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8 September 1997

PHYSICS LETTERS A

Physics Letters A 234 (1997) 1–4

Scattering in the presence of field discontinuities at boundaries

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Received 21 February 1997; accepted for publication 11 June 1997
Communicated by P.R. Holland

Abstract

This paper is concerned with the validity of scalar models frequently used in analyzing scattering of electromagnetic fields by bodies with sharp boundaries. The fact that the electromagnetic field vectors generally undergo discontinuous jumps at the boundaries is usually ignored. We derive a modified equation which takes the field discontinuities into account and we discuss some of its consequences. © 1997 Published by Elsevier Science B.V.

1. Introduction

It is well-known that at boundaries of media with which an electromagnetic field interacts, some components of the field vectors are discontinuous. However, because of the complexity of a full vector treatment, electromagnetic scattering problems are often analyzed by the use of scalar theory; and, moreover, the scalar field and its normal derivatives are, as a rule, assumed to be continuous at the boundary of the scatterer.

In this paper we present some results of the consequences of ignoring the field discontinuities at sharp boundaries of scattering bodies.

2. Discontinuities of the electric field at the boundary of a dielectric scatterer

Suppose that a monochromatic electromagnetic field of angular frequency ω (time dependence $\exp(-\mathrm{i}\omega t)$) is scattered by a homogeneous dielectric medium, surrounded by free space. It follows from Maxwell's equations that the normal component of the electric displacement vector $\boldsymbol{D}$ is continuous at the boundary (Ref. [1], Eq. (1.17) with $\sigma=0$). Hence if $\boldsymbol{D}^+$ and $\boldsymbol{D}^-$ represent the value of $\boldsymbol{D}$ just outside and just inside the medium and $\boldsymbol{n}$ denotes the unit outward normal to the boundary we have

$$\boldsymbol{n}\cdot(\boldsymbol{D}^+-\boldsymbol{D}^-)=0. \tag{1}$$

If $\boldsymbol{E}^+$ and $\boldsymbol{E}^-$ are the corresponding values of the electric field (see Fig. 1), we evidently have

$$\boldsymbol{D}^+=\boldsymbol{E}^+,\quad \boldsymbol{D}^-=\boldsymbol{E}^-+4\pi\boldsymbol{P}^-, \tag{2}$$

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PII S0375-9601(97)00459-3

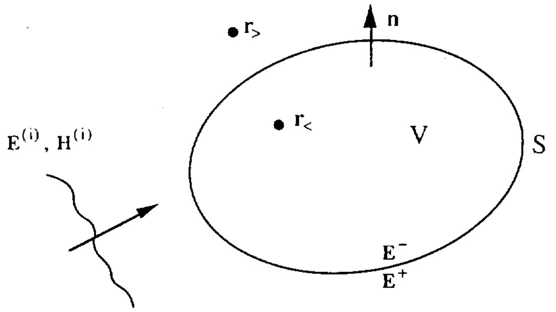

Fig. 1. Illustrating scattering on a dielectric medium occupying a volume V, bounded by a surface S. $E^{(i)}$, $H^{(i)}$ represent the electric and magnetic fields incident on the medium.

where P^- denotes the induced polarization just inside the scatterer.

It follows from Eqs. (1) and (2) that

$$n \cdot \Delta E = 4\pi n \cdot P^-, \tag{3}$$

where

$$\Delta E = E^+ - E^-. \tag{4}$$

Since P^- does not vanish identically it follows that, in general, the normal component of the electric field has a discontinuity (or saltus) at the surface bounding the medium [2]. This discontinuity is not known a priori; it can only be determined after the scattering problem has been solved.

3. Scalar scattering in the presence of field discontinuities at the boundary of the scattering medium

As we already mentioned, discontinuities of the field at the boundaries of scattering media are usually not taken into account in scalar treatments. We will now derive a correction term which must be added to the usual equation for the scattered field to take such discontinuities into account. Because the full calculations are rather lengthy we will only indicate the main steps, based on certain general formulas which were derived in an earlier paper [2], dealing with quantum mechanical potential scattering. It was shown in that paper that the following two equations relating to potential scattering hold:

$$0 = -\frac{1}{4\pi}\int_V \psi(r')U(r')G(r_>, r')\, \mathrm{d}^3r' - \frac{1}{4\pi}\Sigma^{(-)}(r_>) \tag{5}$$

and

$$\psi(r_>) = \psi^{(i)}(r_>) + \frac{1}{4\pi}\Sigma^{(+)}(r_>), \tag{6}$$

where

$$\Sigma^{(\pm)}(r) = \int_{S^\pm}\left[\psi(r')\frac{\partial G(r, r')}{\partial n'} - G(r, r')\frac{\partial \psi(r')}{\partial n'}\right]\mathrm{d}S^\pm. \tag{7}$$

Except for a slight change in the notation, Eqs. (5)–(7) are Eqs. (2.7d), (2.14) and (2.8a) of Ref. [2]. In the above formulas $\psi^{(i)}(r)$ represents the incident field, $\psi(r)$ represents the total field (i.e. incident + scattered), $U(r)$ represents the scattering potential and

$$G(r, r') = \mathrm{e}^{\mathrm{i}k|r-r'|}/|r-r'| \tag{8}$$

is the outgoing free-space Green's function. The vector $r_>$ represents a point outside the scattering volume, $\partial/\partial n'$ denotes differentiation along the outward normal to S and the superscripts $\pm$ indicate that the integration is taken along a surface S^+ just outside or along a surface S^- just inside of S (Fig. 2).

From Eqs. (5) and (6) it follows that

$$\psi(r_>) = \psi^{(i)}(r_>) - \frac{1}{4\pi}\int_V \psi(r')U(r')G(r_>, r')\mathrm{d}^3r' + \mathscr{S}(r_>), \tag{9}$$

where

$$\mathscr{S}(r_>) = \frac{1}{4\pi}\left[\Sigma^{(+)}(r_>) - \Sigma^{(-)}(r_>)\right], \tag{10}$$

[2] The situation is different for potential scattering in non-relativistic quantum mechanics. The Schrödinger wave function and its normal derivative at the boundary of a (finite) potential step or potential well are necessarily continuous because the probability density and the probability current are assumed to be continuous functions of space and time.

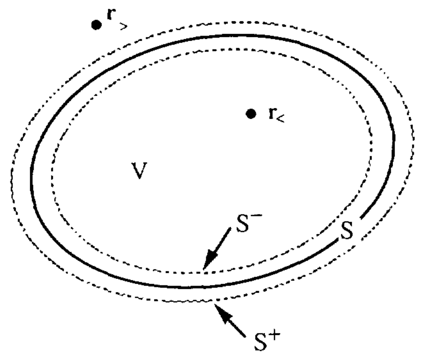

Fig. 2. Illustrating the meaning of the surface S^+ and S^- in the integral (7).

or, more explicitly, using the definitions (7) of $\Sigma^{\pm}$, and proceeding to the limits as the surfaces S^- and S^+ approach the surface S,

$$\mathscr{S}(\boldsymbol{r}_>) = \frac{1}{4\pi}\int_S \left\{ \Delta\psi(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r}_>, \boldsymbol{r}')}{\partial n'} - G(\boldsymbol{r}_>, \boldsymbol{r}')\Delta\left[\frac{\partial\psi(\boldsymbol{r}')}{\partial n'}\right]\right\} \mathrm{d}S, \tag{11}$$

where

$$\Delta\psi(\boldsymbol{r}') = \psi^{(+)}(\boldsymbol{r}') - \psi^{(-)}(\boldsymbol{r}'), \tag{12a}$$

$$\Delta\left[\frac{\partial\psi(\boldsymbol{r}')}{\partial n'}\right] = \frac{\partial\psi^{(+)}(\boldsymbol{r}')}{\partial n'} - \frac{\partial\psi^{(-)}(\boldsymbol{r}')}{\partial n'}, \tag{12b}$$

represent the discontinuities (salti) of the total field ψ and of its normal derivative $\partial\psi/\partial n'$ across the surface S bounding the scattering volume.

The formula (9) is a generalization of the usual integral equation of potential scattering (see, for example, Ref. [3]) when the field and (or) its normal derivative have discontinuities at the boundary of the scatterer and the field point is located outside the scattering volume.

When the field and its normal derivative on the boundary S of the scatterer are continuous, $\Delta\psi(\boldsymbol{r}') = \Delta[\partial\psi(\boldsymbol{r}')/\partial n'] = 0$ on S and then $\Sigma^+ = \Sigma^-$. In this case we see from Eq. (10) that $\mathscr{S}(\boldsymbol{r}_>) = 0$ and Eq. (9) reduces to the usual integral equation of potential scattering. Evidently Eq. (9) is a generalization of that equation when the field and (or) its normal derivative are discontinuous across the boundary of the scattering volume.

4. Effects of discontinuities on the far field

We will now use Eq. (9) to estimate the effect of the discontinuities on the field at points far away from the scatterer.

Suppose that the point $\boldsymbol{r}_>$ is in the far zone, in a direction specified by a unit vector $\boldsymbol{u}$, i.e. $\boldsymbol{r}_> = r_> \boldsymbol{u}$ (see Fig. 3). The asymptotic form of the free-space Green's function as $kr_> \to \infty$, with the unit vector $\boldsymbol{u}$ being kept fixed is [4]

$$G(\boldsymbol{r}, \boldsymbol{r}') \sim \mathrm{e}^{-\mathrm{i}k\boldsymbol{u}\cdot\boldsymbol{r}'}\mathrm{e}^{\mathrm{i}kr}/r, \tag{13}$$

where, for the sake of simplicity, we have now written r in place of $r_>$. If $\boldsymbol{n}'$ denotes the unit outward normal to S, we readily find from Eq. (13) that

$$\partial G(\boldsymbol{r}, \boldsymbol{r}')/\partial n' \sim \mathrm{i}k\boldsymbol{n}' \cdot \boldsymbol{u}\, \mathrm{e}^{-\mathrm{i}k\boldsymbol{u}\cdot\boldsymbol{r}'}\mathrm{e}^{\mathrm{i}kr}/r. \tag{14}$$

On using the asymptotic approximations (13) and (14) in Eq. (11) it follows that the contribution of the surface integral (11) to the integral in Eq. (9) when the field point is in the far zone is (writing now $\mathscr{S}^{(\infty)}$ in place of $\mathscr{S}$):

$$\mathscr{S}^{(\infty)}(r\boldsymbol{u}) = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\int_S \left\{ \mathrm{i}k\boldsymbol{n}' \cdot \boldsymbol{u}\Delta\psi - \Delta\left[\frac{\partial\psi}{\partial n'}\right]\right\} \times \mathrm{e}^{-\mathrm{i}k\boldsymbol{u}\cdot\boldsymbol{r}'}\mathrm{d}S. \tag{15}$$

Since this surface term contains the unknown discontinuities of the field and of its normal derivative, it is not easy to determine quantitatively its contribution to the scattered field. However, for the special case of s-wave scattering by a homogeneous sphere, one

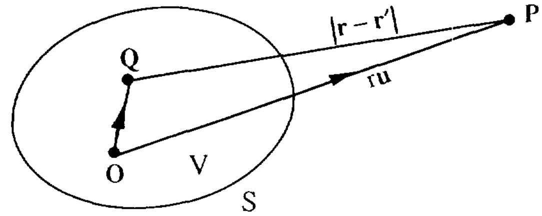

Fig. 3. Illustrating the notation pertaining to the formula (13). $\boldsymbol{u}$ is the unit vector $\boldsymbol{r}/|\boldsymbol{r}|$.

can obtain an order of magnitude estimate for this contribution. We have, in this case,

$$U(r) = U_0, \quad \text{when } r < R,$$
$$= 0, \quad \text{when } r > R, \tag{16}$$

where R is the radius of the scatterer. As is well known, the field in this case is given by the expression (see, for example, Ref. [5])

$$\psi(r) = Cj_0(kr_<), \quad \text{when } r < R,$$
$$= Aj_0(kr_<) + Bn_0(k_> r), \quad \text{when } r > R, \tag{17}$$

where $j_0(x)$ and $n_0(x)$ are the spherical Bessel function and the spherical Neumann function respectively of order zero, $k_<$ and $k_>$ are the wave numbers inside and outside the spherical potential and A, B and C are constants. We assume that the discontinuities $\Delta\psi$ and $\Delta\partial\psi/\partial n'$ are proportional to ψ and $\partial\psi/\partial n'$ respectively, with the proportionality factors being of the order of unity. (The assumption concerning $\Delta\psi$ is seen to be plausible from Eq. (2).) If we bear in mind that the scattering potential is proportional to k^2, with proportionality factor also of the order of unity (see, for example Ref. [6]) we obtain from Eq. (9) with $\mathscr{S}$ replaced by $\mathscr{S}^{(\infty)}$ the following order of magnitude estimate for the ratio of the surface term to the volume term (denoted now by $\mathscr{V}$) in Eq. (9):

$$\frac{\mathscr{S}^{(\infty)}}{\mathscr{V}} = \mathrm{O}\left(\frac{1}{kR}\right), \quad k_> R \approx 1. \tag{18}$$

(The constraint $k_> R \approx 1$ ensures that one is dealing with s-wave scattering.) The order of magnitude relation (18) implies that when the linear dimensions of the scatterer are of the order of the wavelength of the incident field or smaller, the discontinuities of the field and of its normal derivatives across the surface may significantly affect the scattered field in the far zone.

We stress that whilst the formula (9) is exact the order of magnitude estimate (18) has been obtained only for s-wave scattering.

Acknowledgement

This research was supported by the National Science Foundation, the New York State Foundation for Science and Technology and by the Army Research Office under the University Research Initiative Program. T.D. Visser is supported by the Technology Foundation (STW).

References

[1] J.D. Jackson, Classical electrodynamics, 2nd Ed. (Wiley, New York, 1962) Sec. 1.5.
[2] D.N. Pattanayak and E. Wolf, Phys. Rev. D 13 (1976) 913.
[3] P. Roman, Advanced quantum mechanics (Addison-Wesley, Reading, MA, 1965) Sec. 3.2.
[4] L. Mandel and E. Wolf, Optical coherence and quantum optics (Cambridge Univ. Press, Cambridge, 1995) Eq. (3.2-86).
[5] C.J. Joachain, Quantum collision theory (North-Holland, Amsterdam, 1983).
[6] E. Wolf and R.P. Porter, Radio Sci. 21 (1986) 627, Eq. (6).