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Focal shifts of converging diffracted waves of any state of spatial coherence

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Abstract

We analyze the focusing of wave fields of any state of coherence by systems with low Fresnel numbers. We study the optical intensity on the axis in the focal region. The dependence of the focal shift and of the maximum on-axis intensity on the state of coherence is examined for some model fields. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The usual treatment of the focusing of light [1, Chapter 8.8] assumes both that the field is fully coherent and that the focusing system has a Fresnel number which is much greater than unity. Under these two conditions the so-called Debye approximation is valid, and the intensity distribution is found to be symmetrical about the geometrical focal plane.

For low Fresnel-number systems the Debye approximation is not valid [2]. In such focusing configurations the *focal shift* phenomenon appears, i.e. the maximum intensity is no longer at the geometrical focus, but occurs at a point which is located closer to the aperture [3–5].

The focusing of partially coherent light has been the subject of relatively few studies. Wang et al. [6] generalized the Debye theory to include the focusing of partially coherent light by high Fresnel-number systems. Some other papers (e.g. Refs. [7,8]) have dealt with the related subject of imaging of Gaussian Schell-model sources. In the present paper we examine the focusing of partially coherent wave fields by systems with low Fresnel numbers. It is found that in such wave fields the focal shift also occurs, but that it depends not only on the Fresnel number of the focusing system, but also on the state of coherence of the light. One also finds that the maximum spectral intensity on-axis can exceed the spectral intensity at the geometrical focus of a fully coherent focused beam. Some related results were reported by Lü et al. [9].

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2. Partially coherent focused fields

To begin with, let us consider a monochromatic converging spherical wave diffracted at a circular aperture of radius a. The focusing configuration is depicted in Fig. 1. We assume that

$$f \gg a \gg \lambda, \tag{1}$$

where f is the radius of the spherical wave in the aperture, and λ is the wavelength. We note that these assumptions alone do *not* imply that the Fresnel number $N = a^2/\lambda f$ is necessarily small.

A monochromatic field $U^{(0)}$ in the aperture on a sphere S of radius f centered at the geometrical focus F is given by (suppressing a periodic time-dependent factor $\exp[-\mathrm{i}2\pi\nu t]$)

$$U^{(0)}(Q) = A(\boldsymbol{\rho})\frac{\mathrm{e}^{-\mathrm{i}kf}}{f}. \tag{2}$$

Here A is the amplitude, $\boldsymbol{\rho} = (x, y)$ is the two-dimensional transverse vector specifying the position of the point Q on S, and $k = 2\pi\nu/c = 2\pi/\lambda$, with ν denoting the frequency, and c the speed of light in vacuum. The focused field at a point P along the z-axis is, according to the Huygens–Fresnel principle, given by the expression [1, Chapter 8.2]

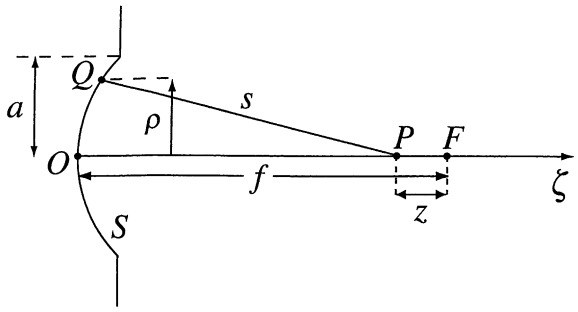

Fig. 1. The focusing geometry. S is a reference sphere in the aperture, centered at the geometrical focus, a is the radius of the aperture, assumed to be circular, and f denotes the distance from any point on the reference sphere to the geometrical focus F. P is a point on the z-axis, and Q a point on the sphere S in the aperture. s is the distance QP. (ξ, η, ζ) are coordinates with respect to a right-handed Cartesian coordinate system with origin O, whereas (x, y, z) are coordinates with respect to a parallel Cartesian coordinate system with origin F, with the ζ- and z-axis taken along the axis of rotational symmetry.

$$U(P) = \frac{-\mathrm{i}k}{2\pi}\int_S U^{(0)}(Q)\frac{e^{\mathrm{i}ks}}{s}\mathrm{d}S, \tag{3}$$

where the integration extends over the portion S of the spherical wave which fills the aperture, and s denotes the distance QP.

For a partially coherent wave field one must consider, instead of the field $U^{(0)}(Q)$, the cross-spectral density function of the field at points Q_1, Q_2 on the spherical surface S, viz.

$$W^{(0)}(Q_1, Q_2, \nu) = W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu) \equiv \langle U^{(0)^*}(\boldsymbol{\rho}_1, \nu)U^{(0)}(\boldsymbol{\rho}_2, \nu)\rangle, \tag{4}$$

where the angular brackets denote the average taken over a statistical ensemble of monochromatic realizations $\{U(\boldsymbol{\rho})\mathrm{e}^{-\mathrm{i}2\pi\nu t}\}$ [10, Section 4.7]. The cross-spectral density of the focused field at two axial field points $P(\boldsymbol{r}_1)$ and $P(\boldsymbol{r}_2)$ is given by the formula

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \nu) = \langle U^*(\boldsymbol{r}_1, \nu)U(\boldsymbol{r}_2, \nu)\rangle, \tag{5}$$

$$= \left(\frac{k}{2\pi}\right)^2 \int_S\int_S W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu) \times \frac{\mathrm{e}^{-\mathrm{i}k(s_1-s_2)}}{s_1 s_2}\mathrm{d}S_1\,\mathrm{d}S_2. \tag{6}$$

From now on we suppress the dependence of the various quantities on the frequency ν. We suppose that the field on the sphere S is a Schell-model field [10, Section 5.3.2] with a spectral degree of coherence that is Gaussian, i.e.

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{1}{f^2}\langle A^*(\boldsymbol{\rho}_1)A(\boldsymbol{\rho}_2)\rangle, \tag{7}$$

$$= \frac{1}{f^2}|A(\boldsymbol{\rho}_1)||A(\boldsymbol{\rho}_2)|\mathrm{e}^{-(\rho_2-\rho_1)^2/2\sigma_g^2}, \tag{8}$$

where Eq. (2) was used. The parameter σ_g denotes a positive constant which is a measure of the effective spectral coherence length of the field in the aperture.

The (spectral) intensity of the focused field at an axial point $\boldsymbol{r}$ is given by the expression

$$I(\boldsymbol{r}) = W(\boldsymbol{r}, \boldsymbol{r}). \tag{9}$$

We assume that on the sphere S in the aperture, the spectral intensity of the field is independent of position, i.e. that

$$|A(\boldsymbol{\rho}_1)||A(\boldsymbol{\rho}_2)|/f^2 = A_0^2/f^2 \tag{10}$$

(cf. Eq. (2)).

Next, we will use approximate expressions for the factors s_1 and s_2 which appear in Eq. (6). Let (ξ, η, ζ) be the coordinates of a point Q on the sphere S with respect to a Cartesian coordinate system with origin O and with the ζ-direction along the axis of rotational symmetry, and let $f + z$ be the distance OP (see Fig. 1). Then

$$s = [\rho^2 + (f + z - \zeta)^2]^{1/2}, \tag{11}$$

with

$$\rho^2 = \xi^2 + \eta^2. \tag{12}$$

Now for all points on S we have

$$\rho^2 + (f - \zeta)^2 = f^2. \tag{13}$$

Because we assumed that $f \gg a$ it follows from Eq. (13) that

$$\zeta \approx \rho^2/2f. \tag{14}$$

On substituting from Eq. (14) into Eq. (11) and neglecting terms of order $\rho^4/4f^2$ which may be assumed to be small compared to $|z|\rho^2/f$, we obtain the approximation

$$s = s(\rho) \approx [(f + z)^2 - z\rho^2/f]^{1/2}. \tag{15}$$

Next we substitute from Eqs. (10) and (15) into Eq. (9). We then obtain for the axial intensity at a distance z from the geometrical focus the expression

$$I(z) = \left(\frac{kA_0}{2\pi f}\right)^2 \int_0^{2\pi}\int_0^a\int_0^{2\pi}\int_0^a \mathrm{e}^{-(\rho_2^2+\rho_1^2-2\rho_1\rho_2\cos[\phi_1-\phi_2])/2\sigma_g^2} \times \mathrm{e}^{-\mathrm{i}k[s_1(\rho_1)-s_2(\rho_2)]} \frac{\rho_1\rho_2}{s_1(\rho_1)s_2(\rho_2)} \mathrm{d}\phi_1\,\mathrm{d}\rho_1\,\mathrm{d}\phi_2\,\mathrm{d}\rho_2, \tag{16}$$

where we used the relation $\mathrm{d}x\,\mathrm{d}y = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$. We note that in Eq. (16) only one factor depends on ϕ_1 and ϕ_2. We therefore consider the integral

$$\mathscr{A} = \int_0^{2\pi}\int_0^{2\pi} \mathrm{e}^{a\cos(\phi_1-\phi_2)}\mathrm{d}\phi_1\,\mathrm{d}\phi_2. \tag{17}$$

On changing variables from ϕ_1 and ϕ_2 to

$$\beta = \phi_1 - \phi_2, \tag{18}$$

$$\gamma = \phi_2, \tag{19}$$

and using the relation $\mathrm{d}\beta\,\mathrm{d}\gamma = \mathrm{d}\phi_1\,\mathrm{d}\phi_2$, it follows that

$$\mathscr{A} = 2\pi \int_{-\phi_2}^{2\pi-\phi_2} \mathrm{e}^{a\cos\beta}\,\mathrm{d}\beta, \tag{20}$$

$$= 2\pi \int_0^{2\pi} \mathrm{e}^{a\cos\beta}\,\mathrm{d}\beta, \tag{21}$$

$$= 4\pi \int_0^{\pi} \mathrm{e}^{a\cos\beta}\,\mathrm{d}\beta. \tag{22}$$

The integral in Eq. (22) is well known and has the value [11]

$$\mathscr{A} = 4\pi^2\mathscr{I}_0(a), \tag{23}$$

with $\mathscr{I}_0$ being the modified Bessel function of order zero. On substituting from Eq. (23) into Eq. (16) and using the relation $a = \rho_1\rho_2/\sigma_g^2$, we find that

$$I(z) = \left(\frac{kA_0}{f}\right)^2 \int_0^a\int_0^a \mathrm{e}^{-(\rho_2^2+\rho_1^2)/2\sigma_g^2}\mathscr{I}_0\left(\frac{\rho_1\rho_2}{\sigma_g^2}\right) \times \mathrm{e}^{-\mathrm{i}k[s_1(\rho_1)-s_2(\rho_2)]}\frac{\rho_1\rho_2}{s_1(\rho_1)s_2(\rho_2)}\mathrm{d}\rho_1\,\mathrm{d}\rho_2. \tag{24}$$

Since the intensity is real valued, Eq. (24) can be simplified to the form

$$I(z) = \left(\frac{kA_0}{f}\right)^2 \int_0^a\int_0^a \mathrm{e}^{-(\rho_2^2+\rho_1^2)/2\sigma_g^2}\mathscr{I}_0\left(\frac{\rho_1\rho_2}{\sigma_g^2}\right) \times \cos[k[s_1(\rho_1) - s_2(\rho_2)]] \times \frac{\rho_1\rho_2}{s_1(\rho_1)s_2(\rho_2)}\mathrm{d}\rho_1\,\mathrm{d}\rho_2. \tag{25}$$

It is useful to introduce a normalization factor $I_{\mathrm{coh}}(0)$, defined as

$$I_{\mathrm{coh}}(0) = \lim_{\sigma_g\to\infty} I(0) \tag{26}$$

$$= \left(\frac{kA_0}{f}\right)^2 \int_0^a\int_0^a \frac{\rho_1\rho_2}{f^2}\,\mathrm{d}\rho_1\,\mathrm{d}\rho_2, \tag{27}$$

$$= \frac{k^2A_0^2a^4}{4f^4}, \tag{28}$$

$$= \left(\frac{A_0 \pi N}{f}\right)^2. \tag{29}$$

Here we made use of Eq. (15) and the fact that $\mathscr{I}_0(0) = 1$. Evidently, $I_{\mathrm{coh}}(0)$ represents the intensity at the geometrical focus $z = 0$ when the field is fully coherent.

3. Focal shifts and excess intensity

In Fig. 2(a)–(d) the normalized axial intensity distribution $I(z)/I_{\mathrm{coh}}(0)$, calculated from Eqs. (25) and (29), is shown for systems with different Fresnel numbers and for various values of the (scaled) coherence parameter σ_g/a. For all finite values of σ_g/a the peak intensity is seen to be smaller than that for the fully coherent case ($\lim \sigma_g/a \to \infty$). For decreasing values of σ_g/a the peak intensity is found to decrease and the secondary maxima become less pronounced. This result is in agreement with the findings of Ref. [6] where the same trend was found for the focusing of partially coherent cylindrical waves. It is seen that the focal shift increases as the Fresnel number decreases, and depends on the state of coherence of the light.

The curves in Fig. 2 indicate the existence of focal shifts in partially coherent focused wave

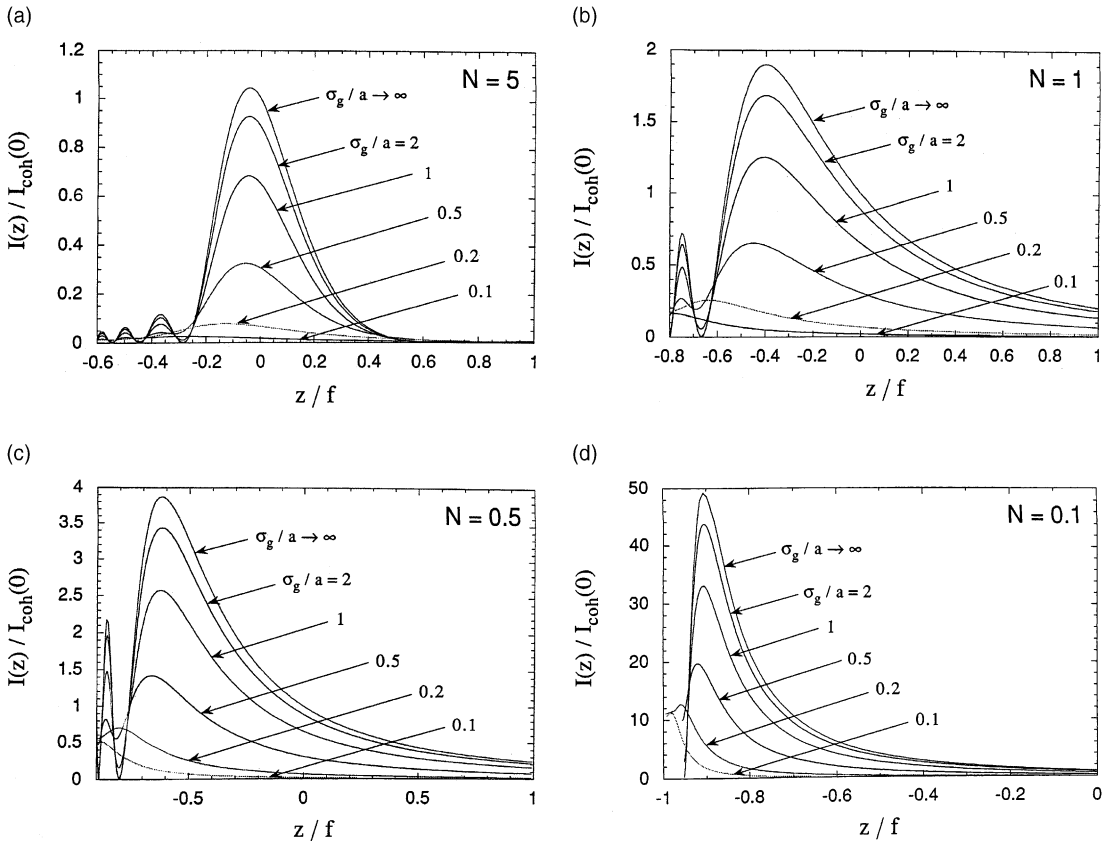

Fig. 2. The normalized axial intensity distribution in partially coherent wave fields, generated by a Gaussian Schell-model field distribution in the aperture, focused by systems with different Fresnel numbers, for various values of the scaled coherence parameter σ_g/a. In (a) the parameters are $f = 1.264$ m and $a = 0.2$ cm; in (b) $f = 6.32$ m and $a = 0.2$ cm; in (c) $f = 3.16$ m and $a = 0.1$ cm; in (d) $f = 15.8$ m and $a = 0.1$ cm. In all the figures $\lambda = 6.328$ nm.

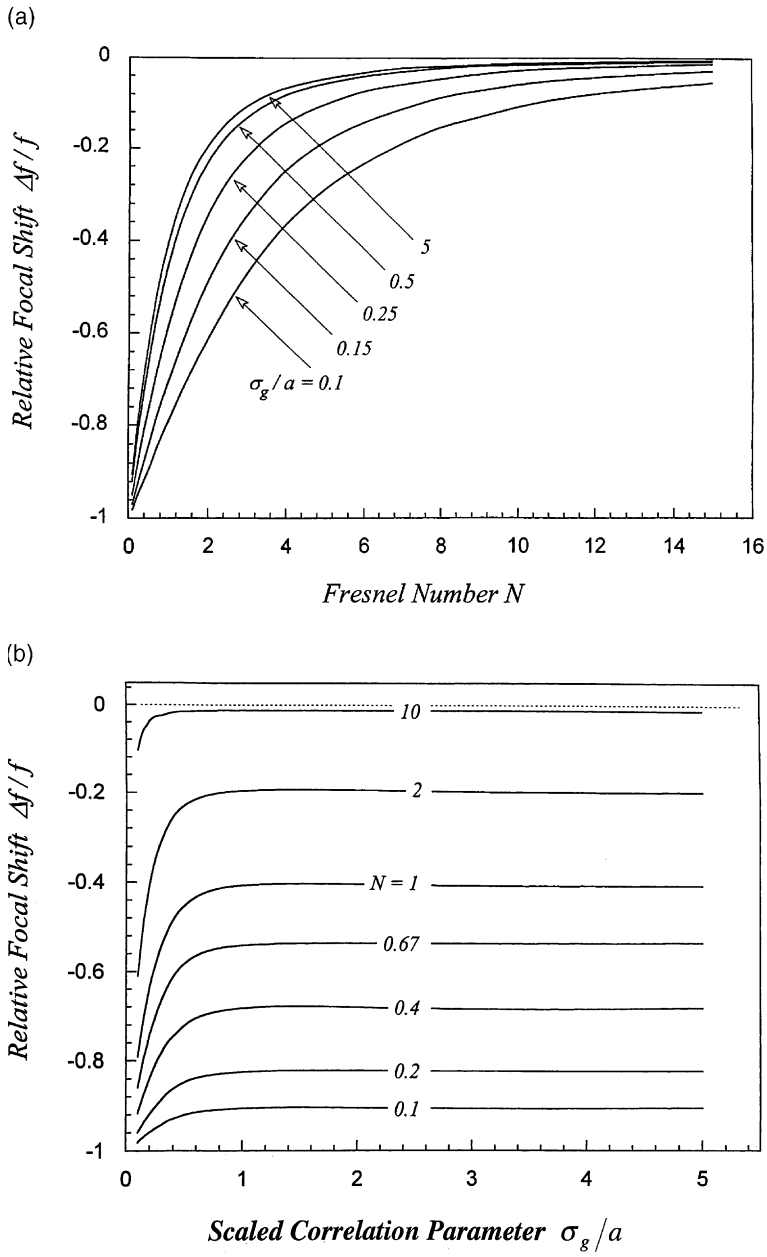

Fig. 3. The relative focal shift $\Delta f/f$ of focused waves (a) as a function of the Fresnel number N for various values of the scaled coherence parameter σ_g/a, and (b) as a function of σ_g/a for selected values of N.

fields when the Fresnel number N is sufficiently small. A more quantitative measure of the focal shift is evident from Fig. 3; more specifically, the relative focal shift $\Delta f/f$, where Δf is the focal shift (i.e. the distance of the location of maximum on-axis intensity from the geometrical focus), is shown in Fig. 3(a) as a function of N, for several values of the scaled coherence parameter σ_g/a,

whereas in Fig. 3(b) $\Delta f/f$ is shown as a function of the scaled coherence parameter σ_g/a for selected values of N; ranging from 0.1 to 10. It is seen that the focal shift is negative, i.e. the maximum intensity occurs closer to the aperture, and that it increases with decreasing degrees of coherence.

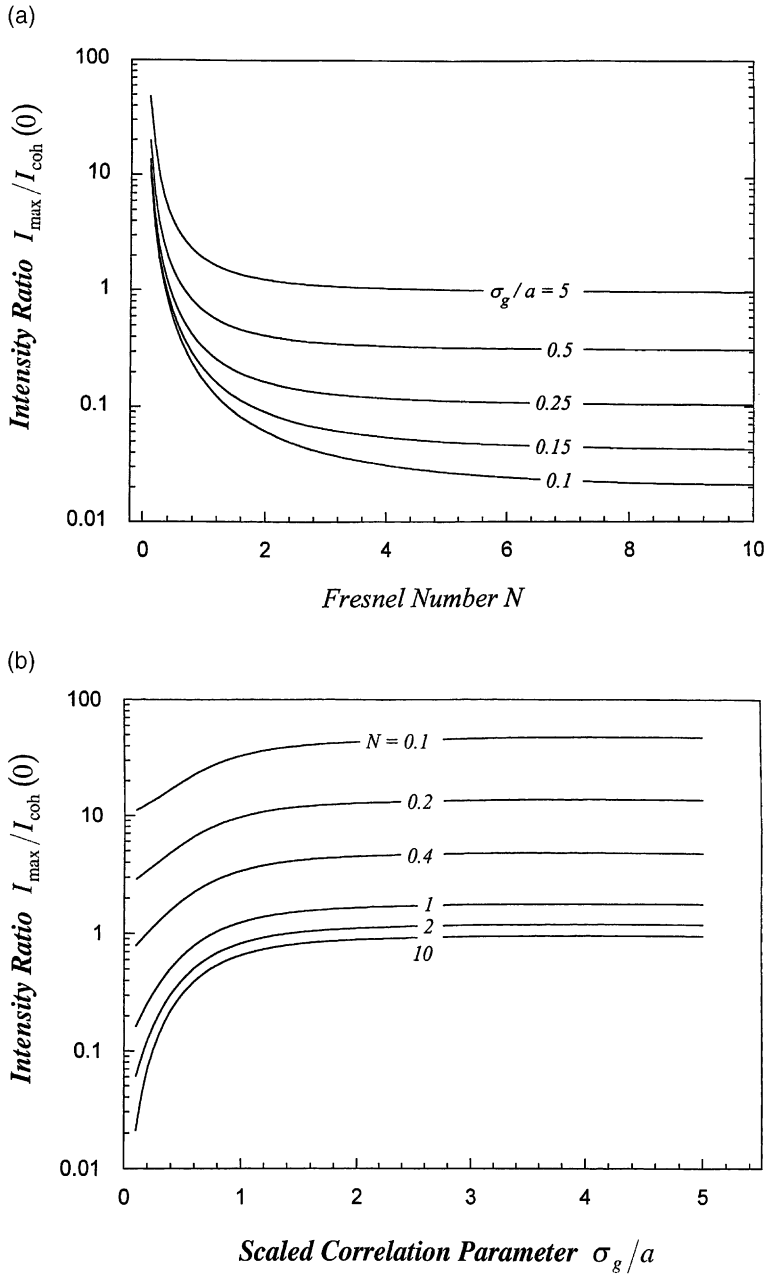

Fig. 4. The spectral intensity ratio $I_{max}/I_{coh}(0)$ in the focal region (a) as a function of N for various values of the scaled correlation parameter σ_g/a, and (b) as a function of σ_g/a for selected values of N.

Another quantitative measure characterizing the spectral intensity in the focal region is the intensity ratio $I_{\max}/I_{\mathrm{coh}}(0)$, where $I_{\max}$ is the maximum value of the on-axis intensity distribution and $I_{\mathrm{coh}}(0)$ denotes, as before, the intensity at the geometrical focus for the fully coherent wave. The behavior of the intensity ratio $I_{\max}/I_{\mathrm{coh}}(0)$ is shown in Fig. 4(a) as a function of the Fresnel number N for various values of the scaled coherence parameter σ_g/a. In Fig. 4(b) the same ratio is shown as a function of N for several selected values of σ_g/a. It is seen that for a coherent focused field $I_{\max}/I_{\mathrm{coh}}(0)$ is larger than unity (i.e. there is an *excess intensity*), and that it can exceed unity even for partially coherent fields when the Fresnel number N is sufficiently small.

4. Conclusions

In this paper we have examined the behavior of the spectral intensity along the axis of a focused partially coherent Schell-model wave field. We obtained curves showing the dependence of the focal shift (i.e. the distance of the intensity maximum from the geometrical focus) on the state of coherence of the focused wave and on the Fresnel number of the focusing geometry.

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References

[1] M. Born, E. Wolf, Principles of Optics, seventh (expanded) ed., Cambridge University Press, Cambridge, 1999.
[2] E. Wolf, Y. Li, Opt. Commun. 39 (1981) 205.
[3] Y. Li, E. Wolf, Opt. Commun. 39 (1981) 211.
[4] Y. Li, E. Wolf, J. Opt. Soc. Am. A 1 (1984) 801.
[5] J.J. Stamnes, Waves in Focal Regions, A. Hilger, Bristol, 1986 (Section 12.2).
[6] W. Wang, A.T. Friberg, E. Wolf, J. Opt. Soc. Am. A 14 (1997) 491.
[7] A.T. Friberg, J. Turunen, J. Opt. Soc. Am. A 5 (1988) 713.
[8] Q. He, J. Turunen, A.T. Friberg, Opt. Commun. 67 (1988) 245.
[9] B. Lü, B. Zhang, B. Cai, J. Mod. Opt. 42 (1995) 289.
[10] L. Mandel, E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, Cambridge, 1995.
[11] G.N. Watson, A Treatise on the Theory of Bessel Functions, second ed., Cambridge University Press, Cambridge, 1966, p. 79 (formula 9, example 4 with $\nu = 0$).