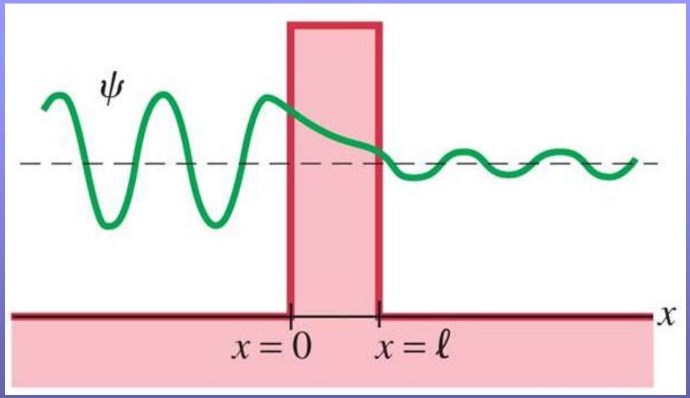
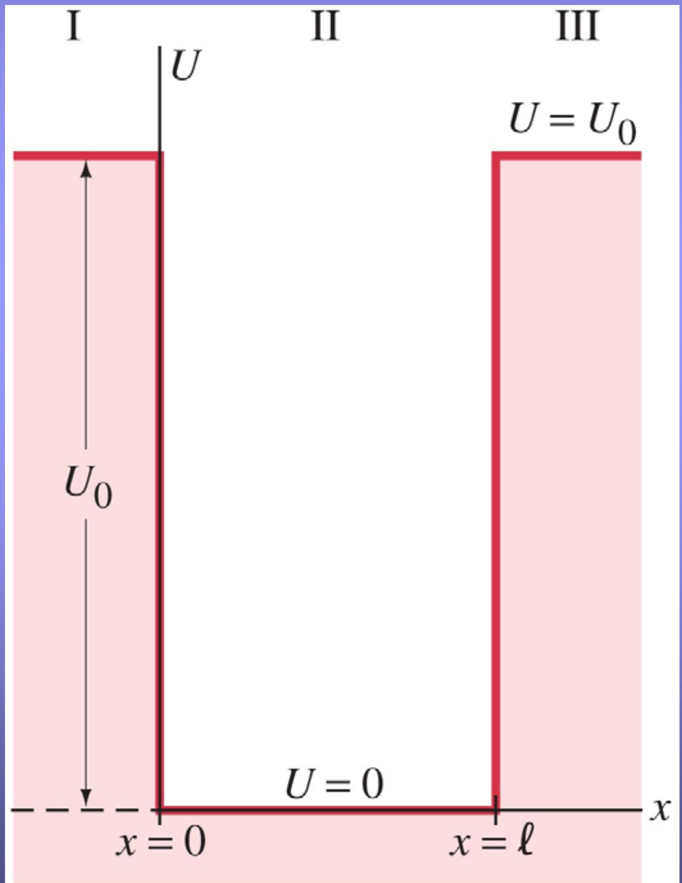
Quantum Tunneling

One of the mysteries of quantum mechanics



Finite Potential Well

A finite potential well has a potential of zero between $x = 0$ and $x = \ell$, but outside that range the potential is a constant U_0.



The potential outside the well is no longer zero; it falls off exponentially.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U_0\psi = E\psi$$

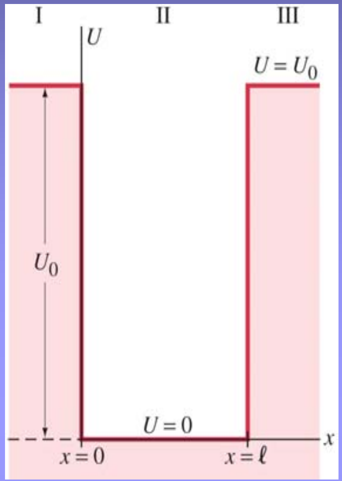
Solve in regions I, II, and III
and use for boundary conditions
Continuity:

$$\psi_I(0) = \psi_{II}(0) \qquad \psi_{II}(\ell) = \psi_{III}(\ell)$$

$$\frac{d\psi_I}{dx}(0) = \frac{d\psi_{II}}{dx}(0) \qquad \frac{d\psi_{II}}{dx}(\ell) = \frac{d\psi_{III}}{dx}(\ell)$$

Bound states: $E < E_0$
Continuum states: $E > E_0$

Finite Potential Well



If $E < U_0$ in the "forbidden regions"

$$\frac{d^2\psi}{dx^2} - \left[\frac{2m(U_0 - E)}{\hbar^2}\right]\psi = 0$$ **with** $$G^2 = \frac{2m(U_0 - E)}{\hbar^2}$$

General solution: $\psi_{I,III} = Ce^{Gx} + De^{-Gx}$

Region I $x < 0$ hence $D = 0$ and similarly for *C*

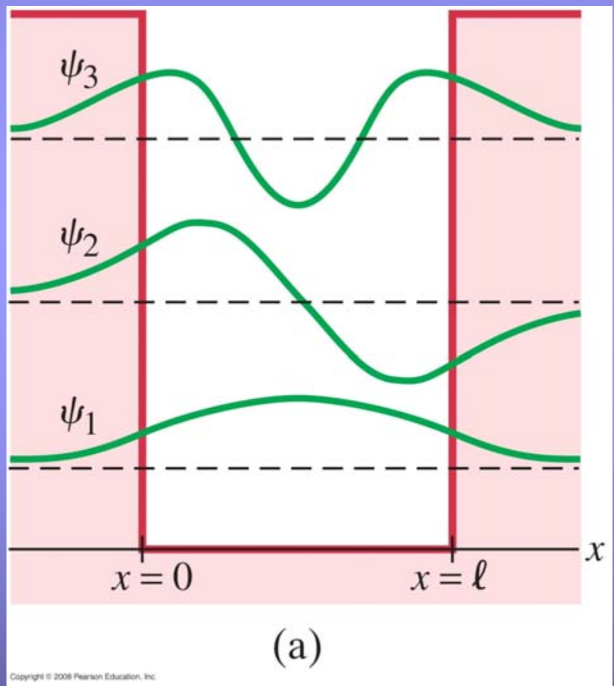
$\psi_I = Ce^{Gx}$ should match $\psi_{II} = A\sin kx + B\cos kx$

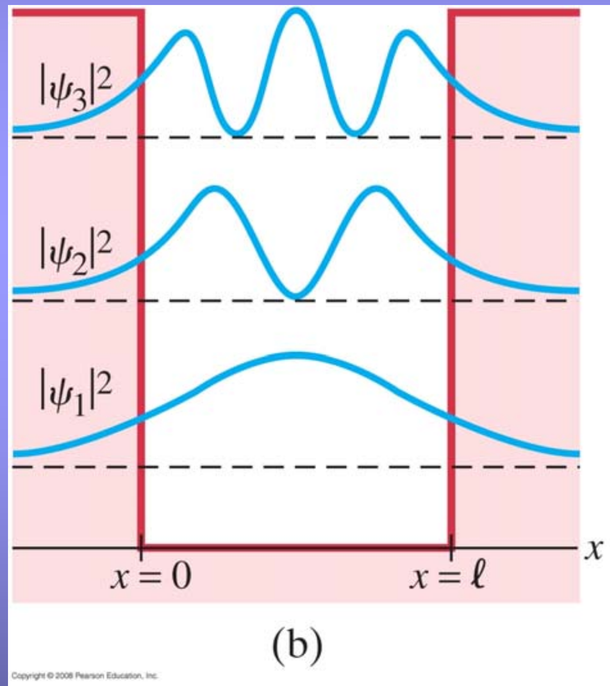
Finite value at $x = 0$ exponentially decaying into the finite walls

Finite Potential Well

These graphs show the wave functions and probability distributions for the first three energy states.

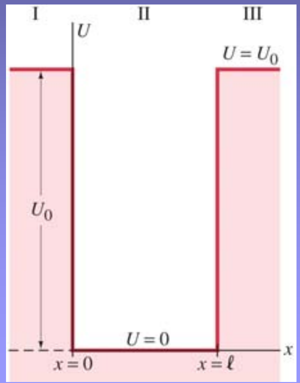
Nonclassical effects





Partile can exist in the forbidden region

Finite Potential Well



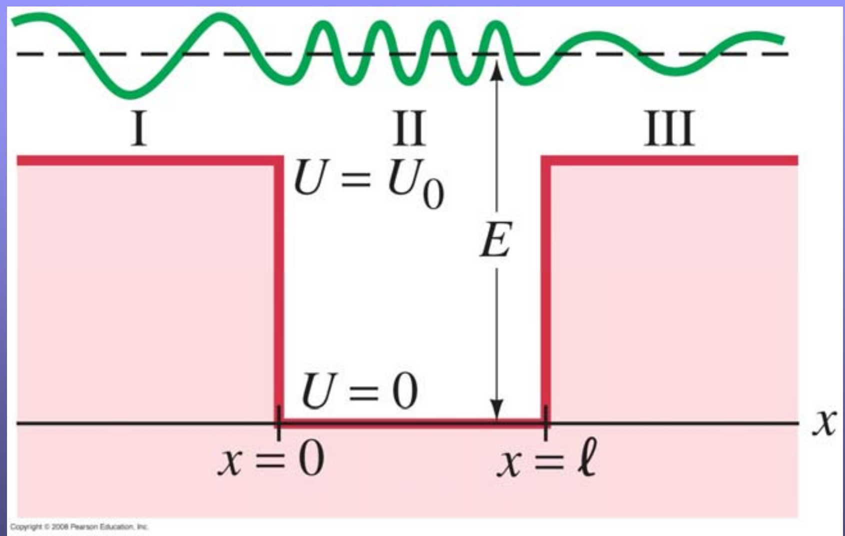
If $E > U_0$ free particle condition

$$\frac{d^2\psi}{dx^2} + \left[\frac{2m(E-U_0)}{\hbar^2}\right]\psi = 0$$

In regions I and III

$$\frac{d^2\psi}{dx^2} + \left[\frac{2mE}{\hbar^2}\right]\psi = 0$$

In region II



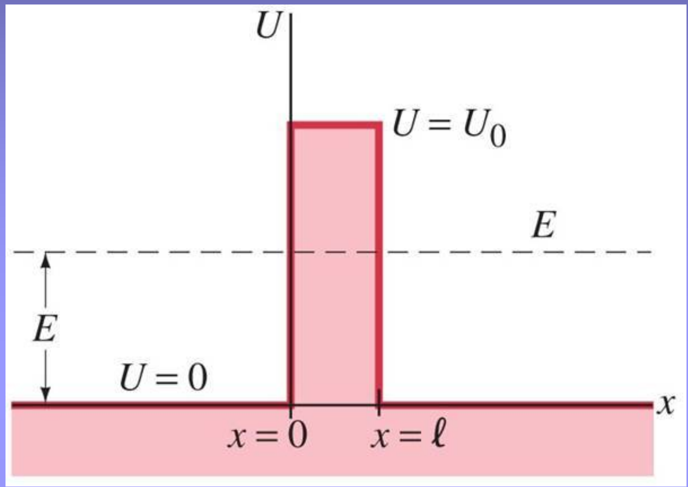
In both cases oscillating free partcile wave function:

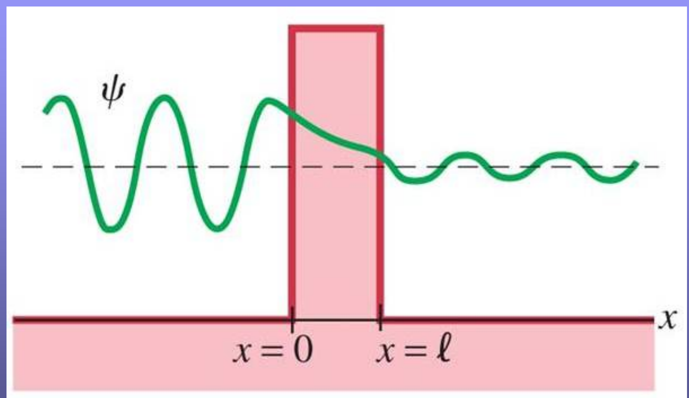
I,III: $$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m(E-U_0)}}$$

II: $$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

$$E = \frac{1}{2}mv^2 + U_0 = \frac{p^2}{2m} + U_0$$

Tunneling Through a Barrier





In region $x < 0$ oscillating wave

$$\frac{d^2\psi}{dx^2} + \left[\frac{2mE}{\hbar^2}\right]\psi = 0 \qquad \lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

Also in region $x > \ell$

Wave with same wavelength $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$

In the barrier:

$$\frac{d^2\psi}{dx^2} - \left[\frac{2m(U_0 - E)}{\hbar^2}\right]\psi = 0 \qquad \psi_b = Ce^{Gx} + De^{-Gx}$$

Approximation: assume that the decaying function is dominant $\psi_b = De^{-Gx}$

Transmission: $$T = \frac{|\psi(x=\ell)|^2}{|\psi(x=0)|^2} = \frac{\left(De^{-Gx}\right)^2}{D^2} = e^{-2G\ell}$$

Tunneling Through a Barrier

The probability that a particle tunnels through a barrier can be expressed as a transmission coefficient, T, and a reflection coefficient, R (where $T + R = 1$). If T is small,

$$T \approx e^{-2Gl}$$

$$G = \sqrt{\frac{2m\left(U_0 - E\right)}{\hbar^2}}$$

The smaller E is with respect to U_0, the smaller the probability that the particle will tunnel through the barrier.