SOLUTIONS PROBLEM SBT III

① $p = \gamma m v = \frac{v}{c^2}\gamma m c^2 = E\frac{v}{c^2} \Rightarrow v = \frac{pc^2}{E}$

$E^2 = p^2c^2 + m^2c^4 \Rightarrow E = \sqrt{p^2c^2+m^2c^4}$

$\Rightarrow v = \frac{pc^2}{\sqrt{p^2c^2+m^2c^4}} = \frac{pc}{\sqrt{p^2+m^2c^2}}$

② $E = \gamma m c^2 \Rightarrow \gamma = E/mc^2 \Rightarrow \frac{1}{\sqrt{1-v^2/c^2}} = \frac{E}{mc^2}$

$\left(\frac{E}{mc^2}\right)^2 = \frac{1}{1-v^2/c^2} = \frac{c^2}{c^2-v^2} = \frac{c^2}{(c-v)(c+v)} = \frac{c^2}{2c(c-v)} = \frac{c}{2(c-v)}$

approximate $(c-v)(c+v) \approx 2c(c-v)$

$\Rightarrow c - v = \Delta v = \frac{c}{2}\left(\frac{mc^2}{E}\right)^2$

b) Electrons accelerated in SOLEIL synchrotron of 3 GeV

$\left.\begin{array}{l} mc^2 = 511\ \mathrm{keV}/c^2 \\ E = 3\ \mathrm{GeV} \end{array}\right\} \Rightarrow \frac{E}{mc^2} = \frac{3\cdot 10^9}{511\cdot 10^3} = 0.59\ 10^4$

$\Delta v = \frac{c}{2}\left(\frac{1}{0.59\cdot 10^4}\right)^2 \simeq 4.35\ \mathrm{m/s}$

c) Protons accelerated in the Large Hadron Collider to 7 TeV

$\left.\begin{array}{l} mc^2 = 936\ \mathrm{MeV}/c^2 \\ E = 7\ \mathrm{TeV} \end{array}\right\} \quad \frac{E}{mc^2} = \frac{7\cdot 10^{12}}{936\cdot 10^6} \simeq 7.5\cdot 10^3$

$\Delta v = \frac{c}{2}\left(\frac{1}{7.5\cdot 10^3}\right)^2 \simeq 2.68\ \mathrm{m/s}$

COMPTON EFFECT

Keep for simplicity "p" in equations.

Energy Conservation

$$\overbrace{\underset{\substack{\text{photon}\\\text{energy}}}{p_\gamma c} + \underset{\substack{\text{rest energy}\\\text{electron}}}{m_e c^2}}^{\text{before}} = \overbrace{\underset{\substack{\text{photon}\\\text{energy}}}{p_{\gamma'} c} + \underset{\substack{\text{total energy}\\\text{electron}}}{\sqrt{p_e^2 c^2 + m_e^2 c^4}}}^{\text{after collision}}$$

$$\Rightarrow \quad p_\gamma + m_e c = p_{\gamma'} + \sqrt{p_e^2 + m_e^2 c^2}$$

$$\Rightarrow \left(p_\gamma + m_e c - p_{\gamma'}\right)^2 = p_e^2 + m_e^2 c^4$$

$$\Rightarrow \quad p_\gamma^2 + \underbrace{m_e^2 c^2 + p_{\gamma'}^2 + 2 p_\gamma p_{\gamma'} + 2 p_\gamma m_e c - 2 p_{\gamma'} m_e c = p_e^2 + m_e^2 c^4}_{\text{drop this.}}$$

$$\Rightarrow \quad p_e^2 = p_\gamma^2 + p_{\gamma'}^2 + 2 p_\gamma p_{\gamma'} + 2 p_\gamma m_e c - 2 p_{\gamma'} m_e c$$

Momentum Conservation along x-axis

$$\underbrace{p_\gamma}_{\text{before}} = \overbrace{p_{\gamma'} \cos\phi + p_e \cos\theta}^{\text{after.}}$$

$$\Rightarrow p_\gamma - p_{\gamma'} \cos\phi = p_e \cos\theta \qquad \Rightarrow \left(p_\gamma - p_{\gamma'} \cos\phi\right)^2 = p_e^2 \cos^2\theta$$

Momentum Conservation along y-axis.

$$p_{\gamma'} \sin\phi = p_e \sin\theta \qquad \Rightarrow \left(p_{\gamma'} \sin\phi\right)^2 = p_e^2 \sin^2\theta$$

Add the last two equations:

$$p_e^2\left(\cos^2\theta + \sin^2\theta\right) = p_\gamma^2 + p_{\gamma'}^2 \cos^2\phi - 2 p_\gamma p_{\gamma'} \cos\phi + p_{\gamma'}^2 \sin^2\phi$$

Simplify the equation from momentum conservation:

Momentum: $p_e^2 = p_\gamma^2 + p_\gamma'^2 - 2 p_\gamma p_\gamma' \cos\phi$

From Energy: $p_e^2 = p_\gamma^2 + p_\gamma'^2 - 2 p_\gamma p_\gamma' + 2 p_\gamma m_e c - 2 p_\gamma' m_e c$

Equate

$$-2 p_\gamma p_\gamma' \cos\phi = 2 p_\gamma p_\gamma' + (2 p_\gamma - 2 p_\gamma') m_e c$$

$$(p_\gamma - p_\gamma') m_e c = p_\gamma p_\gamma' (1 - \cos\phi)$$

$$\frac{p_\gamma - p_\gamma'}{p_\gamma p_\gamma'} = \frac{1 - \cos\phi}{m_e c}$$

$$\frac{1}{p_\gamma'} - \frac{1}{p_\gamma} = \frac{1 - \cos\phi}{m_e c}$$

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\phi)$$

$$\boxed{\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\phi)}$$

Kinetic Energy in the classical limit.

$$K = (\gamma - 1) m c^2 = \frac{m c^2}{\sqrt{1 - v^2/c^2}}$$

Binomial theorem (Taylor expansion)

$$(1+x)^n = 1 + n x + n(n-1)\frac{x^2}{2} + \cdots$$

for $n = -1/2$ and $x = -\frac{v^2}{c^2}$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots$$

$$K = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots - 1\right) m c^2 =$$

$$\lim_{v \to 0} K = \frac{1}{2} m v^2$$