Coherence singularities in the field generated by partially coherent sources

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We analyze the coherence singularities that occur in the far field that is generated by a broad class of partially coherent sources. It is shown that for rotationally symmetric planar quasihomogeneous sources the coherence singularities form a two-dimensional surface in a reduced three-dimensional space. We illustrate our results by studying the topology of the coherence singularity of a partially coherent vortex beam. We find that the geometry of the phase singularity can be associated with conic sections such as ellipses, lines, and hyperbolas.

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I. INTRODUCTION

There is a growing interest in the structure of wave fields in the vicinity of points where certain field parameters are undefined or "singular." This has given rise to the new subdiscipline of *singular optics* [1,2]. In the past few years, many different types of singular behavior have been identified. For example, *phase singularities* occur at positions where the field amplitude vanishes and hence the phase is undefined [3]. *Polarization singularities* arise at locations where the field is circularly or linearly polarized [4–10]. There either the orientation angle of the polarization ellipse or its handedness is undefined. Also the *Poynting vector* can exhibit singular behavior at points where its modulus is zero, and hence its orientation is undefined [11]. Topological reactions of such singularities were studied in [12,13].

Optical coherence theory [14] deals with the statistical properties of light fields. In this theory, correlation functions play a central role [15]. A form of singular behavior that is slightly more abstract than those mentioned above occurs in two-point correlation functions. At pairs of points at which the field (at a particular frequency) is completely uncorrelated, the phase of the correlation function is singular [16–21]. When the field at two such points is combined in an interference experiment, no fringes are produced. These *coherence singularities* are points in six-dimensional space. Their relationship to other types of singularities has only recently been clarified [22–26].

Thus far, only one study has been devoted to the description of the multidimensional structure of a specific coherence singularity, namely, that of a vortex beam propagating through turbulence [27]. In the present article, we analyze the more general case of coherence singularities in the far zone of the field generated by a broad class of partially coherent sources. These *quasihomogeneous sources* are often encountered in practice. We analyze the generic structure of the coherence singularities and also discuss the practical case of a rotationally symmetric source. We illustrate our results by applying them to a recently described new type of "dark core" or vortex beam. For this beam all different cross sections of the singularity are shown to be conic sections in a suitable coordinate system.

II. PARTIALLY COHERENT SOURCES

Consider a partially coherent planar secondary source, situated in the plane $z=0$, that emits radiation into the half space $z>0$ (see Fig. 1). In the space-frequency domain, the source is characterized by its *cross-spectral density function* [14]

$$W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\langle U^{(0)*}(\boldsymbol{\rho}_1,\omega)U^{(0)}(\boldsymbol{\rho}_2,\omega)\rangle. \tag{1}$$

Here $U^{(0)}(\boldsymbol{\rho},\omega)$ represents the source field at frequency ω at position $\boldsymbol{\rho}=(x,y)$, the asterisk indicates complex conjugation, and the angled brackets denote an ensemble average. The *spectral degree of coherence* is the normalized form of the cross-spectral density, i.e.,

$$\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\frac{W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)}{\sqrt{S^{(0)}(\boldsymbol{\rho}_1,\omega)S^{(0)}(\boldsymbol{\rho}_2,\omega)}}, \tag{2}$$

with

$$S^{(0)}(\boldsymbol{\rho},\omega)=W^{(0)}(\boldsymbol{\rho},\boldsymbol{\rho},\omega) \tag{3}$$

the *spectral density* (or "intensity at frequency ω") of the source. For *Schell-model* sources [14] the spectral degree of coherence only depends on position through the difference $\boldsymbol{\rho}_1-\boldsymbol{\rho}_2$, i.e.,

$$\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\mu^{(0)}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2,\omega). \tag{4}$$

The field in an arbitrary transverse plane $z>0$ is given by the expression

$$U(\boldsymbol{\rho},z,\omega)=\int_{(z=0)}U^{(0)}(\boldsymbol{\rho}',\omega)G(\boldsymbol{\rho}-\boldsymbol{\rho}',z,\omega)d^2\rho', \tag{5}$$

where $G(\boldsymbol{\rho}-\boldsymbol{\rho}',z,\omega)$ is an appropriate free-space Green's function [14], Sec. 5.2. On substituting from Eq. (5) in Eq.

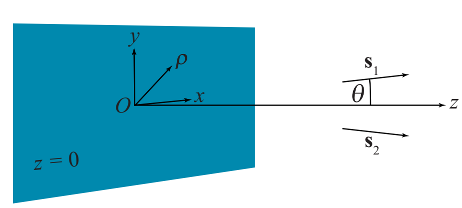

FIG. 1. (Color online) Illustrating the notation.

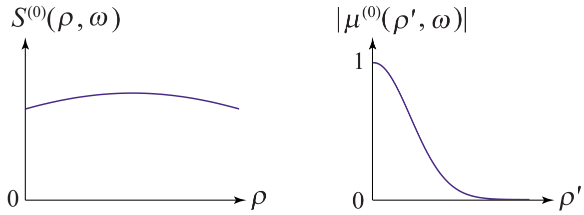

FIG. 2. (Color online) Illustrating the concept of quasihomogeneity.

(1), while interchanging the order of integration and ensemble averaging, it follows that the cross-spectral density at two arbitrary points $(\boldsymbol{\rho}_1, z_1)$ and $(\boldsymbol{\rho}_2, z_2)$ satisfies the equation

$$W(\boldsymbol{\rho}_1,z_1,\boldsymbol{\rho}_2,z_2,\omega) = \int\int_{(z=0)} W^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',\omega)G^*(\boldsymbol{\rho}_1-\boldsymbol{\rho}_1',z_1,\omega) \times G(\boldsymbol{\rho}_2-\boldsymbol{\rho}_2',z_2,\omega)d^2\rho_1'd^2\rho_2'. \quad (6)$$

The spectral density and the spectral degree of coherence at arbitrary points are given by formulas that are quite similar to Eqs. (2) and (3), *viz.*,

$$\mu(\boldsymbol{\rho}_1,z_1,\boldsymbol{\rho}_2,z_2,\omega) = \frac{W(\boldsymbol{\rho}_1,z_1,\boldsymbol{\rho}_2,z_2,\omega)}{\sqrt{S(\boldsymbol{\rho}_1,z_1,\omega)S(\boldsymbol{\rho}_2,z_2,\omega)}} \quad (7)$$

and

$$S(\boldsymbol{\rho},z,\omega) = W(\boldsymbol{\rho},z,\boldsymbol{\rho},z,\omega). \quad (8)$$

Coherence singularities are phase singularities of the spectral degree of coherence. They occur at pairs of points at which the field at frequency ω is completely uncorrelated, i.e.,

$$\mu(\boldsymbol{\rho}_1,z_1,\boldsymbol{\rho}_2,z_2,\omega) = 0. \quad (9)$$

III. QUASIHOMOGENEOUS SOURCES

An important subclass of Schell-model sources is formed by so-called quasihomogeneous sources [14]. For such sources the spectral density $S^{(0)}(\boldsymbol{\rho},\omega)$ varies much more slowly with $\boldsymbol{\rho}$ than the spectral degree of coherence $\mu^{(0)}(\boldsymbol{\rho}',\omega)$ varies with $\boldsymbol{\rho}'$. This behavior, that often occurs in practice, is sketched in Fig. 2.

For quasihomogeneous sources the field in the source plane and the field in the far zone are related by two *reciprocity relations*, namely,

$$S^{(\infty)}(\mathbf{s},\omega) = (2\pi k)^2\tilde{S}^{(0)}(0,\omega)\tilde{\mu}^{(0)}(k\mathbf{s}_\perp,\omega)\cos^2\theta/r^2, \quad (10)$$

$$\mu^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2,\omega) = \frac{\tilde{S}^{(0)}[k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}),\omega]}{\tilde{S}^{(0)}(0,\omega)}\exp[ik(r_2-r_1)], \quad (11)$$

with the two-dimensional spatial Fourier transforms given by the expressions

$$\tilde{S}^{(0)}(\mathbf{f},\omega) = \frac{1}{(2\pi)^2}\int S^{(0)}(\boldsymbol{\rho},\omega)e^{-i\mathbf{f}\cdot\boldsymbol{\rho}}d^2\rho, \quad (12)$$

$$\tilde{\mu}^{(0)}(\mathbf{f},\omega) = \frac{1}{(2\pi)^2}\int \mu^{(0)}(\boldsymbol{\rho},\omega)e^{-i\mathbf{f}\cdot\boldsymbol{\rho}}d^2\rho. \quad (13)$$

In these formulas $k=\omega/c$ is the wave number associated with frequency ω, c being the speed of light in vacuum, $\mathbf{s}_\perp$ is the projection of the unit direction vector $\mathbf{s}$ onto the xy plane, and θ is the angle that the $\mathbf{s}$ direction makes with the z axis (see Fig. 1). The superscript (∞) indicates points in the far zone. Equation (10) states that the far-field spectral density of a planar secondary quasihomogeneous source is proportional to the Fourier transform of its spectral degree of coherence. Equation (11) expresses that the far-field spectral degree of coherence of such a source is, apart from a geometrical factor, given by the Fourier transform of its spectral density.

Even though Eq. (11) is quite general, it allows us to draw several conclusions. First, the far-field spectral degree of coherence depends on the spectral density of the source, but is independent of its spectral degree of coherence. Second, the dependence of the far-field spectral degree of coherence on the two distances r_1 and r_2 enters only through the factor $\exp[ik(r_2-r_1)]$. This means that coherence singularities occur along certain pairs of observation directions $\mathbf{s}_1 = (\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)$ and $\mathbf{s}_2 = (\sin\theta_2\cos\phi_2, \sin\theta_2\sin\phi_2, \cos\theta_2)$ for which the prefactor in Eq. (11) vanishes. Since $\tilde{S}^{(0)}(0,\omega)$ is both finite and real, this yields the two constraints

$$\mathrm{Re}\{\tilde{S}^{(0)}[k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}),\omega]\} = 0, \quad (14)$$

$$\mathrm{Im}\{\tilde{S}^{(0)}[k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}),\omega]\} = 0, \quad (15)$$

where Re and Im denote the real and imaginary parts, respectively. These two conditions imply that generically (i.e., when they are independent and commensurate), the coherence singularities form a two-dimensional surface in the four-dimensional $(\theta_1,\phi_1,\theta_2,\phi_2)$ space.

Let us next consider the specialized case of a source whose spectral density is mirror symmetric with respect to both the x and the y axes. In that case the factor $\tilde{S}^{(0)}[k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp}],\omega)$ that appears in Eq. (11) is real valued for all values of its spatial argument [28] and hence condition (15) is lifted. This means that the coherence singularity is a three-dimensional volume in $(\theta_1,\phi_1,\theta_2,\phi_2)$ space. Furthermore, if the spectral density of the source is rotationally symmetric, the spectral degree of coherence in the far zone depends on the observation angles ϕ_1 and ϕ_2 only through their difference $\phi_2-\phi_1$. We therefore conclude that for planar secondary rotationally symmetric quasihomogeneous sources the coherence singularities are two-dimensional surfaces in the reduced $(\theta_1,\theta_2,\phi_2-\phi_1)$ space. An example of such a source is examined in Sec. IV.

IV. PARTIALLY COHERENT LAGUERRE-GAUSS SOURCE

We illustrate our results thus far with the analysis of a partially coherent source that generates a Laguerre-Gauss beam [29]. For this source we have

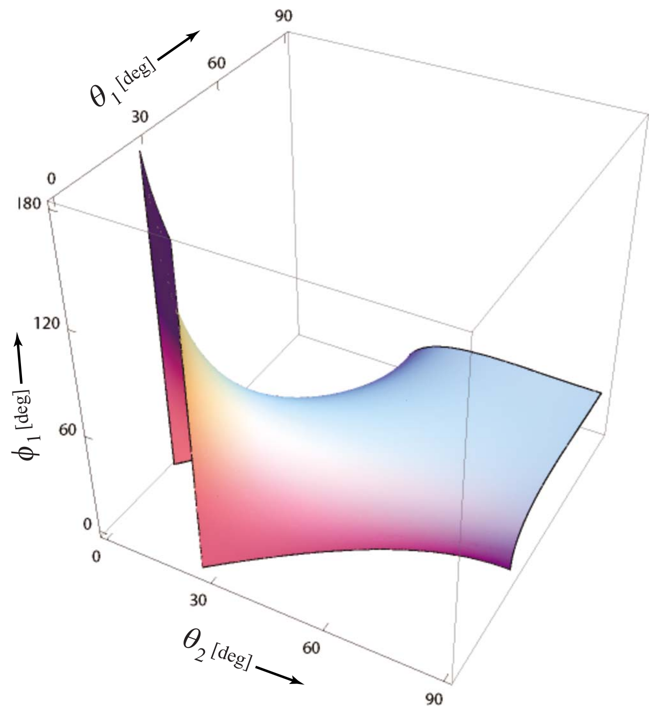

FIG. 3. (Color online) A two-dimensional coherence singularity in $(\theta_1, \theta_2, \phi_1)$ space. In this example $k^2\sigma_S^2=10$.

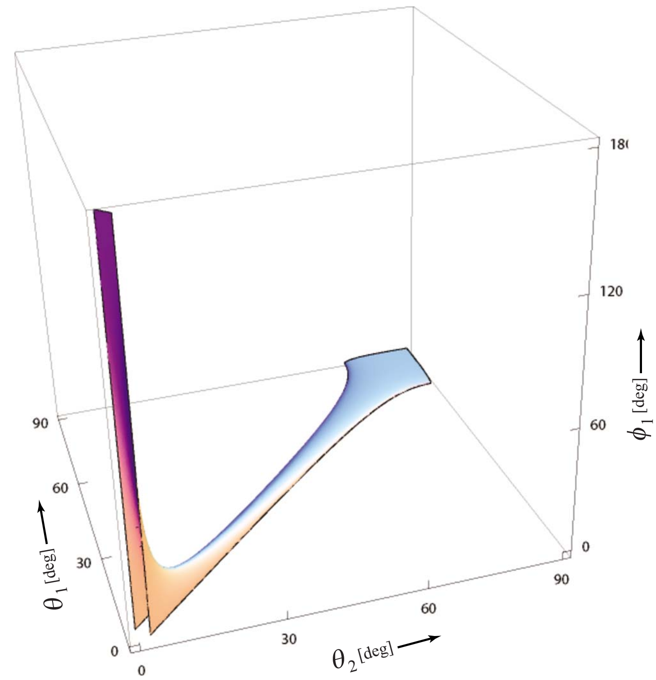

FIG. 4. (Color online) A two-dimensional coherence singularity in $(\theta_1, \theta_2, \phi_1)$ space. In this example $k^2\sigma_S^2=1000$.

$$S^{(0)}(\boldsymbol{\rho},\omega) = A^2\rho^2 \exp(-\rho^2/2\sigma_S^2), \tag{16}$$

$$\mu^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1,\omega) = \exp[-(\rho_2-\rho_1)^2/2\sigma_\mu^2], \tag{17}$$

with A a real number, $\rho=|\boldsymbol{\rho}|$, and σ_S and σ_μ the effective widths of the spectral density and of the spectral degree of coherence, respectively. If $\sigma_\mu \ll \sigma_S$ the source is quasihomogeneous. Since

$$\tilde{S}^{(0)}(\mathbf{f},\omega) = (2-f^2\sigma_S^2)\sigma_S^4 A^2 \exp(-f^2\sigma_S^2/2)/2\pi, \tag{18}$$

application of the reciprocity relation (11) yields

$$\mu^{(\infty)}(r_1\mathbf{s}_1, r_2\mathbf{s}_2,\omega) = [1-k^2(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp})^2\sigma_S^2/2] \times\exp[-k^2(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp})^2\sigma_S^2/2] \times\exp[ik(r_2-r_1)]. \tag{19}$$

Because

$$(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp})^2 = \sin^2\theta_1 + \sin^2\theta_2 - 2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2), \tag{20}$$

it follows from Eq. (19) that coherence singularities occur for those values of θ_1, ϕ_1, θ_2, and ϕ_2 for which

$$\sin^2\theta_1 + \sin^2\theta_2 - 2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) = 2/k^2\sigma_S^2. \tag{21}$$

As remarked at the end of Sec. III, the dependence of the spectral degree of coherence on the two angles ϕ_1 and ϕ_2 is through their difference $\phi_1-\phi_2$. From now on we set, without loss of generality, $\phi_2=0$.

An example of the topology of the coherence singularity is shown in Fig. 3, from which it can be seen that it forms a saddlelike surface. Let us consider the $\theta_1=\theta_2$ cross section. For small values of these two angles, there is no value of ϕ_1 that corresponds with a point on the surface, i.e., there exist no pairs of points at which the field is completely uncorrelated. When the angles are gradually increased to a critical value $\theta_1=\theta_2=\theta_c$ a coherence singularity occurs at $\phi_1=180°$ [29]. For larger values $(\theta_1=\theta_2>\theta_c)$ a value of $\phi_1<180°$ corresponds to a point on the surface. Since Eq. (21) shows a dependence of the singularity on cos ϕ_1, this means that the initial singularity has unfolded into two pairs of singularities: one for ϕ_1 and one for $-\phi_1$. It is noted that in this example $k^2\sigma_S^2=10$ for illustrative purposes. In Fig. 4 the more realistic value of 1000 was used. It can be seen that the topological features of the coherence singularity remain unchanged.

V. CONIC SECTIONS

In order to study the coherence singularity depicted in Fig. 3 in more detail, it is instructive to rewrite Eq. (21) in the form

$$x^2 + y^2 + 2xyz + G = 0, \tag{22}$$

where

$$x = \sin\theta_1, \tag{23}$$

$$y = \sin\theta_2, \tag{24}$$

$$z = -\cos\phi_1, \tag{25}$$

$$G = -2/k^2\sigma_S^2. \tag{26}$$

Although Eq. (22) is *not* a quadratic surface in (x,y,z) space,

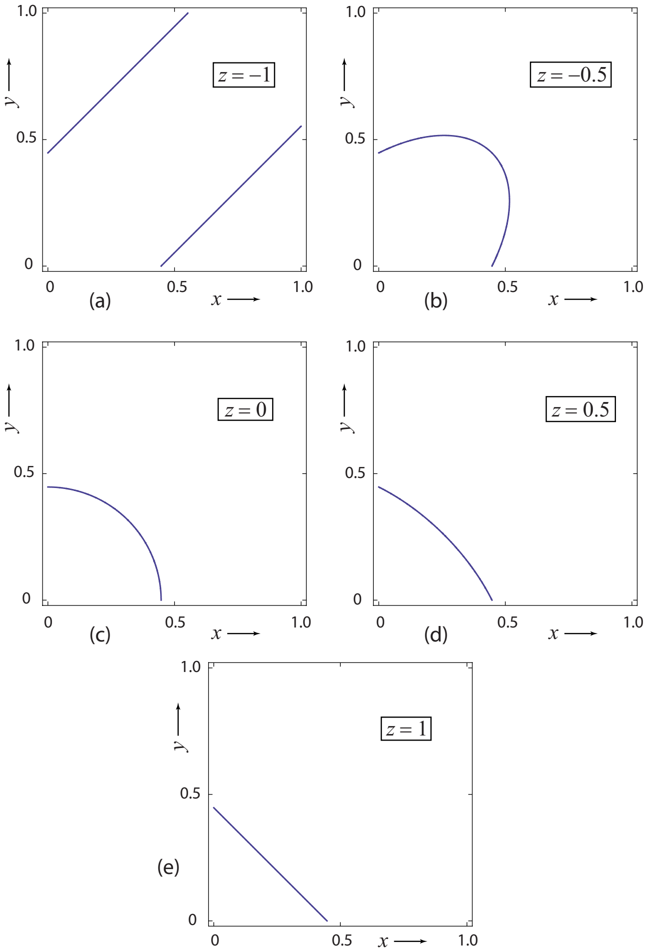

FIG. 5. (Color online) Cross sections of the coherence singularity in the x,y plane for selected values of z, *viz.*, (a) $z=-1$, (b) $z=-0.5$, (c) $z=0$, (d) $z=0.5$, and (e) $z=1$.

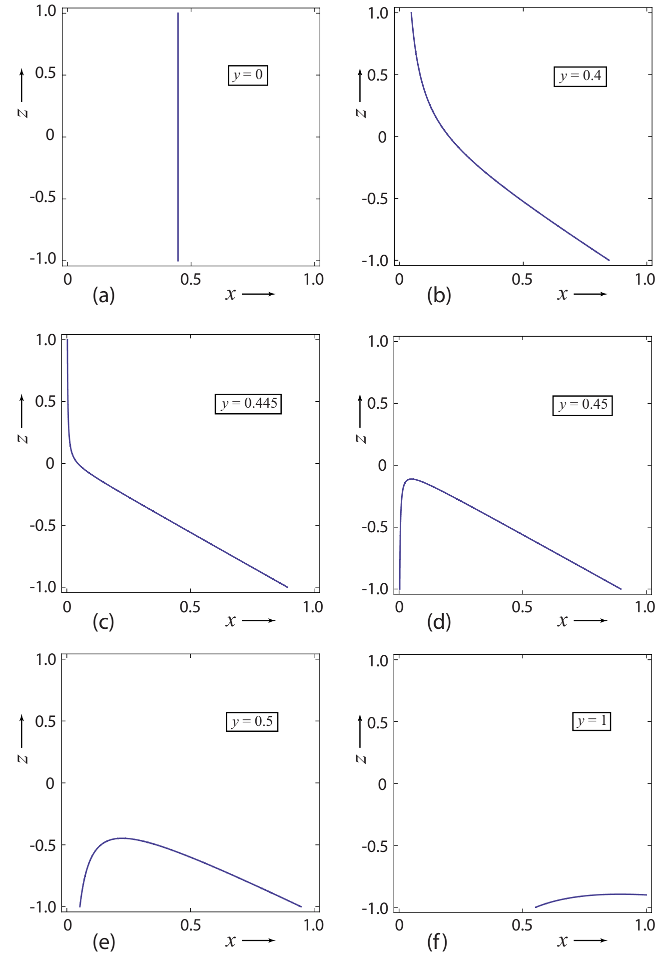

FIG. 6. (Color online) Cross sections of the coherence singularity in the x,z plane for selected values of y, *viz.*, (a) $y=0$, (b) $y=0.4$, (c) $y=0.445$, (d) $y=0.45$, (e)$y=0.5$, and (f) $y=1$.

both the horizontal and the vertical cross sections of the coherence singularity are quadratic curves. Horizontal cross sections (i.e., fixing the value of z and hence of ϕ_1) are conic sections in (x,y) space [30]; since x and y are limited to the interval $[0,1]$, only parts of these conic sections are realized. More specifically, if $z=-1$ the cross section takes the form of two parallel lines. On increasing z it becomes an ellipse (with a circle as a special case when $z=0$), and finally, for $z=1$, it becomes two parallel lines again (only one of which lies in the physical domain of x and y). Various cross sections of the coherence singularity are shown in Fig. 5 for selected values of z. Because of the interchangeable roles of x and y in Eq. (22), the cross sections are symmetric about the line $x=y$.

According to Eq. (22) vertical cross sections of the coherence singularities (e.g., fixing the value of y and hence of θ_2) are conic sections in (x,z) space; since $0 \leq x \leq 1$ and $-1 \leq z \leq 1$, only parts of these conic sections are realized. More specifically, if $y=0$ the cross section has the form of two parallel lines (only one of which lies in the physical domain of x and z). On increasing y it becomes a branch of a hyperbola, two intersecting lines, and again a branch of a hyperbola. This is illustrated in Fig. 6. This concludes our identification of various cross section of the coherence singularity with a variety of conic curves.

VI. CONCLUSIONS

We have analyzed the topology of coherence singularities that occur in the far field generated by quasihomogeneous sources. As an specific example we examined the coherence singularity of a partially coherent vortex beam. Its cross sections were found to be different kinds of conic sections in a modified coordinate system.

ACKNOWLEDGMENTS

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