Physica A 153 (1988) 234–251
North-Holland, Amsterdam

DEVIATIONS FROM EXPONENTIAL DECAY IN QUANTUM MECHANICS

K.J.F. GAEMERS and T.D. VISSER
Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Received 4 July 1988

The usual quantum mechanical explanation of exponential decay is not exact. The approximations made lead to an average lifetime which can also be obtained directly using Fermi's Golden Rule. The question whether or not quantum mechanical decay is always exponential was recently raised again in connection with the possible decay of the proton. In this paper we derive a new criterion for the length of the non-exponential decay era. A formalism for general quantum systems is used to derive an exact expression for the survival probability in terms of a spectral density function. This expression is used for numerical studies. We find that in general, for short times, the decay is not exponential. In some cases a quadratic law is followed instead.

1. Introduction

The decay of unstable quantum systems has been studied almost from the beginning of the formulation of quantum mechanics. Dirac was the first to give an explanation of exponential decay within the framework of this theory [1]. Later Weisskopf and Wigner studied the related problem of the line shape [2]. Recently the quantum mechanical decay formalism has been studied in connection with speculations on proton decay experiments [3–5]. The basis for these discussions was the fact that a quantum system cannot decay exponentially for short times. Depending on the length of the time interval during which the decay is not exponential it is quite possible that the calculated proton lifetime is of the order of 10^{33} years while it is nevertheless impossible to observe proton decay because at the present time the decay is not yet in its exponential phase. This point of view has been advocated by Khalfin [3]. In a paper by Chiu et al. [4] one finds arguments against the statements made in ref. [3].

Another complication in the discussion of the decay phenomena has to do with the possible influence of measurements made on the system while it decays. This point of view can be found in the paper by Ghirardi et al. [6] and in a paper by Ekstein and Siegert [7].

Here it is our aim to study the decay law for small and intermediate times. We consider the survival amplitude and survival probability in the following way.

If the system under consideration is in an initial state $|\psi\rangle$ at time $t=0$ we find from the Schrödinger equation the state of the system at a time $t>0$ as

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\psi(0)\rangle \,. \tag{1}$$

The survival probability $\mathcal{P}(t)$ to find the system at time t still in the initial state is given by

$$\mathcal{P}(t) = |S(t)|^2 = |\langle\psi(0)|\,\mathrm{e}^{-\mathrm{i}Ht}|\psi(0)\rangle|^2 \,, \tag{2}$$

where H denotes the total Hamiltonian and the survival amplitude is written as $S(t)$.

The paper is organised as follows. In section 2 we give a general formalism for the calculation of $S(t)$ and $\mathcal{P}(t)$, we use this formalism for a numerical study of the behaviour of the survival amplitude for a few model systems (section 3). Finally in section 4 we draw some conclusions about the relevance of our results for real physical systems.

2. Formalism

In this section we will give a general formalism for the treatment of the decay of a pure state. On the basis of this formalism we derive an exact expression for the survival amplitude in terms of a spectral density function $\sigma(\epsilon)$. This function depends both on the total Hamiltonian H and on the choice of the (pure) state at $t=0$. Although the expressions that we derive look very much like expressions that can be found in the literature we like to state here that this similarity is only superficial and we will point out the differences with existing formalisms.

The starting point for our considerations is the same as that of A. Peres [8]. We consider a quantum system described by a Hamiltonian H. If at $t=0$ the system is in a state $|\psi\rangle$ with $\langle\psi|\psi\rangle = 1$ we can introduce the one-dimensional orthogonal projection operator

$$P = |\psi\rangle\langle\psi| \tag{3}$$

and its complement

$$Q = 1 - P . \tag{4}$$

With these two projection operators we construct from the full Hamiltonian H two new operators H_0 and V as follows:

$$H_0 = PHP + QHQ , \qquad V = PHQ + QHP . \tag{5}$$

From these definitions, it is clear that

$$H = H_0 + V . \tag{6}$$

It should be noted at this point that this splitting of H into an H_0 and V does not mean that H_0 can be interpreted directly as an unperturbed Hamiltonian nor that V is a simple perturbation. In the standard perturbation treatment the splitting of the total Hamiltonian is independent of the initial state. Here however H_0 and V both depend on $|\psi\rangle$. Now we also decompose the Hilbert space $\mathscr{H}$ for the system into two orthogonal subspaces,

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 , \tag{7}$$

where $\mathscr{H}_0$ is the one-dimensional subspace onto which P projects and $\mathscr{H}_1$ is its orthogonal complement which is left invariant under the action of Q.

2.1. Properties of H_0 and V

From the definitions given in eq. (5) we see that the action of H_0 on the Hilbert space $\mathscr{H}$ is such that $\mathscr{H}_0$ and $\mathscr{H}_1$ are both invariant subspaces and also that the action of V on $\mathscr{H}_0$ has a resulting vector in $\mathscr{H}_1$ and the action of V on any vector from $\mathscr{H}_1$ results in a vector in $\mathscr{H}_0$.

We choose a basis in $\mathscr{H}_1$ such that H_0 is diagonal,

$$H_0|\epsilon, \alpha\rangle = \epsilon|\epsilon, \alpha\rangle , \tag{8}$$

for any

$$|\epsilon, \alpha\rangle \in \mathscr{H}_1 . \tag{9}$$

The ϵ is the corresponding eigenvalue of H_0 and the parameters α serve to label possible degeneracies. For the sake of the following arguments we assume that the spectrum of H_0 restricted to $\mathscr{H}_1$ is purely continuous. It will be seen later that the formalism is equally valid for other cases. We take the following continuum normalisation for the eigenstates:

$$\langle \epsilon', \alpha' | \epsilon, \alpha \rangle = \frac{1}{\rho(\epsilon, \alpha)} \, \delta(\epsilon - \epsilon')\delta(\alpha - \alpha') \tag{10}$$

so that the operator Q can be written as

$$Q = \int |\epsilon, \alpha\rangle \, \rho(\epsilon, \alpha) \langle \epsilon, \alpha| \, \mathrm{d}\epsilon \, \mathrm{d}\alpha \, . \tag{11}$$

With these constructions we also find that the initial state $|\psi\rangle$ is a discrete eigenstate of H_0,

$$H_0|\psi\rangle = \epsilon_0|\psi\rangle \, , \tag{12}$$

and also that

$$\langle \psi|V|\psi\rangle = 0 \, . \tag{13}$$

The only nonzero matrix elements of V are given by

$$\langle \epsilon, \alpha|V|\psi\rangle \quad \text{and} \quad \langle \psi|V|\epsilon, \alpha\rangle \, . \tag{14}$$

From here on our treatment differs from ref. [8].

2.2. *The survival amplitude*

We now introduce the resolvent operators associated with H_0 and H,

$$R_0(z) = (z - H_0)^{-1} \, , \qquad R(z) = (z - H)^{-1} \, . \tag{15}$$

A well known identity is

$$R(z) = R_0(z) + R_0(z)VR(z) \, , \tag{16}$$

which we iterate once to get the relation

$$R(z) = R_0(z) + R_0(z)VR_0(z) + R_0(z)VR_0(z)VR(z) \, . \tag{17}$$

The survival amplitude $S(t)$ which was defined in the introduction as

$$S(t) = \langle \psi| \, \mathrm{e}^{-\mathrm{i}Ht} |\psi\rangle \tag{18}$$

can now be given as a contour integral,

$$S(t) = \frac{1}{2\pi i} \oint_{\mathscr{C}} e^{-izt} \langle \psi | R(z) | \psi \rangle dz \,, \tag{19}$$

where the contour $\mathscr{C}$ runs counter-clockwise around the spectrum of H. Due to the fact that V only connects vectors from $\mathscr{H}_0$ and $\mathscr{H}_1$ as in (13) and (14) we find, using (17), the following relation for the matrix element of $R(z)$:

$$\langle \psi | R(z) | \psi \rangle = \frac{1}{z - \epsilon_0} [1 + \langle \psi | V R_0(z) V | \psi \rangle \langle \psi | R(z) | \psi \rangle] \,. \tag{20}$$

If we introduce the complex function $\Pi(z)$ as

$$\Pi(z) = \langle \psi | V R_0(z) V | \psi \rangle \tag{21}$$

we can re-express the survival amplitude as

$$S(t) = \frac{1}{2\pi i} \oint_{\mathscr{C}} e^{-izt} \frac{1}{z - \epsilon_0 - \Pi(z)} dz \,. \tag{22}$$

This is an exact expression for $S(t)$ in terms of $\Pi(z)$.

2.3. *Properties of $\Pi(z)$*

Using the basis of eigenvectors of H_0 we can write $\Pi(z)$ as

$$\Pi(z) = \int_{\Sigma} \frac{\rho(\epsilon, \alpha) |\langle \epsilon, \alpha | V | \psi \rangle|^2}{z - \epsilon} d\epsilon \, d\alpha \,. \tag{23}$$

The ϵ integration runs over Σ which is the spectrum of the operator H_0 restricted to $\mathscr{H}_1$. From this expression we see that it is advantageous to define a function $\sigma(\epsilon)$ of ϵ alone as

$$\sigma(\epsilon) = \int \rho(\epsilon, \alpha) |\langle \epsilon, \alpha | V | \psi \rangle|^2 d\alpha \,. \tag{24}$$

This function depends in a complicated and implicit way on the initial state $|\psi\rangle$ and on the full Hamiltonian. It is an important ingredient since the decay amplitude can be given completely in terms of this function. The function $\Pi(z)$ can now be written as

$$\Pi(z) = \int_{\Sigma} \frac{\sigma(\epsilon)}{z - \epsilon} d\epsilon \,. \tag{25}$$

If, contrary to what was assumed earlier, H_0 has discrete eigenstates as well with corresponding eigenvalues ϵ_i with $i = 1, 2, \ldots$, then $\Pi(z)$ must be generalised to

$$\Pi(z) = \sum_i \frac{\sigma_i}{z - \epsilon_i} + \int_{\Sigma_c} \frac{\sigma_c(\epsilon)}{z - \epsilon} \, d\epsilon \tag{26}$$

in an obvious way. For the sequel however we will not use this form.

From expression (25) we see that $\Pi(z)$ is an analytic function on the entire z-plane except for a branch-cut on the real axis along Σ. From expression (24) we see that $\sigma(\epsilon)$ is positive semi-definite. If we make the assumption that Σ is bounded from below, we also see that $\Pi(z)$ is real for z real and smaller than this lower bound. So we find that $\Pi(z)$ has the Schwartz reflection property

$$\Pi(z^*) = \Pi^*(z) . \tag{27}$$

Furthermore since $\sigma(\epsilon) \geq 0$ we see that the sign of the imaginary part of $\Pi(z)$ is always opposite to the sign of the imaginary part of z, or

$$\mathrm{Im}\, \Pi(z) \begin{cases} <0 & \text{for } \mathrm{Im}\, z > 0 , \\ >0 & \text{for } \mathrm{Im}\, z < 0 . \end{cases} \tag{28}$$

If we now look at $\Pi(z)$ just above the real axis we may write

$$\lim_{\epsilon \downarrow 0} \Pi(x + i\epsilon) = D(x) - iA(x) , \tag{29}$$

where $A(x) \geq 0$ for all x. The functions A and D can be given in terms of $\sigma(\epsilon)$ as

$$D(x) = P \int_{\Sigma} \frac{\sigma(\epsilon)}{x - \epsilon} \, d\epsilon , \qquad A(x) = \pi \sigma(x) . \tag{30}$$

With the help of these functions we will be able to give the survival amplitude as a simple Fourier transform.

2.4. *The exponential approximation*

Using the defined functions A and D we can rewrite expression (22) as

$$S(t) = \frac{1}{\pi} \int_{\Sigma} \frac{e^{-ixt} A(x)}{(x - \epsilon_0 - D(x))^2 + (A(x))^2} \, dx . \tag{31}$$

We will use this form later on to evaluate $S(t)$ numerically. At this point we will assume that the functions $A(x)$ and $D(x)$ are small on the interval Σ in such a way that the integrand in (31) is sharply peaked around ϵ_0. We can now approximate the integrand by replacing $A(x)$ and $D(x)$ by their (constant) values at $x = \epsilon_0$. If we also extend the integration interval from Σ to the full real axis, the integral (31) can easily be performed to give

$$S(t) = \mathrm{e}^{-\mathrm{i}(\epsilon_0 + D(\epsilon_0))t - A(\epsilon_0)t} \tag{32}$$

so that the decay probability is

$$\mathscr{P}(t) = |S(t)|^2 = \mathrm{e}^{-2A(\epsilon_0)t} \tag{33}$$

which is a pure exponential. We will denote this as the exponential approximation. Using expression (30) we find for the inverse lifetime of the state $|\psi\rangle$

$$\tau^{-1} = \Gamma = 2\pi \int \rho(\epsilon_0, \alpha)|\langle \epsilon_0, \alpha|V|\psi\rangle|^2 \, \mathrm{d}\alpha \tag{34}$$

which is sometimes known as Fermi's Golden Rule.* The quantity Γ is known as the decay width. It should be stressed here that in the usual derivation of this result, the operator V is the perturbation and has no implicit dependence on the initial state $|\psi\rangle$ contrary to what is done here. The function $\rho(\epsilon_0, \alpha)$ is sometimes called the density of final states.

The exponential approximation works very well if the denominator in expression (22) has a complex zero very close to the real axis.

2.5. The quadratic domain

We can define the spread in the energy of the initial state as

$$(\Delta H^2) = \langle \psi|H^2|\psi\rangle - \langle \psi|H|\psi\rangle^2 \tag{35}$$

which, using the properties of H_0 and V, reduces to

$$(\Delta H)^2 = \langle \psi|V^2|\psi\rangle \tag{36}$$

and in turn can be written by means of expression (24) as

$$(\Delta H)^2 = \int_\Sigma \sigma(\epsilon) \, \mathrm{d}\epsilon \, . \tag{37}$$

* This expression is, incidentally, derived by Dirac [1], Fermi only gave it its name.

We note at this point that $(\Delta H)^2$ and Γ as calculated in the exponential approximation both depend in a linear way on $\sigma(\epsilon)$.

We will now give a derivation for a lower bound on $\mathscr{P}(t)$ in which the quantity $(\Delta H)^2$ plays a fundamental rôle. In the Heisenberg picture the state of the system is given by the (time-independent) $|\psi\rangle$. The time evolution of the projection operator P as given in eq. (3) is described by the equation of motion

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,P = [P(t), H]\,. \tag{38}$$

If we define the standard deviation for P in the usual way we have

$$\Delta P = \{\langle\psi|P(t)^2|\psi\rangle - \langle\psi|P(t)|\psi\rangle^2\}^{1/2} \tag{39}$$

and since $P(t)$ is a projection operator we may also write

$$\Delta P = \{\langle\psi|P(t)|\psi\rangle - \langle\psi|P(t)|\psi\rangle^2\}^{1/2}\,. \tag{40}$$

The expectation value appearing in this formula is nothing but the survival probability

$$\mathscr{P}(t) = \langle\psi|P(t)|\psi\rangle \tag{41}$$

so that

$$\Delta P = \{\mathscr{P}(t) - \mathscr{P}(t)^2\}^{1/2}\,. \tag{42}$$

We now derive from expression (38) the uncertainty relation

$$\Delta P\cdot\Delta H \geqslant \frac{1}{2}\left|\left\langle\psi\left|\frac{\mathrm{d}P}{\mathrm{d}t}\right|\psi\right\rangle\right| = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathscr{P}(t) \tag{43}$$

which becomes after using expression (42)

$$\Delta H\cdot\{\mathscr{P}(t) - \mathscr{P}(t)^2\}^{1/2} \geqslant \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathscr{P}(t)\,. \tag{44}$$

This inequality gives after a simple integration

$$1 \geqslant \mathscr{P}(t) \geqslant \cos^2(\Delta H\cdot t)\,, \tag{45}$$

a result first derived by Mandelstam and Tamm [9], and later in an independent way by Fleming [10]. For times much smaller than $(\Delta H)^{-1}$ the bound can be expanded as

$$\mathscr{P}(t) \geq 1 - (\Delta H)^2 \cdot t^2 . \tag{46}$$

Using the Schrödinger equation for infinitesimal times one can show that this is in fact a good approximation. For this reason we denote the times for which this holds as the quadratic domain.

We have in the exponential approximation

$$\mathscr{P}(t) = \mathrm{e}^{-\Gamma t} \quad \text{with } \Gamma = 2\pi\sigma(\epsilon_0) , \tag{47}$$

while on the other hand the inequality (45) holds. We can now make an accurate estimate of the time interval during which exponential decay cannot hold. We define the time t_q by requiring that the exponential in (47) equals the Mandelstam Tamm bound. Since t_q depends only on Γ and ΔH we must have on dimensional grounds

$$t_q = \frac{1}{\Gamma} T\left(\frac{\Gamma}{\Delta H}\right) . \tag{48}$$

A first approximation gives

$$t_q \approx \frac{\Gamma}{(\Delta H)^2} . \tag{49}$$

We will now make this more quantitative. The variable t_q is defined as the solution of the equation:

$$\mathrm{e}^{-\Gamma t} = \cos^2(\Delta H \cdot t) . \tag{50}$$

We introduce the new variable $T = t \cdot (\Delta H)^2/\Gamma$, so that eq. (50) can now be written as

$$\mathrm{e}^{-A^2 T} = \cos^2(A \cdot T) , \tag{51}$$

where $A = \Gamma/(\Delta H)$. This equation can easily be solved numerically so that we get T_q as a function of the parameter A. In fig. 1 we plot this function for values of A between 0 and 2. We see that for values of A below ≈ 0.5, T_q is well approximated by one. From this result we can conclude that under the condition that $\Gamma/(\Delta H) < 0.5$, $t_q \approx \Gamma/(\Delta H)^2$ and thus that for times smaller than this value the decay cannot be exponential. Since Γ and $(\Delta H)^2$ are both linear in $\sigma(\epsilon)$, t_q as defined above hardly depends on the coupling strength.

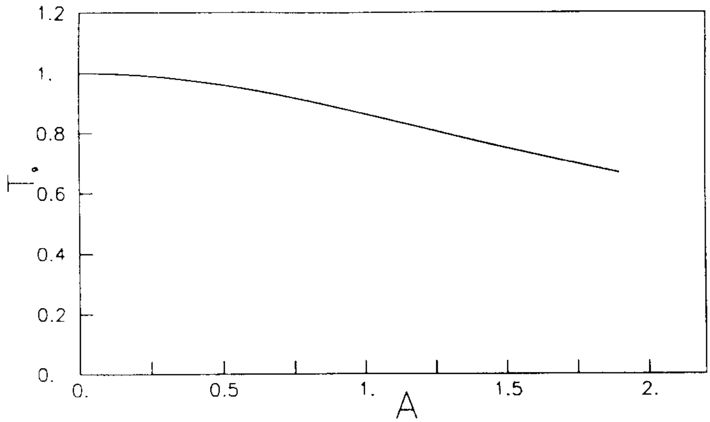

Fig. 1. The scaled length of the quadratic domain T_q as a function of the variable $A = \Gamma/(\Delta H)$.

2.6. Model calculations

In order to study deviations from exponential decay at times of order t_q we will introduce a class of model systems that can be treated using numerical Fourier transform techniques. The starting point is the choice of an interval on the real axis which we identify as Σ and of a function $\sigma(\epsilon)$ on this interval. We point out here that a choice of Σ and a corresponding $\sigma(\epsilon)$ completely specifies a (formal) quantum system. This means that given $\sigma(\epsilon)$ we can construct a Hilbert space and operators H_0 and V that satisfy the conditions given in the beginning of this section.

We would like to study the following three cases in detail (numerical results are given in the next section).

(i) The spectrum Σ consists of a finite interval on the real axis. Here we take this interval to be $[0, 1]$. For the function $\sigma(\epsilon)$ we take the polynomial form

$$\sigma(\epsilon) = g^2 \epsilon^n (1 - \epsilon)^m , \tag{52}$$

where we introduce the parameter g^2 in order to be able to vary the strength of the interaction responsible for the decay. For this case the quantity $(\Delta H)^2$ is always finite.

(ii) The spectrum Σ consists of an infinite interval on the real axis for which we take $[1, \infty]$. The function $\sigma(\epsilon)$ is chosen as

$$\sigma(\epsilon) = g^2 \frac{(\epsilon - 1)^m}{\epsilon^n} . \tag{53}$$

We also want the quantity $(\Delta H)^2$ to be finite which means that we must impose $n - m \geq 2$.

(iii) The spectrum Σ and the form of $\sigma(\epsilon)$ are as in case (ii) but now we want to consider the case where $(\Delta H)^2$ is infinite but where $\Pi(z)$ is still finite which means that now we must impose $n - m = 1$.

For the first two cases we expect a quadratic domain as given by our analysis whereas we expect a more complicated behaviour in the third case. We do not know whether there exists a general characterisation of the survival amplitude in terms of a power-law or otherwise for this diverging $(\Delta H)^2$ case.

With the choices for Σ and $\sigma(x)$ given above it is possible to give analytic expressions for $\Pi(z)$, as worked out in detail in the appendix.

We now turn to the question of the normalization of $S(t)$, and shall prove, under the condition that the integrand in eq. (22) has no poles on the first Riemann sheet, that $S(0) = 1$.

Suppose that $\Pi(z)$ has a cut from 0 to 1 in the complex plane. Consider a circular contour $\mathscr{C}_1$ (with a counter-clockwise orientation) centered at the origin with radius R. If we take $R > 1$ the contour will enclose the entire cut. Now calculate $S(0)$ as follows. Introduce a new variable $v = 1/z$. In the v-plane the cut runs from 1 to infinity. The new contour, which we call $\mathscr{C}_2$, is a circle of radius $1/R$, still centered at the origin, but with its orientation now clockwise. Thus we have

$$S(0) = \frac{1}{2\pi i} \int_{\mathscr{C}_1} \frac{1}{z - \epsilon_0 - \Pi(z)} \, dz \tag{54}$$

$$= \frac{1}{2\pi i} \int_{\mathscr{C}_2} \frac{-1}{v(1 - v\epsilon_0 - v\Pi(1/v))} \, dv \,. \tag{55}$$

The residue at $v = 0$ equals

$$\lim_{v \to 0} \frac{-1}{2\pi i(1 - v\epsilon_0 - v\Pi(1/v))} = \frac{-1}{2\pi i} \,. \tag{56}$$

We assume that $\lim_{z\to\infty} \Pi(z)$ is finite. So now, because of the orientation of $\mathscr{C}_2$ we have $S(0) = 1$. This proof is only valid if there are no poles on the real axis outside the cut Σ. However this is not always the case. From relation (28) it is clear that we can only have poles on the real axis. If we take for example

$$\sigma(\epsilon) = \begin{cases} g^2 \,, & \text{for } 0 \leqslant \epsilon \leqslant 1 \,, \\ 0 \,, & \text{otherwise} \,, \end{cases}$$

we then have $\Pi(z) = g^2 \ln(z/(z-1))$. This function has a cut from 0 to 1. The denominator of (22) now has two roots on the real axis, one left and one right of the cut. The functions that we will consider in the next section have always been chosen such (by varying the coupling constant g) that they have no poles on the real axis outside Σ. This proof can easily be extended to the case where the spectrum Σ is infinite.

3. Model systems

The choice of the spectral density function $\sigma(\epsilon)$, together with ϵ_0, completely determines the survival amplitude $S(t)$. We recall that

$$S(t) = \frac{1}{\pi} \int_{\Sigma} \frac{e^{-ixt} A(x)}{(x - \epsilon_0 - D(x))^2 + (A(x))^2} \, dx \tag{57}$$

with $D(x)$ and $A(x)$ as given in (30). The expression for $S(t)$ was evaluated numerically using the routine $D01ANF$ from the NAG library [11]. It is now desirable to consider functions $\sigma(x)$ that enable us to perform the defining integral for $\Pi(z)$ analytically. Of each of the three cases from the previous section we discuss typical examples.

(i) *Finite spectra*. An example of the survival probability in this case is shown in fig. 2, with $\sigma(\epsilon) = 10^{-2}\epsilon^2(1-\epsilon)^2$ and $\epsilon_0 = 0.15$. For $t < t_q$ the quadratic approximation is very good, whereas for greater times $\mathcal{P}(t)$ is closely following the exponential curve (see fig. 3). If we turn down the coupling constant g^2 from 10^{-2} to 10^{-4}, the decay will go slower but the length of the quadratic domain remains unchanged as expected. (See fig. 4.)

(ii) *Infinite spectra I*. We take Σ to be the interval $[1, \infty]$. For $\sigma(\epsilon)$ we choose

$$\sigma(\epsilon) = 10^{-3} \frac{(\epsilon - 1)}{\epsilon^3} \, . \tag{58}$$

With this choice $(\Delta H)^2$ is again finite. In fig. 5 an example is shown with $\epsilon_0 = 1.25$. Again we see that for short times the decay is very nearly quadratic. For longer times it is exponential with the expected lifetime. From all this it is clear that cases (i) and (ii) are not fundamentally different.

(iii) *Infinite spectra II*. Σ is again chosen to be the interval $[1, \infty]$, but now we choose the function $\sigma(\epsilon)$ so that $(\Delta H)^2$ diverges. We take

$$\sigma(\epsilon) = g^2 \frac{\epsilon - 1}{\epsilon^2} \, . \tag{59}$$

The integral defining $\Pi(z)$ still exists so the survival amplitude $S(t)$ can be evaluated. The numerical integration needed for $S(t)$ runs from $x = 1$ to a cutoff value at $x = b$. In doing this we actually make Σ finite, and so $(\Delta H)^2$

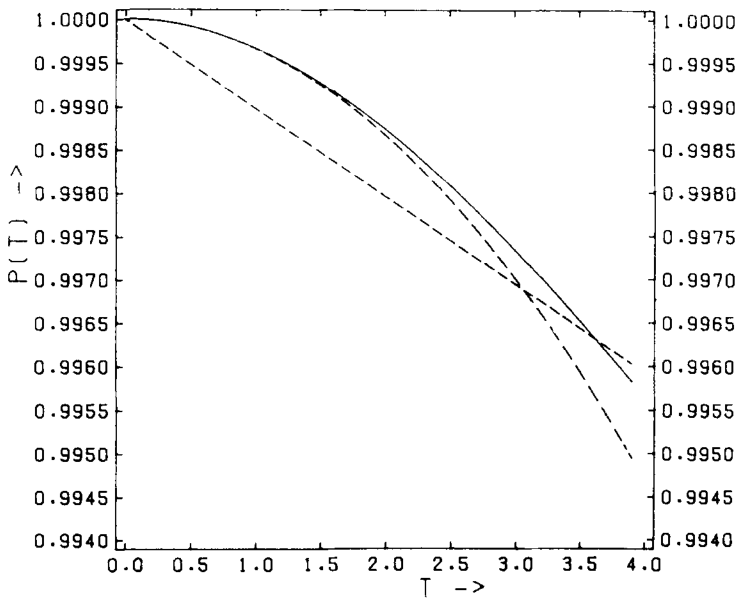

Fig. 2. The survival probability for the model function $\sigma(\epsilon) = 10^{-2}\epsilon^2(1-\epsilon)^2$ and $\epsilon_0 = 0.15$. The exact result is presented by the continuous curve. The almost straight dashed curve is the exponential approximation whereas the parabola shaped dashed curve is the Mandelstam–Tamm bound.

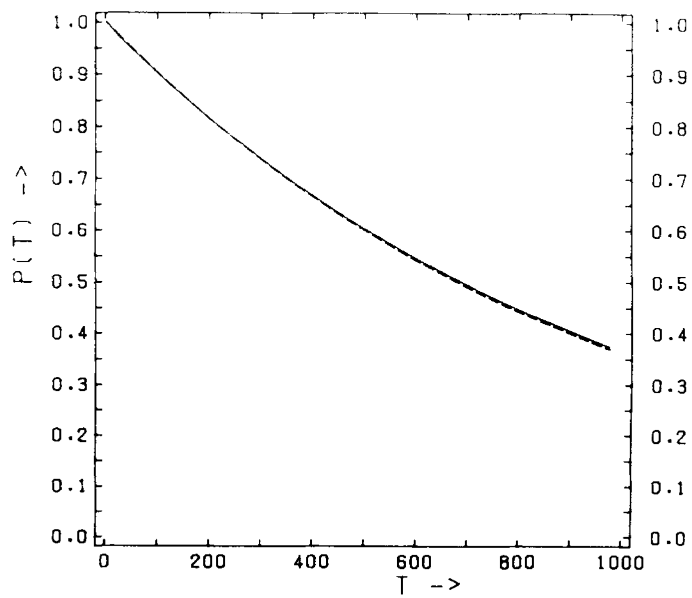

Fig. 3. The survival probability for the model function $\sigma(\epsilon) = 10^{-2}\epsilon^2(1-\epsilon)^2$ and $\epsilon_0 = 0.15$. Here the exact result (full curve) and the exponential approximation (dashed curve) are given for times of the order of the average lifetime of the state.

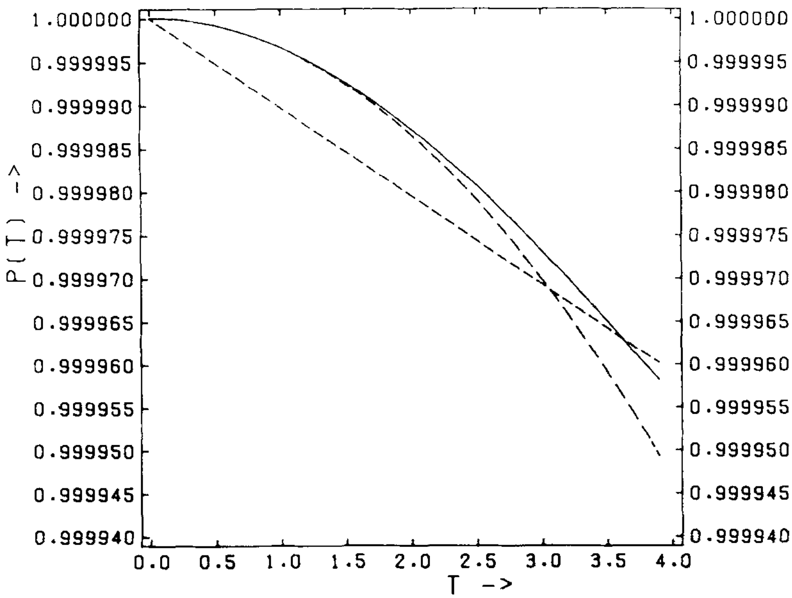

Fig. 4. The survival probability for the model function $\sigma(\epsilon) = 10^{-4}\epsilon^2(1-\epsilon)^2$ and $\epsilon_0 = 0.15$. The exact result is presented by the continuous curve. The almost straight dashed curve is the exponential approximation whereas the parabola shaped dashed curve is the Mandelstam–Tamm bound.

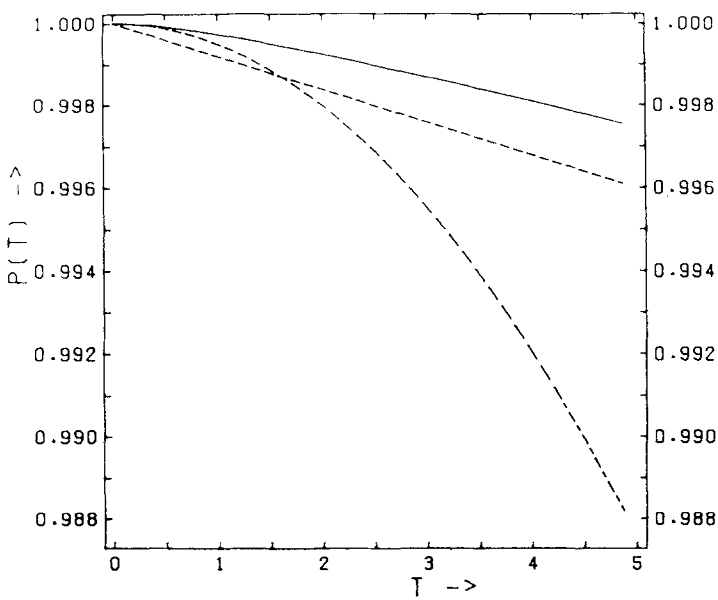

Fig. 5. The survival probability for the model function $\sigma(\epsilon) = 10^{-3}(\epsilon-1)/\epsilon^3$ and $\epsilon_0 = 1.25$. The exact result is presented by the continuous curve. The almost straight dashed curve is the exponential approximation whereas the parabola shaped dashed curve is the Mandelstam–Tamm bound.

becomes finite too. Consequently this procedure would give a wrong behaviour for $S(t)$. The cure is to estimate the neglected tail and add this to $S(t)$, which then becomes

$$S(t)=\frac{1}{\pi}\int_{1}^{b}\frac{e^{-ixt}A(x)}{(x-\epsilon_0-D(x))^2-(A(x))^2}\,dx+\Delta S(t)\,. \tag{60}$$

(The first term alone would yield quadratic decay for short times.) The second term is given by

$$\Delta S(t)=\frac{1}{\pi}\int_{b}^{\infty}\frac{e^{-ixt}A(x)}{(x-\epsilon_0-D(x))^2-(A(x))^2}\,dx\,, \tag{61}$$

where we take b arbitrary but large. In this example we have

$$A(x)=g^2\,\pi\left(\frac{1}{x}-\frac{1}{x^2}\right)$$

and (62)

$$D(x)=g^2\left(\frac{-1}{x}+\left(\frac{1}{x}-\frac{1}{x^2}\right)\ln|1-x|\right).$$

Because these functions fall off rapidly, we approximate

$$\Delta S(t)\approx g^2\int_{b}^{\infty}\frac{e^{-ixt}}{x^3}\,dx \tag{63}$$

which can be evaluated as

$$\frac{g^2}{2}\left[\frac{e^{-ibt}}{b^2}-it\,\frac{e^{-ibt}}{b}+t^2\left(\mathrm{Ci}\,(bt)+i\left(\frac{\pi}{2}-\mathrm{Si}(bt)\right)\right)\right]. \tag{64}$$

We can expand expression (64) for short times and add the result to the first term of (60) which by itself gives a quadratic decay with

$$(\Delta H)^2\approx g^2\left(\ln b+\frac{1}{b}-1\right). \tag{65}$$

Eventually we get for the total survival probability an expression of the form

$$\mathcal{P}(t)\approx 1-g^2t^2(C-\ln t). \tag{66}$$

Note that the logarithmic b dependence has dropped out as it should have. The constant C is a combination of several terms from the expansion of Ci and Si. The $t^2 \ln t$ behaviour is characteristic for all models of class (iii). We see that here again the decay, although not quadratic, starts off horizontally, as in cases (i) and (ii). This was to be expected from the work of Khalfin [3]. For longer times we find in a numerical study again the expected exponential behaviour.

4. Conclusions

In this paper we have considered the behaviour of quantum states that are unstable. We have presented a general formalism where the survival amplitude can be expressed in terms of a spectral function $\Pi(z)$ which in turn depends on the total Hamiltonian and on the initial state $|\psi\rangle$. We use this formalism to obtain numerical results on several model systems where it is possible to show clearly the quadratic behaviour of the survival probability for short times. It is also demonstrated that for model systems with $(\Delta H)^2$ finite the derived lower bound on the time interval during which the decay cannot be exponential is actually of the same order of magnitude as the time scale for which there is a cross-over from quadratic to exponential behaviour. Although the model systems described in section 2 were introduced because they allow analytical evaluation of the function $\Pi(z)$ it is nevertheless clear that for many systems of physical importance such as unstable elementary particles the corresponding function $\sigma(\epsilon)$ can be well approximated by the functions we have introduced here.

Acknowledgements

We would like to thank Sjors Wiersma for his help in the numerical part of this work, and also J. Uffink and J. Hilgevoord for bringing ref. [9] to our notice.

Appendix A

The function $\Pi(z)$ for model systems

Here we will calculate some of the integrals that are needed for the numerical study. We start by defining the polynomials $B_{n,m}(x)$ as

$$B_{n,m}(x) = x^n(1-x)^m \tag{67}$$

and the associated functions $Q_{n,m}(z)$ as

$$Q_{n,m}(z)=\int_0^1 \frac{B_{n,m}(x)}{z-x}\,\mathrm{d}x\,. \tag{68}$$

If we use the following notation for the well known Euler B-function:

$$B(n+1,m+1)=\int_0^1 x^n(1-x)^m\,\mathrm{d}x\,, \tag{69}$$

we find the following recursion relation for the $Q_{n,m}$:

$$Q_{n+1,m}(z)=zQ_{n,m}(z)-B(n+1,m+1)\,. \tag{70}$$

By means of a change of integration variable in eq. (68) we can also derive the following relation:

$$Q_{n,m}(z)=-Q_{m,n}(1-z)\,. \tag{71}$$

For the simplest case $n=0$ and $m=0$ we find

$$Q_{0,0}(z)=\ln\left(\frac{z}{z-1}\right). \tag{72}$$

From this starting point and the relations (70) and (71) it is easy to see that the functions $Q_{n,m}$ must have the general form

$$Q_{n,m}(z)=z^n(1-z)^m\ln\left(\frac{z}{z-1}\right)+P_{n,m}(z)\,, \tag{73}$$

where $P_{n,m}(z)$ is a polynomial in z, which can be expressed in closed form as

$$P_{n,m}(z)=\int_0^1 \frac{x^n(1-x)^m-z^n(1-z)^m}{z-x}\,\mathrm{d}x\,. \tag{74}$$

Using this form for $P_{n,m}$ we can easily evaluate these polynomials for arbitrary n and m with the help of the algebraic manipulation program SCHOONSCHIP [12]. In table I we give these polynomials for $n+m\leqslant 6$.

In the case of infinite spectra we consider the following model functions in complete analogy with the previous examples:

$$R_{n,m}=\int_1^\infty \frac{(x-1)^m}{x^n(z-x)}\,\mathrm{d}x\,. \tag{75}$$

Table I
The polynomials $P_{n,m}$ for $n + m \leq 6$.

n	m	$P_{n,m}(z)$
1	1	$(2z - 1)/2$
1	2	$(-6z^2 + 9z - 2)/6$
2	1	$(6z^2 - 3z - 1)/6$
1	3	$(12z^3 - 30z^2 + 22z - 3)/12$
2	2	$(-12z^3 + 18z^2 - 4z - 1)/12$
3	1	$(12z^3 - 6z^2 - 2z - 1)/12$
1	4	$(-60z^4 + 210z^3 - 260z^2 + 125z - 12)/60$
2	3	$(60z^4 - 150z^3 + 110z^2 - 15z - 3)/60$
3	2	$(-60z^4 + 90z^3 - 20z^2 - 5z - 2)/60$
4	1	$(60z^4 - 30z^3 - 10z^2 - 5z - 3)/60$
1	5	$(60z^5 - 270z^4 + 470z^3 - 385z^2 + 137z - 10)/60$
2	4	$(-60z^5 + 210z^4 - 260z^3 + 125z^2 - 12z - 2)/60$
3	3	$(60z^5 - 150z^4 + 110z^3 - 15z^2 - 3z - 1)/60$
4	2	$(-60z^5 + 90z^4 - 20z^3 - 5z^2 - 2z - 1)/60$
5	1	$(60z^5 - 30z^4 - 10z^3 - 5z^2 - 3z - 2)/60$

Here we impose the condition $n - m > 0$ with n and m integers. This function can be obtained by means of a transformation of variable. If one uses $u = 1/x$ it can easily be seen that the integral for $R_{n,m}$ can be expressed in terms of the functions $Q_{n,m}$.

This completes our calculation of the model functions.

References

[1] P.A.M. Dirac, Proc. Roy. Soc. (London) A 114 (1927) 243.
[2] V. Weisskopf and E. Wigner, Z. Phys. 63 (1930) 54.
[3] L.A. Khalfin, Phys. Lett. B 112 (1982) 223.
[4] C.B. Chiu, B. Misra and E.C.G. Sudarshan, Phys. Lett. B 117 (1982) 34.
[5] G.N. Fleming, Phys. Lett. B 125 (1983) 287.
[6] L. Fonda, G.C. Ghirardi and A. Rimini, Rep. Prog. Phys. 41 (1978) 587.
[7] H. Ekstein and A.J.F. Siegert, Ann. Phys. (NY) 68 (1971) 509.
[8] A. Peres, Ann. Phys. (NY) 129 (1980) 33.
[9] L. Mandelstam and Ig. Tamm, J. Phys. IX (1945) 249.
[10] G.N. Fleming, Nuov. Cim. 16A (1973) 232.
[11] NAG Fortran Library Manual, Mark 11 (1984), Oxford.
[12] M. Veltman, SCHOONSCHIP.
H. Strubbe, Comput. Phys. Commun. 8 (1974) 1.