + \int_0^\infty dx\ (\epsilon\,\ln x + \frac{1}{2}\epsilon^2\,\ln^2 x + \ldots)\exp(-x) && \text{(D.14)}\\ &= 1 - \epsilon\,\gamma_E + \epsilon^2\left(\frac{1}{2}\gamma_E^2 + \frac{1}{12}\pi^2\right) + \ldots, && \text{(D.15)}\end{aligned}$$

where $\gamma_E \approx 0.5772$ is Euler's constant. One has

$$\begin{aligned}\Gamma(z) &\overset{z\approx 0}{=} \frac{1}{z} - \gamma_E + \mathscr{O}(z), && \text{(D.16)}\\ \Gamma(z) &\overset{z\approx -n}{=} \frac{(-)^n}{n!(z+n)} - \gamma_E + 1 + \ldots + \frac{1}{n} + \mathscr{O}(z+n), && \text{(D.17)}\end{aligned}$$

D.4 Dimensional regularization

Minkowski space integrals can easily be turned into Euclidean integrals using

$$\begin{aligned}i\,k_E^0 &= k^0, && \text{(D.18)}\\ i\,d^n k_E &= d^n k && \text{(D.19)}\\ -k_E^2 &= k^2. && \text{(D.20)}\end{aligned}$$

Basically n-dimensional Euclidean integrals are performed via the angular decomposition and integration

$$\begin{aligned}&\int d^n x = \int_0^\infty dr\ r^{n-1}\int_0^\pi d\theta_{n-1}\ \sin^{n-2}\theta_{n-1}\int_0^\pi d\theta_{n-2}\ \sin^{n-3}\theta_{n-2}\ldots\int_0^{2\pi} d\theta_1, && \text{(D.21)}\\ &\int_0^\pi d\theta\ \sin^m\theta = \sqrt{\pi}\,\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}. && \text{(D.22)}\end{aligned}$$

Radial and one-dimensional integrals are

$$\begin{aligned}\int d^n x\ f(r) &= \frac{2\,\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int dr\ r^{n-1}\,f(r), && \text{(D.23)}\\ \int dx\ \frac{x^\beta}{(x^2+m^2)^\alpha} &= \frac{1}{2}\,\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\alpha - \frac{\beta+1}{2}\right)}{\Gamma(\alpha)\,(m^2)^{\alpha-(\beta+1)/2}}. && \text{(D.24)}\end{aligned}$$

A basic integral is

$$\begin{aligned}\int d^n k\,\frac{1}{(k^2 - M^2 + i\epsilon)^s} &= i\int d^n k_E\,\frac{1}{(-k_E^2 - M^2)^s} && \text{(D.25)}\\ &= i\,(-)^s\,\pi^{n/2}\,\frac{\Gamma\left(s-\frac{n}{2}\right)}{\Gamma(s)}\,(M^2)^{n/2-s} && \text{(D.26)}\\ &= \frac{i\,(-)^s\,\pi^{n/2}}{\left(s-\frac{n}{2}\right)}\,\frac{\Gamma\left(s-\frac{n}{2}\right)}{\Gamma(s)}\,(M^2)^{n/2-s} && \text{(D.27)}\\ &\approx \frac{i\,(-)^s\,\pi^s}{\left(s-\frac{n}{2}\right)}\,\frac{1}{\Gamma(s)}\left[1 - \left(s-\frac{n}{2}\right)[\gamma_E + \ln(\pi\,M^2)] + \ldots\right], && \text{(D.28)}\end{aligned}$$

where the last expansion is near $n = 2s$, or with $n = 2s - \epsilon$,

$$\int d^{2s-\epsilon}k\,\frac{1}{(k^2 - M^2 + i\epsilon)^s} \approx \frac{i\,(-)^s\,\pi^s}{\Gamma(s)}\left[\frac{2}{\epsilon} - \gamma_E - \ln(\pi\,M^2) + \mathscr{O}(\epsilon)\right]. \tag{D.29}$$

Note that one needs to be careful with factors like

$$M^\epsilon = e^{\epsilon \ln M} = 1 + \frac{\epsilon}{2} \ln M^2 + \mathscr{O}(\epsilon),$$
$$\frac{1}{(2\pi)^n} = \frac{1}{(2\pi)^{2s-\epsilon}} = \frac{1}{(2\pi)^s} \left[1 + \frac{\epsilon}{2} \ln(4\pi^2) + \mathscr{O}(\epsilon)\right].$$

Thus one has

$$M^\epsilon \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - M^2 + i\epsilon)^s} \overset{n=2s-\epsilon}{\approx} \frac{i\,(-)^s}{(4\pi)^s \Gamma(s)} \left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) + \mathscr{O}(\epsilon)\right]. \tag{D.30}$$

Including numerators we have for instance

$$\int d^n k\, \frac{1}{((k-p)^2 - M^2 + i\epsilon)^s} = i\,(-)^s\, \pi^{n/2}\, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)}\, M^{n-2s} \tag{D.31}$$
$$\int d^n k\, \frac{k_\mu}{((k-p)^2 - M^2 + i\epsilon)^s} = -ip_\mu\,(-)^s\, \pi^{n/2}\, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)}\, M^{n-2s} \tag{D.32}$$
$$\int d^n k\, \frac{k_\mu k_\nu}{((k-p)^2 - M^2 + i\epsilon)^s} = i\left[p_\mu p_\nu - \frac{1}{2}\frac{\Gamma\left(s - \frac{n}{2} - 1\right)}{\Gamma\left(s - \frac{n}{2}\right)}\, M^2\, g_{\mu\nu}\right] \times (-)^s\, \pi^{n/2}\, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)}\, M^{n-2s}. \tag{D.33}$$

Actually the above integrals can be obtained by realizing that the proper averaging requires

$$k_\mu k_\nu \longrightarrow \frac{1}{n}\, k^2\, g_{\mu\nu},$$
$$k_\mu k_\nu k_\rho k_\sigma \longrightarrow \frac{1}{n(n+2)}\, (k^2)^2\, (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})\,.$$

This leads to

$$\int d^n k\, \frac{1}{(k^2 - M^2 + i\epsilon)^s} = i\,(-)^s\, \pi^{n/2}\, \frac{\Gamma\left(s - \frac{n}{2}\right)}{\Gamma(s)}\, M^{n-2s}$$
$$\int d^n k\, \frac{k^2}{(k^2 - M^2 + i\epsilon)^s} = -i\,(-)^s\, \pi^{n/2}\, \frac{n}{2}\, \frac{\Gamma\left(s - \frac{n}{2} - 1\right)}{\Gamma(s)}\, M^{n-2s-2}$$
$$\int d^n k\, \frac{k_\mu k_\nu}{(k^2 - M^2 + i\epsilon)^s} = -i\,(-)^s\, \pi^{n/2}\, \frac{1}{2}\, g_{\mu\nu}\, \frac{\Gamma\left(s - \frac{n}{2} - 1\right)}{\Gamma(s)}\, M^{n-2s-2}$$
$$\int d^n k\, \frac{(k^2)^2}{(k^2 - M^2 + i\epsilon)^s} = i\,(-)^s\, \pi^{n/2}\, \frac{n(n+2)}{4}\, \frac{\Gamma\left(s - \frac{n}{2} - 2\right)}{\Gamma(s)}\, M^{n-2s-4}$$
$$\int d^n k\, \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2 - M^2 + i\epsilon)^s} = i\,(-)^s\, \pi^{n/2}\, \frac{1}{4}\, (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho})\, \frac{\Gamma\left(s - \frac{n}{2} - 2\right)}{\Gamma(s)}\, M^{n-2s-4}$$

Finally because $g_{\mu\nu} g^{\mu\nu} = g^\mu_\mu = n$ one has in n dimensions

$$\gamma^\mu \gamma_\mu = n, \tag{D.34}$$
$$\gamma_\mu \gamma^\rho \gamma^\mu = -(n-2)\,\gamma^\rho, \tag{D.35}$$
$$\gamma_\mu \gamma^\rho \gamma^\sigma \gamma^\mu = 4\, g^{\rho\sigma} - (4-n)\,\gamma^\rho \gamma^\sigma, \tag{D.36}$$
$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = -2\,\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\,\gamma^\nu \gamma^\rho \gamma^\sigma. \tag{D.37}$$

D.5 Some useful relations involving distributions

The integral representation of the step functions are

$$\theta(\pm x) = \pm \int \frac{d\alpha}{2\pi\, i}\, \frac{e^{i\alpha x}}{\alpha \mp i\epsilon}, \;=\; \mp \int \frac{d\alpha}{2\pi\, i}\, \frac{e^{-i\alpha x}}{\alpha \pm i\epsilon}, \tag{D.38}$$
$$\epsilon(\pm x) = \mathrm{P} \int \frac{d\alpha}{\pi\, i}\, \frac{e^{i\alpha x}}{\alpha} = \theta(x) - \theta(-x). \tag{D.39}$$

The derivatives satisfy $\theta'(x) = \delta(x)$ and $\epsilon'(x) = 2\,\delta(x)$.
Other useful relations are

$$\frac{1}{x - i\epsilon} = P\,\frac{1}{x} + i\pi\,\delta(x), \tag{D.40}$$

$$\ln(x - i\epsilon) = \ln|x| - i\pi\,\theta(-x). \tag{D.41}$$

In order to treat endpoint singularities in $\int dx\ f(x)/(1-x)$ near $x = 1$ one usually employs the "+" prescription,

$$(F(x))_+ \equiv \lim_{\beta\to 0}\left[F(x)\,\theta(1-\beta-x) + \delta(1-\beta-x)\int_0^{1-\beta} dy\,F(y)\right], \tag{D.42}$$

satisfying $\int_0^1 dx(F(x))_+ = 0$. Note that for an integral not running between 0 and 1 one has

$$\begin{aligned}\int_x^1 dz\ \frac{G(z)}{(1-z)_+} &= \int_x^{1-\beta} dz\ \frac{G(z)}{(1-z)} - G(1-\beta)\int_0^{1-\beta} dz\ \frac{1}{1-z}\\ &= \int_x^1 dz\ \frac{G(z)-G(1)}{1-z} - G(1)\int_0^x dz\ \frac{1}{1-z}\\ &= \int_x^1 dz\ \frac{G(z)-G(1)}{1-z} + G(1)\ \ln(1-x)\end{aligned} \tag{D.43}$$

One thus has regularized via the endpoint of the integration leading to

$$\frac{1}{1-x} = -\ln\beta\ \delta(1-x) + \frac{1}{(1-x)_+} + \mathscr{O}(\beta). \tag{D.44}$$

One can also use

$$\frac{1}{(1-x)^{1-\epsilon}} = -\frac{1}{\epsilon}\,\delta(1-x) + \frac{1}{(1-x)_+} + \mathscr{O}(\epsilon).$$

To see how this works, consider for a small ϵ

$$\begin{aligned}\int_x^1 dx\,\frac{G(z)}{(1-z)^{1-\epsilon}} &= \int_x^1 dx\,\frac{G(z)-G(1)}{(1-z)^{1-\epsilon}} + G(1)\int_x^1 dx\,\frac{1}{(1-z)^{1-\epsilon}}\\ &= \int_x^1 dx\,\frac{G(z)-G(1)}{(1-z)} + G(1)\int_0^{1-x} dy\ y^{\epsilon-1}\\ &= \int_x^1 dx\,\frac{G(z)-G(1)}{(1-z)} - G(1)\,\frac{1}{\epsilon} + G(1)\ \ln(1-x).\end{aligned}$$

Another useful "+" function is

$$\int_x^1 dz\ G(z)\left(\frac{\ln(1-z)}{1-z}\right)_+ = \int_x^1 dz\ \frac{(G(z)-G(1))\ln(1-z)}{1-z} + \frac{1}{2}\,G(1)\ \ln^2(1-x). \tag{D.45}$$

Let's also investigate the PV-prescription using the definition (see e.g. G. Leibbrandt, Rev. Mod. Phys. 59 (1987) 1067)

$$\mathrm{PV}\frac{1}{x^n} = \frac{1}{2}\left(\frac{1}{(x+i\epsilon)^n} + \frac{1}{(x-i\epsilon)^n}\right). \tag{D.46}$$

For $n = 1$ this is identical to the symmetric prescription

$$\mathrm{PV}\int dx\ \frac{F(x)}{x} = \frac{1}{2}\int dx\ \left(\frac{F(x)}{x+i\epsilon} + \frac{F(x)}{x+i\epsilon}\right) = \int^{-\delta} dx\ \frac{F(x)}{x} + \int_{\delta} dx\ \frac{F(x)}{x}. \tag{D.47}$$

For finite boundaries one finds

$$\mathrm{PV}\int_a^b dx\ \frac{F(x)}{x-x_0} = \int_a^b dx\ \frac{F(x)-F(x_0)}{x-x_0} + F(x_0)\ \ln\left|\frac{b-x_0}{a-x_0}\right|. \tag{D.48}$$

Comparing with the + prescription yields

$$\begin{aligned}
\mathrm{PV}\int_x^{1-\beta} dz\ \frac{F(z)}{1-z} &= \int_x^{1-\beta} dz\ \frac{F(z)-F(1)}{1-z} - F(1)\ \ln\left|\frac{\beta}{1-x}\right| \\
&= \int_x^1 dz\ \frac{F(z)}{(1-z)_+} - F(1)\ \ln|\beta|\,. \qquad \text{(D.49)}
\end{aligned}$$

For $n>1$ we can use

$$\mathrm{PV}\int_a^b dx\ \frac{F(x)}{(x-x_0)^n} = \frac{1}{n-1}\left(\frac{F(a)}{(a-x_0)^{n-1}} - \frac{F(b)}{(b-x_0)^{n-1}} + \mathrm{PV}\int_a^b dx\ \frac{F'(x)}{(x-x_0)^{n-1}}\right). \qquad \text{(D.50)}$$

As specific example

$$\mathrm{PV}\int_a^b dx\ \frac{F(x)}{(x-x_0)^2} = \frac{b\,F(a)-a\,F(b)}{(a-x_0)(b-x_0)} + \mathrm{PV}\int_a^b dx\ \frac{F'(x)}{x-x_0} \qquad \text{(D.51)}$$

D.6 Theta functions

Often it is useful to attack loop-integrals via lightcone variables leading to specific integrals of the type

$$\Theta^m_{n_1 n_2\ldots}(x_1,x_2,\ldots) = \int \frac{d\alpha}{2\pi\, i}\ \frac{\alpha^m}{(\alpha x_1 - 1 + i\epsilon)^{n_1}(\alpha x_2 - 1 + i\epsilon)^{n_2}\ldots}. \qquad \text{(D.52)}$$

In these integrals m can be reduced via

$$\Theta^m_{111\ldots}(x_1,x_2,x_3,\ldots) = \frac{1}{(x_1-x_2)}\left[\Theta^{(m-1)}_{11\ldots}(x_2,x_3,\ldots) - \Theta^{(m-1)}_{11\ldots}(x_1,x_3,\ldots)\right]. \qquad \text{(D.53)}$$

Proof:

$$\begin{aligned}
\Theta^m_{111}(x_1,x_2,x_3) &= \int \frac{d\alpha}{2\pi\, i}\ \frac{\alpha}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)(\alpha x_3 - 1 + i\epsilon)} \\
&= \frac{1}{(x_1-x_2)}\int \frac{d\alpha}{2\pi\, i}\ \frac{(\alpha x_1 - 1)-(\alpha x_2 - 1)}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)(\alpha x_3 - 1 + i\epsilon)}
\end{aligned}$$

The lower index can be lowered together with an upper index via

$$\Theta^m_{n_1\ldots}(x_1,\ldots) = -(n_1-1)\frac{d}{dx_1}\,\Theta^{(m-1)}_{(n_1-1)\ldots}(x_1,\ldots). \qquad \text{(D.54)}$$

Proof:

$$\begin{aligned}
\Theta^1_{21}(x_1,x_2) &= \int \frac{d\alpha}{2\pi\, i}\ \frac{\alpha}{(\alpha x_1 - 1 + i\epsilon)^2(\alpha x_2 - 1 + i\epsilon)} \\
&= -\frac{d}{dx_1}\int \frac{d\alpha}{2\pi\, i}\ \frac{1}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)}
\end{aligned}$$

The reduction of $\Theta^0_{111\ldots}$ to $\Theta_{11\ldots}$ is achieved via

$$\Theta^0_{111\ldots}(x_1,x_2,x_3,\ldots) = \frac{x_2\,\Theta^0_{11\ldots}(x_2,x_3) - x_1\,\Theta^0_{11\ldots}(x_1,x_3)}{x_1-x_2}. \qquad \text{(D.55)}$$

Proof:

$$\begin{aligned}
\Theta^0_{111\ldots}(x_1,x_2,x_3) &= \int \frac{d\alpha}{2\pi\, i}\ \frac{1}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)(\alpha x_1 - 1 + i\epsilon)} \\
&= \int \frac{d\alpha}{2\pi\, i}\ \frac{\alpha x_1 - (\alpha x_1 - 1)}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)(\alpha x_1 - 1 + i\epsilon)} \\
&= x_1\,\Theta^1_{111\ldots}(x_1,x_2,x_3) - \Theta^0_{11}(x_2,x_3)
\end{aligned}$$

and using the previous relation.

Finally we can reduce $n = 2$ (for $m = 0$) via

$$\int_{x_0}^{x_1} dy\ \Theta^0_{21}(y, x_2) = (x_1 - x_0)\,\Theta^0_{111}(x_0, x_1, x_2) \tag{D.56}$$

Proof:

$$\begin{aligned}\int_{x_0}^{x_1} dy\ \Theta^0_{21}(y, x_2) &= \int_{x_0}^{x_1} dy \int \frac{d\alpha}{2\pi\, i}\ \frac{1}{(\alpha y - 1 + i\epsilon)^2(\alpha x_2 - 1 + i\epsilon)} \\ &= \int \frac{d\alpha}{2\pi\, i}\ \frac{1}{\alpha} \int_{\alpha x_0}^{\alpha x_1} d(\alpha y)\ \frac{1}{(\alpha y - 1 + i\epsilon)^2(\alpha x_2 - 1 + i\epsilon)}\end{aligned}$$

and then using that

$$\frac{1}{(\alpha x_0 - 1)} - \frac{1}{(\alpha x_1 - 1)} = \frac{\alpha(x_1 - x_0)}{(\alpha x_0 - 1)(\alpha x_1 - 1)}$$

Further reduction for $n = 2$ and $m = 0$ then is possible, giving

$$\Theta^0_{21}(x_1, x_2) = -\frac{d}{dx_1}\left[x_1\,\Theta^0_{11}(x_1, x_2)\right]. \tag{D.57}$$

The only integral to be calculated actually is $\Theta^0_{11}(x_1, x_2)$, which is easily done via a contour integration in the complex plane. The explicit results for the simplest functions are then

$$\Theta^0_{11}(x_1, x_2) = \frac{\theta(x_1)\theta(-x_2) - \theta(-x_1)\theta(x_2)}{x_1 - x_2} = \frac{\theta(x_1) - \theta(x_2)}{x_1 - x_2}, \tag{D.58}$$

$$\Theta^0_2(x) = \delta(x), \tag{D.59}$$

$$\Theta^0_{111}(x_1, x_2, x_3) = \frac{x_2}{(x_1 - x_2)}\,\Theta^0_{11}(x_2, x_3) - \frac{x_1}{(x_1 - x_2)}\,\Theta^0_{11}(x_1, x_3) \tag{D.60}$$

$$= \frac{(x_2 - x_3)\,x_1\,\theta(x_1) + (x_3 - x_1)\,x_2\,\theta(x_2) + (x_1 - x_2)\,x_3\,\theta(x_3)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}, \tag{D.61}$$

$$\Theta^0_{21}(x_1, x_2) = \frac{x_2}{(x_1 - x_2)}\,\Theta^0_{11}(x_1, x_2) - \frac{x_1}{(x_1 - x_2)}\,\delta(x_1), \tag{D.62}$$

$$\Theta^1_{21}(x_1, x_2) = \frac{1}{(x_1 - x_2)}\,\Theta^0_{11}(x_1, x_2) - \frac{1}{(x_1 - x_2)}\,\delta(x_1), \tag{D.63}$$

$$\Theta^1_{111}(x_1, x_2, x_3) = \frac{1}{(x_1 - x_2)}\left[\Theta^0_{11}(x_2, x_3) - \Theta^0_{11}(x_1, x_3)\right] \tag{D.64}$$

$$= \frac{(x_2 - x_3)\,\theta(x_1) + (x_3 - x_1)\,\theta(x_2) + (x_1 - x_2)\,\theta(x_3)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}. \tag{D.65}$$

We also will use the following equality for Θ^0_{11} functions, seen using a principal value prescription,

$$\int dy\ \frac{x}{x - y}\left[\Theta^0_{11}(y, y - x) + \Theta^0_{11}(x, x - y)\right] = 0. \tag{D.66}$$

The singularities in the two terms come from $y \uparrow x$ (first term) and $y \downarrow x$ (right term) respectively.
Notes on the $i\epsilon$ prescription:

$$\Theta^0_{11}(x_1, x_2) = \int \frac{d\alpha}{2\pi\, i}\ \frac{1}{(\alpha x_1 - 1 + i\epsilon)(\alpha x_2 - 1 + i\epsilon)} = \frac{1}{x_1 - x_2}\left[\theta(x_1)\,\theta(-x_2) - \theta(-x_1)\theta(x_2)\right]$$

$$\Theta^0_{\underline{1}1}(x_1, x_2) = \int \frac{d\alpha}{2\pi\, i}\ \frac{1}{(\alpha x_1 - 1 - i\epsilon)(\alpha x_2 - 1 + i\epsilon)} = \frac{1}{x_1 - x_2}\left[\theta(x_1)\,\theta(x_2) - \theta(-x_1)\theta(-x_2)\right]$$

Including arbitrary pole positions we get

$$\int \frac{d\alpha}{2\pi\, i}\ \frac{1}{(\alpha x_1 - A_1 + i\epsilon)(\alpha x_2 - A_2 + i\epsilon)} \tag{D.67}$$

$$= \frac{x_1 - x_2}{x_1 A_2 - x_2 A_1}\, \Theta_{11}^{0}(x_1, x_2) \tag{D.68}$$

$$= \frac{\theta(x_1) - \theta(x_2)}{x_1 A_2 - x_2 A_1} = \frac{\theta(x_1)\,\theta(-x_2) - \theta(-x_1)\,\theta(x_2)}{x_1 A_2 - x_2 A_1}, \tag{D.69}$$

$$\int \frac{d\alpha}{2\pi\, i}\ \frac{1}{(\alpha x_1 - A_1 + i\epsilon)(\alpha x_2 - A_2 + i\epsilon)(\alpha x_1 - A_3 + i\epsilon)} \tag{D.70}$$

$$= \frac{1}{x_1 A_2 - x_2 A_1}\left[x_2\, \frac{x_2 - x_3}{x_2 A_3 - x_3 A_2}\, \Theta_{11}^{0}(x_2, x_3) - x_1\, \frac{x_1 - x_3}{x_1 A_3 - x_3 A_1}\, \Theta_{11}^{0}(x_1, x_3)\right] \tag{D.71}$$

$$= \frac{(x_2 A_3 - x_3 A_2)\, x_1\, \theta(x_1) + (x_3 A_1 - x_1 A_3)\, x_2\, \theta(x_2) + (x_1 A_2 - x_2 A_1)\, x_3\, \theta(x_3)}{(x_1 A_2 - x_2 A_1)(x_2 A_3 - x_3 A_2)(x_3 A_1 - x_1 A_3)}, \tag{D.72}$$

$$\int \frac{d\alpha}{2\pi\, i}\ \frac{\alpha}{(\alpha x_1 - A_1 + i\epsilon)(\alpha x_2 - A_2 + i\epsilon)(\alpha x_1 - A_3 + i\epsilon)} \tag{D.73}$$

$$= \frac{1}{x_1 A_2 - x_2 A_1}\left[A_2\, \frac{x_2 - x_3}{x_2 A_3 - x_3 A_2}\, \Theta_{11}^{0}(x_2, x_3) - A_1\, \frac{x_1 - x_3}{x_1 A_3 - x_3 A_1}\, \Theta_{11}^{0}(x_1, x_3)\right] \tag{D.74}$$

$$= \frac{(x_2 A_3 - x_3 A_2)\, \theta(x_1) + (x_3 A_1 - x_1 A_3)\, \theta(x_2) + (x_1 A_2 - x_2 A_1)\, \theta(x_3)}{(x_1 A_2 - x_2 A_1)(x_2 A_3 - x_3 A_2)(x_3 A_1 - x_1 A_3)}. \tag{D.75}$$

D.7 IR and UV singularities

In a $D = 0$ divergent integral one can separate the singularities e.g. as follows

$$\int \frac{d^2 k_T}{\boldsymbol{k}_T^2} = \int d^2 k_T \left[\frac{M^2}{\boldsymbol{k}_T^2(\boldsymbol{k}_T^2 + M^2)} + \frac{1}{\boldsymbol{k}_T^2 + M^2}\right]. \tag{D.76}$$

Regularized one sees that

$$M^\epsilon \int \frac{d^{2-\epsilon} k_T}{\boldsymbol{k}_T^2} = \underbrace{M^\epsilon \int d^{2-\epsilon} k_T\ \frac{M^2}{\boldsymbol{k}_T^2(\boldsymbol{k}_T^2 + M^2)}}_{\substack{-2\pi/\epsilon \\ \text{(IR-divergence)}}} + \underbrace{M^\epsilon \int d^{2-\epsilon} k_T\ \frac{1}{\boldsymbol{k}_T^2 + M^2}}_{\substack{2\pi/\epsilon \\ \text{(UV-divergence)}}} \tag{D.77}$$

D.8 Transforms

D.8.1 One dimensional transform

The Fourier transform in one dimension,

$$\tilde{F}(k) = \int dx\ \exp(-ikx)\, F(x) \tag{D.78}$$

$$F(x) = \int \frac{dk}{2\pi}\ \exp(+ikx)\, \tilde{F}(k) \tag{D.79}$$

has as basic orthogonality relations

$$\int_{-\infty}^{\infty} dx\ e^{i(k-k')x} = 4\int_0^{\infty} dx\ \sin(kx)\,\sin(k'x) = 2\pi\,\delta(k-k'), \tag{D.80}$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{-ik(x-x')} = 4\int_0^{\infty} \frac{dk}{2\pi}\ \sin(kx)\,\sin(kx') = \delta(x-x'). \tag{D.81}$$

Some examples of Bessel transformed pairs are

$$F(x) \Longleftrightarrow \tilde{F}(k) \tag{D.82}$$

$$e^{ipx} \Longleftrightarrow 2\pi\,\delta(p-k) \tag{D.83}$$

$$\delta(x-x^0) \Longleftrightarrow \frac{1}{2\pi}\, e^{-ikx_0} \tag{D.84}$$

$$\theta(x)\, e^{-\mu x} \Longleftrightarrow \frac{-i}{k-i\mu} \tag{D.85}$$

$$\theta(-x)\, e^{\mu x} \Longleftrightarrow \frac{i}{k+i\mu} \tag{D.86}$$

$$\epsilon(x)\, e^{-\mu|x|} \Longleftrightarrow \frac{-2ik}{k^2+\mu^2} \tag{D.87}$$

$$e^{-\mu|x|} \Longleftrightarrow \frac{2\mu}{k^2+\mu^2} \tag{D.88}$$

$$e^{-x^2/R^2} \Longleftrightarrow (\pi R^2)^{1/2}\, e^{-k^2/4R^2}. \tag{D.89}$$

D.8.2 Three dimensional transform

The Fourier transform for the 3-dimensional expansion

$$F(\boldsymbol{r}) = \sum_{L=0}^{\infty}\sum_{M=-L}^{+L} \frac{F_L^M(r)}{r}\, Y_L^M(\hat{\boldsymbol{r}}), \tag{D.90}$$

using

$$\exp(i\boldsymbol{k}\cdot\boldsymbol{r}) = \sum_\ell i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}) = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} i^\ell\, j_\ell(kr)\, Y_\ell^m(\hat{\boldsymbol{r}})\, Y_\ell^{m*}(\hat{\boldsymbol{k}}), \tag{D.91}$$

gives the Fourier transform $\tilde{F}(\boldsymbol{k})$ of $F(\boldsymbol{r})$,

$$\tilde{F}(\boldsymbol{k}) = \int d^3r\ \exp(-i\boldsymbol{k}\cdot\boldsymbol{r})\, F(\boldsymbol{r}) = 2\pi \sum_{L=0}^{\infty}\sum_{M=-L}^{+L} (-i)^L\, \frac{\tilde{F}_L^M(k)}{k}\, Y_\ell^m(\hat{\boldsymbol{k}}). \tag{D.92}$$

Note for functions $F(\boldsymbol{r})$ only depending on the length $r = |\boldsymbol{r}|$ one has

$$\tilde{F}(k) = 4\pi \int r^2 dr\ j_0(kr)\, F(r) = \frac{4\pi}{k} \int r dr\ \sin(kr)\, F(r). \tag{D.93}$$

The radial functions form Bessel transforms with spherical Bessel functions,

$$\tilde{F}_L^M(k) = \int_0^\infty dr\ 2kr\, j_L(kr)\, F_L^M(r) \tag{D.94}$$

$$F_L^M(r) = \int_0^\infty \frac{dk}{2\pi}\ 2kr\, j_L(kr)\, \tilde{F}_L^M(k) \tag{D.95}$$

$$\int_0^\infty dr\ 2kr\, j_\ell(kr)\, 2k'r\, j_\ell(k'r) = 2\pi\, \delta(k-k'), \tag{D.96}$$

$$\int_0^\infty \frac{dk}{2\pi}\ 2kr\, j_\ell(kr)\, 2kr'\, j_\ell(kr') = \delta(r-r'). \tag{D.97}$$

Equivalently one has for the combinations relevant in three dimensions,

$$\frac{2\pi\, \tilde{F}_L^M(k)}{k} = \int_0^\infty 4\pi r^2 dr\ j_L(kr)\, \frac{F_L^M(r)}{r} \tag{D.98}$$

$$\frac{F_L^M(r)}{r} = \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3}\ j_L(kr)\, \frac{2\pi\, \tilde{F}_L^M(k)}{k}. \tag{D.99}$$

Some examples of Bessel transformed pairs and corresponding 3-dimensional Fourier transforms are

Bessel transforms:

$$F_\ell(r) \iff \tilde{F}_\ell(k)$$
$$2\,pr\, j_\ell(pr) \iff 2\pi\, \delta(k-p)$$
$$F_0(r) \iff \tilde{F}_0(k)$$
$$e^{-\mu\, r} \iff \frac{2\,k}{(k^2+\mu^2)},$$
$$r\, e^{-\mu\, r} \iff \frac{4k\,\mu}{(k^2+\mu^2)^2}.$$
$$r\, e^{-r^2/2R^2} \iff \frac{\sqrt{\pi}}{2}\, k\, R^3\, e^{-k^2R^2/2}$$

Fourier transforms:

$$\frac{F_\ell(r)}{r} \iff \frac{2\pi}{k}\tilde{F}_\ell(k)$$
$$2\,p\, j_\ell(pr) \iff \frac{4\pi^2}{k}\,\delta(k-p)$$
$$\frac{F_0(r)}{r} \iff \frac{2\pi}{k}\,\tilde{F}_0(k)$$
$$\frac{e^{-\mu\, r}}{r} \iff \frac{4\pi}{(k^2+\mu^2)},$$
$$e^{-\mu\, r} \iff \frac{8\pi\,\mu}{(k^2+\mu^2)^2}.$$
$$e^{-r^2/R^2} \iff (\pi\, R^2)^{3/2}\, e^{-k^2R^2/4}$$

D.8.3 Two dimensional transform

The Fourier transform for the 2-dimensional expansion

$$F(\boldsymbol{b}) = \sum_{M=-\infty}^{\infty} \frac{F^M(b)}{\sqrt{b}}\, e^{iM\varphi_b}, \tag{D.100}$$

using

$$\exp(i\boldsymbol{k}_T\cdot\boldsymbol{b}) = \sum_{m=-\infty}^{\infty} i^m\, J_m(k_T b)\, e^{im(\varphi_b-\varphi_k)}, \tag{D.101}$$

gives

$$\tilde{F}(\boldsymbol{k}_T) = \int d^2b\ \exp(-i\boldsymbol{k}_T\cdot\boldsymbol{b})\, F(\boldsymbol{b}) = 2\pi \sum_{M=-\infty}^{\infty} (-i)^M\, \frac{\tilde{F}_M(k_T)}{\sqrt{k}}\, e^{iM\varphi_k}. \tag{D.102}$$

If $F(\boldsymbol{b})$ only depends on $b = |\boldsymbol{b}|$ one has

$$\tilde{F}(k_T) = 2\pi \int bdb\ J_0(k_T b)\, F(b). \tag{D.103}$$

The function $F(\boldsymbol{b})$ is the Fourier transform of $\tilde{F}(\boldsymbol{k}_T)$,

$$F(\boldsymbol{b}) = \int \frac{d^2k_T}{(2\pi)^2}\ \exp(+i\boldsymbol{k}_T\cdot\boldsymbol{b})\, \tilde{F}(\boldsymbol{k}_T). \tag{D.104}$$

The radial functions are Bessel transforms,

$$\tilde{F}^M(k_T) = \int_0^\infty db\ \sqrt{k_T b}\, J_M(k_T b)\, F^M(b) \tag{D.105}$$

$$F^M(b) = \int_0^\infty dk_T\ \sqrt{k_T b}\, J_M(k_T b)\, \tilde{F}^M(k_T) \tag{D.106}$$

$$\int_0^\infty db\ \sqrt{k_T b}\, J_m(k_T b)\, \sqrt{k_T' b}\, J_m(k_T' b) = \delta(k_T - k_T'), \tag{D.107}$$

$$\int_0^\infty dk_T\ \sqrt{k_T b}\, J_m(k_T b)\, \sqrt{k_T b'}\, J_m(k_T b') = \delta(b - b'). \tag{D.108}$$

Equivalently one has

$$\frac{2\pi\, \tilde{F}^M(k_T)}{\sqrt{k_T}} = \int_0^\infty 2\pi b db\ J_M(k_T b)\, \frac{F^M(b)}{\sqrt{b}}, \tag{D.109}$$

$$\frac{F^M(b)}{\sqrt{b}} = \int_0^\infty \frac{2\pi k_T dk_T}{(2\pi)^2}\ J_M(k_T b)\, \frac{2\pi\, \tilde{F}^M(k_T)}{\sqrt{k_T}}. \tag{D.110}$$

Some examples of Bessel transformed pairs are

$$F^m(b) = \sqrt{p_T b}\, J_m(p_T b) \quad \text{and} \quad \tilde{F}^m(k_T) = \delta(k_T - p_T), \tag{D.111}$$

$$\frac{F^0(b)}{\sqrt{b}} = \frac{e^{-\mu b}}{b} \quad \text{and} \quad \frac{F^0(k_T)}{\sqrt{k_T}} = \frac{1}{\sqrt{k_T^2 + \mu^2}}. \tag{D.112}$$

$$\frac{F^0(b)}{\sqrt{b}} = \frac{\cos(\mu b)}{b} \quad \text{and} \quad \frac{F^0(k_T)}{\sqrt{k_T}} = \frac{\theta(k_T - \mu)}{\sqrt{k_T^2 - \mu^2}}, \tag{D.113}$$

$$\frac{F^0(b)}{\sqrt{b}} = \frac{\sin(\mu b)}{b} \quad \text{and} \quad \frac{F^0(k_T)}{\sqrt{k_T}} = \frac{\theta(\mu - k_T)}{\sqrt{k_T^2 - \mu^2}}. \tag{D.114}$$

$$\frac{F^0(b)}{\sqrt{b}} = \frac{1}{\pi R}\, e^{-b^2/2R^2} \quad \text{and} \quad \frac{F^0(k_T)}{\sqrt{k_T}} \propto e^{-k_T^2 R^2/2}. \tag{D.115}$$

Using the tensor representation for transverse momenta,

$$k_T^{(\pm m)} \equiv k_T^m\, e^{\pm i m \varphi}, \tag{D.116}$$

Appendix E

Spin

E.1 Density matrices: definition and example for spin 1/2

In many applications in quantum mechanics one does not have a pure state to begin with. An impure state is described with a density operator

$$\rho = \sum_i |i\rangle\, p_i\, \langle i|, \tag{E.1}$$

where $|i\rangle$ are pure states, not necessarily orthogonal, and p_i are the probabilities (e.g. a beam of spin 1/2 electrons with 50 % spin along the z-axis, 25 % spin along x-axis and 25 % spin along y-axis). We do know

$$0 \leq p_i \leq 1 \qquad \text{and} \qquad \sum_i p_i = 1.$$

It is straightforward to obtain the following properties:

- $\mathrm{Tr}\,\rho = \sum_i p_i = 1$,
- $\langle A\rangle = \sum_i p_i \langle i|A|i\rangle = \mathrm{Tr}\,(\rho A) = \mathrm{Tr}\,(A\rho)$ and $\mathrm{Tr}\,(\rho^2) \leq 1$,
- ρ is a positive definite, self-adjoint ($\rho^\dagger = \rho$) operator,
- For a pure state $\rho^2 = \rho \Longleftrightarrow \mathrm{Tr}\,(\rho^2) = 1$. In that case ρ is a projection operator.

E.2 Spherical tensor operators

More general, for spin s one can use the spherical tensor operators R^L_M, defined as

$$\begin{aligned}
\langle s, m|R^L_M|s, m'\rangle = \left(R^L_M\right)_{mm'} &= \sqrt{2L+1}\; C^{s\;L\;s}_{m'Mm} \\
&= (-)^L \sqrt{2L+1}\; C^{L\;s\;s}_{Mm'm} \\
&= (-)^{s-m'} \sqrt{2s+1}\; C^{s\;\;s\;\;L}_{m-m'M} \\
&= (-)^{s-m} \sqrt{(2s+1)(2L+1)} \begin{pmatrix} s & s & L \\ m & -m' & -M \end{pmatrix}.
\end{aligned} \tag{E.2}$$

The last expression involves the so-called $3j$-symbol. The properties of these tensor operators are:

- $\mathrm{Tr}\; R^L_M = (2s+1)\,\delta_{L0}\,\delta_{M0}$,
- $\mathrm{Tr}\;(R^{L'}_{M'} {R^L_M}^\dagger) = (2s+1)\,\delta_{LL'}\,\delta_{MM'}$,
- R^L_M are real $(2s+1)\times(2s+1)$ matrices,
- ${R^L_M}^\dagger = (-)^M R^L_{-M}$.

Expanding the density matrix ρ,

$$\rho \equiv \frac{1}{2s+1}\sum_{L=0}^{2s}\sum_{M=-L}^{L} \rho^{L\,*}_M R^L_M = \frac{1}{2s+1}\sum_{L=0}^{2s}\sum_{M=-L}^{L} \rho^L_M R^{L\,\dagger}_M, \tag{E.3}$$

the quantities ρ^L_M are (complex) numbers satisfying

- $\rho^L_M = \mathrm{Tr}\,(\rho R^L_M)$,
- $\rho^{L\,*}_M = (-)^M \rho^L_{-M}$, i.e. ρ^L_0 real (because ρ is self-adjoint),
- $\rho^0_0 = 1$ (because Tr $\rho = 1$)
- $\sum_{L,M} |\rho^L_M|^2 \le 2s+1$ (because Tr $\rho^2 \le 1$).

E.3 Spin 1/2

Besides the tensor operators a Cartesian set is used, the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{E.4}$$

The explicit tensor operators and their relation to the Cartesian operators are

$$R^0_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}, \tag{E.5}$$

$$R^1_1 = \begin{pmatrix} 0 & -\sqrt{2} \\ 0 & 0 \end{pmatrix} = -\frac{1}{\sqrt{2}}(\sigma_x + i\sigma_y), \tag{E.6}$$

$$R^1_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z, \tag{E.7}$$

$$R^1_{-1} = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_x - i\sigma_y), \tag{E.8}$$

The R^1_m are precisely the spherical components of the spin vector $\boldsymbol{\sigma}$. The explicit form of the density matrix is

$$\rho = \frac{1}{2}\left(\mathbf{1} + \boldsymbol{\sigma}\cdot\boldsymbol{P}\right) = \frac{1}{2}\begin{pmatrix} 1+P_z & P_x - i\,P_y \\ P_x + i\,P_y & 1-P_z \end{pmatrix} \tag{E.9}$$

(E.10)

$$= \frac{1}{2}\sum_{L=0}^{1}\sum_{M=-L}^{L} \rho^L_M R^{L\,\dagger}_M = \frac{1}{2}\begin{pmatrix} 1+\rho^1_0 & \rho^1_{-1}\sqrt{2} \\ -\rho^1_1\sqrt{2} & 1-\rho^1_0 \end{pmatrix}. \tag{E.11}$$

The vector $\boldsymbol{P}$ is called the polarization vector or also the spin vector of a state. For a pure state $|\boldsymbol{P}| = 1$, for an unpolarized state $|\boldsymbol{P}| = 0$ corresponding with $\rho = \frac{1}{2}\,\mathbf{1}$. The numbers ρ^1_m are just the spherical components of the polarization vector,

$$\rho^1_1 = -\frac{1}{\sqrt{2}}(P_x + iP_y), \tag{E.12}$$

$$\rho^1_0 = P_z, \tag{E.13}$$

$$\rho^1_{-1} = \frac{1}{\sqrt{2}}(P_x - iP_y). \tag{E.14}$$

One has $\mathrm{Tr}\,\rho^2 = \frac{1}{2}(1+\boldsymbol{P}^2) \le 1$. The degree of polarization is $0 \le |\boldsymbol{P}| \le 1$.

We note that any matrix $\tilde{M}$ in the spin-space can be transformed into a function depending on the polarization vector $\boldsymbol{P}$, which we write as

$$(P^i) = (S^1_T, S^2_T, S_L). \tag{E.15}$$

Explicitly we have

$$M(S) = \mathrm{Tr}\,\left(\rho(S)\,\tilde{M}\right). \tag{E.16}$$

where the density matrix is written as

$$\rho(S) = \frac{1}{2}\begin{pmatrix} 1+S_L & S_T^1 - i\,S_T^2 \\ S_T^1 + i\,S_T^2 & 1-S_L \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}\,S_L & \frac{1}{\sqrt{2}}\,S_T^- \\ -\frac{1}{\sqrt{2}}\,S_T^+ & \frac{1}{2}-\frac{1}{2}\,S_L \end{pmatrix}. \tag{E.17}$$

The equivalence is

$$M(S) = M_O + S_L\,M_L + S_T^1\,M_T^1 + S_T^2\,M_T^2 = M_O + S_L\,M_L - S_T^+\,M_T^- - S_T^-\,M_T^+ \tag{E.18}$$

$$\tilde{M}_{mm'} = \begin{pmatrix} M_O + M_L & M_T^-\sqrt{2} \\ -M_T^+\sqrt{2} & M_O - M_L \end{pmatrix} = \begin{pmatrix} M_O + M_L & M_T^1 - i\,M_T^2 \\ M_T^1 + i\,M_T^2 & M_O - M_L \end{pmatrix}. \tag{E.19}$$

if the matrix $\tilde{M}$ is written on the basis of eigenstates of σ_z.

The parameters in the density matrix in Eq. E.17 can be given an explicit probabilistic interpretation. Introducing $p_m(\theta,\phi)$ as the probabilities to have spin-component m along the direction specified by θ and ϕ, we have

$$P_z = S_L = p_{1/2}(\hat{z}) - p_{-1/2}(\hat{z}) \tag{E.20}$$

$$P_x = S_T^1 = p_{1/2}(\hat{x}) - p_{-1/2}(\hat{x}) \tag{E.21}$$

$$P_y = S_T^2 = p_{1/2}(\hat{y}) - p_{-1/2}(\hat{y}), \tag{E.22}$$

showing all these parameters to lie in the interval $[-1,1]$.

E.4 Spin 1

A Cartesian set transforming like a vector is given by the matrices

$$\Sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{E.23}$$

A set of tensor operators of rank two (symmetric and traceless) is the set $\Sigma_{ij} = \frac{1}{2}\,\Sigma_{\{i}\Sigma_{j\}} - \frac{2}{3}\,\delta_{ij}\,\mathbf{1}$,

$$\Sigma_{xx} = \frac{1}{6}\begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}, \quad \Sigma_{yy} = \frac{1}{6}\begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix}, \quad \Sigma_{zz} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E.24}$$

$$\Sigma_{xy} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \Sigma_{xz} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \Sigma_{yz} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}. \tag{E.25}$$

The explicit tensor operators and their relation to the Cartesian operators are

$$R^0_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1} \tag{E.26}$$

$$R^1_1 = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = -\sqrt{\frac{3}{4}}\,(\Sigma_x + i\,\Sigma_y)\,, \tag{E.27}$$

$$R^1_0 = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \sqrt{\frac{3}{2}}\,\Sigma_z \tag{E.28}$$

$$R^1_{-1} = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \sqrt{\frac{3}{4}}\,(\Sigma_x - i\,\Sigma_y)\,, \tag{E.29}$$

$$R^2_2 = \sqrt{3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{3}{4}}\,(\Sigma_{xx} - \Sigma_{yy} + 2i\,\Sigma_{xy}) \tag{E.30}$$

$$R^2_1 = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = -\sqrt{3}\,(\Sigma_{xz} + i\,\Sigma_{yz}) \tag{E.31}$$

$$R^2_0 = \sqrt{\tfrac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sqrt{\frac{9}{2}}\,\Sigma_{zz} \tag{E.32}$$

$$R^2_{-1} = \sqrt{\tfrac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \sqrt{3}\,(\Sigma_{xz} - i\,\Sigma_{yz}) \tag{E.33}$$

$$R^2_{-2} = \sqrt{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \sqrt{\frac{3}{4}}\,(\Sigma_{xx} - \Sigma_{yy} - 2i\,\Sigma_{xy}) \tag{E.34}$$

Rather than the spin-basis (states $|1,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$) one can use the Cartesian basis $\boldsymbol{\epsilon}_x$, $\boldsymbol{\epsilon}_y$ and $\boldsymbol{\epsilon}_z$ for the spin states, with unitary transformation

$$\begin{pmatrix} \quad \text{cartesian} \quad \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 0 & +1/\sqrt{2} \\ -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \quad \text{spin} \quad \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & i/\sqrt{2} & 0 \end{pmatrix} \tag{E.35}$$

This leads to the natural basis,

$$\Sigma_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Sigma_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{E.36}$$

Note that expressed in Gell-Mann matrices, $\Sigma_x = \lambda_7$, $\Sigma_y = -\lambda_5$ and $\Sigma_z + \lambda_2$. A set of tensor operators of rank two (symmetric and traceless) is the set $\Sigma_{ij} = \frac{1}{2}\,\Sigma_{\{i}\Sigma_{j\}} - \frac{2}{3}\,\delta_{ij}\,\mathbf{1}$,

$$\Sigma_{xx} = \frac{1}{3} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Sigma_{yy} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Sigma_{zz} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{E.37}$$

$$\Sigma_{xy} = -\frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_{xz} = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Sigma_{yz} = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{E.38}$$

In terms of $SU(3)$ generators, $\Sigma_{zz} = \lambda_8/\sqrt{3} = Y_I/3$, etc., while $\Sigma_{xy} = -\lambda_1/2$, $\Sigma_{xz} = -\lambda_4/2$ and $\Sigma_{yz} = -\lambda_6/2$. Together with the vector operators $\boldsymbol{\Sigma}$ the (five) tensor operators span the $SU(3)$ algebra.

The explicit form of the density matrix can be written in one of the following forms,

$$\rho = \frac{1}{3}\left(\mathbf{1} + \frac{3}{2}\,\Sigma_i P_i + 3\,\Sigma_{ij} T_{ij}\right) \tag{E.39}$$

$$= \begin{pmatrix} \frac{1}{3}+\frac{P_z}{2}+\frac{T_{zz}}{2} & \frac{P_x - i\,P_y}{2\sqrt{2}} + \frac{T_{xz}-i\,T_{yz}}{\sqrt{2}} & \frac{T_{xx}-T_{yy}-2i\,T_{xy}}{2} \\ \frac{P_x+i\,P_y}{2\sqrt{2}} + \frac{T_{xz}+i\,T_{yz}}{\sqrt{2}} & \frac{1}{3} - T_{zz} & \frac{P_x-i\,P_y}{2\sqrt{2}} - \frac{T_{xz}-i\,T_{yz}}{\sqrt{2}} \\ \frac{T_{xx}-T_{yy}+2i\,T_{xy}}{2} & \frac{P_x+i\,P_y}{2\sqrt{2}} - \frac{T_{xz}+i\,T_{yz}}{\sqrt{2}} & \frac{1}{3}-\frac{P_z}{2}+\frac{T_{zz}}{2} \end{pmatrix} \tag{E.40}$$

$$= \frac{1}{3}\sum_{L=0}^{2}\sum_{M=-L}^{L} \rho^L_M R^{L\,\dagger}_M \tag{E.41}$$

$$= \frac{1}{3}\begin{pmatrix} 1+\rho^1_0\sqrt{\frac{3}{2}}+\rho^2_0\sqrt{\frac{1}{2}} & \rho^1_{-1}\sqrt{\frac{3}{2}}+\rho^2_{-1}\sqrt{\frac{3}{2}} & \rho^2_{-2}\sqrt{3} \\ -\rho^1_1\sqrt{\frac{3}{2}}-\rho^2_1\sqrt{\frac{3}{2}} & \rho^2_0\sqrt{2} & \rho^1_{-1}\sqrt{\frac{3}{2}}-\rho^2_{-1}\sqrt{\frac{3}{2}} \\ \rho^2_2\sqrt{3} & -\rho^1_1\sqrt{\frac{3}{2}}+\rho^2_1\sqrt{\frac{3}{2}} & 1-\rho^1_0\sqrt{\frac{3}{2}}+\rho^2_0\sqrt{\frac{1}{2}} \end{pmatrix}, \tag{E.42}$$

where T is a traceless, symmetric tensor. One identifies

$$\rho^1_1 = -\sqrt{\frac{3}{4}}\,(P_x + iP_y)\,, \tag{E.43}$$

$$\rho^1_0 = \sqrt{\frac{3}{2}}\,P_z, \tag{E.44}$$

$$\rho^1_{-1} = \sqrt{\frac{3}{4}}\,(P_x - iP_y)\,, \tag{E.45}$$

$$\rho^2_2 = \sqrt{\frac{3}{4}}\,(T_{xx} - T_{yy} + 2i\,T_{xy}) \tag{E.46}$$

$$\rho^2_1 = -\sqrt{3}\,(T_{xz} + i\,T_{yz})\,, \tag{E.47}$$

$$\rho^2_0 = \sqrt{\frac{9}{2}}\,T_{zz}, \tag{E.48}$$

$$\rho^2_{-1} = \sqrt{3}\,(T_{xz} - i\,T_{yz})\,, \tag{E.49}$$

$$\rho^2_{-2} = \sqrt{\frac{3}{4}}\,(T_{xx} - T_{yy} - 2i\,T_{xy}) \tag{E.50}$$

One has $\mathrm{Tr}\,\rho^2 = \frac{1}{3}(1 + \frac{3}{2}\,P^iP^i + 3\,T^{ij}T^{ij}) \le 1$. The degree of polarization is defined as $0 \le [\frac{3}{4}\,P_iP_i + \frac{3}{2}\,T_{ij}T_{ij}]^{1/2} \le 1$.

Again one can establish the equivalence between a spin-dependent function $M(S)$ and a matrix $\tilde{M}$ in spin-space. If

$$(P^i) = (S^1_T, S^2_T, S_L) \tag{E.51}$$

$$(T^{ij}) = \frac{1}{2}\begin{pmatrix} S^{11}_{TT} + S_{LL} & S^{12}_{TT} & S^1_{LT} \\ S^{21}_{TT} & S^{22}_{TT} + S_{LL} & S^2_{LT} \\ S^1_{LT} & S^2_{LT} & -2\,S_{LL} \end{pmatrix}, \tag{E.52}$$

with $S^{22}_{TT} = -S^{11}_{TT}$ and $S^{12}_{TT} = S^{21}_{TT}$. Explicitly we have

$$M(S) = \mathrm{Tr}\,\left(\rho(S)\,\tilde{M}\right). \tag{E.53}$$

where the density matrix is written as

$$\rho(S) = \begin{pmatrix} \frac{1}{3} + \frac{1}{2}\,S_L - \frac{1}{2}\,S_{LL} & \frac{1}{2}\,(S_T^- + S_{LT}^-) & \frac{1}{\sqrt{2}}\,S_{TT}^{1-} \\ -\frac{1}{2}\,(S_T^+ + S_{LT}^+) & \frac{1}{3} + S_{LL} & \frac{1}{2}\,(S_T^- - S_{LT}^-) \\ -\frac{1}{\sqrt{2}}\,S_{TT}^{1+} & -\frac{1}{2}\,(S_T^+ - S_{LT}^+) & \frac{1}{3} - \frac{1}{2}\,S_L - \frac{1}{2}\,S_{LL} \end{pmatrix}. \tag{E.54}$$

Note that $\frac{1}{\sqrt{2}}\,S_{TT}^{1-} = \frac{1}{2}(S_{TT}^{11} - i\,S_{TT}^{12}) = \frac{1}{4}(S_{TT}^{11} - S_{TT}^{22} - 2i\,S_{TT}^{12})$. The equivalence is

$$M(S) = M_O + S_L\,M_L + S_{LL}M_{LL} + S_T^i\,M_T^i + S_{LT}^i\,M_{LT}^i + S_{TT}^{1i}\,M_{TT}^{1i} \tag{E.55}$$

$$\tilde{M}_{mm'} = \begin{pmatrix} M_O + M_L - \frac{1}{2}\,M_{LL} & M_T^- + M_{LT}^- & M_{TT}^{1-}\sqrt{2} \\ -(M_T^+ + M_{LT}^+) & M_O + M_{LL} & M_T^- - M_{LT}^- \\ -M_{TT}^{1+}\sqrt{2} & -(M_T^+ - M_{LT}^+) & M_O - M_L - \frac{1}{2}\,M_{LL} \end{pmatrix} \tag{E.56}$$

if the matrix $\tilde{M}$ is written on the basis of eigenstates of Σ_z.

The parameters in the density matrix in Eq. E.54 can be given an explicit probabilistic interpretation. Introducing $p_m(\theta,\phi)$ as the probabilities to have spin-component m along the direction specified by θ and ϕ, we see from the diagonal elements

$$S_{LL} = \frac{1}{2}\Big[p_1(\hat{z}) + p_{-1}(\hat{z})\Big] - p_0(\hat{z}), \tag{E.57}$$

$$S_L = p_1(\hat{z}) - p_{-1}(\hat{z}), \tag{E.58}$$

implying $-1/3 \le S_{LL} \le 2/3$ and $-1 \le S_L \le 1$.

E.5 Reaction parameters

In general one can express the initial state density matrix as

$$\rho_i \equiv \frac{1}{2s_i+1}\sum_{\ell,m}(\rho_i)^\ell_m {R^\ell_m}^\dagger, \tag{E.59}$$

and obtain from that, given a transition matrix T, the final state density matrix

$$\rho_f = \frac{\mathscr{T}\,\rho_i\,\mathscr{T}^\dagger}{\mathrm{Tr}\,(\mathscr{T}\,\rho_i\,\mathscr{T}^\dagger)} = \frac{\sum_{\ell,m}(\rho_i)^\ell_m\,\mathscr{T}\,{R^\ell_m}^\dagger\,\mathscr{T}^\dagger}{\sum_{\ell,m}(\rho_i)^\ell_m\;\mathrm{Tr}\left(\mathscr{T}\,{R^\ell_m}^\dagger\,\mathscr{T}^\dagger\right)} \tag{E.60}$$

$$= \frac{1}{2s_f+1}\sum_{\ell',m'}(\rho_f)^{\ell'}_{m'}{R^{\ell'}_{m'}}^\dagger, \tag{E.61}$$

with

$$(\rho_f)^{\ell'}_{m'} = \frac{\sum_{\ell,m}(\rho_i)^\ell_m\;\mathrm{Tr}\left(\mathscr{T}\,{R^\ell_m}^\dagger\,\mathscr{T}^\dagger\,R^{\ell'}_{m'}\right)}{\sum_{\ell,m}(\rho_i)^\ell_m\;\mathrm{Tr}\left(\mathscr{T}\,{R^\ell_m}^\dagger\,\mathscr{T}^\dagger\right)}. \tag{E.62}$$

Defining spin transfer parameters

$$\langle \ell',m'|\ell,m\rangle \equiv \frac{\mathrm{Tr}\left(\mathscr{T}\,{R^\ell_m}^\dagger\,\mathscr{T}^\dagger\,R^{\ell'}_{m'}\right)}{\mathrm{Tr}\,(\mathscr{T}\,\mathscr{T}^\dagger)}, \tag{E.63}$$

one gets

$$(\rho_f)^{\ell'}_{m'} = \frac{\sum_{\ell,m}(\rho_i)^\ell_m\;\langle \ell',m'|\ell,m\rangle}{\sum_{\ell,m}(\rho_i)^\ell_m\;\langle 0,0|\ell,m\rangle}. \tag{E.64}$$

To illustrate the reaction parameters, let us consider the reaction parameters for a process with one spin 1/2 particle in initial and one spin 1/2 particle in the final state. in that case it is actually more common to use the Euclidean vector notation instead of the tensor operators. With the 2×2 scattering matrix given by $\mathscr{T}$ and

$$\rho_i = \frac{1}{2}\left(\mathbf{1} + \boldsymbol{\sigma}\cdot\boldsymbol{P}_{\rm in}\right),$$

one finds

$$\rho_f = \frac{\mathscr{T}\,\rho_i\,\mathscr{T}^\dagger}{\mathrm{Tr}\,(\mathscr{T}\,\rho_i\,\mathscr{T}^\dagger)} = \frac{\mathscr{T}\,\mathscr{T}^\dagger + P^i_{\rm in}\,\mathscr{T}\,\sigma^i\,\mathscr{T}^\dagger}{\mathrm{Tr}\left(\mathscr{T}\,\mathscr{T}^\dagger + P^i_{\rm in}\,\mathscr{T}\,\sigma^i\,\mathscr{T}^\dagger\right)} \tag{E.65}$$

or introducing the C^{ij} spin transfer parameters,

$$C^{ij} = \frac{\mathrm{Tr}\left(\mathscr{T}\,\sigma^i\,\mathscr{T}^\dagger\,\sigma^j\right)}{\mathrm{Tr}\,(\mathscr{T}\,\mathscr{T}^\dagger)}, \tag{E.66}$$

with arguments $i, j = 0, 1, 2, 3$ using also the matrix $\sigma^0 \equiv \mathbf{1}$ one obtains:

$$\rho_f = \frac{1}{2}\left(\mathbf{1} + \boldsymbol{\sigma}\cdot\boldsymbol{P}_{\rm out}\right),$$

with

$$P^i_{\rm out} = \frac{C^{0i} + P^j_{\rm in}\,C^{ji}}{C^{00} + P^k_{\rm in}\,C^{k0}} \tag{E.67}$$

($C^{00} = 1$). Note that the final state density matrix ρ_f can be used as the input density matrix for a decay process, in this way enabling polarimetry.

E.6 Spin vectors

In situations such as deep inelastic scattering one likes to work with spin vectors in the cross sections just as with the momenta, e.g. one considers the hadronic tensor $W^{\mu\nu}(q, P, S)$, where q is the virtual photon momentum, P is the target momentum and S the spin vector. We will discuss the meaning of such spin vectors in initial and final state.

We start with a hadron in the initial state with momentum P, which is specified by a *density matrix*, for an ensemble of initial states indicated with α,

$$\begin{aligned}(\rho_i)_{\alpha\beta}(P,S) &= \sum_i \langle P;\alpha|i\rangle\,p_i\,\langle i|P;\beta\rangle \\ &\overset{\text{H rest frame}}{\equiv} \frac{1}{2}\Big(\delta_{\alpha\beta} + \boldsymbol{\sigma}_{\alpha\beta}\cdot\boldsymbol{S}\Big)\end{aligned} \tag{E.68}$$

Note that this in a general frame generalizes to a vector $S^\mu(P)$ which satisfies $P \cdot S = 0$ and $-1 \le S^2 \le 0$.

Then, let's assume that a particle h is produced in the final state with momentum P_h, which is analyzed via its decay products, e.g. a Λ decaying into $p\pi^-$. In that case the probability of finding a specific final state configuration (f) is contained in the *decay matrix*

$$\begin{aligned}R^{(\rm decay)}_{\alpha'\beta'}(f) &\propto \sum_{\lambda_f} \mathscr{T}^*_{\beta'\to\lambda_f}(f)\,\mathscr{T}_{\alpha'\to\lambda_f}(f) \\ &= \sum_{\lambda_f} \langle P_h;\alpha'|\mathscr{T}^\dagger|f;\lambda_f\rangle\,\langle f;\lambda_f|\mathscr{T}|P_h;\beta'\rangle.\end{aligned} \tag{E.69}$$

In this matrix one has summed over all final state spins or helicities (λ_f) in the decay channel (in the example the proton polarizations). The decay matrix depends on the phase space (f) of the decay channel (in the example $\theta^{cm}_{p\pi^+}$). This and other examples are given below. In general we may write

$$\begin{aligned}R^{(\rm decay)}_{\alpha'\beta'}(P_h, f) &\overset{\text{h rest frame}}{\equiv} w(f)\left(\delta_{\alpha'\beta'} + \boldsymbol{\sigma}_{\alpha'\beta'}\cdot\boldsymbol{A}_h(f)\right) \\ &\overset{\text{general frame}}{=} (2S_h+1)\,w(f)\,\rho_{\alpha'\beta'}(P_h, A_h(P_h,f)),\end{aligned} \tag{E.70}$$

normalized to $\sum_f w(f) = 1$ and $\sum_f \boldsymbol{A}_h(P_h,f)\,w(f) = 0$, or $\sum_f R_{\alpha'\beta'}(f) = \delta_{\alpha'\beta'}$. Note that the summation over f is just symbolic for all kinematic variables (usually angles) appearing in the decay. Sometimes one only integrates over a particular subset in f, in which case the righthandside is multiplied with a function depending on the remaining variables. This defines A_h as the analyzing power of the decay channel. Note that A_h depends on P_h and f, i.e. $A_h(P_h,f)$. In general it satisfies $P_h \cdot A_h = 0$. For a decaying particle h with a polarization state determined by a density matrix $\rho(P_h,S_h)$ defined in the same way as $\rho_{\rm in}$, with $P_h \cdot S_h = 0$ and $-1 \le S_h^2 \le 0$, the probability of a final state is given by

$$W(h\to f) = {\rm Tr}\left(\rho(P_h,S_h)\,R^{(\rm decay)}(P_h,f)\right) \overset{\text{h rest frame}}{=} = w(f)\,[1+\boldsymbol{S}_h\cdot \boldsymbol{A}_h(f)]$$
$$\overset{\text{general frame}}{=} = w(f)\,[1 - S_h(P_h)\cdot A_h(P_h,f)]\,. \tag{E.71}$$

Starting with some (general) initial state, the result for a semi-inclusive measurement in which the decay products of hadron h are detected employs the production matrix,

$$\begin{aligned} R^{(\rm prod)}_{\alpha\beta;\alpha'\beta'}(P;P_h) &= \mathscr{T}^*_{\beta\to\beta'}(P,P_h)\,\mathscr{T}_{\alpha\to\alpha'}(P,P_h) \\ &= \langle P;\beta|\mathscr{T}^\dagger|P_h;\beta'\rangle\,\langle P_h;\alpha'|\mathscr{T}|P;\alpha\rangle. \end{aligned} \tag{E.72}$$

The S- and S_h-dependent matrix elements can be introduced as

$$R^{(\rm prod)}_{\alpha'\beta'}(P,S;P_h) \equiv \rho_{\beta\alpha}(P,S)\,R^{(\rm prod)}_{\alpha\beta;\alpha'\beta'}(P,P_h), \tag{E.73}$$

$$\frac{1}{2S_h+1}\,R^{(\rm prod)}_{\alpha\beta}(P;P_h,S_h) \equiv R^{(\rm prod)}_{\alpha\beta;\alpha'\beta'}(P,P_h)\,\rho_{\beta'\alpha'}(P_h,S_h), \tag{E.74}$$

$$\frac{1}{2S_h+1}\,R^{(\rm prod)}(P,S;P_h,S_h) \equiv \rho_{\beta\alpha}(P,S)\,R^{(\rm prod)}_{\alpha\beta;\alpha'\beta'}(P,P_h)\,\rho_{\beta'\alpha'}(P_h,S_h), \tag{E.75}$$

in which the spin vectors (thus) can only appear linearly. Note that because of the definition of the decay matrix it is convenient to absorb a multiplicity factor in the final state spin-dependence. If this is not done, it will explicitly appear in the following expressions.

The probability to produce the final state configuration f via the decay of h is

$$\begin{aligned} W(i\to h\to f) &= (\rho_{\rm in})_{\beta\alpha}(P,S)\,R^{(\rm prod)}_{\alpha\beta,\alpha'\beta'}(P;P_h)\,R^{(\rm decay)}_{\beta'\alpha'}(P_h,f) \\ &= R^{(\rm prod)}_{\alpha'\beta'}(P,S;P_h)\,R^{(\rm decay)}_{\beta'\alpha'}(P_h,f). \end{aligned} \tag{E.76}$$

Using the parametrization of $R^{(\rm decay)}$ in terms of the analyzing power one has

$$W(i\to h\to f) = R^{(\rm prod)}\,(P,S;P_h,A_h(P_h,f))\;w(f), \tag{E.77}$$

while summing (or integrating) over the decay products of h one is not able to measure the polarization of h, and finds

$$\sum_f W(i\to h\to f) = \sum_{\alpha'} R^{(\rm prod)}_{\alpha'\alpha'}(P,S;P_h) = R^{(\rm prod)}(P,S;P_h,0). \tag{E.78}$$

When the production is described as the product of a distribution and fragmentation part as in the case of the hadronic tensor in deep inelastic scattering,

$$\begin{aligned} R^{(\rm prod)}\,(P,S,P_h,S_h) &= {\rm Tr}_D\,[\Phi(P,S)*H*\Delta(P_h,S_h)] \\ &= {\rm Tr}_D\,[(\Phi_O(P)+S\cdot\Phi_S(P))*H*(\Delta_O(P_h)+S_h\cdot\Phi_{S_h}(P_h))] \end{aligned} \tag{E.79}$$

where the ${\rm Tr}_D$ indicates tracing (and possibly integrating) over internal space, e.g. tracing in Dirac space and integrating over momenta for produced and fragmenting quarks including a hard part (H).

E.7 Examples of analyzing power in decays

E.7.1 Λ decay

The decay amplitude for $\Lambda \to N\pi$ is given by

$$\mathscr{T}_{m\to\lambda}(s,\Omega) = \sqrt{\frac{1}{2\pi}}\,A_\lambda\,D^{(1/2)*}_{m\lambda}(\Omega). \tag{E.80}$$

Explicitly,

$$\mathscr{T}_{1/2\to 1/2}(s,\Omega) = \sqrt{\frac{1}{2\pi}}\,A_+(s)\,\cos(\theta/2)$$
$$\mathscr{T}_{1/2\to -1/2}(s,\Omega) = -\sqrt{\frac{1}{2\pi}}\,A_-(s)\,\sin(\theta/2)\,e^{i\,\phi}$$
$$\mathscr{T}_{-1/2\to 1/2}(s,\Omega) = \sqrt{\frac{1}{2\pi}}\,A_+(s)\,\sin(\theta/2)\,e^{-i\,\phi}$$
$$\mathscr{T}_{-1/2\to -1/2}(s,\Omega) = \sqrt{\frac{1}{2\pi}}\,A_-(s)\,\cos(\theta/2)$$

For a spin up or down Λ one then finds respectively

$$\begin{aligned} W_+(\theta,\phi) &= \frac{1}{2\pi}\left(|A_+|^2\,\cos^2(\theta/2) + |A_-|^2\,\sin^2(\theta/2)\right) \\ &= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right) + \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\cos\theta \qquad \text{(E.81)} \\ W_-(\theta,\phi) &= \frac{1}{2\pi}\left(|A_+|^2\,\sin^2(\theta/2) + |A_-|^2\,\cos^2(\theta/2)\right) \\ &= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right) - \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\cos\theta. \qquad \text{(E.82)} \end{aligned}$$

For more general use we calculate the so-called decay matrix (summed over the final state helicities λ),

$$R_{mn}(s,\Omega) \propto \sum_\lambda \mathscr{T}^*_{m\to\lambda}(s,\Omega)\,\mathscr{T}_{n\to\lambda}(s,\Omega),$$

which is equal to

$$\begin{aligned} R(\theta,\phi) &= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right) + \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\begin{pmatrix} \cos\theta & \sin\theta\,e^{-i\,\phi} \\ \sin\theta\,e^{i\,\phi} & -\cos\theta \end{pmatrix} \\ &= \frac{1}{4\pi}\left(|A_+|^2 + |A_-|^2\right)\mathbf{1} + \frac{1}{4\pi}\left(|A_+|^2 - |A_-|^2\right)\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}^{\rm cm}, \qquad \text{(E.83)} \end{aligned}$$

appropriately normalized giving

$$R(\theta,\phi) = \frac{1}{4\pi}\left(\mathbf{1} + \alpha\,\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}^{\rm cm}\right) \qquad \text{(E.84)}$$

where $\alpha = 0.642$ for the decay $\Lambda \to p\pi^-$. Thus one has

$$\boldsymbol{A}_h\left(\Omega^{\rm cm}_{p\pi^-}\right) \overset{\Lambda\ \text{rest-frame}}{=} \alpha\,\hat{\boldsymbol{p}}^{\rm cm}. \qquad \text{(E.85)}$$

Given a specific polarization for the decaying hadron ($h = \Lambda$) via a density matrix $\rho(S_h)$ one finds a CM distribution

$$\begin{aligned} W(S_h;\theta,\phi) &= \text{Tr}\left[\rho(S_h)\,R(\theta,\phi)\right] \\ &= 1 + \boldsymbol{S}_h\cdot A_h(\theta,\phi) \\ &= 1 + \alpha\,(S_L\,\cos\theta + S_T^1\,\sin\theta\,\cos\phi + S_T^2\,\sin\theta\,\sin\phi). \qquad \text{(E.86)} \end{aligned}$$

In covariant form (with $P_\Lambda = P_p + P_\pi$) one obtains for the analyzing power,

$$\begin{aligned} A^\mu_\Lambda(P_p,P_\pi) &= \alpha\left[\frac{M_\Lambda^2 - M_p^2 + M_\pi^2}{M_\Lambda\sqrt{\Delta(M_\Lambda^2,M_p^2,M_\pi^2)}}\,P_p^\mu - \frac{M_\Lambda^2 + M_p^2 - M_\pi^2}{M_\Lambda\sqrt{\Delta(M_\Lambda^2,M_p^2,M_\pi^2)}}\,P_\pi^\mu\right], \qquad \text{(E.87)} \\ &= 0.642\left[1.710\,\frac{P_p^\mu}{M_\Lambda} - 9.38\,\frac{P_\pi^\mu}{M_\Lambda}\right] \end{aligned}$$

with $\Delta(M_\Lambda^2,M_p^2,M_\pi^2) = \left(M_\Lambda^2 - (M_p+M_\pi)^2\right)\left(M_\Lambda^2 - (M_p-M_\pi)^2\right) = 4M_\Lambda^2\,|\boldsymbol{p}^{\rm cm}|^2$. This illustrates how the analyzing power in the final state is determined from momenta of the decay products. For instance the helicity $A_{\Lambda\,L} = M_\Lambda\, A_\Lambda^-/P_\Lambda^-$ is given by

$$A_{\Lambda\,L} = \alpha \left[\frac{M_\Lambda^2 - M_p^2 + M_\pi^2}{\sqrt{\Delta(M_\Lambda^2, M_p^2, M_\pi^2)}}\,\frac{z_p}{z_\Lambda} - \frac{M_\Lambda^2 + M_p^2 - M_\pi^2}{\sqrt{\Delta(M_\Lambda^2, M_p^2, M_\pi^2)}}\,\frac{z_\pi}{z_\Lambda} \right], \tag{E.88}$$

which takes the values $A_{\Lambda\,L} = \alpha$ for the maximal value of z_p, namely $(z_p)_{\rm max} = 0.936\, z_\Lambda$ and the value $A_{\Lambda\,L} = -\alpha$ for $(z_p)_{\rm min} = 0.756\, z_\Lambda$. Furthermore we have

$$A^\mu_{\Lambda T} = A^\mu_{\Lambda\perp} - A_{\Lambda L}\,\frac{P^\mu_{\Lambda\perp}}{M_\Lambda}. \tag{E.89}$$

E.7.2 ρ decay

The decay amplitude for $\rho \to \pi\pi$ is given by

$$\mathscr{T}_m(s,\Omega) = \sqrt{\frac{3}{4\pi}}\,A(s)\,D^{(1)*}_{m0}(\Omega). \tag{E.90}$$

Explicitly,

$$\mathscr{T}_1(s,\Omega) = -\sqrt{\frac{3}{8\pi}}\,A\,\sin\theta\,e^{i\,\phi}$$
$$\mathscr{T}_0(s,\Omega) = \sqrt{\frac{3}{4\pi}}\,A\,\cos\theta$$
$$\mathscr{T}_{-1}(s,\Omega) = \sqrt{\frac{3}{8\pi}}\,A\,\sin\theta\,e^{-i\,\phi}$$

Next we calculate the decay matrix,

$$R_{mn}(s,\Omega) \propto \mathscr{T}^*_m(s,\Omega)\,\mathscr{T}_n(s,\Omega),$$

which is equal to

$$R(\theta,\phi) \propto \frac{|A(s)|^2}{4\pi} \begin{pmatrix} \frac{3}{2}\,\sin^2\theta & -\frac{3}{2\sqrt{2}}\,\sin 2\theta\,e^{-i\,\phi} & -\frac{3}{2}\,\sin^2\theta\,e^{-2i\,\phi} \\ -\frac{3}{2\sqrt{2}}\,\sin 2\theta\,e^{i\,\phi} & 3\,\cos^2\theta & \frac{3}{2\sqrt{2}}\,\sin 2\theta\,e^{-i\,\phi} \\ -\frac{3}{2}\,\sin^2\theta\,e^{-2i\,\phi} & \frac{3}{2\sqrt{2}}\,\sin 2\theta\,e^{i\,\phi} & \frac{3}{2}\,\sin^2\theta \end{pmatrix}. \tag{E.91}$$

After normalization this gives

$$R(\theta,\phi) = \frac{1}{4\pi}\left(\mathbf{1} + 3\,\Sigma_{ij}\,A_{ij}\right), \tag{E.92}$$

with the tensor

$$(A_h)_{ij}\,(\Omega^{\rm cm}_{\pi\pi}) \overset{\rho\ \text{rest-frame}}{=} \frac{1}{3}\,\delta_{ij} - \hat{p}^{\rm cm}_i \hat{p}^{\rm cm}_j. \tag{E.93}$$

and the vector part being zero.

Given a specific polarization for the decaying hadron ($h = \rho$) via a density matrix $\rho(S_h)$ one finds a CM distribution

$$\begin{aligned} W(S_h;\theta,\phi) &= {\rm Tr}\left[\rho(S_h)\,R(\theta,\phi)\right] \\ &= \frac{1}{4\pi}\left[1 + 3\,(T_h)_{ij}(A_h)_{ij}(\theta,\phi)\right] \\ &= \frac{3}{8\pi}\Bigg(\frac{2}{3} - \frac{2}{3}\,S_{LL}\,(3\cos^2\theta - 1) - S^1_{LT}\,\sin 2\theta\,\cos\phi - S^2_{LT}\,\sin 2\theta\,\sin\phi \\ &\qquad - S^{11}_{TT}\,\sin^2\theta\,\cos 2\phi - S^{12}_{TT}\,\sin^2\theta\,\sin 2\phi\Bigg). \end{aligned} \tag{E.94}$$

In covariant form (with $P_h = P_1 + P_2$) we have

$$\hat{p}^{\mu}_{\rm cm} = \frac{P_1^{\mu} - P_2^{\mu}}{\sqrt{M_\rho^2 - 4M_\pi^2}}, \tag{E.95}$$

giving

$$\begin{aligned}
A_h^{\mu\nu} &= \frac{1}{M_\rho^2 - 4M_\pi^2}\left[\frac{1}{2}\,P_h^{\{\mu}P_h^{\nu\}} - P_1^{\{\mu}P_1^{\nu\}} - P_2^{\{\mu}P_2^{\nu\}}\right] - \frac{1}{3}\left(g^{\mu\nu} - \frac{P_h^{\mu}P_h^{\nu}}{M_\rho^2}\right) \\
&= \frac{2(M_\rho^2 - M_\pi^2)}{3M_\rho^2(M_\rho^2 - 4M_\pi^2)}\,P_h^{\{\mu}P_h^{\nu\}} - \frac{1}{M_\rho^2 - 4M_\pi^2}\left[P_1^{\{\mu}P_1^{\nu\}} + P_2^{\{\mu}P_2^{\nu\}}\right] - \frac{1}{3}\,g^{\mu\nu} \\
&= \frac{2(M_\rho^2 - M_\pi^2)}{3M_\rho^2(M_\rho^2 - 4M_\pi^2)}\,S^{\mu\alpha\nu\beta}\,P_{h\,\alpha}P_{h\,\beta} - \frac{1}{M_\rho^2 - 4M_\pi^2}\left[S^{\mu\alpha\nu\beta}\,P_{1\,\alpha}P_{1\,\beta} + S^{\mu\alpha\nu\beta}\,P_{2\,\alpha}P_{2\,\beta}\right] \\
&= \frac{2(M_\rho^2 - M_\pi^2)}{3M_\rho^2(M_\rho^2 - 4M_\pi^2)}\,S^{\mu P_h \nu P_h} - \frac{1}{M_\rho^2 - 4M_\pi^2}\left[S^{\mu P_1 \nu P_1} + S^{\mu P_2 \nu P_2}\right].
\end{aligned} \tag{E.96}$$

E.8 Comment on BLT sumrule

Spin sum rules

There are two spin sum rules that relate integrals over distribution functions to (local) QCD operators. One for *longitudinal spin*,

$$\begin{aligned}\frac{1}{2} &= \frac{1}{2}\sum_q \int_{-1}^{1} dx \Delta q(x) + \int_0^1 dx\, \Delta G(x) + L^q + L^G \\ &= \frac{1}{2}\sum_q \int_0^1 dx\, (\Delta q(x) + \Delta\overline{q}(x)) + \int_0^1 dx\, \Delta G(x) + L^q + L^G, \end{aligned} \tag{E.97}$$

and one for transverse spin (O. Teryaev, B. Pire and J. Soffer, hep-ph/9806502; P.G. Ratcliffe, hep-ph/9811348)

$$\begin{aligned}\frac{1}{2} &= \frac{1}{2}\sum_q \int_{-1}^{1} dx g_T^q(x) + \int_0^1 dx\, \Delta G_T(x) + L_T^q + L_T^G \\ &= \frac{1}{2}\sum_q \int_0^1 dx\, \left(g_T^q(x) + g_T^{\bar{q}}(x)\right) + \int_0^1 dx\, \Delta G_T(x) + L_T^q + L_T^G \end{aligned} \tag{E.98}$$

In fact these two sum rules are expected to have the same contributions, at least for the quark spin part, where the equality is just the Burkhard-Cottingham sumrule. The equalities for the various terms are a consequence of Lorentz invariance. At the operator level the transverse spin sum rule involves quark-quark-gluon operators, exactly what one would expect since partons correspond to the quanta of good quark and gluon fields in front form quantization.

Tensor charge

There is a sumrule for transverse spin polarization. It relates the integral over $h_1^q(x) = \delta q(x) = \Delta_T q(x)$ to the tensor charge (local operator is $\overline{\psi}\, \sigma^{\mu\nu}\gamma_5\, \psi$),

$$\sum_q \int_{-1}^{1} dx\, \delta q(x) = \sum_q \int_0^1 dx\, \left(\delta q(x) - \delta\overline{q}(x)\right) = g_T. \tag{E.99}$$

Interpretation as spin densities

The leading twist distribution functions $f_1^q(x) = q(x)$, $g_1^1(x) = \Delta q(x)$ and $h_1^q(x) = \delta q(x)$ can be interpreted as spin densities. They are 'quadratic' operators for good fields, $\psi_+(x) \equiv P_+\psi(x) = \frac{1}{2}\gamma^+\gamma^-\psi(x)$ after taking (spin) projections, $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$ and $P_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^1\gamma_5)$. These spin projectors commute with P_+. One has

$$q(x) = q_R(x) + q_L(x) = q_\uparrow(x) + q_\downarrow(x), \tag{E.100}$$

$$\Delta q(x) = q_R(x) - q_L(x) \tag{E.101}$$

$$\delta q(x) = q_\uparrow(x) - q_\downarrow(x), \tag{E.102}$$

Transverse spin

(a) From the interpretation one expects

$$\sum_q \int_0^1 dx\, \left(\delta q(x) + \delta\overline{q}(x)\right) \tag{E.103}$$

to have a meaning as 'transverse spin'. It certainly is a measure for transverse polarization of quarks and antiquarks, but there is no local operator to which it can be equated.

One can write down an operator expression, but it is nonlocal. The starting point is $\theta(x)\, \delta q(x) - \theta(-x)\, \delta q(-x)$

(b) For an ensemble of *free* quarks (and gluons) the quantity entering in the transverse spin sumrule is related to transverse spin density. One has

$$g_T(x) = \frac{m}{Mx}\, h_1(x). \tag{E.104}$$

Appendix F

Kinematics in hard processes

F.1 kinematics with a hard $2 \to 2$ subprocess

Consider the kinematics of a hard process, $H_1 + H_2 \to h_1 + h_2 + X$, including momenta

$$H_1(P_1) + H_2(P_2) \to h_1(K_1) + h_2(K_2) + X. \tag{F.1}$$

This is an inclusive process, for which we will use the variables,

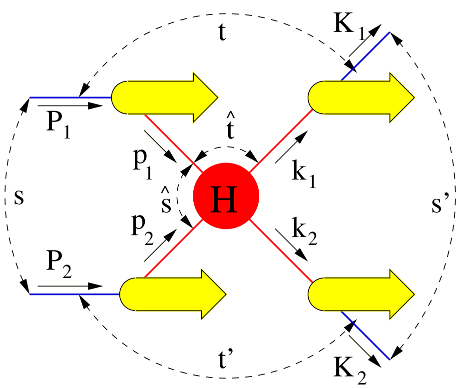
$$s = (P_1 + P_2)^2 \simeq 2\,P_1 \cdot P_2, \tag{F.2}$$
$$s' = (K_1 + K_2)^2 \simeq 2\,K_1 \cdot K_2, \tag{F.3}$$
$$t = (P_1 - K_1)^2 \simeq -2\,P_1 \cdot K_1, \tag{F.4}$$
$$t' = (P_2 - K_2)^2 \simeq -2\,P_2 \cdot K_2, \tag{F.5}$$
$$u' = (P_1 - K_2)^2 \simeq -2\,P_1 \cdot K_2, \tag{F.6}$$
$$u = (P_2 - K_1)^2 \simeq -2\,P_2 \cdot K_1, \tag{F.7}$$

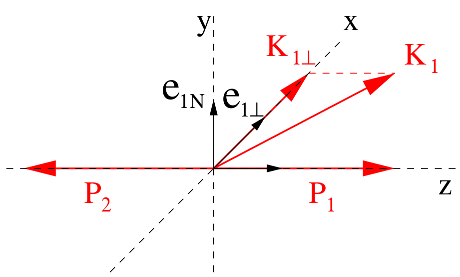
All these invariants are assumed to be of order s. The corrections are $\mathscr{O}(1)$, that means suppressed by two orders of the hard scale, $(\sqrt{s})$ indicated with the $\simeq$ symbol between the entries. The dot-products can be used to expand the final state vectors K_1 and K_2 in terms of P_1 and P_2 and orthogonal parts,

$$K_1 \simeq -\frac{u}{s}\,P_1 - \frac{t}{s}\,P_2 + K_{1\perp}, \tag{F.8}$$
$$K_2 \simeq -\frac{t'}{s}\,P_1 - \frac{u'}{s}\,P_2 + K_{2\perp}, \tag{F.9}$$

where $K_{1\perp} \cdot P_1 = K_{1\perp} \cdot P_2 = 0$ and $K_{2\perp} \cdot P_1 = K_{2\perp} \cdot P_2 = 0$. Note that $K_{1\perp}^2 \simeq -tu/s$ and $K_{2\perp}^2 \simeq -t'u'/s$ with $\mathscr{O}(1)$ corrections, and

$$K_{1\perp} \cdot K_{2\perp} \simeq \frac{ss' - tt' - uu'}{2s}. \tag{F.10}$$





In practice often used is the Feynman-x, $x_{1F} \equiv K_{1z}^{\rm cm}/K_{1z}^{\rm cm(max)} \simeq 2\,K_{1z}^{\rm cm}/\sqrt{s} = \simeq (t-u)/s$. Scaling also the transverse momentum in the CM frame and using $x_{1T} \equiv 2|K_{1\perp}^{\rm cm}|/\sqrt{s} \simeq 2\sqrt{tu}/s$, or the CM scattering angle θ_1 one has $\epsilon_1 \equiv 2\,E_{K_1}^{\rm cm}/\sqrt{s} \simeq x_{1F}/\cos\theta_1 \simeq \sqrt{x_{1F}^2+x_{1T}^2}$. In terms of the CM (pseudo)-rapidity η_1 one has $x_{1F} \equiv x_{1T}\sinh\eta_1$ and $\epsilon_1 \simeq x_{1T}\cosh\eta_1$. It is related to θ_1 via $\eta_1 \simeq -\ln\tan(\theta_1/2)$ or $\sinh\eta_1 \simeq \cot\theta_1$. One has

$$t \simeq -2\,E_{K_1}^{\rm cm}\sqrt{s}\,\sin^2(\theta_1/2) \simeq -\frac{s}{2}\left(\sqrt{x_{1F}^2+x_{1T}^2}-x_{1F}\right) = -\frac{s}{2}\,x_{1T}\,e^{-\eta_1}, \tag{F.11}$$

$$u \simeq -2\,E_{K_1}^{\rm cm}\sqrt{s}\,\cos^2(\theta_1/2) \simeq -\frac{s}{2}\left(\sqrt{x_{1F}^2+x_{1T}^2}+x_{1F}\right) = -\frac{s}{2}\,x_{1T}\,e^{+\eta_1}, \tag{F.12}$$

$$t' \simeq -2\,E_{K_2}^{\rm cm}\sqrt{s}\,\cos^2(\theta_2/2) \simeq -\frac{s}{2}\left(\sqrt{x_{2F}^2+x_{2T}^2}+x_{2F}\right) = -\frac{s}{2}\,x_{2T}\,e^{+\eta_2}, \tag{F.13}$$

$$u' \simeq -2\,E_{K_2}^{\rm cm}\sqrt{s}\,\sin^2(\theta_2/2) \simeq -\frac{s}{2}\left(\sqrt{x_{2F}^2+x_{2T}^2}-x_{2F}\right) = -\frac{s}{2}\,x_{2T}\,e^{-\eta_2}, \tag{F.14}$$

$$\tag{F.15}$$

and

$$\frac{s'}{s} = x_{1T}\,x_{2T}\left[\cosh^2\left(\frac{\eta_1-\eta_2}{2}\right)+\cos^2\left(\frac{\phi_1-\phi_2}{2}\right)\right]. \tag{F.16}$$

The final state phase space can be expressed as

$$\begin{aligned}
\frac{d^3K_1}{(2\pi)^3\,2E_{K_1}} &= \frac{1}{(2\pi)^3}\,\frac{dt}{2t}\,d^2K_{1\perp} = \frac{1}{(2\pi)^3}\,\frac{du}{2u}\,d^2K_{1\perp} \\
&= \frac{1}{(2\pi)^3}\,\frac{dt\,du}{4s}\,d\phi_1 = \frac{1}{16\,\pi^2}\,\frac{dt\,du}{4s}\,\frac{d\phi_1}{2\pi} \\
&= \frac{1}{(2\pi)^3}\,\frac{\sqrt{s}}{8\,E_{K_1}^{\rm cm}}\,dx_{1F}\,d|K_{1\perp}^{\rm cm}|^2\,d\phi_1 = \frac{s}{64\,\pi^2}\,\frac{dx_{1F}\,dx_{1T}^2}{\sqrt{x_{1F}^2+x_{1T}^2}}\,\frac{d\phi_1}{2\pi} \\
&\simeq \frac{s}{64\,\pi^2}\,d\eta_1\,dx_{1T}^2\,\frac{d\phi_1}{2\pi}.
\end{aligned} \tag{F.17}$$

Similar relations can be written down for the momentum K_2 involving t', u', x_{2T}, and x_{2F}.

The incoming hadrons produce two partons with momenta p_1 and p_2, the outcoming hadrons are assumed to originate from partons k_1 and k_2, in which case we assume approximate collinearity, implying $p_i^2 \sim p_i\cdot P_i \sim P_i^2 = M_i^2$. These partons participate in a hard process in which the momenta satisfy $p_1+p_2 = k_1+k_2$. For the subprocess we use

$$\hat{s} = (p_1+p_2)^2 = (k_1+k_2)^2 \simeq 2\,p_1\cdot p_2 \simeq 2\,k_1\cdot k_2, \tag{F.18}$$

$$\hat{t} = (p_1-k_1)^2 = (p_2-k_2)^2 \simeq -2\,p_1\cdot k_1 \simeq -2\,p_2\cdot k_2, \tag{F.19}$$

$$\hat{u} = (p_1-k_2)^2 = (p_2-k_1)^2 \simeq -2\,p_1\cdot k_2 \simeq -2\,p_2\cdot k_1, \tag{F.20}$$

adding up to zero, $\hat{s}+\hat{t}+\hat{u} \simeq 0$. For the initial/final state partons, we write

$$p_i = x_i\,P_i + p_{iT} + \sigma_i\,n_i, \tag{F.21}$$

$$k_i = \frac{K_i}{z_i} + k_{iT} + \sigma_i\,n_i, \tag{F.22}$$

where the only condition on the vector n_i is that $P_i\cdot n_i \sim K_i\cdot n_i \sim \sqrt{s}$. The fraction $x_i = p_i\cdot n_i/P_i\cdot n_i$ is a lightcone fraction. The quantity multiplying the vector n_i is the lightcone component conjugate to $p_i\cdot n_i$ and is given by

$$\sigma_i = \frac{p_i^2 - p_{iT}^2 - x_i^2 M_i^2}{2\,x_i\,P_i\cdot n_i} = \frac{p_i\cdot P_i - x_i\,M_i^2}{P_i\cdot n_i}, \tag{F.23}$$

(and similar expressions for K_i), quantities which are of order $1/\sqrt{s}$. Note that we have the exact relations $p_{iT}^2 = (p_i - x_i\,P_i)^2$ and $p_i\cdot p_{iT} = p_{iT}^2$. The integration over parton momenta is

$$d^4p_i = dx_i\,d^2p_{iT}\,d(p_i\cdot P_i). \tag{F.24}$$

If we only consider the (large) momenta, defining

$$q_1 = x_1\, P_1 - \frac{K_1}{z_1} \qquad \text{with} \qquad q_1^2 \simeq \frac{x_1}{z_1}\, t \simeq \hat{t}, \tag{F.25}$$

$$q_2 = x_2\, P_2 - \frac{K_2}{z_2} \qquad \text{with} \qquad q_2^2 \simeq \frac{x_2}{z_2}\, t' \simeq \hat{t}, \tag{F.26}$$

their sum is of $\mathscr{O}(1)$ and given by

$$q_1 + q_2 \approx -q_T, \tag{F.27}$$

where

$$q_T \equiv p_{1T} + p_{2T} - k_{1T} - k_{2T}. \tag{F.28}$$

When we use the $\approx$ symbol the expression gets corrections one order suppressed in the hard scale. The small $\mathscr{O}(1)$ momentum

$$r_\perp = \frac{K_{1\perp}}{z_1} + \frac{K_{2\perp}}{z_2}. \tag{F.29}$$

is actually just the projection of the (small) transverse momenta in the perpendicular plane, $r_\perp \approx q_{T\perp}$, but it requires knowledge of z_1 and z_2. It is convenient to introduce the *transverse energy* $k_\perp$, defined as $k_\perp \equiv |k_{1\perp}| \simeq |k_{2\perp}| \simeq |K_{2\perp}|/z_2 \simeq |K_{1\perp}|/z_1$ and its scaled version $x_\perp = 2\,k_\perp/\sqrt{s}$.

F.2 parton momentum fractions

In the next step we want to see how to obtain the parton momentum fractions from the external momenta using as basis the momentum conservation in the hard subprocess. By taking the product of the constraint $p_1 + p_2 - k_1 - k_2 = 0$ we get (omitting $\mathscr{O}(1)$ corrections) the constraints

$$2P_1 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq x_2\, s + \frac{1}{z_1}\, t + \frac{1}{z_2}\, u' + 2\, P_1 \cdot q_T, \tag{F.30}$$

$$2P_2 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq x_1\, s + \frac{1}{z_1}\, u + \frac{1}{z_2}\, t' + 2\, P_2 \cdot q_T, \tag{F.31}$$

$$2K_1 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq -x_1\, t - x_2\, u - \frac{1}{z_2}\, s' + 2\, K_1 \cdot q_T, \tag{F.32}$$

$$2K_2 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq -x_1\, u' - x_2\, t' - \frac{1}{z_1}\, s' + 2\, K_2 \cdot q_T, \tag{F.33}$$

Instead of the latter two conditions we get for the $\perp$-components

$$\begin{aligned} 2K_{1\perp} \cdot (p_1 + p_2 - k_1 - k_2) = 0 \quad &\simeq \quad 2\,\frac{|K_{1\perp}|^2}{z_1} - 2\,\frac{K_{1\perp} \cdot K_{2\perp}}{z_2} + 2\, K_{1\perp} \cdot q_T, \\ &\simeq \quad 2\,\frac{t\,u}{z_1 s} + \frac{tt' + uu' - ss'}{z_2 s} + 2\, K_{1\perp} \cdot q_T, \end{aligned} \tag{F.34}$$

$$2K_{2\perp} \cdot (p_1 + p_2 - k_1 - k_2) = 0 \quad \simeq \quad 2\,\frac{t'\,u'}{z_2 s} + \frac{tt' + uu' - ss'}{z_1 s} + 2\, K_{2\perp} \cdot q_T, \tag{F.35}$$

At leading order, the Mandelstam variables for the subprocess ($\hat{s}$, $\hat{t}$ and $\hat{u}$) are related to variables in the full process through

$$\hat{s} \approx x_1 x_2\, s \approx \frac{s'}{z_1\, z_2} \approx 4\, k_\perp^2 \left[\cosh^2\left(\frac{\eta_1 - \eta_2}{2}\right) + \cos^2\left(\frac{\phi_1 - \phi_2}{2}\right)\right], \tag{F.36}$$

$$\hat{t} \approx \frac{x_1}{z_1}\, t \approx \frac{x_2}{z_2}\, t' \approx -2k_\perp^2\, e^{-(\eta_1 - \eta_2)/2}\, \cosh\left(\frac{\eta_1 - \eta_2}{2}\right), \tag{F.37}$$

$$\hat{u} \approx \frac{x_1}{z_2}\, u' \approx \frac{x_2}{z_1}\, u \approx -2\, k_\perp^2\, e^{+(\eta_1 - \eta_2)/2}\, \cosh\left(\frac{\eta_1 - \eta_2}{2}\right), \tag{F.38}$$

or

$$\sqrt{ss'} = \sqrt{\frac{z_1 z_2}{x_1 x_2}}\,\hat{s}, \tag{F.39}$$

$$\sqrt{tt'} = -\sqrt{\frac{z_1 z_2}{x_1 x_2}}\,\hat{t}, \tag{F.40}$$

$$\sqrt{uu'} = -\sqrt{\frac{z_1 z_2}{x_1 x_2}}\,\hat{u}, \tag{F.41}$$

which are relations with $\mathscr{O}(\sqrt{s})$ corrections. Looking only at the leading parts ($\sim s$) in the above constraints, we obtain

$$s\,s' \approx \left(\sqrt{t\,t'} + \sqrt{u'\,u}\right)^2, \tag{F.42}$$

and

$$x_1\,z_2 \approx \sqrt{\frac{s'\,t'}{s\,t}} \quad \text{and} \quad x_1\,z_1 \approx \sqrt{\frac{s'\,u}{s\,u'}}, \tag{F.43}$$

$$x_2\,z_2 \approx \sqrt{\frac{s'\,u'}{s\,u}} \quad \text{and} \quad x_2\,z_1 \approx \sqrt{\frac{s'\,t}{s\,t'}}, \tag{F.44}$$

$$\frac{z_1}{z_2} \approx \sqrt{\frac{t\,u}{t'\,u'}}. \tag{F.45}$$

We note that in an analysis of the hadrons in the final state jets one can determine z_1 and z_2 also when the jet momentum is assumed to be known. In that case

$$z_1 \approx \frac{K_1 \cdot K_2}{k_1 \cdot K_2}, \tag{F.46}$$

$$z_2 \approx \frac{K_2 \cdot K_1}{k_2 \cdot K_1}. \tag{F.47}$$

F.3 kinematics in the transverse plane

The approximations above (coming from the constraint $p_1 + p_2 - k_1 - k_2 = 0$) lead to Eq. F.42, which implies that for a two to two hard subprocess

$$K_{1\perp} \cdot K_{2\perp} \approx \sqrt{tt'uu'}/s, \tag{F.48}$$

i.e. the vectors $K_{1\perp}$ and $K_{2\perp}$ are almost parallel. Hence, the directions

$$e^\mu_{1\perp} \equiv \frac{K^\mu_{1\perp}}{|K_{1\perp}|} \simeq \sqrt{\frac{s}{tu}}\,K^\mu_{1\perp} \qquad \text{and} \qquad e^\mu_{2\perp} \equiv \frac{K^\mu_{2\perp}}{|K_{2\perp}|} \simeq \sqrt{\frac{s}{t'u'}}\,K^\mu_{2\perp} \tag{F.49}$$

are opposite in leading order, $e_{1\perp} \approx -e_{2\perp}$. In the following we will keep the small part, thus using Eq. F.10 instead of Eq. F.48, or $e_{1\perp} \cdot e_{2\perp} \simeq -(tt' + uu' - ss')/2\sqrt{tt'uu'}$, with corrections that are of $\mathscr{O}(1/s)$, The vectors

$$e_\perp \equiv \frac{1}{2}\,(e_{1\perp} - e_{2\perp})\,. \qquad \text{and} \qquad \rho_N \equiv (e_{1\perp} + e_{2\perp}), \tag{F.50}$$

are orthogonal ones. The small vector ρ_N has invariant length squared

$$\rho_N^2 = (e_{1\perp} + e_{2\perp})^2 \approx -\frac{(\sqrt{tt'} + \sqrt{uu'})^2 - ss'}{\sqrt{tt'uu'}} \approx -2\sqrt{\frac{ss'}{tt'uu'}}\,\left(\sqrt{tt'} + \sqrt{uu'} - \sqrt{ss'}\right). \tag{F.51}$$

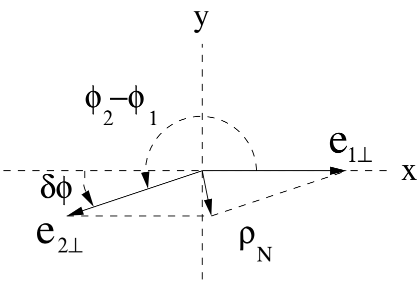
where the last step again makes use of Eq. F.42.

We can define other (spacelike) vector in the transverse plane via

$$e^\mu_{1N} \equiv -\frac{\epsilon^{P_1 P_2 K_1\,\mu}}{P_1 \cdot P_2\,|K_{1\perp}|} \simeq -\frac{2}{\sqrt{stu}}\,\epsilon^{P_1 P_2 K_1\,\mu}, \tag{F.52}$$

$$e^\mu_{2N} \equiv -\frac{\epsilon^{P_1 P_2 K_2\,\mu}}{P_1 \cdot P_2\,|K_{2\perp}|} \simeq -\frac{2}{\sqrt{st'u'}}\,\epsilon^{P_1 P_2 K_2\,\mu}, \tag{F.53}$$

and $e_N \equiv (e_{1N} - e_{2N})/2$. We again note that e_{1N} and e_{2N} are approximately opposite.



Connecting the vectors to the angles, we have using the angle $\delta\phi = \phi_2 - \phi_1 - \pi$ in the frame shown

$$e_{1\perp} = (1,\ 0) \qquad e_{1N} = (0,\ 1)$$
$$e_{2\perp} = -(\cos\delta\phi,\ \sin\delta\phi) \qquad e_{2N} = (\sin\delta\phi,\ -\cos\delta\phi)$$

$$e_\perp \equiv \tfrac{1}{2}(e_{1\perp} - e_{2\perp}) = \cos\left(\tfrac{1}{2}\delta\phi\right)\ \left(\cos\left(\tfrac{1}{2}\delta\phi\right), \sin\left(\tfrac{1}{2}\delta\phi\right)\right)$$
$$e_N \equiv \tfrac{1}{2}(e_{1N} - e_{2N}) = \cos\left(\tfrac{1}{2}\delta\phi\right)\ \left(-\sin\left(\tfrac{1}{2}\delta\phi\right), \cos\left(\tfrac{1}{2}\delta\phi\right)\right)$$

$$\rho_N \equiv e_{1\perp} + e_{2\perp} = 2\,\sin\left(\tfrac{1}{2}\delta\phi\right)\ \left(\sin\left(\tfrac{1}{2}\delta\phi\right),\ -\cos\left(\tfrac{1}{2}\delta\phi\right)\right)$$
$$= -2\,\tan\left(\tfrac{1}{2}\delta\phi\right)\, e_N \ \approx\ -\delta\phi\, e_N,$$

and from the expression for $e_{1\perp}\cdot e_{2\perp}$,

$$\delta\phi^2 \approx 4\,\sin^2\left(\frac{\delta\phi}{2}\right) \approx 2\sqrt{\frac{ss'}{tt'uu'}}\,\left(\sqrt{tt'} + \sqrt{uu'} - \sqrt{ss'}\right).$$

The vector $r_\perp = k_{1\perp} + k_{2\perp}$ (including the lengths) can have any direction in the transverse plane. One has

$$r_{\perp x} \approx |k_{1\perp}| - |k_{2\perp}| \simeq \frac{1}{z_1}\sqrt{\frac{tu}{s}} - \frac{1}{z_2}\sqrt{\frac{t'u'}{s}}, \tag{F.54}$$
$$r_{\perp y} \approx -k_\perp\,\delta\phi \tag{F.55}$$

where $k_\perp = |K_{2\perp}|/z_2 \approx |K_{1\perp}|/z_1$.

An interesting quantity is the volume spanned by the four vectors P_1, P_2, K_1 and K_2. From the above definition of vectors one immediately sees that

$$\epsilon^{P_1\,P_2\,K_1\,K_2} \simeq \frac{s}{2}\,|K_{1\perp}|\,|K_{2\perp}|\,\sin\delta\phi \simeq \frac{1}{2}\,\sqrt{tt'uu'}\,\sin\delta\phi. \tag{F.56}$$

We can also calculate

$$\begin{aligned}\left(\epsilon^{P_1\,P_2\,K_1\,K_2}\right)^2 &\simeq \frac{1}{16}\left(2\,ss'tt' + 2\,ss'tt' + 2\,tt'uu' - s^2\,s'^2 - t^2\,t'^2 - u^2\,u'^2\right)\\ &\simeq \frac{1}{16}\left(-\sqrt{ss'} + \sqrt{tt'} + \sqrt{uu'}\right)\left(\sqrt{ss'} + \sqrt{tt'} - \sqrt{uu'}\right)\\ &\quad\times\left(\sqrt{ss'} - \sqrt{tt'} + \sqrt{uu'}\right)\left(\sqrt{ss'} + \sqrt{tt'} + \sqrt{uu'}\right).\end{aligned} \tag{F.57}$$

Since the first of these terms is explicitly $\mathscr{O}(\sqrt{s})$ one can use the leading expressions for the other terms to obtain

$$\left(\epsilon^{P_1\,P_2\,K_1\,K_2}\right)^2 = \frac{1}{2}\sqrt{s\,s'\,t\,t'\,u\,u'}\left(\sqrt{tt'} + \sqrt{uu'} - \sqrt{ss'}\right) \approx \frac{1}{4}\,tt'\,uu'\,\delta\phi^2 \tag{F.58}$$

The normal direction can be expanded as

$$\begin{aligned}e_{1N}^\mu &\approx -\frac{2}{\sqrt{stu}}\,\epsilon^{P_1 P_2 K_1\,\mu}\\ &\approx \frac{2}{\sqrt{stu}}\,\frac{z_1}{x_1\,x_2}\left[-\epsilon^{p_1 p_2 k_1\,\mu} + \epsilon^{p_{1T} p_2 k_1\,\mu} + \epsilon^{p_1 p_{2T} k_1\,\mu} + \epsilon^{p_1 p_2 k_{1T}\,\mu}\right]\end{aligned} \tag{F.59}$$

Finally we give the results for the delta-function expressing momentum conservation for the partons using

$$\delta^4(R) \simeq \frac{s}{2}\,\delta(R\cdot P_1)\,\delta(R\cdot P_2)\,\delta(R\cdot e_{1\perp})\,\delta(R\cdot e_{1N}), \tag{F.60}$$

(similarly with $e_{2\perp}$ and e_{2N} or $e_\perp$ and e_N) using for $R = p_1 + p_2 - k_1 - k_2$. We find

$$R \cdot e_{1\perp} \simeq -\frac{|K_{2\perp}|}{z_2}\, e_{1\perp} \cdot e_{2\perp} + e_{1\perp} \cdot q_T, \tag{F.61}$$

$$R \cdot e_{2\perp} \simeq -\frac{|K_{1\perp}|}{z_1}\, e_{1\perp} \cdot e_{2\perp} + e_{2\perp} \cdot q_T, \tag{F.62}$$

$$R \cdot e_{\perp} \approx \frac{|K_{2\perp}|}{z_2} - \frac{|K_{1\perp}|}{z_1} \approx \frac{1}{z_2}\sqrt{\frac{t'u'}{s}} - \frac{1}{z_1}\sqrt{\frac{tu}{s}}, \tag{F.63}$$

$$R \cdot e_{1N} \simeq \frac{|K_{2\perp}|}{z_2}\, \sin\delta\phi + e_{1N} \cdot q_T, \tag{F.64}$$

$$R \cdot e_{2N} \simeq -\frac{|K_{1\perp}|}{z_1}\, \sin\delta\phi + e_{2N} \cdot q_T, \tag{F.65}$$

$$R \cdot e_{N} \approx e_N \cdot q_T + \frac{1}{z_1}\sqrt{\frac{tu}{s}}\, \sin\delta\phi \approx e_N \cdot q_T + \frac{1}{z_2}\sqrt{\frac{t'u'}{s}}\, \sin\delta\phi \tag{F.66}$$

The parton momentum conservation delta-function thus can be written as

$$\begin{aligned}
\delta^4(p_1 + p_2 - k_1 - k_2) &= 2\,s\,\delta\left(x_2\,s + \frac{1}{z_1}\,t + \frac{1}{z_2}\,u'\right)\,\delta\left(x_1\,s + \frac{1}{z_1}\,u + \frac{1}{z_2}\,t'\right) \\
&\quad \times \delta\left(\frac{1}{z_2}\sqrt{\frac{t'u'}{s}} - \frac{1}{z_1}\sqrt{\frac{tu}{s}}\right)\,\delta\left(e_N \cdot q_T + \frac{1}{z_2}\sqrt{\frac{t'u'}{s}}\,\sin\delta\phi\right) \qquad \text{(F.67)} \\
&= \frac{2}{\sqrt{s\,t\,u}}\,\delta\left(x_1 - \frac{1}{z_2}\sqrt{\frac{s't'}{s\,t}}\right)\,\delta\left(x_2 - \frac{1}{z_2}\sqrt{\frac{s'u'}{s\,u}}\right) \\
&\quad \times \delta\left(\frac{1}{z_1} - \frac{1}{z_2}\sqrt{\frac{t'u'}{t\,u}}\right)\,\delta\left(e_N \cdot q_T + \frac{1}{z_1}\sqrt{\frac{t\,u}{s}}\,\sin\delta\phi\right) \qquad \text{(F.68)} \\
&= \frac{2}{\sqrt{s\,t\,u}}\,\delta\left(x_1 - \frac{x_{1T}}{z_1}\,e^{\eta_1} - \frac{x_{2T}}{z_2}\,e^{\eta_2}\right)\,\delta\left(x_2 - \frac{x_{1T}}{z_1}\,e^{-\eta_1} - \frac{x_{2T}}{z_2}\,e^{-\eta_2}\right) \\
&\quad \times \delta\left(\frac{1}{z_1} - \frac{1}{z_2}\,\frac{x_{2T}}{x_{1T}}\right)\,\delta\left(e_N \cdot q_T + \frac{\sqrt{s}}{2}\,\frac{x_{2T}}{z_2}\,\sin\delta\phi\right). \qquad \text{(F.69)} \\
&= \frac{2}{\sqrt{s\,t\,u}}\,\delta\left(x_1 - \frac{x_\perp}{2}\,(e^{\eta_1} + e^{\eta_2})\right)\,\delta\left(x_2 - \frac{x_\perp}{2}\,(e^{-\eta_1} + e^{-\eta_2})\right) \\
&\quad \times \delta\left(\frac{1}{z_1} - \frac{x_\perp}{x_{1T}}\right)\,\delta\left(e_N \cdot q_T + \frac{\sqrt{s}}{2}\,x_\perp\,\sin\delta\phi\right), \qquad \text{(F.70)} \\
&= \frac{2}{\sqrt{s\,t\,u}}\,\delta\left(x_1 - x_\perp\,e^{(\eta_1+\eta_2)/2}\,\cosh\left(\frac{\eta_1 - \eta_2}{2}\right)\right) \\
&\quad \times\,\delta\left(x_2 - x_\perp\,e^{-(\eta_1+\eta_2)/2}\,\cosh\left(\frac{\eta_1 - \eta_2}{2}\right)\right) \\
&\quad \times \delta\left(\frac{1}{z_1} - \frac{x_\perp}{x_{1T}}\right)\,\delta\left(e_N \cdot q_T + \frac{\sqrt{s}}{2}\,x_\perp\,\sin\delta\phi\right), \qquad \text{(F.71)}
\end{aligned}$$

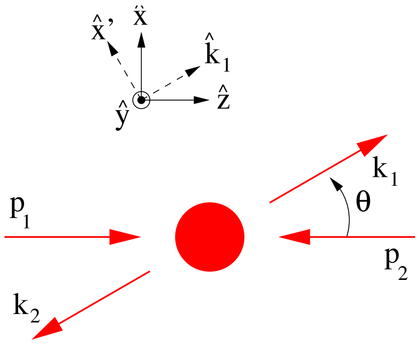
where $x_\perp = 2\,k_\perp/\sqrt{s}$. Note that the transverse energy can be easily brought into the phase space integration which involves

$$\frac{1}{x_{1T}}\,dz_1^{-1}\,dz_2^{-1}\,dx_{1T}\,dx_{2T}\,\delta\left(\frac{1}{z_1} - \frac{x_\perp}{x_{1T}}\right) = \frac{1}{x_{1T}\,x_{2T}}\,dz_1^{-1}\,dz_2^{-1}\,dx_\perp\,dx_{1T}\,dx_{2T}\,\delta\left(\frac{1}{z_1} - \frac{x_\perp}{x_{1T}}\right)\,\delta\left(\frac{1}{z_2} - \frac{x_\perp}{x_{2T}}\right) \tag{F.72}$$

which makes the kinematics nicely symmetric.

F.4 Explicit frame dependence and n-dependence

In order to illustrate the n-dependence we start with looking at the momenta in a two-to-two hard scattering process, $p_1 + p_2 = k_1 + k_2$. In the CM system we have



The Mandelstam variables for the (assumed massless) partons are:

$$\begin{aligned} \hat{s} &= 2\,p_1 \cdot p_2, \\ -\hat{t} &= 2\,k_1 \cdot p_1 = \hat{s}\,\sin^2\left(\tfrac{1}{2}\theta\right), \\ -\hat{u} &= 2\,k_1 \cdot p_2 = \hat{s}\,\cos^2\left(\tfrac{1}{2}\theta\right), \end{aligned}$$

For the scattering angle in the parton CM system one has:

$$\cos\theta = \frac{\hat{t}-\hat{u}}{\hat{s}}, \qquad \sin\theta = \frac{2\sqrt{\hat{t}\hat{u}}}{\hat{s}}.$$

The explicit parton momenta are

$$p_1 = \tfrac{1}{2}\sqrt{\hat{s}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\quad p_2 = \tfrac{1}{2}\sqrt{\hat{s}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix},\quad k_1 = \tfrac{1}{2}\sqrt{\hat{s}}\begin{pmatrix}1\\ \sin\theta\\0\\ \cos\theta\end{pmatrix},\quad k_2 = \tfrac{1}{2}\sqrt{\hat{s}}\begin{pmatrix}1\\ -\sin\theta\\0\\ -\cos\theta\end{pmatrix}, \tag{F.73}$$

or in lightcone coordinates, $[p^-, p^+, p_\perp]$,

$$p_1 = \sqrt{\frac{\hat{s}}{2}}\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad p_2 = \sqrt{\frac{\hat{s}}{2}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad k_1 = \sqrt{\frac{\hat{s}}{2}}\begin{bmatrix}-\hat{t}/\hat{s}\\ -\hat{u}/\hat{s}\\ \sqrt{2\hat{t}\hat{u}}/\hat{s}\\ 0\end{bmatrix},\quad k_2 = \sqrt{\frac{\hat{s}}{2}}\begin{bmatrix}-\hat{u}/\hat{s}\\ -\hat{t}/\hat{s}\\ -\sqrt{2\hat{t}\hat{u}}/\hat{s}\\ 0\end{bmatrix}. \tag{F.74}$$

For the momenta P_1 and P_2 we have (using $m_{1T}^2 \equiv x_1^2 M_1^2 - p_{1T}^2$),

$$P_1 = \frac{1}{x_1}\sqrt{\frac{\hat{s}}{2}}\begin{bmatrix} m_{1T}^2/\hat{s}\\ 1\\ -p_{1T}^x\sqrt{2}/\sqrt{\hat{s}}\\ -p_{1T}^y\sqrt{2}/\sqrt{\hat{s}}\end{bmatrix},\quad P_2 = \frac{1}{x_2}\sqrt{\frac{\hat{s}}{2}}\begin{bmatrix} 1\\ m_{2T}^2/\hat{s}\\ -p_{2T}^x\sqrt{2}/\sqrt{\hat{s}}\\ -p_{2T}^y\sqrt{2}/\sqrt{\hat{s}}\end{bmatrix}. \tag{F.75}$$

Using (p^0, p^x, p^y, p^z) coordinates, one has for P_1 and the (over an angle θ) rotated momentum K_1 (fraction $x_1 \to 1/z_1$),

$$P_1 = \frac{1}{x_1}\frac{\sqrt{\hat{s}}}{2}\begin{pmatrix}(\hat{s}+m_{1T}^2)/\hat{s}\\ -2\,p_{1T}^x/\sqrt{\hat{s}}\\ -2\,p_{1T}^y/\sqrt{\hat{s}}\\ (\hat{s}-m_{1T}^2)/\hat{s}\end{pmatrix},\quad K_1 = z_1\frac{\sqrt{\hat{s}}}{2}\begin{pmatrix}(\hat{s}+m_{h1T}^2)/\hat{s}\\ 2\sqrt{\hat{t}\hat{u}}(\hat{s}-m_{h1T}^2)/\hat{s}^2 - 2(\hat{t}-\hat{u})k_{1T}^{x'}/\hat{s}\sqrt{\hat{s}}\\ -2\,k_{1T}^y/\sqrt{\hat{s}}\\ (\hat{t}-\hat{u})(\hat{s}-m_{h1T}^2)/\hat{s}^2 + 4\sqrt{\hat{t}\hat{u}}\,k_{1T}^{x'}/\hat{s}\sqrt{\hat{s}}\end{pmatrix}, \tag{F.76}$$

where $m_{h1T}^2 \equiv z_1^{-2}\,M_{h1}^2 - k_{1T}^2$. In lightcone coordinates we get for K_1 and K_2,

$$K_1 = z_1\sqrt{\frac{\hat{s}}{2}}\begin{bmatrix}-\hat{t}/\hat{s} - 2\sqrt{\hat{t}\hat{u}}\,k_{1T}^{x'}/\hat{s}\sqrt{\hat{s}} - \hat{u}\,m_{h1T}^2/\hat{s}^2\\ -\hat{u}/\hat{s} + 2\sqrt{\hat{t}\hat{u}}\,k_{1T}^{x'}/\hat{s}\sqrt{\hat{s}} - \hat{t}\,m_{h1T}^2/\hat{s}^2\\ \sqrt{2\hat{t}\hat{u}}/\hat{s} - (\hat{t}-\hat{u})k_{1T}^{x'}\sqrt{2}/\hat{s}\sqrt{\hat{s}} - \sqrt{2\hat{t}\hat{u}}\,m_{h1T}^2/\hat{s}^2\\ -k_{1T}^y\sqrt{2}/\sqrt{\hat{s}}\end{bmatrix}, \tag{F.77}$$

$$K_2 = z_2\sqrt{\frac{\hat{s}}{2}}\begin{bmatrix}-\hat{u}/\hat{s} - 2\sqrt{\hat{t}\hat{u}}\,k_{2T}^{x'}/\hat{s}\sqrt{\hat{s}} - \hat{t}\,m_{h2T}^2/\hat{s}^2\\ -\hat{t}/\hat{s} + 2\sqrt{\hat{t}\hat{u}}\,k_{2T}^{x'}/\hat{s}\sqrt{\hat{s}} - \hat{u}\,m_{h2T}^2/\hat{s}^2\\ -\sqrt{2\hat{t}\hat{u}}/\hat{s} - (\hat{t}-\hat{u})k_{2T}^{x'}\sqrt{2}/\hat{s}\sqrt{\hat{s}} + \sqrt{2\hat{t}\hat{u}}\,m_{h2T}^2/\hat{s}^2\\ -k_{2T}^y\sqrt{2}/\sqrt{\hat{s}}\end{bmatrix}. \tag{F.78}$$

We note that with $n \sim p_2$ one finds exactly $x_1 = p_1 \cdot n/P_1 \cdot n = p_1 \cdot p_2/P_1 \cdot p_2$, while one has

$$p_1 = x_1\,P_1 + p_{1T} + \frac{m_{1T}^2}{x_1\,\hat{s}}\,p_2, \tag{F.79}$$

with $p_{1T} = p_{1T}^x \hat{x} + p_{1T}^y \hat{y}$. Note that the coefficient of p_2 is given by $\sigma_1/(\hat{s}/2) = m_{1T}^2/x_1 = p_1 \cdot P_1$. Similarly with $n \sim k_2$ one finds exactly $1/z_1 = k_1 \cdot n/K_1 \cdot n = k_1 \cdot k_2/K_1 \cdot k_2$, while one has

$$k_1 = z_1^{-1}\, K_1 + k_{1T} + z_1\, \frac{m_{h1T}^2}{\hat{s}}\, k_2, \tag{F.80}$$

with $k_{1T} = k_{1T}^{x'} \hat{x}' + k_{1T}^y \hat{y}$. Without reference to k_2 one could have chosen $n \sim p_1 + p_2 - k_1 = p_{\rm cm} - k_1$ or in a multiparton 2-to-many process, with p_1 and p_2 the initial state parton momenta and k one of the final state parton momenta, one can use (with $\hat{t} \equiv 2k \cdot p_1$ and $\hat{u} \equiv 2k \cdot p_2$),

$$\tilde{k} = 2\,\frac{k \cdot p_{\rm cm}}{p_{\rm cm}^2}\, p_{\rm cm} - k = \frac{2k \cdot (p_1 + p_2)}{\hat{s}}\,(p_1 + p_2) - k = -\frac{\hat{t} + \hat{u}}{\hat{s}}\,(p_1 + p_2) - k, \tag{F.81}$$

which in the CM system is just the space-inverted vector. Note that the vector $x_1\, P_1 + x_2\, P_2$ can mostly be used instead of $p_{\rm cm}$, since the difference is $\mathcal{O}(\infty)$.

To see what is happening for a different n, we start with the hadronic vector K in the jet with momentum k, given in the form as above

$$K = z\, k - z \begin{bmatrix} k_T^{x'}\ \sin\theta/\sqrt{2} \\ -k_T^{x'}\ \sin\theta/\sqrt{2} \\ k_T^{x'}\ \cos\theta \\ k_T^y \end{bmatrix} - z\, \frac{m_{hT}^2}{\hat{s}}\, \tilde{k} \approx z\, k - z \begin{pmatrix} 0 \\ k_T^{x'}\ \cos\theta \\ k_T^y \\ -k_T^{x'}\ \sin\theta \end{pmatrix},$$

where z and k_T correspond with choice $n = \tilde{k}/k \cdot \tilde{k}$. Using the choice $n = p_2$ for K one finds

$$\begin{aligned} z' = \frac{K \cdot p_2}{k \cdot p_2} &= z \left(1 + 2\sqrt{\frac{\hat{t}}{\hat{u}}}\, \frac{k_T^{x'}}{\sqrt{\hat{s}}} + \frac{\hat{t}}{\hat{u}}\, \frac{m_{hT}^2}{\hat{s}} \right) = z \left(1 + 2\, \tan\left(\tfrac{\theta}{2}\right) \frac{k_T^{x'}}{\sqrt{\hat{s}}} + \tan^2\left(\tfrac{\theta}{2}\right), \frac{m_{hT}^2}{\hat{s}} \right) \\ &\approx z \left(1 + 2\sqrt{\frac{\hat{t}}{\hat{u}}}\, \frac{k_T^{x'}}{\sqrt{\hat{s}}} \right) = z \left(1 + 2\, \tan\left(\tfrac{\theta}{2}\right) \frac{k_T^{x'}}{\sqrt{\hat{s}}} \right), \end{aligned} \tag{F.82}$$

which differs at $\mathcal{O}(1/\sqrt{\hat{s}})$ from z but causes $\mathcal{O}(1)$ corrections to the x-, z- and time-component of k_T'. We get

$$K \approx z'\, k - z \begin{pmatrix} k_T^{x'}\ \tan(\theta/2) \\ k_T^{x'} \\ k_T^y \\ -k_T^{x'}\ \tan(\theta/2) \end{pmatrix}.$$

The transverse momentum vector acquires a piece along k, but its length does not change.

F.5 Limiting cases

Limiting cases are:

- $H_1(P_1) + H_2(P_2) \to h_1(K_1) + j(k_2) + X$:

$$z_2 = 1,\ k_{2T} = 0 \qquad (k_2 = K_2). \tag{F.83}$$

- $H_1(P_1) + H_2(P_2) \to \ell_1(k_1) + \ell_2(k_2) + X$ (Drell-Yan like process):

$$z_1 = z_2 = 1,\ k_{1T} = k_{2T} = 0, \tag{F.84}$$

$$q = k_1 + k_2, \tag{F.85}$$

$$q_T = q - x_1\, P_1 - x_2\, P_2 = p_{1T} + p_{2T}, \tag{F.86}$$

- $\ell_1(p_1) + \ell_2(p_2) \to h_1(K_1) + h_2(K_2) + X$ (Annihilation type of process):

$$x_1 = x_2 = 1,\ p_{1T} = p_{2T} = 0, \tag{F.87}$$

$$q = p_1 + p_2, \tag{F.88}$$

$$q_T = \frac{K_1}{z_1} + \frac{K_2}{z_2} - q = -k_{1T} - k_{2T}, \tag{F.89}$$

- $\ell(p_1) + H(P) \to \ell'(k_1) + h(K) + X$ (Leptoproduction type of process):

$$x_1 = z_1 = 1,\ x_2 = x,\ z_2 = z,\ p_{1T} = k_{1T} = 0, \tag{F.90}$$
$$q = p_1 - k_1, \tag{F.91}$$
$$q_T = \frac{K}{z} - x\,P - q = p_T - k_T. \tag{F.92}$$

F.6 kinematics of jet-jet production

Consider the kinematics of the hard process, $H_1 + H_2 \to j_1 + j_2 + X$, including momenta

$$H_1(P_1) + H_2(P_2) \to j_1(k_1) + j_2(k_2) + X. \tag{F.93}$$

This is an inclusive process, for which we will use the same variables as in the general case. The n-vectors to expand internal momenta will be choosen from P_1 and P_2, so the internal transverse momenta could also be labeled by $\perp$-indices. We will maintain T-indices for the partons in hadrons H_1 and H_2 (which are order 1 rather than the $\mathcal{O}(\sqrt{s})$ momenta $k_{1\perp}$ and $k_{2\perp}$.

The incoming hadrons produce two partons with momenta p_1 and p_2, the outcoming jets are identified with partons k_1 and k_2. These partons participate in a hard process in which the momenta satisfy $p_1 + p_2 = k_1 + k_2$. We now have

$$q_T = k_1 + k_2 - x_1\,P_1 - x_2\,P_2 = p_{1T} + p_{2T}. \tag{F.94}$$

or

$$q_T = k_{1\perp} + k_{2\perp} = r_\perp. \tag{F.95}$$

This is a $\mathcal{O}(1)$ vector and it is convenient to introduce the *transverse energy* $k_\perp$ (order Q) given by $k_\perp = |k_{1\perp}| \simeq |k_{2\perp}|$ and its scaled version $x_\perp = 2\,k_\perp/\sqrt{s}$, although one might also have to consider the difference in azimuthal asymmetries.

The relations with parton momenta simplify. By taking the product of the constraint $p_1+p_2-k_1-k_2 = 0$ we get (omitting $\mathscr{O}(1)$ corrections) the constraints

$$2P_1 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq x_2\,s + t + u' + 2\,P_1 \cdot q_T, \tag{F.96}$$
$$2P_2 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq x_1\,s + u + t' + 2\,P_2 \cdot q_T, \tag{F.97}$$
$$2k_1 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq -x_1\,t - x_2\,u - s' + 2\,k_1 \cdot q_T, \tag{F.98}$$
$$2k_2 \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq -x_1\,u' - x_2\,t' - s' + 2\,k_2 \cdot q_T, \tag{F.99}$$

Instead of the latter two conditions we get for the $\perp$-components

$$\begin{aligned} 2k_{1\perp} \cdot (p_1 + p_2 - k_1 - k_2) = 0 &\simeq 2\,|k_{1\perp}|^2 - 2\,k_{1\perp} \cdot k_{2\perp} + 2\,k_{1\perp} \cdot q_T, \\ &\simeq 2\,\frac{t\,u}{s} + \frac{tt' + uu' - ss'}{s} + 2\,k_{1\perp} \cdot q_T, \end{aligned} \tag{F.100}$$
$$2k_{2\perp} \cdot (p_1 + p_2 - k_1 - k_2) = 0 \simeq 2\,\frac{t'\,u'}{s} + \frac{tt' + uu' - ss'}{s} + 2\,k_{2\perp} \cdot q_T, \tag{F.101}$$

At leading order, the Mandelstam variables for the subprocess ($\hat{s}$, $\hat{t}$ and $\hat{u}$) are now related to variables in the full process through

$$\hat{s} \approx x_1 x_2\,s \approx s' \approx 4\,k_\perp^2\,\left[\cosh^2\left(\frac{\eta_1 - \eta_2}{2}\right) + \cos^2\left(\frac{\phi_1 - \phi_2}{2}\right)\right], \tag{F.102}$$
$$\hat{t} \approx x_1\,t \approx x_2\,t' \approx -2k_\perp^2\,e^{-(\eta_1-\eta_2)/2}\,\cosh\left(\frac{\eta_1 - \eta_2}{2}\right), \tag{F.103}$$
$$\hat{u} \approx x_1\,u' \approx x_2\,u \approx -2\,k_\perp^2\,e^{+(\eta_1-\eta_2)/2}\,\cosh\left(\frac{\eta_1 - \eta_2}{2}\right), \tag{F.104}$$

or

$$\sqrt{ss'} = \frac{\hat{s}}{\sqrt{x_1 x_2}} \tag{F.105}$$

$$\sqrt{tt'} = -\frac{\hat{t}}{\sqrt{x_1 x_2}}, \tag{F.106}$$

$$\sqrt{uu'} = -\frac{\hat{u}}{\sqrt{x_1 x_2}}, \tag{F.107}$$

which are relations with $\mathscr{O}(\sqrt{s})$ corrections. At leading order it of course again follows Eq. F.42. and we now have

$$x_1 \approx \sqrt{\frac{s'\,t'}{s\,t}} \approx \sqrt{\frac{s'\,u}{s\,u'}}, \tag{F.108}$$

$$x_2 \approx \sqrt{\frac{s'\,u'}{s\,u}} \approx \sqrt{\frac{s'\,t}{s\,t'}}, \tag{F.109}$$

$$\sqrt{\frac{t\,u}{t'\,u'}} \approx 1. \tag{F.110}$$

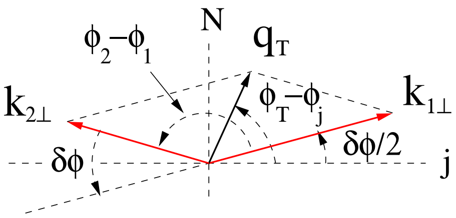
As before let us consider the kinematics in the transverse plane. The approximations above (coming from the constraint $p_1 + p_2 - k_1 - k_2 = 0$) lead to Eq. F.42, which implies that

$$k_{1\perp} \cdot k_{2\perp} \approx 1 \tag{F.111}$$

i.e. the vectors $k_{1\perp}$ and $k_{2\perp}$ are almost parallel and opposite, $k_{1\perp} \approx -k_{2\perp}$ and similarly its directions $e_{1\perp} \approx -e_{2\perp}$. While the directions $e_{1\perp}$ and $e_{2\perp}$ make a small angle $\delta\phi$,

$$\delta\phi^2 \approx 2\sqrt{\frac{ss'}{tt'uu'}}\left(\sqrt{tt'} + \sqrt{uu'} - \sqrt{ss'}\right).$$

and their sum vector is in the direction N, the sum vector $r_\perp = q_T = k_{1\perp} + k_{2\perp}$ can have any direction.



In the figure the jet direction $\phi_j \equiv (\phi_1+\phi_2-\pi)/2$ has been chosen to the right (j- or x-direction with orthogonal N- or y-direction).

$$\begin{aligned} r_{\perp x} &= (|k_{1\perp}| - |k_{2\perp}|)\cos\left(\tfrac{1}{2}\delta\phi\right) \\ &\simeq (x_{1T} - x_{2T})\frac{\sqrt{s}}{2}. \simeq \sqrt{\frac{tu}{s}} - \sqrt{\frac{t'u'}{s}} \end{aligned} \tag{F.112}$$

$$\begin{aligned} r_{\perp y} &= (|k_{1\perp}| + |k_{2\perp}|)\sin\left(\tfrac{1}{2}\delta\phi\right) \\ &\simeq -k_\perp\,\delta\phi \simeq -x_\perp\frac{\sqrt{s}}{2}\delta\phi \end{aligned} \tag{F.113}$$

where $k_\perp = (|k_{1\perp}| + |k_{2\perp}|)/2 = x_\perp\sqrt{s}/2$.

Finally we give again the results for the delta-function expressing momentum conservation for the partons using

$$\delta^4(R) \simeq \frac{s}{2}\,\delta(R\cdot P_1)\,\delta(R\cdot P_2)\,\delta^2(R_\perp), \tag{F.114}$$

and $R = p_1 + p_2 - k_1 - k_2$. We find (note that $q_T = r_\perp$)

$$\delta^4(p_1+p_2-k_1-k_2) = 2\,s\,\delta\left(x_2\,s + t + u'\right)\,\delta\left(x_1\,s + u + t'\right)\,\delta^2\left(p_{1T} + p_{2T} - k_{1\perp} - k_{2\perp}\right) \tag{F.115}$$

$$= \frac{2}{\sqrt{s\,t\,u}}\,\delta\left(x_1 - \frac{1}{z_2}\sqrt{\frac{s't'}{s\,t}}\right)\,\delta\left(x_2 - \frac{1}{z_2}\sqrt{\frac{s'u'}{s\,u}}\right)\,\delta^2\left(p_{1T} + p_{2T} - q_T\right) \tag{F.116}$$

$$= \frac{2}{\sqrt{s\,t\,u}}\,\delta\left(x_1 - x_{1T}\,e^{\eta_1} - x_{2T}\,e^{\eta_2}\right)\,\delta\left(x_2 - x_{1T}\,e^{-\eta_1} - x_{2T}\,e^{-\eta_2}\right)\,\delta^2\left(p_{1T} + p_{2T} - q_T\right) \tag{F.117}$$

We note that in the phase-space integration in the transverse plane, we have variables x_{1T}, x_{2T}, ϕ_1 and ϕ_2 or $x_\perp$, ϕ_j, ϕ_T and $Q_T \equiv |q_T|$. We have the relations

$$\frac{2Q_T}{\sqrt{s}}\,\cos(\phi_T-\phi_j) = (x_{1T}-x_{2T})\,\cos\left(\tfrac{1}{2}\delta\phi\right), \tag{F.118}$$

$$\frac{2Q_T}{\sqrt{s}}\,\sin(\phi_T-\phi_j) = (x_{1T}+x_{2T})\,\sin\left(\tfrac{1}{2}\delta\phi\right), \tag{F.119}$$

from which we get (up to mass corrections exact)

$$4Q_T^2/s \;=\; (x_{1T}-x_{2T})^2\,\cos^2\left(\tfrac{1}{2}\delta\phi\right) + (x_{1T}+x_{2T})^2\,\sin^2\left(\tfrac{1}{2}\delta\phi\right), \tag{F.120}$$

$$Q_T^2\,\cos\left(2(\phi_T-\phi_j)\right) \;=\; Q_T^2 - 8\,k_\perp^2\,\sin^2\left(\tfrac{1}{2}\delta\phi\right), \tag{F.121}$$

$$\begin{aligned} Q_T^4\,\cos\left(4(\phi_T-\phi_j)\right) &= Q_T^4 - 32\,Q_T^2\,k_\perp^2\,\sin^2\left(\tfrac{1}{2}\delta\phi\right) + 128\,k_\perp^4\,\sin^4\left(\tfrac{1}{2}\delta\phi\right) \\ &= Q_T^4 - 16\,Q_T^2\,k_\perp^2 + 16\,Q_T^2\,k_\perp^2\,\cos(\delta\phi) \\ &\quad + 48\,k_\perp^4 - 64\,k_\perp^4\,\cos(\delta\phi) + 16\,k_\perp^4\,\cos(2\delta\phi) \end{aligned} \tag{F.122}$$

F.6.1 Kinematics for GPD's

Consider the amplitude for $\gamma^*(q) + P \to \gamma(q') + P'$ and the handbag diagram underlying this process, i.e. with $\gamma^*(q) + \text{parton}(k) \to \gamma(q') + \text{parton}'(k')$. A convenient parametrization of momenta satisfying $P^2 = P'^2 = M^2$ and using

$$\Delta \equiv P - P' \quad \text{with} \quad \Delta^2 = t \le 0, \tag{F.123}$$

$$\overline{P} \equiv (P+P')/2 \quad \text{with} \quad \overline{P}^2 = \overline{M}^2 = M^2 - \tfrac{1}{4}t \ge M^2. \tag{F.124}$$

Introducing lightlike vectors p and n satisfying $p\cdot n = 1$ we write

$$\Delta \equiv 2\xi\,p + \frac{t-\Delta_\perp^2}{4\xi}\,n + \Delta_\perp = 2\xi\,p - \xi\left(M^2 - \tfrac{1}{4}t\right)\,n + \Delta_\perp = 2\xi\,p - \xi\,\overline{M}^2\,n + \Delta_\perp \tag{F.125}$$

$$P \equiv (1+\xi)\,p + \frac{M^2 - \frac{1}{4}\Delta_\perp^2}{2(1+\xi)}\,n + \frac{1}{2}\Delta_\perp = (1+\xi)\,p + (1-\xi)\frac{\overline{M}^2}{2}\,n + \frac{1}{2}\Delta_\perp, \tag{F.126}$$

$$P' = (1-\xi)\,p + \frac{M^2 - \frac{1}{4}\Delta_\perp^2}{2(1-\xi)}\,n - \frac{1}{2}\Delta_\perp = (1-\xi)\,p + (1+\xi)\frac{\overline{M}^2}{2}\,n - \frac{1}{2}\Delta_\perp, \tag{F.127}$$

$$\overline{P} = p + \frac{\overline{M}^2}{2}\,n = p + \frac{M^2 - \frac{1}{4}t}{2}\,n = p + \frac{M^2 - \frac{1}{4}\Delta_\perp^2}{2(1-\xi^2)}\,n. \tag{F.128}$$

The latter equation implies that $\Delta_\perp^2 = t(1-\xi^2) + 4M^2\xi^2$ which has been used to rewrite the first three expressions. A slightly different way of writing is $-\Delta_\perp^2 + 4\overline{M}^2\xi^2 = -t$. It implies that for $-t \to 0$ one must have $\xi \to 0$ and $-\Delta_\perp^2 \to 0$, while for small t one has $-t \approx -\Delta_\perp^2 + 4M^2\xi^2$.

Turning to the kinematics for q and q' we can make the choice that $q_\perp = 0$, implying

$$q = -2\xi\left(1 + \frac{\Delta_\perp^2}{Q^2}\right)\,p + \frac{Q^2}{4\xi\left(1 + \frac{\Delta_\perp^2}{Q^2}\right)}\,n \;\approx\; -2\xi\,p + \frac{Q^2}{4\xi}\,n, \tag{F.129}$$

$$q' = -\frac{2\xi\Delta_\perp^2}{Q^2}\,p + \frac{Q^2}{4\xi}\,n + \Delta_\perp. \tag{F.130}$$

Furthermore, one has (with $\{\overline{M}^2, M^2, |\Delta_\perp^2|, |t|\} \ll Q^2$)

$$\xi \approx \frac{Q^2}{4\overline{P}\cdot q} = \frac{Q^2}{2(P+P')\cdot q}. \tag{F.131}$$

and for the light-like vectors in terms of $\overline{P}$ and q

$$p \;\approx\; \overline{P} - \frac{\overline{M}^2}{2\,\overline{P}\cdot q}\,q \approx \overline{P} \tag{F.132}$$

$$n \;\approx\; \frac{q}{\overline{P}\cdot q} + \frac{Q^2}{2\,(\overline{P}\cdot q)^2}\,\overline{P} \approx \frac{q + 2\xi\overline{P}}{\overline{P}\cdot q}. \tag{F.133}$$

The parton momenta and photon momenta, satisfying $\Delta = q' - q = k - k'$ are choosen

$$k = (x+\xi)\,p - \frac{\left(k_\perp + \frac{1}{2}\Delta_\perp\right)^2}{2(x+\xi)}\,n + k_\perp + \frac{1}{2}\Delta_\perp, \tag{F.134}$$

$$k' = (x-\xi)\,p - \frac{\left(k_\perp + \frac{1}{2}\Delta_\perp\right)^2}{2(x-\xi)}\,n - k_\perp - \frac{1}{2}\Delta_\perp, \tag{F.135}$$

where x is simply one of the partonic momentum integration variables, as is the transverse momentum $k_\perp$. One has for the spectator and struck partons the momenta

$$P - k = P' - k' = (1-x)\,p + (\ldots)\,n, \tag{F.136}$$

$$k + q \approx (x-\xi)\,p + \frac{Q^2}{4\xi}\,n \quad \text{and} \quad \frac{i}{\not k + \not q} = \frac{i}{2}\frac{\not n}{x - \xi + i\epsilon}, \tag{F.137}$$

$$k' - q \approx (x+\xi)\,p - \frac{Q^2}{4\xi}\,n \quad \text{and} \quad \frac{i}{\not k' - \not q} = \frac{i}{2}\frac{\not n}{x + \xi - i\epsilon}. \tag{F.138}$$

The two possibilities for the struck parton correspond to the two diagrams (handbag and crossed handbag).

Appendix G

Reference material

A (non-exhaustive) list of references considering material related to these notes is in the following bibtex-items:

key	citation
Adams:1991cs	[1]
Adams:1991rw	[2]
Adams:1991ru	[3]
Adams:2003fx	[4]
Adams:1993ip	[5]
Adler:2003pb	[6]
Ageev:2006da	[7]
Airapetian:2001eg	[8]
Airapetian:2004tw	[9]
Alexakhin:2005iw	[10]
Ali:1992qj	[11]
Ali:1995vw	[12]
Altarelli:1978id	[13]
Altarelli:1978tq	[15]
Altarelli:1979ub	[14]
Anselmino:1994tv	[16]
Anselmino:1998yz	[18]
Anselmino:2006yq	[17]
Bacchetta:1999kz	[19]
Bacchetta:2000jk	[23]
Bacchetta:2001di	[22]
Bacchetta:2002	[24]
Bacchetta:2002tk	[21]
Bacchetta:2003rz	[29]
Bacchetta:2004it	[28]
Bacchetta:2004zf	[27]
Bacchetta:2005rm	[20]
Bacchetta:2006tn	[26]
Bacchetta:2007sz	[?]
Bacchetta:2008xw	[25]
Balitsky:1987bk	[30]
Barone:1998dx	[31]
Barone:2001sp	[32]
Bashinsky:1998if	[33]
Belitsky:1997ay	[34]
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key	citation
Boer:1997mf	[40]
Boer:1997bw	[44]
Boer:1997nt	[42]
Boer:1997qn	[41]
Boer:1998	[39]
Boer:1999si	[43]
Boer:1999si	[43]
Boer:2003cm	[?]
Boer:2003tx	[45]
Boer:2007nd	[38]
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Bomhof:2004aw	[48]
Bomhof:2006dp	[49]
Bomhof:2006ra	[47]
Bomhof:2007su	[50]
Bomhof:2007xt	[51]
Bomhof:2007	[46]
Bravar:1996ki	[52]
Brodsky:2002cx	[53]
Brodsky:2002rv	[54]
Bunce:1976yb	[55]
Cahn:1978se	[56]
Chen:1994ar	[57]
Close:1988br	[58]
Collins:1981tt	[67]
Collins:1981uw	[66]
Collins:1982wa	[68]
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Diehl:1998sm	[?]
Donohue:1980tn	[73]
Efremov:1978cu	[75]
Efremov:1978xm	[76]
Efremov:1980kz	[77]
Efremov:1981sh	[78]
Efremov:1984ip	[79]
Efremov:2004ph	[74]
Eguchi:2006mc	[80]
Ellis:1978ty	[83]
Ellis:1982cd	[82]
Ellis:1982wd	[81]
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Jaffe:1993xb	[102]
Jakob:1996xt	[105]
Jakob:1997wg	[106]
Ji:1990br	[108]
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Ji:2004wu	[110]
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Kouvaris:2006zy	[122]
Lam:1978pu	[123]
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Mulders:1995dh	[133]
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Qiu:1991pp	[136]
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Qiu:2007ey	[140]
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Ralston:1980pp	[142]
Ratcliffe:1998se	[143]
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Signal:1989yc	[146]
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Sivers:1990fh	[148]
Soper:1976jc	[149]
Soper:1979fq	[150]
Stratmann:1992gu	[151]
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Tangerman:1995hw	[153]
Teryaev:1998uy	[154]
Vogelsang:1998yd	[155]
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