Journal of the Optical Society of America A
OPTICS, IMAGE SCIENCE, AND VISION

Generalized Hanbury Brown-Twiss effect for Stokes parameters

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Received 29 November 2018; accepted 7 January 2019; posted 15 January 2019 (Doc. ID 353041); published 13 February 2019

The classic experiments by Hanbury Brown and Twiss (HBT) were concerned with the correlation of intensity fluctuations at two different positions in a wave field. We generalize the HBT effect that occurs in random electromagnetic beams by examining its polarization-resolved version. This leads naturally to the concept of correlations of fluctuations of the four Stokes parameters. We calculate the correlations of such "Stokes fluctuations" for the case of Gaussian statistics. When the two points of observation coincide, these correlations reduce to "Stokes scintillations." Our work reveals a new layer of complexity in random beams by showing that the HBT effect and the scintillation coefficient are just two of many correlations that are present. We illustrate that, in general, the fluctuations of the various Stokes parameters are all correlated by studying beams and sources with different polarization states. © 2019 Optical Society of America

https://doi.org/10.1364/JOSAA.36.000362

1. INTRODUCTION

In their landmark experiments, Hanbury Brown and Twiss measured the correlation of intensity fluctuations at two detectors. Observing how this correlation decreases as the detectors become more separated allowed them to determine the angular diameter of radio stars [1–3]. The eponymous HBT effect has since been applied in nuclear physics [4], quantum optics [5], and atomic physics [6,7]. In classical optics [8,9], it has been used for a variety of inverse problems [10,11], ghost imaging [12], optical coherence tomography [13], and holography [14].

In order to analyze a polarization-resolved version of the HBT effect, we recall that the state of polarization of an electromagnetic beam is characterized by the four Stokes parameters that, for fully polarized radiation, correspond to a unique point on the Poincaré sphere [15]. The correlation between the various instantaneous Stokes parameters ([16], Section 3.1.6.6) has been studied by several groups. Friberg *et al.* [17,18] used this correlation to introduce the concepts of polarization time and polarization length. These are measures of a duration of time and a distance, respectively, over which the state of polarization remains essentially unchanged. In [19,20], such Stokes correlations were investigated experimentally.

In an astronomical HBT experiment, the value of the time-averaged intensity at the two detectors is typically the same and independent of the distance that separates them. As described above, this is not true for the *correlation* of the intensity fluctuations that are observed. Interestingly, these fluctuation correlations contain information about the source that cannot be otherwise obtained [11]. The first Stokes parameter, denoted S_0, describes the total spectral density at a specific location. This means that the HBT coefficient is equivalent to the two-point correlation of the fluctuations of S_0. It seems natural to generalize this approach and examine the *correlation between the fluctuations* of the various Stokes parameters. Here, we develop a framework to explore all possible correlations of what we call *Stokes fluctuations*, (not to be confused with correlations of the Stokes parameters themselves). For the case of Gaussian statistics, these can be expressed in terms of second-order correlation functions [21]. Our results show that, in general, the fluctuations of any Stokes parameter at one particular point of observation are correlated with the fluctuations of all the other Stokes parameters at another point of observation. When the two observation points coincide, these fluctuations become *Stokes scintillations*. For the first Stokes parameter, S_0, this is equivalent to the scintillation coefficient of the beam. Generalizing the two concepts of the Hanbury Brown–Twiss effect and the scintillation coefficient reveals the existence of a level of correlations that has previously not been explored. Examples of the correlations of Stokes fluctuations and Stokes scintillations are presented for beams and sources with different states of polarization. Our results may be applied to cases where polarization fluctuations play a significant role, e.g., in studies on light–matter interactions [22], semiconductor lasers [23], supercontinuum generation [24], beam propagation [25], and cosmology [26].

2. STOKES FLUCTUATIONS

Consider an electromagnetic beam that propagates along the z direction. The longitudinal component of its electric field vector can then be neglected, and only the two orthogonal, transverse Cartesian components, E_x and E_y, need to be taken into account. The polarization properties of the beam are characterized by its four spectral Stokes parameters, which, at a position $\mathbf{r}$ at frequency ω, are defined as [15]

$$S_0(\mathbf{r},\omega) = E_x^*(\mathbf{r},\omega)E_x(\mathbf{r},\omega) + E_y^*(\mathbf{r},\omega)E_y(\mathbf{r},\omega), \tag{1}$$

$$S_1(\mathbf{r},\omega) = E_x^*(\mathbf{r},\omega)E_x(\mathbf{r},\omega) - E_y^*(\mathbf{r},\omega)E_y(\mathbf{r},\omega), \tag{2}$$

$$S_2(\mathbf{r},\omega) = E_x^*(\mathbf{r},\omega)E_y(\mathbf{r},\omega) + E_y^*(\mathbf{r},\omega)E_x(\mathbf{r},\omega), \tag{3}$$

$$S_3(\mathbf{r},\omega) = \mathrm{i}[E_y^*(\mathbf{r},\omega)E_x(\mathbf{r},\omega) - E_x^*(\mathbf{r},\omega)E_y(\mathbf{r},\omega)]. \tag{4}$$

These real-valued parameters can be determined by intensity measurements (that are more robust than amplitude and phase measurements) using standard polarimetry techniques [27]. Their physical interpretation is discussed in ([15], Section 10.9.3).

For the case of a stochastic beam, the Stokes parameters are random quantities. The fluctuations around their average value (i.e., Stokes fluctuations) are defined as

$$\Delta S_n(\mathbf{r},\omega) = S_n(\mathbf{r},\omega) - \langle S_n(\mathbf{r},\omega)\rangle \qquad (n = 0,1,2,3), \tag{5}$$

where $S_n(\mathbf{r},\omega)$ is the Stokes parameter pertaining to a single realization of the beam, and $\langle S_n(\mathbf{r},\omega)\rangle$ denotes its ensemble average.

We will now examine how these Stokes fluctuations are correlated. All possible pairs of their 2-point correlations can be captured by introducing a 4-by-4 matrix $\mathbf{C}(\mathbf{r}_1,\mathbf{r}_2,\omega)$, with elements

$$C_{nm}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle \Delta S_n(\mathbf{r}_1,\omega)\Delta S_m(\mathbf{r}_2,\omega)\rangle \quad (n,m = 0,1,2,3). \tag{6}$$

It immediately follows that this matrix satisfies the symmetry relation

$$C_{nm}(\mathbf{r}_1,\mathbf{r}_2) = C_{mn}(\mathbf{r}_2,\mathbf{r}_1). \tag{7}$$

Furthermore, it is clear from the definition of $S_0(\mathbf{r},\omega)$, given by Eq. (1), that the element $C_{00}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ represents the correlation of the intensity fluctuations at $\mathbf{r}_1$ and $\mathbf{r}_2$. It is therefore identical to the traditional Hanbury Brown–Twiss coefficient. The three other diagonal elements, $C_{pp}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ with $p = 1, 2$, or 3, represent the autocorrelation of the fluctuations of S_p at two positions. The 12 off-diagonal elements represent 2-point cross-correlations of the Stokes fluctuations. The $\mathbf{C}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ matrix is a generalization of the HBT coefficient, a correlation of the fluctuations of just a single Stokes parameter, namely, S_0, to all possible correlations between the four Stokes parameters.

The second-order statistical properties of a partially coherent electromagnetic beam are described by its cross-spectral density matrix, which is defined as [21]

$$\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \begin{pmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \end{pmatrix}, \tag{8}$$

where all the matrix elements are functions of the same three variables and given by the expression

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle, \qquad (i,j = x,y). \tag{9}$$

As before, the angular brackets indicate an average taken over an ensemble of beam realizations. The expectation value of the Stokes parameters can be expressed in terms of the cross-spectral density matrix. It readily follows from the definitions in Eqs. (1)–(4) together with Eq. (9) that

$$\langle S_0(\mathbf{r},\omega)\rangle = W_{xx}(\mathbf{r},\mathbf{r},\omega) + W_{yy}(\mathbf{r},\mathbf{r},\omega), \tag{10}$$

$$\langle S_1(\mathbf{r},\omega)\rangle = W_{xx}(\mathbf{r},\mathbf{r},\omega) - W_{yy}(\mathbf{r},\mathbf{r},\omega), \tag{11}$$

$$\langle S_2(\mathbf{r},\omega)\rangle = W_{xy}(\mathbf{r},\mathbf{r},\omega) + W_{yx}(\mathbf{r},\mathbf{r},\omega), \tag{12}$$

$$\langle S_3(\mathbf{r},\omega)\rangle = \mathrm{i}[W_{yx}(\mathbf{r},\mathbf{r},\omega) - W_{xy}(\mathbf{r},\mathbf{r},\omega)]. \tag{13}$$

All preceding equations have an explicit frequency dependence, indicating that they are defined for a specific frequency component of the optical field. For brevity, we will no longer display this ω dependence. We next derive a general expression for the correlations of the Stokes fluctuations in terms of the cross-spectral density matrix and the Pauli spin matrices.

3. CORRELATION MATRIX FOR STOKES FLUCTUATIONS

The 2-by-2 identity matrix, $\boldsymbol{\sigma}^0$, and the three Pauli spin matrices are defined as

$$\boldsymbol{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\sigma}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{14}$$

respectively. The Stokes parameters can be expressed in terms of these four matrices as ([27], Section 22.2)

$$S_n(\mathbf{r}) = \mathbf{E}(\mathbf{r})^\dagger \boldsymbol{\sigma}^n \mathbf{E}(\mathbf{r}), \qquad (n = 0,1,2,3), \tag{15}$$

where † denotes the Hermitian conjugate, and

$$\mathbf{E}(\mathbf{r}) = \begin{pmatrix} E_x(\mathbf{r}) \\ E_y(\mathbf{r}) \end{pmatrix}. \tag{16}$$

Hence,

$$S_n(\mathbf{r}) = \sum_{a,b} \sigma_{ab}^n E_a^*(\mathbf{r})E_b(\mathbf{r}), \qquad (a,b = x,y). \tag{17}$$

We can now calculate the elements of the Stokes fluctuations matrix as follows:

$$C_{nm}(\mathbf{r}_1,\mathbf{r}_2) = \langle \Delta S_n(\mathbf{r}_1)\Delta S_m(\mathbf{r}_2)\rangle \tag{18}$$

$$= \langle S_n(\mathbf{r}_1)S_m(\mathbf{r}_2)\rangle - \langle S_n(\mathbf{r}_1)\rangle\langle S_m(\mathbf{r}_2)\rangle \tag{19}$$

$$= \sum_{a,b}\sum_{c,d} \sigma_{ab}^n \sigma_{cd}^m \langle E_a^*(\mathbf{r}_1)E_b(\mathbf{r}_1)E_c^*(\mathbf{r}_2)E_d(\mathbf{r}_2)\rangle - \sum_{a,b}\sigma_{ab}^n\langle E_a^*(\mathbf{r}_1)E_b(\mathbf{r}_1)\rangle \sum_{c,d}\sigma_{cd}^m\langle E_c^*(\mathbf{r}_2)E_d(\mathbf{r}_2)\rangle. \tag{20}$$

Let us now invoke the assumption of Gaussian statistics. This allows us to make use of the Gaussian moment theorem [28] and, hence, express the fourth-order correlations on the right-hand side of Eq. (20) as the sum of products of second-order correlations. Specifically,

$$C_{nm}(\mathbf{r}_1,\mathbf{r}_2)=\sum_{a,b}\sum_{c,d}\sigma^n_{ab}\sigma^m_{cd}[\langle E^*_a(\mathbf{r}_1)E_b(\mathbf{r}_1)\rangle\langle E^*_c(\mathbf{r}_2)E_d(\mathbf{r}_2)\rangle + \langle E^*_a(\mathbf{r}_1)E_d(\mathbf{r}_2)\rangle\langle E^*_c(\mathbf{r}_2)E_b(\mathbf{r}_1)\rangle] - \sum_{a,b}\sum_{c,d}\sigma^n_{ab}\sigma^m_{cd}\langle E^*_a(\mathbf{r}_1)E_b(\mathbf{r}_1)\rangle\langle E^*_c(\mathbf{r}_2)E_d(\mathbf{r}_2)\rangle \tag{21}$$

$$=\sum_{a,b}\sum_{c,d}\sigma^n_{ab}\sigma^m_{cd}W_{ad}(\mathbf{r}_1,\mathbf{r}_2)W^*_{bc}(\mathbf{r}_1,\mathbf{r}_2). \tag{22}$$

Because each $\boldsymbol{\sigma}$ matrix has only two nonzero elements, the total sum in Eq. (22) consists of just four terms for each choice of n and m. We can introduce normalized matrix elements, indicated by the superscript N, by defining

$$C^N_{nm}(\mathbf{r}_1,\mathbf{r}_2)\equiv\frac{C_{nm}(\mathbf{r}_1,\mathbf{r}_2)}{\langle S_0(\mathbf{r}_1)\rangle\langle S_0(\mathbf{r}_2)\rangle}. \tag{23}$$

On making use of Eq. (22), it can be seen that the sum of the four normalized diagonal elements has a clear physical meaning:

$$\sum_{m=0}^{3}C^N_{mm}(\mathbf{r}_1,\mathbf{r}_2)=2\frac{\{|W_{xx}(\mathbf{r}_1,\mathbf{r}_2)|^2+|W_{yy}(\mathbf{r}_1,\mathbf{r}_2)|^2+2\,\mathrm{Re}[W_{xx}(\mathbf{r}_1,\mathbf{r}_2)W^*_{yy}(\mathbf{r}_1,\mathbf{r}_2)]\}}{\langle S_0(\mathbf{r}_1)\rangle\langle S_0(\mathbf{r}_2)\rangle} \tag{24}$$

$$=2\frac{|\mathrm{Tr}\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2)|^2}{\mathrm{Tr}\mathbf{W}(\mathbf{r}_1,\mathbf{r}_1)\mathrm{Tr}\mathbf{W}(\mathbf{r}_2,\mathbf{r}_2)}=2|\eta(\mathbf{r}_1,\mathbf{r}_2)|^2. \tag{25}$$

Here, $\eta(\mathbf{r}_1,\mathbf{r}_2)$ denotes the spectral degree of coherence [21], the magnitude of which indicates the visibility of the interference pattern produced in Young's experiment with pinholes located at $\mathbf{r}_1$ and $\mathbf{r}_2$.

It is worth noting that, whereas the Stokes fluctuation correlations are described by Eq. (22), the average Stokes parameters themselves are related by the inequality

$$\langle S_1(\mathbf{r},\omega)\rangle^2+\langle S_2(\mathbf{r},\omega)\rangle^2+\langle S_3(\mathbf{r},\omega)\rangle^2\le\langle S_0(\mathbf{r},\omega)\rangle^2. \tag{26}$$

The equality holds only for the case of a fully polarized beam.

4. STOKES SCINTILLATIONS

Just as the HBT coefficient is a special case of the correlation of Stokes fluctuations, the scintillation coefficient of a random beam is a special case of *Stokes scintillations*. This can be seen by considering the behavior of the matrix $\mathbf{C}(\mathbf{r}_1,\mathbf{r}_2)$, defined by Eq. (6), when its two spatial arguments coincide. Let us use Eqs. (10)–(13) to express the elements of the cross-spectral density matrix $W_{ij}(\mathbf{r},\mathbf{r})$ in terms of the four ensemble-averaged Stokes parameters. The resulting formulas are

$$W_{xx}(\mathbf{r},\mathbf{r})=\frac{1}{2}[\langle S_0(\mathbf{r})\rangle+\langle S_1(\mathbf{r})\rangle], \tag{27}$$

$$W_{yy}(\mathbf{r},\mathbf{r})=\frac{1}{2}[\langle S_0(\mathbf{r})\rangle-\langle S_1(\mathbf{r})\rangle], \tag{28}$$

$$W_{xy}(\mathbf{r},\mathbf{r})=\frac{1}{2}[\langle S_2(\mathbf{r})\rangle+\mathrm{i}\langle S_3(\mathbf{r})\rangle], \tag{29}$$

$$W_{yx}(\mathbf{r},\mathbf{r})=\frac{1}{2}[\langle S_2(\mathbf{r})\rangle-\mathrm{i}\langle S_3(\mathbf{r})\rangle]. \tag{30}$$

On making use of these expressions in Eq. (22), we find for the four diagonal elements of the Stokes scintillation matrix $D_{nm}(\mathbf{r})\equiv C_{nm}(\mathbf{r},\mathbf{r})$ that

$$D_{00}(\mathbf{r})=\frac{1}{2}[\langle S_0(\mathbf{r})\rangle^2+\langle S_1(\mathbf{r})\rangle^2+\langle S_2(\mathbf{r})\rangle^2+\langle S_3(\mathbf{r})\rangle^2], \tag{31}$$

$$D_{11}(\mathbf{r})=\frac{1}{2}[(\langle S_0(\mathbf{r})\rangle^2+\langle S_1(\mathbf{r})\rangle^2-\langle S_2(\mathbf{r})\rangle^2-\langle S_3(\mathbf{r})\rangle^2], \tag{32}$$

$$D_{22}(\mathbf{r})=\frac{1}{2}[\langle S_0(\mathbf{r})\rangle^2-\langle S_1(\mathbf{r})\rangle^2+\langle S_2(\mathbf{r})\rangle^2-\langle S_3(\mathbf{r})\rangle^2], \tag{33}$$

$$D_{33}(\mathbf{r})=\frac{1}{2}[\langle S_0(\mathbf{r})\rangle^2-\langle S_1(\mathbf{r})\rangle^2-\langle S_2(\mathbf{r})\rangle^2+\langle S_3(\mathbf{r})\rangle^2], \tag{34}$$

whereas the 12 off-diagonal elements are given by the expressions

$$D_{pq}(\mathbf{r})=\langle S_p(\mathbf{r})\rangle\langle S_q(\mathbf{r})\rangle,\qquad (p\neq q;\ \text{and } p,q=0,1,2,3). \tag{35}$$

The matrix $\mathbf{D}$ is seen to be symmetric and has diagonal elements of the form $D_{nn}(\mathbf{r})=\langle[\Delta S_n(\mathbf{r})]^2\rangle$. It is a generalization of the scintillation coefficient, which is the variance of S_0, to all possible variances of the Stokes parameters. The element $D_{00}(\mathbf{r})$ represents the usual intensity scintillation at position $\mathbf{r}$. The other three diagonal elements $D_{pp}(\mathbf{r})$ with $p=1,2,$ or 3 represent the variance of S_1, S_2, and S_3, respectively. The 12 off-diagonal matrix elements describe all possible 1-point cross-correlations of the fluctuations of the Stokes parameters.

We can again introduce normalized matrix elements by defining

$$D^N_{nm}(\mathbf{r})\equiv\frac{D_{nm}(\mathbf{r})}{\langle S_0(\mathbf{r})\rangle^2}. \tag{36}$$

The first element, $D_{00}^N(\mathbf{r})$, is the square of the usual scintillation index [29]. The assumption of Gaussian statistics implies that it is bounded [30], i.e.,

$$1/2 \leq D_{00}^N(\mathbf{r}) \leq 1. \tag{37}$$

On making use of the fact that $\eta(\mathbf{r},\mathbf{r}) = 1$ in Eq. (25), it readily follows that

$$\sum_{m=0}^{3} D_{mm}^N(\mathbf{r}) = 2. \tag{38}$$

This result shows that the normalized, diagonal Stokes scintillations are not independent of each other because their sum equals two. Both Eqs. (25) and (38) hold for any stochastic electromagnetic beam that is generated by a source that is governed by Gaussian statistics.

It is worth noting that the Stokes fluctuations and the Stokes scintillations can be measured using a narrowband spectral filter together with a division-of-amplitude photopolarimeter, see, for example, [31] and the references therein. Having established this general formalism, we next discuss some specific examples.

5. EXAMPLES

I. As a particular example of a fully polarized beam, consider the case where

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2) = E_i^*(\mathbf{r}_1)E_j(\mathbf{r}_2). \tag{39}$$

We note that the matrix elements now factorize, and there is no ensemble averaging involved. On substituting from Eq. (39) into the definition of the degree of polarization, namely, [21]

$$P(\mathbf{r}) = \sqrt{1 - \frac{4\,\mathrm{Det}\,W(\mathbf{r},\mathbf{r})}{[\mathrm{Tr}W(\mathbf{r},\mathbf{r})]^2}}, \tag{40}$$

it is readily seen that $P(\mathbf{r}) = 1$, i.e., the beam is indeed fully polarized. Furthermore, Eq. (22) now takes on the form

$$C_{nm}(\mathbf{r}_1,\mathbf{r}_2) = \sum_{a,b}\sum_{c,d} \sigma_{ab}^n \sigma_{cd}^m E_a^*(\mathbf{r}_1)E_d(\mathbf{r}_2)E_b(\mathbf{r}_1)E_c^*(\mathbf{r}_2) \tag{41}$$

$$= \sum_{a,b} \sigma_{ab}^n E_a^*(\mathbf{r}_1)E_b(\mathbf{r}_1) \times \sum_{c,d} \sigma_{cd}^m E_c^*(\mathbf{r}_2)E_d(\mathbf{r}_2) \tag{42}$$

$$= S_n(\mathbf{r}_1)S_m(\mathbf{r}_2). \tag{43}$$

It is worth pointing out that, for this case, all cross-correlations can be nonzero.

The Stokes scintillation matrix for this case is obtained by setting $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$ in Eq. (43). This means that

$$D(\mathbf{r})_{nm} = S_n(\mathbf{r})S_m(\mathbf{r}). \tag{44}$$

II. As an example of a partial polarization, we study a beam whose cross-spectral density matrix elements are

$$W_{xx}(\mathbf{r}_1,\mathbf{r}_2) \neq 0, \tag{45}$$

$$W_{yy}(\mathbf{r}_1,\mathbf{r}_2) \neq 0, \tag{46}$$

$$W_{xy}(\mathbf{r}_1,\mathbf{r}_2) = W_{yx}(\mathbf{r}_1,\mathbf{r}_2) = 0. \tag{47}$$

The two nonzero diagonal elements need not be equal. A beam of this type can be realized by superposing two independent beams, one that is x polarized and the other y polarized. Their respective amplitudes and transverse coherence widths can be chosen arbitrarily. On substituting into Eq. (40), one finds for the degree of polarization the expression

$$P(\mathbf{r}) = \left|\frac{W_{xx}(\mathbf{r}) - W_{yy}(\mathbf{r})}{W_{xx}(\mathbf{r}) + W_{yy}(\mathbf{r})}\right|, \tag{48}$$

which is between 0 and 1, indicating that the beam is indeed partially polarized. We now have that

$$\mathbf{C}(\mathbf{r}_1,\mathbf{r}_2) = \begin{pmatrix} |W_{xx}|^2 + |W_{yy}|^2 & |W_{xx}|^2 - |W_{yy}|^2 & 0 & 0 \\ |W_{xx}|^2 - |W_{yy}|^2 & |W_{xx}|^2 + |W_{yy}|^2 & 0 & 0 \\ 0 & 0 & 2\,\mathrm{Re}[W_{xx}^* W_{yy}] & 2\,\mathrm{Im}[W_{xx}^* W_{yy}] \\ 0 & 0 & 2\,\mathrm{Im}[W_{xx} W_{yy}^*] & 2\,\mathrm{Re}[W_{xx} W_{yy}^*] \end{pmatrix}, \tag{49}$$

with the $(\mathbf{r}_1,\mathbf{r}_2)$ dependence of the matrix elements suppressed for brevity. We note that, unlike the HBT coefficient, the other Stokes fluctuation correlations are not always positive. For example, if $|W_{yy}(\mathbf{r}_1,\mathbf{r}_2)| > |W_{xx}(\mathbf{r}_1,\mathbf{r}_2)|$, then both C_{01} and C_{10} are negative.

For the Stokes scintillation matrix, we obtain

$$\mathbf{D}(\mathbf{r}) = \begin{pmatrix} \frac{1}{2}[\langle S_0\rangle^2 + \langle S_1\rangle^2] & \langle S_0\rangle\langle S_1\rangle & 0 & 0 \\ \langle S_0\rangle\langle S_1\rangle & \frac{1}{2}[\langle S_0\rangle^2 + \langle S_1\rangle^2] & 0 & 0 \\ 0 & 0 & \frac{1}{2}[\langle S_0\rangle^2 - \langle S_1\rangle^2] & 0 \\ 0 & 0 & 0 & \frac{1}{2}[\langle S_0\rangle^2 - \langle S_1\rangle^2] \end{pmatrix}, \tag{50}$$

with the $\mathbf{r}$ dependence of the various Stokes parameters not being displayed. Notice that beams that are purely x or y polarized are special cases of beams of this kind.

For the sake of completeness, we state without proof that, for the case of a completely unpolarized beam, both $\mathbf{C}(\mathbf{r}_1,\mathbf{r}_2)$ and $\mathbf{D}(\mathbf{r})$ are proportional to the identity matrix.

III. As a final example, we consider the generalized scintillations that occur in a planar Gaussian–Schell model (GSM) source. In that case [21],

$$W_{ij}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = A_i A_j B_{ij} \exp\left[-\left(\frac{\rho_1^2}{4\sigma_i^2} + \frac{\rho_2^2}{4\sigma_j^2}\right)\right] \exp\left[-\frac{(\rho_2 - \rho_1)^2}{2\delta_{ij}^2}\right]. \tag{51}$$

Here, $\boldsymbol{\rho}_i = (x_i, y_i)$ denotes a transverse position in the source plane, A_i is the spectral amplitude of E_i, and B_{ij} is the correlation coefficent between E_i and E_j. The symbols σ_i and δ_{ij} represent effective widths and coherence radii, respectively. We will restrict ourselves to the case $\sigma_x = \sigma_y = \sigma$. On making use of Eq. (51), we find for the normalized, diagonal Stokes scintillations in the source plane the expressions

$$D_{00}^N = \frac{A_x^4 + A_y^4 + 2A_x^2 A_y^2 |B_{xy}|^2}{(A_x^2 + A_y^2)^2}, \quad (52)$$

$$D_{11}^N = \frac{A_x^4 + A_y^4 - 2A_x^2 A_y^2 |B_{xy}|^2}{(A_x^2 + A_y^2)^2}, \quad (53)$$

$$D_{22}^N = \frac{2A_x^2 A_y^2 [1 + |B_{xy}|^2 \cos(2\phi)]}{(A_x^2 + A_y^2)^2}, \quad (54)$$

$$D_{33}^N = \frac{2A_x^2 A_y^2 [1 - |B_{xy}|^2 \cos(2\phi)]}{(A_x^2 + A_y^2)^2}, \quad (55)$$

where ϕ denotes the angle, or phase, of the complex-valued coefficient B_{xy}. We note that, whereas D_{00}^N and D_{11}^N are independent of this angle, the other two Stokes scintillations display a harmonic dependence. It is seen that the scintillations are all uniform, i.e., they are independent of the transverse position $\boldsymbol{\rho}$. Furthermore, it is easily verified that their sum equals 2, in agreement with Eq. (38). This means that a GSM source that is designed to have a reduced scintillation D_{00}^N will have an increased scintillation of some of the other Stokes parameters.

6. CONCLUSIONS

We have developed a formalism to analyze the correlations of the fluctuations of the four Stokes parameters that occur in random electromagnetic beams. These correlations can be interpreted as a polarization-resolved generalization of the Hanbury Brown–Twiss effect. Under the assumption that the source fluctuations obey Gaussian statistics, they can be described in terms of the second-order cross-spectral density matrix. It is found, in general, that the fluctuations of each Stokes parameter at a certain point are correlated with the fluctuations of all other Stokes parameters elsewhere. Furthermore, the sum of the diagonal Stokes fluctuations was shown to be related to the spectral degree of coherence, i.e., to the fringe visibility in Young's experiment.

We also derived expressions for the variance of the Stokes parameters at a single point. These are a natural generalization of the usual scintillation coefficient. The sum of the normalized diagonal scintillation coefficients equals 2, showing that they are not independent. This means that a decreased scintillation of S_0 comes at a cost, namely, an increased variance of other Stokes parameters. Whether or not such an increase is acceptable will be highly context-dependent.

Examples of beams with different states of polarization were presented. In those examples, the Stokes fluctuations and Stokes scintillations matrices take on different forms. For the case of a Gaussian–Schell model source, expressions were derived that show the dependence of the Stokes scintillations on the source parameters.

Observing the Stokes fluctuations and their correlations provides a new way to characterize random electromagnetic beams. Our results may be applied to the wide range of systems, discussed in [22–26], in which polarization fluctuations play an important role.

Funding. Air Force Office of Scientific Research (AFOSR) (FA9550-16-1-0119).

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