The Special Theory of Relativity

Chapter III

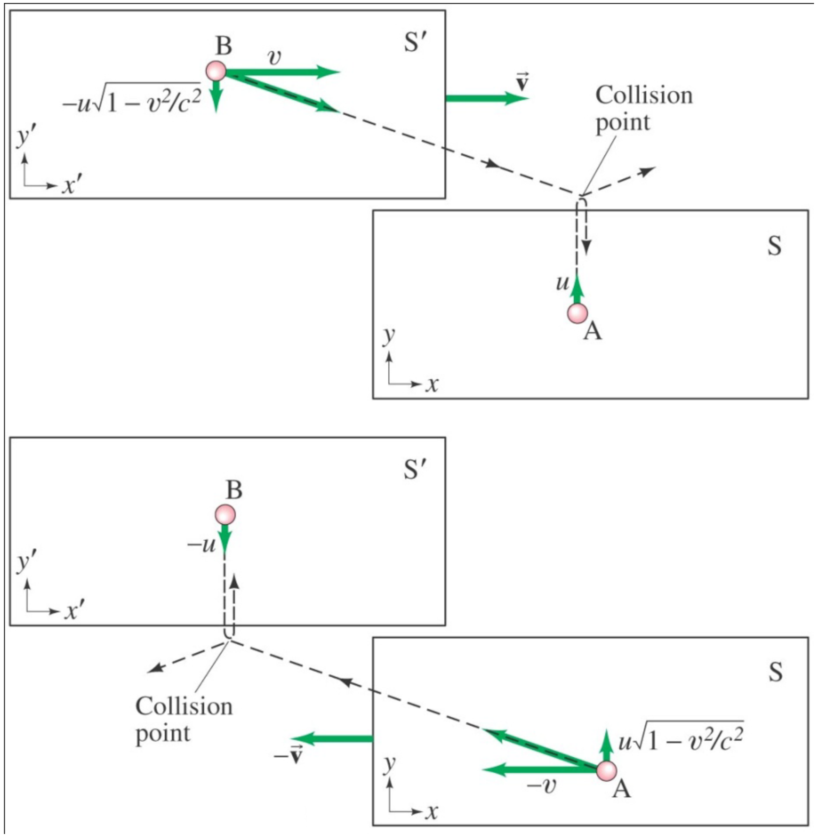
1. Relativistic dynamics
2. Momentum and energy
3. $E=mc^2$
4. Relativistic particle scattering

Excercise

Relativistic Momentum

The formula for relativistic momentum can be derived by requiring that the conservation of momentum during collisions remain valid in all inertial reference frames.

Note: that does NOT mean that the momentum is equal in different reference frames



Result

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = \gamma m\vec{v}$$

Go over this and derive !

Relativistic Force

Newtons second law remains valid (without proof)

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\gamma m\vec{v} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right)$$

1) For every physical law it has to be established how they transform in relativity (under Lorentz transformations)
2) Quantities (like *F*) not the same in reference frames

Relativistic acceleration

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\gamma m\vec{v} = \gamma m\frac{d\vec{v}}{dt} + m\vec{v}\frac{d\gamma}{dt}$$

$$= \gamma m\vec{a} + m\vec{v}\frac{d\gamma}{dt}$$

The force vector does not point in the same direction as the acceleration vector

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} = -\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\cdot\left(\frac{-2v}{c^2}\right) = \gamma^3\frac{v}{c^2}$$

Note: in case of acceleration γ is not constant

Relativistic Mass

From the momentum:

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2 / c^2}} = \gamma m\vec{v}$$

Gamma and the rest mass are combined to form the relativistic mass:

$$m_{rel} = \frac{m}{\sqrt{1 - v^2 / c^2}}$$

Relativistic Energy

Work done to increase the speed of a particle from *v=0* (i-state) to *v=v* (f state):

$$W=\int_i^f F dx=\int_i^f \frac{dp}{dt}dx=\int_i^f \frac{dp}{dt}v dt=\int_i^f v dp=\int_i^f d(pv)-\int_i^f p dv$$

because $vdp=d(pv)-pdv$

$$\int_i^f d(pv)=pv\Big|_i^f=(\gamma mv)v$$

$$-\int_i^f p dv=-\int_0^v \frac{mv}{\sqrt{1-v^2/c^2}}dv=mc^2\sqrt{1-v^2/c^2}\Big|_0^v=mc^2\sqrt{1-v^2/c^2}-mc^2$$

use

$$\frac{d}{dv}\left(\sqrt{1-v^2/c^2}\right)=-\left(\frac{v}{c^2}\right)/\sqrt{1-v^2/c^2}$$

So:

$$W=\gamma mv^2+\frac{mc^2}{\gamma}-mc^2=\frac{\left(\gamma^2v^2+c^2\right)}{\gamma}m-mc^2=(\gamma-1)mc^2$$

Kinetic energy of the particle is

$$K=(\gamma-1)mc^2$$

1) Amount of kinetic energy depends on inertial frame

2) Amount of kinetic energy reduces to classical value at low *v*

3) Note

$$K\neq\frac{1}{2}mv^2$$

Mass and Energy

The kinetic energy $K = (\gamma - 1)mc^2$

Can be written as the total energy: $E = \gamma mc^2 = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$

Where the difference is the rest energy: $E = mc^2$

The last equation is Einstein famous equation
implying that mass is equivalent to energy
The energy of a particle at rest.

Note that mc^2 is the same as seen from all reference frames;
It is an *invariant* upon frame transformation

Mass, Energy, Momentum

Energy $E = \gamma m c^2$ **Momentum** $p = \gamma m v$

Combining these relations gives

$$E^2 = p^2 c^2 + m^2 c^4$$

Hence also the following
Is an invariant under Lorentz
transformations

$$E^2 - p^2 c^2$$

Mass, Energy, Momentum for light particles

Light particles have no "rest" mass ($m=0$), but they have energy

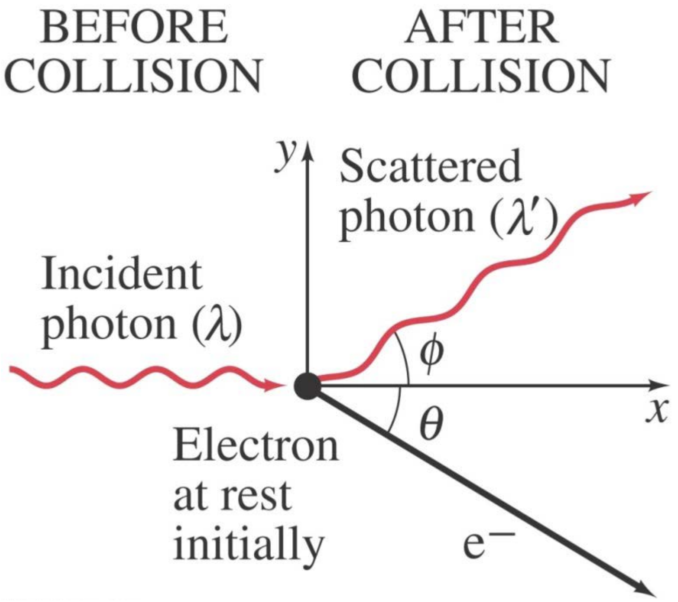
$$E^2 = p^2c^2 + m^2c^4$$ **hence** $$p = \frac{E}{c}$$

Energy in the quantum picture $$E = h\nu = \frac{hc}{\lambda}$$

Hence momentum $$p = \frac{E}{c} = \frac{h}{\lambda}$$

Compton scattering

A photon (is a light particle) collides with an electron and its energy (so its wavelength) must change !!



Before collision

photon $E = hf = \frac{hc}{\lambda}$ $p = \frac{h}{\lambda}$

electron $E_e = m_e c^2$

After collision

photon $E' = \frac{hc}{\lambda'}$ $p' = \frac{h}{\lambda'}$

electron $E_{tot}^{e} = \gamma m_e c^2$ $p_e = \gamma m_e v$

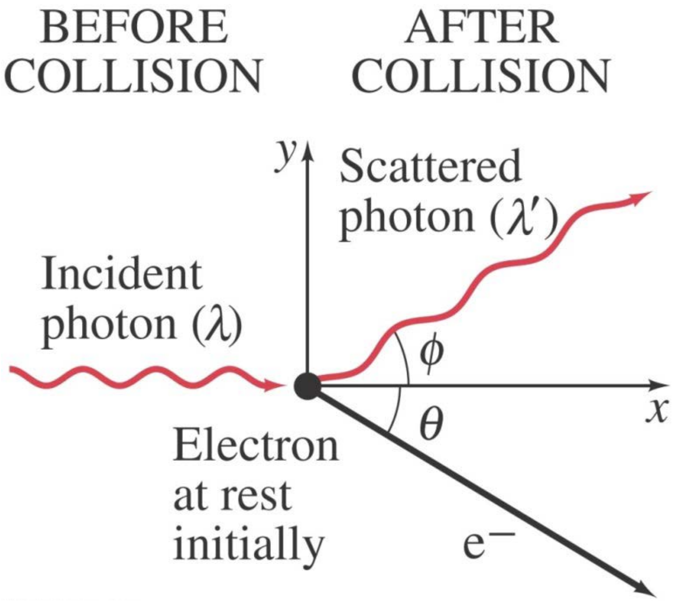
$E_{kin}^{e} = (\gamma - 1) m_e c^2$

Write the momentum conservation equations along the x-coordinate and along the y-coordinate.

Write the energy conservation equation.

Then solve the equations and determine the wavelength λ' for angle ϕ.

Compton scattering



Conservation of energy

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + (\gamma - 1)m_e c^2$$

Conservation of momentum

Along x: $$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\phi + \gamma m_e v \cos\theta$$

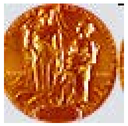
Along y: $$0 = \frac{h}{\lambda'}\sin\phi - \gamma m_e v \sin\theta$$

Three equations with 3 unknowns, eliminate v and θ

Compton scattering:

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\phi)$$

Compton scattering



Arthur Compton

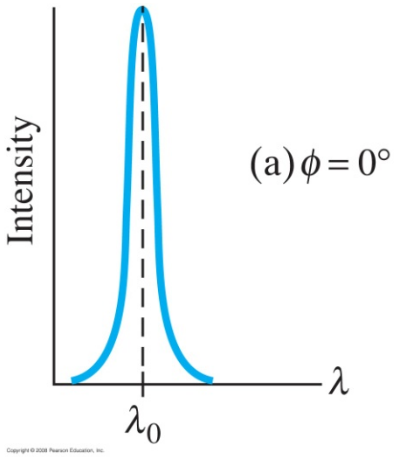
The Nobel Prize in Physics 1927 "for his discovery of the effect named after him"

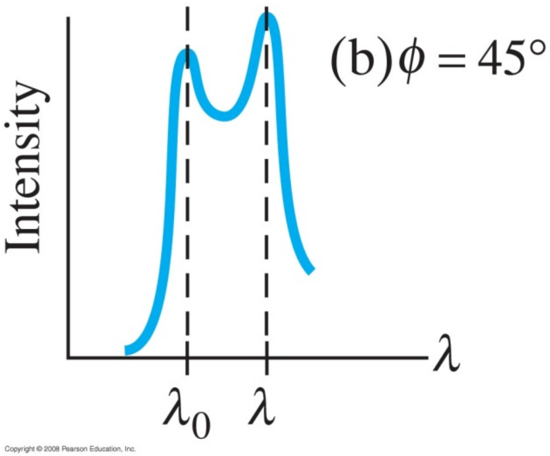
$$\Delta\lambda = \frac{h}{m_e c}(1-\cos\phi) = \lambda_C(1-\cos\phi)$$

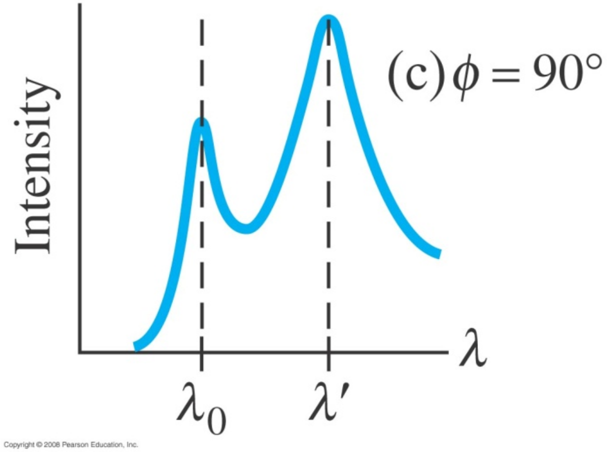
Note that λ_C ~ 0.00243 nm

So the effects is not so well visible with visible light

Compton performed his experiment with x-rays

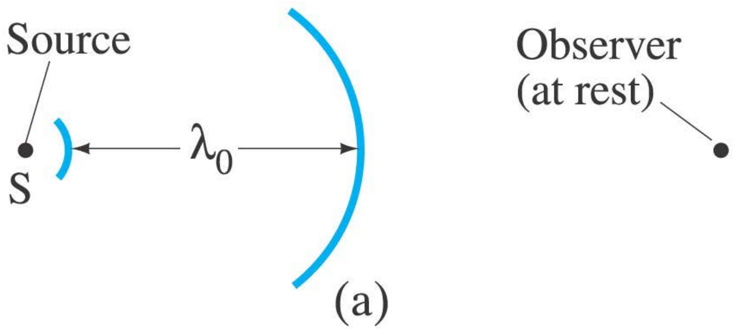




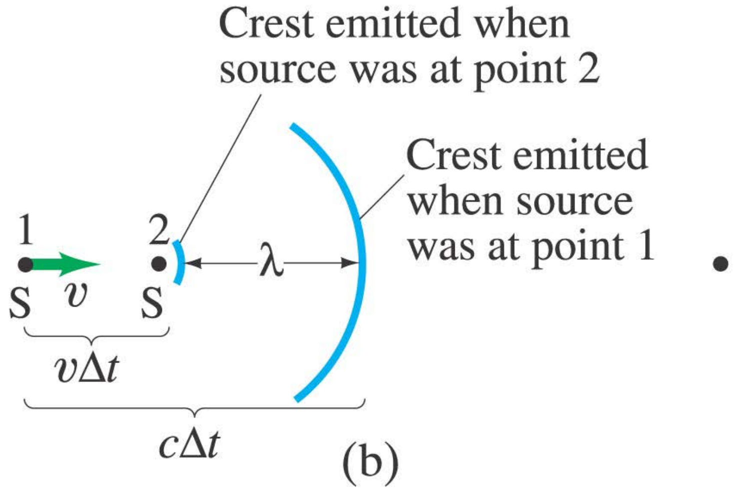


Doppler Shift for Light

The Doppler shift for light for c=constant in all inertial frames.



(a)



(b)

$$\lambda_0 = c\Delta t_0 = c / f_0$$

Two effects in relativistic Doppler

1) Moving waves + 2) Time dilation!!

1) $\lambda = c\Delta t - v\Delta t$

2) $\Delta t = \Delta t_0 / \sqrt{1 - v^2 / c^2}$

$$\lambda = (c - v)\Delta t = \frac{(c - v)\Delta t_0}{\sqrt{1 - v^2 / c^2}} = \lambda_0 \sqrt{\frac{c - v}{c + v}}$$

When source moving toward observer

Doppler Shift for Light

Hence, one can derive the observed frequency and wavelength:

$$\lambda = \lambda_0 \sqrt{\frac{c - v}{c + v}}. \quad \left[\begin{array}{c}\text{source and observer}\\ \text{moving toward}\\ \text{each other}\end{array}\right]$$

$$f = \frac{c}{\lambda} = f_0 \sqrt{\frac{c + v}{c - v}}. \quad \left[\begin{array}{c}\text{source and observer}\\ \text{moving toward}\\ \text{each other}\end{array}\right]$$

If the source and observer are moving away from each other, *v* changes sign.

Remember: higher pitch, blue shift when moving toward each other

Doppler Shift for Light

Speeding through a red light.

A driver claims that he did not go through a red light because the light was Doppler shifted and appeared green. Calculate the speed of a driver in order for a red light to appear green.

$$\lambda = \lambda_0 \sqrt{\frac{c - v}{c + v}}$$

λ = 500nm ; λ_0 =650 nm

v **= 0.26** ***c***