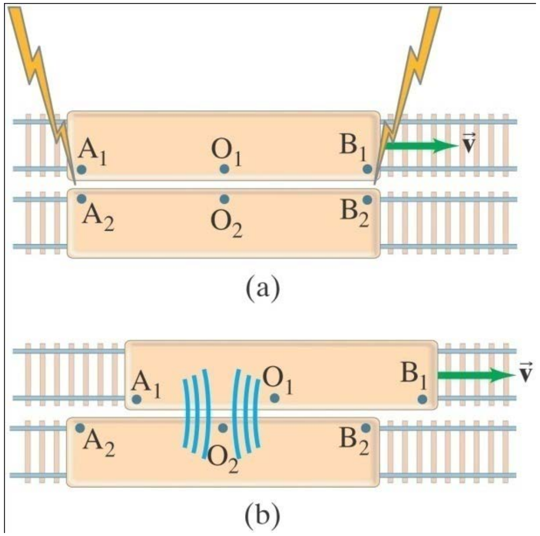
The Special Theory of Relativity

Chapter II

1. Relativistic Kinematics
2. Time dilation and space travel
3. Length contraction
4. Lorentz transformations
5. Paradoxes ?

Simultaneity/Relativity



If one observer sees the events as simultaneous, the other cannot, given that the speed of light is the same for each.

Conclusions:

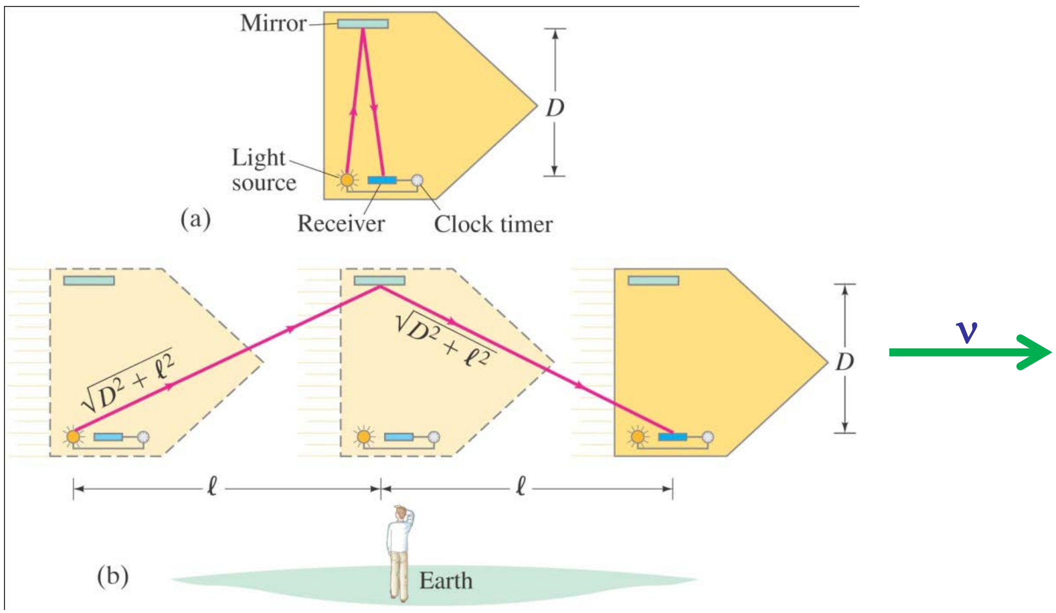
Simultaneity is not an absolute concept

Time is not an absolute concept

It is relative

Time Dilation

How much time does it take for light to Travel up and down in the space ship?



a) Observer in space ship:

$$\Delta t_0 = \frac{2D}{c}$$ **proper time**

b) Observer on Earth:

speed c is the same

apparent distance longer

$$2\ell = v\Delta t$$

Light along diagonal:

$$c = \frac{2\sqrt{D^2+\ell^2}}{\Delta t} = \frac{2\sqrt{D^2+v^2\Delta t^2/4}}{\Delta t}$$

$$\Delta t = \frac{2D}{c\sqrt{1-v^2/c^2}}$$

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} = \gamma\Delta t_0}$$

This shows that moving observers must disagree on the passage of time.

Clocks moving relative to an observer run more slowly as compared to clocks at rest relative to that observer

Time Dilation

Calculating the difference between clock “ticks,” we find that the interval in the moving frame is related to the interval in the clock’s rest frame:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2 / c^2}}$$

Δt_0 is the proper time (in the co-moving frame)

It is the shortest time an observer can measure

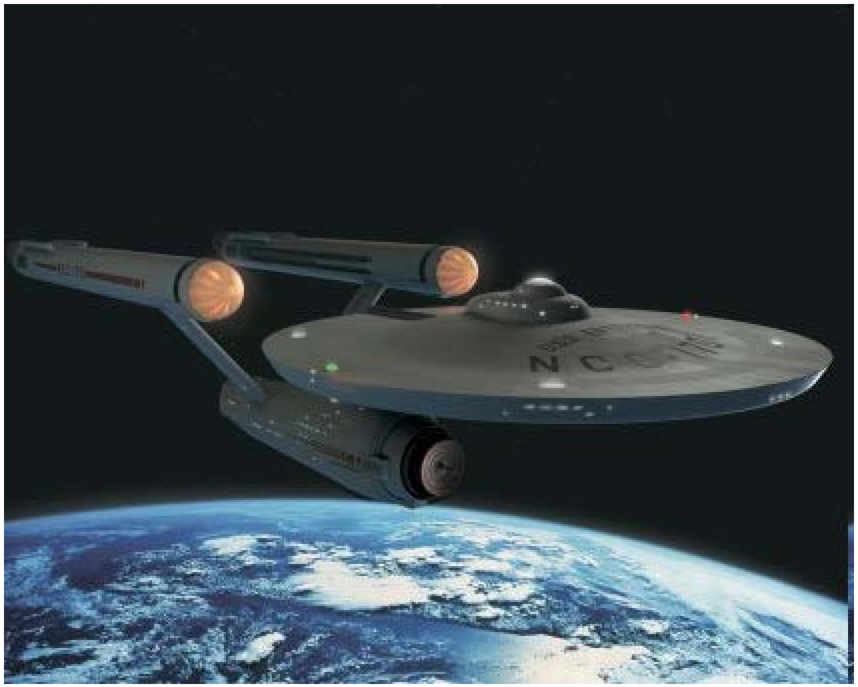
with $\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$ **then** $\Delta t = \gamma \Delta t_0$

Applications:

Lifetimes of muons in the Earth atmosphere

Time dilation on atomic clocks in GPS (v=4 km/s; timing “error” 10^{-10} s)

On Space Travel



100 light years ~ 10^{16} m

If space ship travels at v=0.999 c
then it takes ~100 years to travel.

But in the rest frame of the carrier:

$$\Delta t_0 = \Delta t\sqrt{1-v^2/c^2} \approx 4.5yr$$

The higher the speed the faster you get there;
But not from our frame perspective !

Twin Paradox

Question:

On her 21^{st} birthday an astronaut takes off in a rocket ship at a speed of 12/13 c. After 5 years elapsed on her watch, she turns around and heads back to rejoin with her twin brother, who stayed at home.

How old is each twin at the reunion ?

Twin Paradox

Solution:

The traveling twin has traveled for 5+5=10 years so she will be 31.

As viewed from earth the traveling clock has moved slower by a factor:

$$\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}} = \frac{1}{\sqrt{1 - (12/13)^2}} = \frac{13}{5}$$

So the time elapsed on Earth is 26 years, and her brother will be celebrating his 47th birthday.

Note that the traveling twin has really spent only 10 years of her life.
She has not lived more, her clock ticked slower. Time really has evolved slower.

Twin Paradox

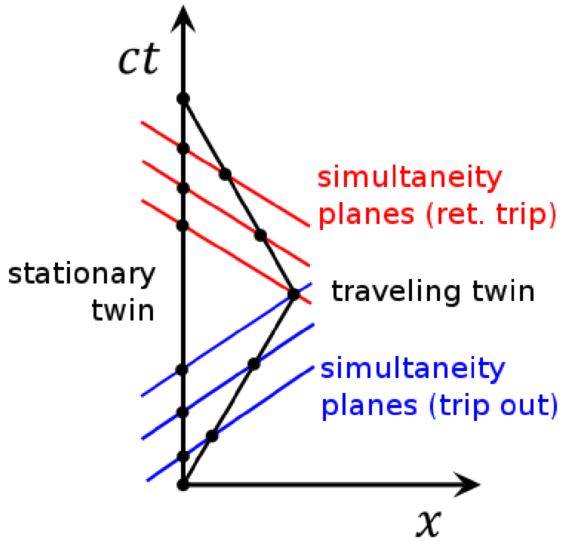
Where is the real paradox?

Think about the problem from the perspective of sister who sees
the Earth moving in her frame of reference, with the consequence that
the time in her brothers frame should evolve more slowly.
Why isn't the brother "younger" ?

The two twins are not equivalent !

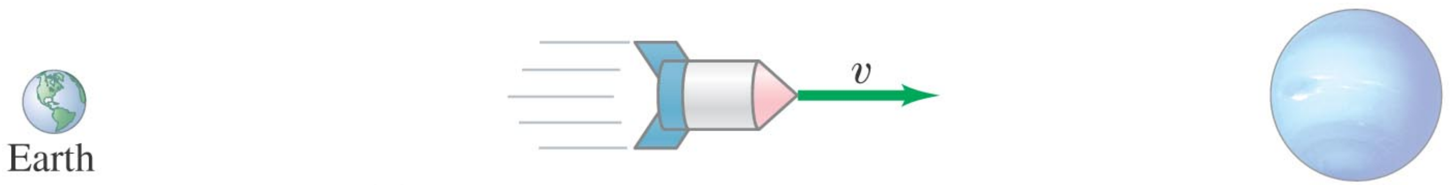
1) The sister is not in an inertial frame of reference ! Well she is, but two times in a different one (von Laue)

2) The space ship turns around which requires acceleration (Langevin)



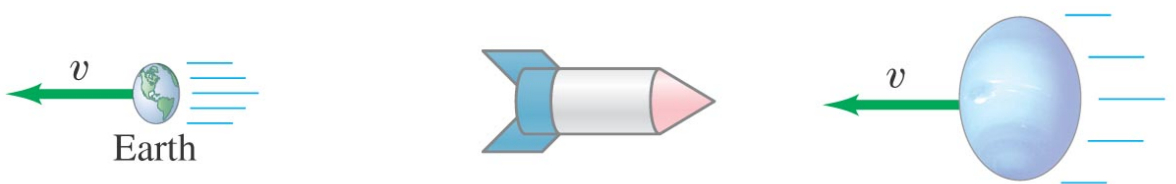
Can be explained by a Minkowski diagram

Length Contraction



(a) Earth at rest

Distance between planets is: ℓ_0 **Time for travel:** $\Delta t = \dfrac{\ell_0}{v}$ **Earth observer**



(b) Spacecraft at rest

Time dilatation

$$\Delta t_0 = \Delta t\sqrt{1 - v^2/c^2} = \frac{\Delta t}{\gamma}$$

Proper time

Space craft observers measure the same speed but less time

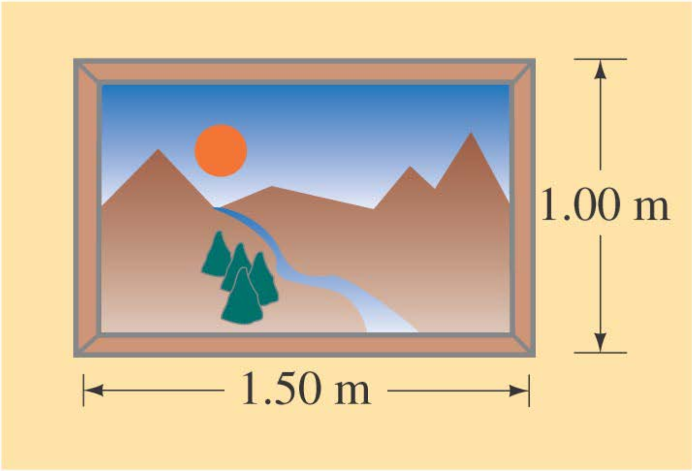
$$\ell = v\Delta t_0 \longrightarrow \ell = \frac{\ell_0}{\gamma}$$

Contraction only along The direction of motion

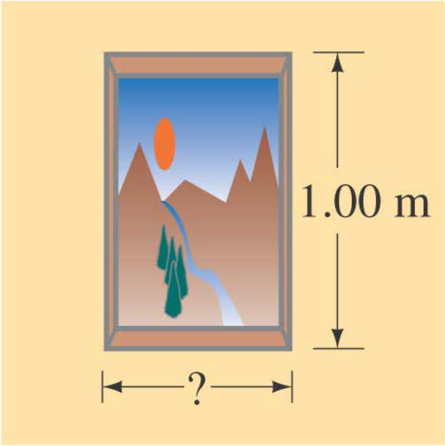
Length Contraction

Only observed in the direction of the motion.

No contraction, or dilation in perpendicular direction



(a)



(b)

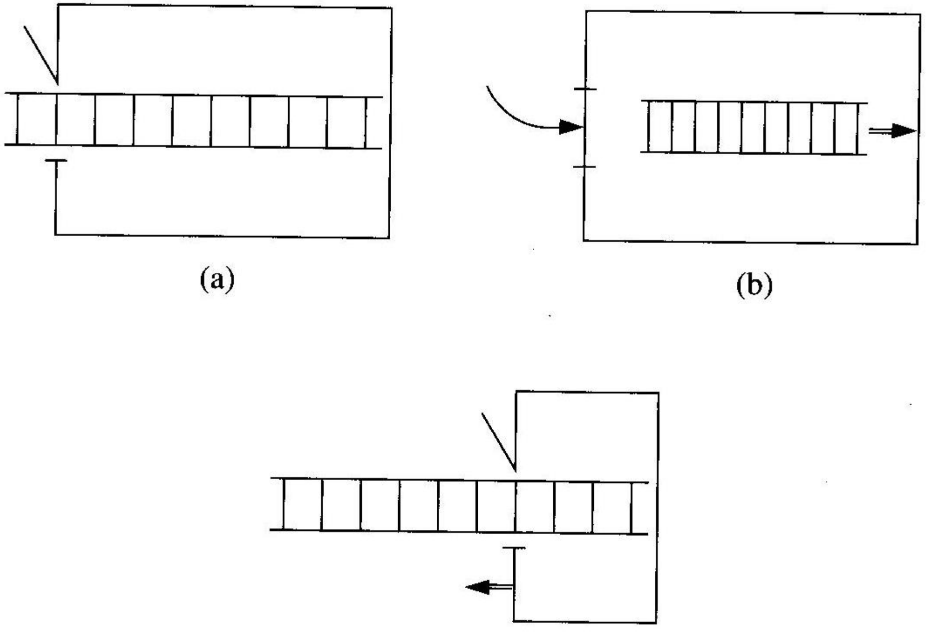
Excercise

The Barn and Ladder Paradox

There once was a farmer who had a ladder too long to store in his barn. He read some relativity and came up with the following idea. He instructed his daughter to run with the ladder fast, such that the ladder would Lorentz contract to fit in the barn. When through the farmer intended to slam the door and hold the ladder fixed inside.

The daughter however pointed out that (in her frame of reference) the barn, and not the ladder would contract, and the fit would be even worse.

Who is right ?



(a) (b) (c)

See: the fantasy train in 36.6 & 36.7

Lorentz Transformations

In relativity, assume a linear transformation:

$$x = \gamma(x' + vt') \qquad y = y' \qquad z = z'$$

γ as a constant to be determined (γ=1 classically).
Inverse transformation with $v \rightarrow -v$

$$x' = \gamma(x - vt)$$

Consider light pulse at common origin of S and S' at $t=t'=0$
measure the distance in $x=ct$ and $x'=ct'$:

$$x' \equiv ct' = \gamma(x - vt) = \gamma(ct - vt) = \gamma(c - v)t \longrightarrow t' = \gamma \frac{(c - v)}{c} t$$

$$x \equiv ct = \gamma(x' + vt') = \gamma(ct' + vt') = \gamma(c + v)t'$$

fill in

Transformation parameter $\qquad \gamma = \dfrac{1}{\sqrt{1 - v^2 / c^2}}$

Lorentz Transformations

Solve further: $$x' = \gamma\left(x - vt\right) = \gamma\left(\gamma\left(x' + vt'\right) - vt\right)$$

Leading to the transformations:

$$x' = \gamma\left(x - vt\right)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

Time dilation and length contraction can be derived
From these Lorentz transformations

Test invariance of Maxwell equations under Lorentz Transformations

Coordinates

$$x' = \gamma\left(x - vt\right)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Partial Derivatives

$$\frac{\partial x'}{\partial x} = \gamma \qquad \frac{\partial x'}{\partial t} = -\gamma v \qquad \frac{\partial t'}{\partial t} = \gamma \qquad \frac{\partial t'}{\partial x} = -\gamma \frac{v}{c^2}$$

Spatial, first

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x} = \gamma \frac{\partial E}{\partial x'} - \gamma \frac{v}{c^2}\frac{\partial E}{\partial t'}$$

Spatial, second

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x}\left(\gamma \frac{\partial E}{\partial x'} - \gamma \frac{v}{c^2}\frac{\partial E}{\partial t'}\right) =$$

$$= \frac{\partial}{\partial x'}\left(\gamma \frac{\partial E}{\partial x'} - \gamma \frac{v}{c^2}\frac{\partial E}{\partial t'}\right)\frac{\partial x'}{\partial x} + \frac{\partial}{\partial t'}\left(\gamma \frac{\partial E}{\partial x'} - \gamma \frac{v}{c^2}\frac{\partial E}{\partial t'}\right)\frac{\partial t'}{\partial x} =$$

$$= \gamma^2 \frac{\partial^2 E}{\partial x'^2} - \frac{2v}{c^2}\gamma^2 \frac{\partial^2 E}{\partial t' \partial x'} + \gamma^2 \frac{v^2}{c^4}\frac{\partial^2 E}{\partial t'^2}$$

Evaluate similarly the temporal term and test invariance of Maxwell's wave equation

Excercise

The addition of velocities in reference frames
I. longitudinal

Observer in frame S determines speed u_x of object in S' (x', t', u_x')

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Derivatives: of coordinates

$$u_x = \frac{dx}{dt} \qquad u_x' = \frac{dx'}{dt'} \qquad \frac{dt'}{dt} = 1 \Big/ \frac{dt}{dt'} = \frac{1}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)}$$

$$\frac{dt}{dt'} = \frac{d}{dt'}\left(\gamma\left(t' + \frac{vx'}{c^2}\right)\right) = \gamma + \gamma\frac{v}{c^2}u_x' = \gamma\left(1 + \frac{vu_x'}{c^2}\right)$$

Then:

$$u_x = \frac{dx}{dt} = \frac{d}{dt}[\gamma(x' + vt')] = \frac{d}{dt'}[\gamma(x' + vt')]\frac{dt'}{dt} = (\gamma u_x' + \gamma v)\cdot\left(\frac{1}{\gamma\left(1 + \frac{vu_x'}{c^2}\right)}\right)$$

$$\boxed{u_x = \frac{u_x' + v}{\left(1 + \frac{vu_x'}{c^2}\right)}}$$

Excercise

The addition of velocities in reference frames
II. Transversal

Observer in frame S determines speed u_x of object in S' (x', t',u_x')

$$y = y'$$

Derivatives:

$$u_y = \frac{dy}{dt} \qquad u_y' = \frac{dy'}{dt'} \qquad \frac{dt'}{dt} = 1\Big/\frac{dt}{dt'} = \frac{1}{\gamma\left(1+\frac{vu_x'}{c^2}\right)}$$

Then:

$$u_y = \frac{dy}{dt} = \frac{dy'}{dt} = \frac{dy'}{dt'}\frac{dt'}{dt} = u_y' \cdot \left(\frac{1}{\gamma\left(1+\frac{vu_x'}{c^2}\right)}\right)$$

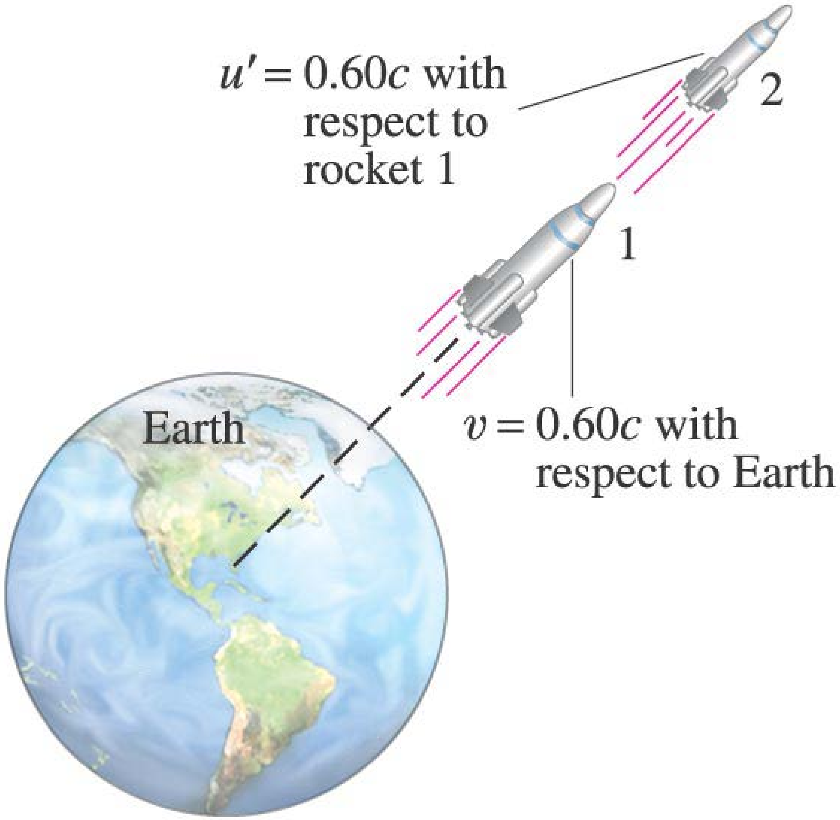
So also u_y and u_z transform; this has to do with the transformation (non-absoluteness) of time

$$\frac{dt'}{dt} \neq 1$$

Excercise

Lorentz Transformations

Calculate the speed of rocket 2 with respect to Earth.

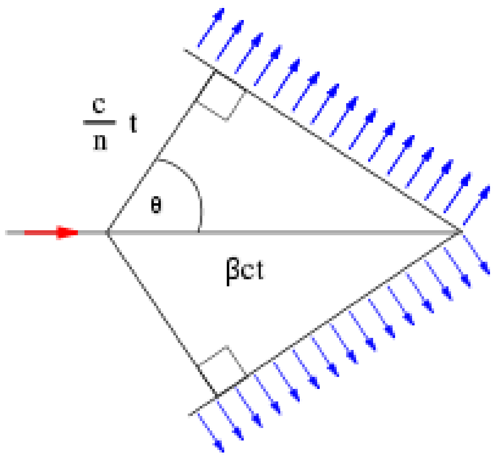


$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}} = 0.88c$$

This equation also yields as result that c is the maximum obtainable speed (in any frame).

Faster than the speed of light ? Cherenkov radiation

A blue light cone



Pavel Cherenkov
Nobel Prize 1958

Particle travels at $x_p = \beta ct$

Waves emitted as (spherical) $x_e = \frac{c}{n}t$

Emittance cone: $\cos\theta = \frac{1}{n\beta}$

Application in the ANTARES detector