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Spectral invariance of Gaussian Schell-model beams

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Abstract: It is well known that in general the spectrum of a beam that is generated by a partially coherent source will change on propagation. Here we derive necessary and sufficient conditions under which the often-used Gaussian Schell-model sources can produce beams whose normalized spectrum is invariant everywhere, or is invariant just along the beam axis. These sources are not necessarily quasi-homogeneous or obeying the scaling law.

1. Introduction

Due to the pioneering work of Wolf it is now generally appreciated that, in general, the spectrum of the field that is generated by a partially coherent source changes on propagation [1–4]. An exception are quasi-homogeneous sources [5, Sec. 5.3.2] that obey the scaling law [6–8]. These are planar, secondary sources that, at points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ and frequency ω, are characterized by a spectral degree of coherence $\mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega)$ that depends on $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ only through their difference, i.e.,

$$\mu^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \mu^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1; \omega). \tag{1}$$

Here the superscript (0) indicates the source plane $z = 0$. Furthermore, at each frequency the spectral density $S^{(0)}(\boldsymbol{\rho}, \omega)$ of these sources varies much slower with $\boldsymbol{\rho}$ than $\mu^{(0)}$ varies with $\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$. If, in addition, the frequency dependence of $\mu^{(0)}$ has the functional form

$$\mu^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1; \omega) = \mu^{(0)}[k(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)], \tag{2}$$

with k the wavenumber associated with frequency ω, then everywhere the normalized spectrum of the radiated field is equal to the normalized source spectrum. Equation (2) is referred to as the scaling law. Examples of quasi-homogeneous sources that satisfy this law, and hence produce spectrally invariant fields, are certain LEDs [9] and thermal sources [6].

Apart from the researches we just mentioned, most studies do not explore the possibility of the source width or the spectral degree of coherence being frequency-dependent. Notable exceptions are [10–12] in which frequency-dependent source parameters are shown to cause significant spectral changes. Taking the opposite approach, we investigate if such a frequency dependence can lead to spectrally invariant beams. A related investigation was reported in [13]. There conditions were derived under which the spatial distribution of the spectral density remains invariant on propagation, apart from a transverse scale modification.

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We report two new classes of Gaussian Schell-model (GSM) sources. One generates beams whose normalized spectrum is invariant everywhere, the other produces beams whose spectrum is invariant just along the propagation axis. These sources need not be of the quasi-homogeneous type, and they do not necessarily satisfy the scaling law. GSM sources are the workhorse of coherence theory [5, Sec. 5.4] and numerous studies have been dedicated to the fields that they generate. Because of their ubiquity, and their use as being representative of a general source, a better understanding of the spectral changes that they may or may not produce is useful. Furthermore, GSM beams are candidates for telecommunication applications because they are known to be more resistant to atmospheric turbulence than their fully coherent counterparts [14]. The GSM sources that we describe in this study would also display this robustness, but with the additional advantage of having a spectrum that remains invariant.

2. Gausian Schell-model sources

A scalar, planar, secondary GSM source (see Fig. 1) is described by a cross-spectral density function

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; k) = \sqrt{S^{(0)}(\boldsymbol{\rho}_1; k)}\sqrt{S^{(0)}(\boldsymbol{\rho}_2; k)}\mu^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1; k). \tag{3}$$

Both the spectral density and the spectral degree of coherence are assumed to have a Gaussian form, i.e.,

$$S^{(0)}(\boldsymbol{\rho}; k) = S_0(k) e^{-\rho^2/2\sigma^2(k)}, \tag{4}$$

$$\mu^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1; k) = e^{-(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2/2\delta^2(k)}. \tag{5}$$

Here $S_0(k)$ is the spectrum at the center O of the source, $\sigma(k)$ denotes the effective source width, and $\delta(k)$ represents the transverse coherence length (or coherence radius). As our notation suggests, we explore the possibility that the latter two quantities depend on the wavenumber. A GSM source will produce a beam-like field if, for each wavenumber k that is present in the source spectrum, the inequality [5, Eq. (5.6–73)],

$$\frac{1}{4\sigma^2(k)} + \frac{1}{\delta^2(k)} \ll \frac{k^2}{2} \tag{6}$$

holds. If $\delta(k) \ll \sigma(k)$ the source is quasi-homogeneous. However, we will not assume quasi-homogeneity.

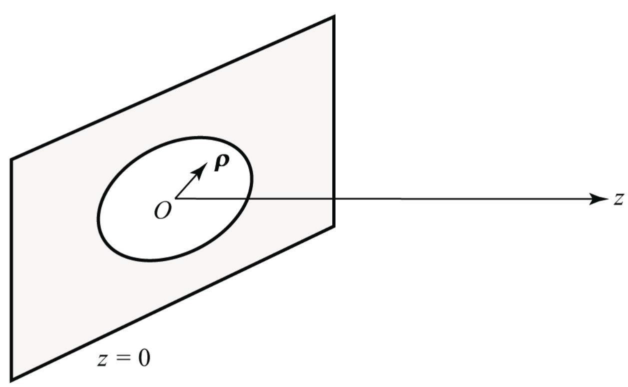

Fig. 1. A planar, secondary Gaussian Schell-model source occupies the plane $z = 0$ and radiates into the half-space $z>0$. The vector $\boldsymbol{\rho}$ denotes a position transverse to the z axis.

The spectral density at an arbitrary point of observation $\mathbf{r} = (\boldsymbol{\rho}, z)$ in the half-space $z \geq 0$ follows from the cross-spectral density function through the relation

$$S(\boldsymbol{\rho}, z; k) = W(\boldsymbol{\rho}, \boldsymbol{\rho}, z; k). \tag{7}$$

It is important to distinguish the spectral density and the normalized spectral density. The latter is defined as

$$s(\boldsymbol{\rho}, z; k) = \frac{S(\boldsymbol{\rho}, z; k)}{\int_0^\infty S(\boldsymbol{\rho}, z; k')\,\mathrm{d}k'}. \tag{8}$$

On propagation to a plane $z \geq 0$ the cross-spectral density function of a GSM beam is given by the scalar version of [15, p. 184, Eq. (10)], with an obvious change in notation and with the penultimate last minus sign corrected:

$$\begin{aligned} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z; k) = {} & \frac{S_0(k)}{\Delta^2(z;k)} \exp\left[-\frac{(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)^2}{8\sigma^2(k)\Delta^2(z;k)}\right] \\ & \times \exp\left[-\frac{(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2}{2\Omega^2(k)\Delta^2(z;k)}\right] \exp\left[\frac{\mathrm{i}k(\rho_2^2 - \rho_1^2)}{2R(z;k)}\right], \end{aligned} \tag{9}$$

where the beam expansion coefficient equals

$$\left(\frac{z}{k\sigma(k)\Omega(k)}\right)^2, \tag{10}$$

with

$$\frac{1}{\Omega^2(k)} = \frac{1}{4\sigma^2(k)} + \frac{1}{\delta^2(k)}, \tag{11}$$

and

$$R(z;k) = z\left[1 + \frac{k^2\sigma^2(k)}{z^2}\Omega^2(k)\right]. \tag{12}$$

The spectral density of the propagated beam is then, according to Eq. (7), given by the expression

$$S(\boldsymbol{\rho}, z; k) = \frac{S_0(k)}{\Delta^2(z;k)} \exp\left[-\frac{\rho^2}{2\sigma^2(k)\Delta^2(z;k)}\right]. \tag{13}$$

Spectral invariance in the $z \geq 0$ half space – Let us first restrict our attention to points on the beam axis ($\boldsymbol{\rho} = \mathbf{0}$). For these points Eq. (13) reduces to

$$S(\mathbf{0}, z; k) = \frac{S_0(k)}{\Delta^2(z;k)}, \tag{14}$$

and hence the normalized spectral density equals

$$s(\mathbf{0}, z; k) = \frac{S_0(k)/\Delta^2(z;k)}{\int_0^\infty S_0(k')/\Delta^2(z;k')\,\mathrm{d}k'}. \tag{15}$$

If the expansion coefficient $\Delta^2(z;k)$ is independent of the wavenumber k, this simplifies to

$$s(\mathbf{0}, z; k) = \frac{S_0(k)}{\int_0^\infty S_0(k')\,\mathrm{d}k'}, \quad \text{for all } z \geq 0. \tag{16}$$

Clearly, this expression has no z dependence. In other words, the normalized spectral density along the entire beam axis is invariant provided that $\Delta^2(z;k)$ has no frequency dependence. It

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can be shown that this condition is not just sufficient but also necessary (see Appendix 1). It is seen from Eq. (10) that this condition is fulfilled if

$$\frac{1}{k^2\sigma^2(k)}\left[\frac{1}{4\sigma^2(k)}+\frac{1}{\delta^2(k)}\right]=C^2, \tag{17}$$

where C is some frequency-independent constant with dimension $[\mathrm{m}^{-1}]$. Solving for $\delta^2(k)$ we find that

$$\delta^2(k)=\frac{4\sigma^2(k)}{4C^2k^2\sigma^4(k)-1}. \tag{18}$$

On using this in Eq. (5) it follows that, in general, this does *not* describe a source whose spectral degree of coherence satisfies the scaling law (2).

The constant C cannot be chosen arbitrarily. The left-hand side of Eq. (18), being positive, implies that

$$C^2>\frac{1}{4k^2\sigma^4(k)}, \tag{19}$$

for all wavenumbers k that are present in the source spectrum. Together with the beam condition (6) we thus find two constraints for C^2, namely

$$\frac{1}{4k^2\sigma^4(k)}<C^2\ll\frac{1}{2\sigma^2(k)}. \tag{20}$$

It is possible to construct a GSM source that generates a beam with an invariant normalized spectrum not just on the axis, but in the entire half-space $z\geq 0$. This only happens when Eq. (18) is satisfied, and in addition the source width $\sigma^2(k)$ does *not* depend on the wavenumber, i.e., when

$$\sigma^2(k)=B^2, \tag{21}$$

with B a constant length. Only in that case, namely, the exponential in Eq. (13), and therefore the entire right-hand side of that equation, is independent of frequency. On substituting from Eq. (21) into Eq. (18) we obtain

$$\delta^2(k)=\frac{4B^2}{4C^2k^2B^4-1}. \tag{22}$$

On using this result in Eq. (3) we thus find that *a secondary, planar GSM source will generate a beam whose normalized spectrum everywhere in the half-space $z\geq 0$ is invariant if and only if its cross-spectral density function is of the form*

$$W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;k)=S_0(k)\exp\left[-\frac{(\rho_1^2+\rho_2^2)}{4B^2}\right]\exp\left[-\frac{(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2}{2\delta^2(k)}\right], \tag{23}$$

with B a constant length, and with the coherence radius $\delta^2(k)$ given by Eq. (22).

As mentioned above, a source described by (23) does not necessarily satisfy the scaling law (2), and furthermore it need not be quasi-homogeneous. The latter can be seen as follows. The inequalities (19) must hold for all wavenumbers present in the source spectrum $S_0(k)$. If we denote the lowest of them by k_{min}, then

$$C^2>\frac{1}{4k_{\mathrm{min}}^2B^4}. \tag{24}$$

Let us now choose a value for C^2 such that

$$C^2=\frac{1}{4k_0^2B^4}, \tag{25}$$

with $k_0<k_{\mathrm{min}}$. We can then use Eq. (22) to plot $\delta(k)$ as a function of the wavenumber k. An example is shown in Fig. 2 for a band-limited source with $k_{\mathrm{min}}<k<k_{\mathrm{max}}$. It is seen that the

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coherence radius $\delta(k)$ and the source width $\sigma = B$ are comparable in magnitude across the entire spectral range. Therefore the source is clearly not quasi-homogeneous, but it nevertheless generates a beam that is spectrally invariant in the entire half space into which the source radiates.

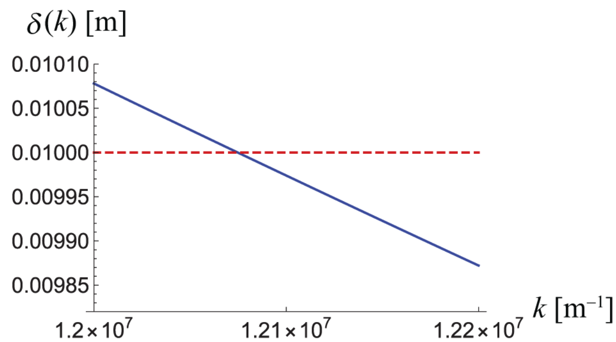

Fig. 2. The coherence radius $\delta(k)$ as function of the wavenumber k (solid blue curve). For comparison's sake the effective source width B is also shown (horizontal dashed line). In this example $k_0 = 5.4 \times 10^6$ m^{-1}, $k_{\mathrm{min}} = 1.20 \times 10^7$ m^{-1}, $k_{\mathrm{max}} = 1.22 \times 10^7$ m^{-1}, and $\sigma = B = 1$ cm. The horizontal axis is restricted to the band-limited source spectrum.

3. Spectral invariance only along the beam axis

We saw above that a necessary and sufficient condition for spectral invariance along the beam axis is given by Eq. (17). One possible alternative solution (not the most general) can be found by assuming that the effective source width and the transverse coherence length have a k dependence that is of the form

$$\sigma^2(k) = ak^{\alpha}, \tag{26}$$

$$\delta^2(k) = bk^{\beta}, \tag{27}$$

respectively, with a and b two positive constants that are independent of frequency, and with the powers α and β to be determined. On using these two Ansätze in Eq. (18) and collecting identical powers in k it follows that $\Delta(z; k)$ will be k-independent if

$$\alpha = \beta, \tag{28}$$

$$2 + 2\alpha = 0. \tag{29}$$

Thus we readily find that

$$\alpha = \beta = -1, \tag{30}$$

$$b = \frac{4a}{4C^2a^2 - 1}. \tag{31}$$

and hence

$$\sigma^2(k) = a/k, \tag{32}$$

$$\delta^2(k) = b/k, \tag{33}$$

with a and b two constant lengths related by Eq. (31). When the source width σ and the coherence radius δ satisfy Eqs. (32) and (33) the expansion coefficient $\Delta(z; k)$ is again independent of frequency, ensuring that the normalized spectrum along the beam axis is invariant. On making use of Eq. (3) we thus conclude that *a sufficient condition for a secondary, planar GSM source to* *generate a beam whose normalized spectrum everywhere on the propagation axis is the same is for its cross-spectral density function to be of the form*

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; k) = S_0(k) \exp\left[-\frac{k(\boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2)}{4a}\right] \times \exp\left[-\frac{k(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2}{2b}\right], \tag{34}$$

with the lengths a and b related by Eq. (31). The presence of a factor k rather than k^2 in the last exponent shows that such sources do not satisfy the scaling law (2).

It is interesting to note that for the case of GSM sources the beam expansion factor Δ^2 is equivalent to the transverse scale factor M that is used in [13] to discuss the so-called shape invariance of polychromatic fields. It can be shown that for such sources the concept of shape invariance is equivalent with spectral invariance on the beam axis.

The spectral density of the field at off-axis points is described by Eq. (13). Even when the expansion coefficient Δ^2 is independent of the wavenumber, the normalized spectrum at those points will *not* be invariant, due to the k-dependence of the product $\sigma^2(k)\Delta^2(z;k)$ in the exponential. The presence of this factor gives rise to a red-shifted spectrum. The magnitude of this shift can be remarkably small as we now show. As z tends to infinity, the first term on the right-hand side of Eq. (10) may be be neglected, and the exponential in Eq. (13) becomes

$$\exp\left[-\frac{\rho^2}{2\sigma^2(k)\Delta^2(z;k)}\right] = \exp\left\{-\frac{\rho^2}{z^2}\frac{k^2}{2}\left[\frac{1}{4\sigma^2(k)} + \frac{1}{\delta^2(k)}\right]^{-1}\right\}. \tag{35}$$

On introducing the polar angle $\theta \approx \tan\theta = \rho/z$ and making use of Eqs. (32) and (33), we get for the spectral density the expression

$$S(\theta;k) = \frac{S_0(k)}{\Delta^2(z;k)} \exp\left(-\frac{2\theta^2 kab}{b+4a}\right), \quad \text{as } z \to \infty, \tag{36}$$

and hence the normalized spectrum given by Eq. (8) equals

$$s(\theta;k) = \frac{S_0(k)\exp\left[-2\theta^2 kab/(b+4a)\right]}{\int_0^\infty S_0(k')\exp\left[-2\theta^2 k'ab/(b+4a)\right]\,\mathrm{d}k'}, \quad \text{as } z \to \infty. \tag{37}$$

As an example, we choose a Gaussian spectrum centered around $\bar{k}$, with a width δ_k, i.e.,

$$S_0(k) = A_0^2 \exp\left[-\frac{(k-\bar{k})^2}{2\delta_k^2}\right], \tag{38}$$

with A_0 a positive constant. Clearly, this spectrum reaches its maximum at wavenumber $k_{\max} = \bar{k}$. The relative shift of this maximum

$$\Delta_{\max}(\theta) = \frac{k_{\max}(\theta) - \bar{k}}{\bar{k}} \tag{39}$$

of the far-zone normalized spectral density is illustrated in Fig. 3. It is seen that the shift is indeed always negative, i.e., it is red-shifted, and increases when the polar angle θ increases. Also, the shift is less than 1 part per million over the entire width of the beam. This is orders of magnitude less than the spectral changes reported in, for example, [11].

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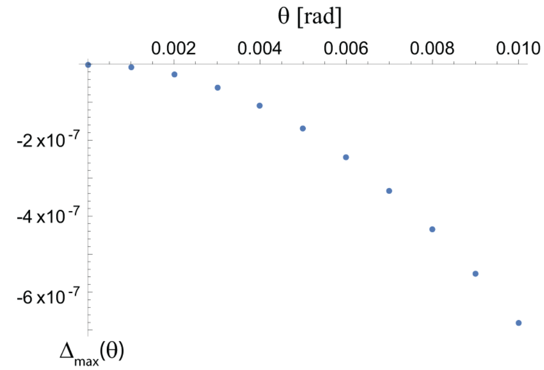

Fig. 3. The shift $\Delta_{\max}(\theta)$ of the maximum of the far-zone normalized spectrum for selected values of the observation angle θ. In this example $\bar{k} = 1.50 \times 10^7\ \mathrm{m}^{-1}$ (corresponding to a peak wavelength $\lambda = 419$ nm), $\delta_k = 1 \times 10^4\ \mathrm{m}^{-1}$, $a = 1$ mm, $C = 700\,\mathrm{m}^{-1}$, and hence $b = 4.16$ mm. The angular half-width of the spectral density distribution is 6 mrad.

4. Concluding remarks

Although in general the field that is generated by a Gausian Schell-model source will have a spectrum that changes on propagation, we have derived two conditions under which such a source generates a beam that is spectrally invariant. Both conditions prescribe a certain frequency dependence of the source width and its spatial correlation length. The first condition describes a source with a normalized spectral density that is the same in the entire half space into which the source radiates. The second condition describes a source that produces an invariant normalized spectral density along the beam axis. Such sources do not necessarily belong to the previously studied class of quasi-homogeneous sources that satisfy the scaling law.

Our results may be useful in applications where the advantages of partial coherence, such as reduced speckle and increased robustness with respect to atmospheric turbulence, are required but where the Wolf effect is detrimental.

Appendix 1

Consider a source that produces a field whose normalized spectral density, as defined by Eq. (15), is invariant along the beam axis. If we then equate $s(\mathbf{r};k)$ at $\mathbf{r} = (\mathbf{0}, 0)$ and $\mathbf{r} = (\mathbf{0}, z)$, we obtain

$$\frac{S_0(k)}{\int_0^\infty S_0(k')\,\mathrm{d}k'} = \frac{S_0(k)/\Delta^2(z;k)}{\int_0^\infty S_0(k'')/\Delta^2(z;k'')\,\mathrm{d}k''}, \tag{40}$$

where we made use of the fact that

$$\Delta^2(0;k) = 1. \tag{41}$$

We thus get that

$$\Delta^2(z;k) = \frac{\int_0^\infty S_0(k')\,\mathrm{d}k'}{\int_0^\infty S_0(k'')/\Delta^2(z;k'')\,\mathrm{d}k''}. \tag{42}$$

The right-hand side of Eq. (42) does not depend on the wavenumber k, and therefore the left-hand side does not either. In other words, spectral invariance along the z axis implies that the expansion coefficient Δ^2 must be independent of the wavenumber.

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Disclosures

The authors declare no conflicts of interest.

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