Optics Communications 295 (2013) 11–16

Contents lists available at SciVerse ScienceDirect

Optics Communications

journal homepage: www.elsevier.com/locate/optcom

Far-zone properties of electromagnetic beams generated by quasi-homogeneous sources

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ARTICLE INFO

Article history:
Received 30 October 2012
Received in revised form
10 January 2013
Accepted 11 January 2013
Available online 29 January 2013

Keywords:
Quasi-homogeneous sources
Electromagnetic beams
Coherence
Diffraction
Polarization
Spectral changes

ABSTRACT

We derive expressions for the far-zone properties of electromagnetic beams generated by a broad class of partially coherent sources, namely those of the quasi-homogeneous type. We use these reciprocity relations to study the intensity distribution, the state of coherence and the polarization properties of such beams.

1. Introduction

The fully coherent, monochromatic beams that are often encountered in the literature are idealizations. In practice, optical fields are partially coherent. This may be due to several causes. The source may emit several modes, or it may fluctuate due to mechanical vibrations or quantum noise. In addition, if the field propagates through a fluctuating random medium such as the atmosphere, its coherence will degrade. Partially coherent beams have some interesting properties. For example, they may have the same directionality as nearly coherent laser beams, while not giving rise to unwanted speckle [1, Section 5.4.2]. Equally important, the state of coherence of a field can be controlled to optimize it for certain applications such as propagation through atmospheric turbulence [2], optical coherence tomography [3], and the trapping of low refractive index particles [4]. Accounts of partially coherent fields and their applications are given, for example, in [1,5,6].

The majority of the studies dealing with partially coherent beams is concerned with beams that are generated by so-called Gaussian Schell model (GSM) sources, see for example [7–11]. Another important class of partially coherent sources, which partially overlaps with those of the GSM type, is formed by so-called *quasi-homogeneous planar sources* [1, Section 5.3.2]. Such sources are characterized, at each frequency ω, by (a) a spectral degree of coherence that is homogeneous, meaning that it only depends on the directional distance between two source points $\boldsymbol{\rho}'_1$ and $\boldsymbol{\rho}'_2$, i.e., $\mu^{(0)}(\boldsymbol{\rho}'_1,\boldsymbol{\rho}'_2,\omega) \equiv \mu^{(0)}(\boldsymbol{\rho}'_2-\boldsymbol{\rho}'_1,\omega)$, and (b) by a spectral density $S^{(0)}(\boldsymbol{\rho}',\omega)$ that varies much more slowly with $\boldsymbol{\rho}'$ than $|\mu^{(0)}(\boldsymbol{\rho}'_2-\boldsymbol{\rho}'_1,\omega)|$ varies with $\boldsymbol{\rho}'_2-\boldsymbol{\rho}'_1$. The properties of such sources and the fields that they generate have been extensively studied. In particular, several *reciprocity relations*, which express far-zone properties of the field in terms of the Fourier transforms of source properties, have been derived [12–18]. Most of these studies, however, were limited to scalar fields. Notable exceptions are [19,20], in which the far-zone properties of fields generated by quasi-homogeneous, electromagnetic sources were studied. However, the analysis there was limited to sources with a *uniform state of polarization*, i.e., sources whose state of polarization is the same at every point. The fields produced by *primary* and *isotropic* quasi-homogeneous electromagnetic sources were described in [21]. In the present paper no restrictions on the symmetry, on the state of coherence or on the state of polarization of the source are assumed. Our results are, therefore, generally valid. We extend the concept of quasi-homogeneity to sources that generate *electromagnetic beams*. The source can have an arbitrary shape, an arbitrary state of coherence or state of polarization. We derive

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http://dx.doi.org/10.1016/j.optcom.2013.01.008

new reciprocity relations which involve the spectral density and the degree of coherence of the beams in the far-zone. These results are then used to study the changes in the spectrum, in the state of coherence and in the state of polarization that such beams undergo on propagation.

2. Partially coherent electromagnetic beams

The state of coherence and polarization of a random electromagnetic beam that propagates along the z-axis may be characterized, in the space–frequency domain, by a 2×2 *electric cross-spectral density matrix* [5, Chapter 9]

$$\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\begin{pmatrix} W_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega)\\ W_{yx}(\mathbf{r}_1,\mathbf{r}_2,\omega) & W_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega)\end{pmatrix}, \tag{1}$$

where

$$W_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle \quad (i,j=x,y). \tag{2}$$

Here $E_i(\mathbf{r},\omega)$ is a Cartesian component of the electric field, at a point $\mathbf{r}$ at frequency ω, of a typical realization of the statistical ensemble representing the beam, and the angled brackets indicate an ensemble average. From this matrix several important quantities can be derived.

The *spectral density* of the field is given by the expression

$$S(\mathbf{r},\omega)=\mathrm{Tr}\,\mathbf{W}(\mathbf{r},\mathbf{r},\omega), \tag{3}$$

where Tr denotes the trace.

The *spectral degree of coherence* of the field at two points $\mathbf{r}_1$ and $\mathbf{r}_2$ is defined by the formula:

$$\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)=\frac{\mathrm{Tr}\,\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)}{[\mathrm{Tr}\,\mathbf{W}(\mathbf{r}_1,\mathbf{r}_1,\omega)\mathrm{Tr}\,\mathbf{W}(\mathbf{r}_2,\mathbf{r}_2,\omega)]^{1/2}}. \tag{4}$$

It can be shown that the modulus of the spectral degree of coherence is bounded by zero and unity, i.e.,

$$0\le|\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)|\le 1. \tag{5}$$

The upper bound represents full coherence, whereas the lower bound indicates a complete absence of coherence.

The *spectral degree of polarization*, i.e., the ratio of the intensity of the polarized portion of the beam to its total intensity, at a point $\mathbf{r}$ can be shown to be [5, Chapter 9]

$$P(\mathbf{r},\omega)=\sqrt{1-\frac{4\,\mathrm{Det}\,\mathbf{W}(\mathbf{r},\mathbf{r},\omega)}{[\mathrm{Tr}\,\mathbf{W}(\mathbf{r},\mathbf{r},\omega)]^2}}, \tag{6}$$

where Det denotes the determinant. We will make use of the definitions (3), (4) and (6) to study the far-zone behavior of beams generated by quasi-homogeneous sources.

3. Quasi-homogeneous, planar electromagnetic sources

Let us consider a planar, secondary source that produces an electromagnetic beam with its axis along the positive z-direction (see Fig. 1). We first consider the two *diagonal* elements of its electric cross-spectral density matrix $\mathbf{W}^{(0)}$. They can be expressed in the form [5, Section 9.4.2]

$$W_{ii}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',\omega)=\sqrt{S_i^{(0)}(\boldsymbol{\rho}_1',\omega)\,S_i^{(0)}(\boldsymbol{\rho}_2',\omega)}\;\mu_{ii}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',\omega)\quad (i=x,y). \tag{7}$$

Here $S_i^{(0)}(\boldsymbol{\rho}',\omega)$ is the spectral density associated with a Cartesian component E_i of the electric field vector, and $\mu_{ii}^{(0)}$ is the correlation coefficient of E_i at the two positions $\boldsymbol{\rho}_1'$ and $\boldsymbol{\rho}_2'$. The superscript (0) refers to quantities in the source plane, taken to be the plane $z=0$.

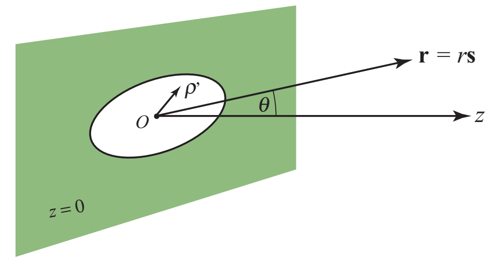

Fig. 1. Illustrating the notation. The vector $\boldsymbol{\rho}'=(x,y)$ labels a point in the source plane $z=0$. The line from the origin O to an observation point $\mathbf{r}=r\mathbf{s}$, with $|\mathbf{s}|=1$, makes an angle θ with the positive z-axis.

If the source is *homogeneous*, the correlation coefficients $\mu_{ii}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',\omega)$ depend only on the difference $\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1'$, i.e.,

$$\mu_{ii}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2',\omega)=\mu_{ii}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1',\omega). \tag{8}$$

A source is said to be *quasi-homogeneous* if the modulus of the correlation coefficient $\mu_{ii}^{(0)}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1',\omega)$ varies much more rapidly with its argument $\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1'$, than the spectral density $S_i^{(0)}(\boldsymbol{\rho},\omega)$ varies with $\boldsymbol{\rho}$. Since both $S_x^{(0)}(\boldsymbol{\rho},\omega)$ and $S_y^{(0)}(\boldsymbol{\rho},\omega)$ are "slow" functions when compared to $\mu_{xx}^{(0)}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1',\omega)$ and $\mu_{yy}^{(0)}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1',\omega)$, respectively, we can write

$$W_{ii}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2')\approx S_i^{(0)}\left(\frac{\boldsymbol{\rho}_1'+\boldsymbol{\rho}_2'}{2}\right)\,\mu_{ii}^{(0)}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1'), \tag{9}$$

where for brevity we have omitted the ω-dependence of the various quantities.

Next we make the change of variables

$$\boldsymbol{\rho}^{(+)}=\frac{\boldsymbol{\rho}_1'+\boldsymbol{\rho}_2'}{2}, \tag{10}$$

$$\boldsymbol{\rho}^{(-)}=\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1'. \tag{11}$$

The Jacobian of this transformation is unity, and the inverse transformation is given by the expressions

$$\boldsymbol{\rho}_1'=\boldsymbol{\rho}^{(+)}-\boldsymbol{\rho}^{(-)}/2, \tag{12}$$

$$\boldsymbol{\rho}_2'=\boldsymbol{\rho}^{(+)}+\boldsymbol{\rho}^{(-)}/2. \tag{13}$$

For the purpose of later analysis we now derive an equation for the four-dimensional, spatial Fourier transformation of $W_{ij}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2')$, which is defined as

$$\tilde{W}_{ij}^{(0)}(\mathbf{f}_1,\mathbf{f}_2)=\frac{1}{(2\pi)^4}\iint_{z=0}W_{ij}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2')e^{-\mathrm{i}[\mathbf{f}_1\cdot\boldsymbol{\rho}_1'+\mathbf{f}_2\cdot\boldsymbol{\rho}_2']}\,\mathrm{d}^2\rho_1'\,\mathrm{d}^2\rho_2'. \tag{14}$$

It is readily seen that $\tilde{W}_{ii}^{(0)}(\mathbf{f}_1,\mathbf{f}_2)$ factorizes into the product of two two-dimensional Fourier transforms, viz.,

$$\tilde{W}_{ii}^{(0)}(\mathbf{f}_1,\mathbf{f}_2)=\tilde{S}_i^{(0)}(\mathbf{f}_1+\mathbf{f}_2)\;\tilde{\mu}_{ii}^{(0)}\left(\frac{\mathbf{f}_2-\mathbf{f}_1}{2}\right), \tag{15}$$

where

$$\tilde{S}_i^{(0)}(\mathbf{f})=\frac{1}{(2\pi)^2}\int_{z=0}S_i^{(0)}(\boldsymbol{\rho}')e^{-\mathrm{i}\mathbf{f}\cdot\boldsymbol{\rho}'}\,\mathrm{d}^2\rho' \tag{16}$$

and

$$\tilde{\mu}_{ii}^{(0)}(\mathbf{f})=\frac{1}{(2\pi)^2}\int_{z=0}\mu_{ii}^{(0)}(\boldsymbol{\rho}')e^{-\mathrm{i}\mathbf{f}\cdot\boldsymbol{\rho}'}\,\mathrm{d}^2\rho'. \tag{17}$$

We note that the relation $\mu_{ii}^{(0)}(\boldsymbol{\rho}')=\mu_{ii}^{(0)*}(-\boldsymbol{\rho}')$ implies that $\tilde{\mu}_{ii}^{(0)}(\mathbf{f})$ is real-valued. We will make use of Eqs. (15)–(17) in the next section.

4. Two reciprocity relations

The elements of the cross-spectral density matrix in the far-zone, which we label by the superscript (∞), are related to those in the source plane, labeled by the superscript (0), by the formula

$$W_{ij}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2)=(2\pi k)^2\cos\theta_1\cos\theta_2\,\frac{e^{ik(r_2-r_1)}}{r_1r_2}\tilde{W}_{ij}^{(0)}(-k\mathbf{s}_{1_\perp},k\mathbf{s}_{2_\perp}), \tag{18}$$

where $\mathbf{s}_{i_\perp}$ is the two-dimensional projection, considered as a vector, of $\mathbf{s}_i$ onto the xy-plane. Eq. (18) is a straightforward generalization of a similar expression for scalar fields [1, Section 5.3.1]. A derivation of it is presented in Appendix A.

On substituting Eq. (15) into Eq. (18) we obtain the formulas

$$W_{ii}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2)=(2\pi k)^2\cos\theta_1\cos\theta_2\,\frac{e^{ik(r_2-r_1)}}{r_1r_2}\times\tilde{S}_i^{(0)}[k(\mathbf{s}_{2_\perp}-\mathbf{s}_{1_\perp})]\,\tilde{\mu}_{ii}^{(0)}[k(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})/2]. \tag{19}$$

On making use of Eqs. (19) in expression (3), we find the following expression for the far-zone spectral density:

$$S^{(\infty)}(r\mathbf{s})=\left(\frac{2\pi k\cos\theta}{r}\right)^2[\tilde{S}_x^{(0)}(0)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_\perp)+\tilde{S}_y^{(0)}(0)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_\perp)]. \tag{20}$$

Eq. (20) is a *reciprocity relation* that shows that the far-zone spectral density of an electromagnetic beam generated by a planar, secondary, quasi-homogeneous source, is a linear combination of the two Fourier transforms of the correlation coefficients of the electric field components. This relation takes on a particularly simple form for an on-axis observation point [i.e., $\mathbf{s}=(0,0,1)$], viz.,

$$S^{(\infty)}(0,0,z)=\left(\frac{2\pi k}{z}\right)^2[\tilde{S}_x^{(0)}(0)\tilde{\mu}_{xx}^{(0)}(0)+\tilde{S}_y^{(0)}(0)\tilde{\mu}_{yy}^{(0)}(0)]. \tag{21}$$

Next we derive a reciprocity relation for the spectral degree of coherence. On substituting Eq. (19) into expression (4) that defines this quantity, we find that

$$\begin{aligned}\eta^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2)=&\left\{\tilde{S}_x^{(0)}[k(\mathbf{s}_{2_\perp}-\mathbf{s}_{1_\perp})]\tilde{\mu}_{xx}^{(0)}\left[\frac{k}{2}(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})\right]\right.\\&\left.+\tilde{S}_y^{(0)}[k(\mathbf{s}_{2_\perp}-\mathbf{s}_{1_\perp})]\tilde{\mu}_{yy}^{(0)}\left[\frac{k}{2}(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})\right]\right\}\\&\times[\tilde{S}_x^{(0)}(0)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{1_\perp})+\tilde{S}_y^{(0)}(0)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{1_\perp})]^{-1/2}\\&\times[\tilde{S}_x^{(0)}(0)\tilde{\mu}_{xx}^{(0)}(k\mathbf{s}_{2_\perp})+\tilde{S}_y^{(0)}(0)\tilde{\mu}_{yy}^{(0)}(k\mathbf{s}_{2_\perp})]^{-1/2}e^{ik(r_2-r_1)}.\end{aligned} \tag{22}$$

Since $\mu_{ii}^{(0)}$ is a "fast" function of its argument, its Fourier transform $\tilde{\mu}_{ii}^{(0)}$ is a "slow" function. Hence

$$\tilde{\mu}_{ii}^{(0)}(k\mathbf{s}_{1_\perp})\approx\tilde{\mu}_{ii}^{(0)}(k\mathbf{s}_{2_\perp})\approx\tilde{\mu}_{ii}^{(0)}\left[\frac{k(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})}{2}\right]. \tag{23}$$

On making use of these approximations in Eq. (22) we obtain the formula

$$\begin{aligned}\eta^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2)=&\left\{\tilde{S}_x^{(0)}[k(\mathbf{s}_{2_\perp}-\mathbf{s}_{1_\perp})]\tilde{\mu}_{xx}^{(0)}\left[\frac{k}{2}(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})\right]\right.\\&\left.+\tilde{S}_y^{(0)}[k(\mathbf{s}_{2_\perp}-\mathbf{s}_{1_\perp})]\tilde{\mu}_{yy}^{(0)}\left[\frac{k}{2}(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})\right]\right\}\\&\times\left\{\tilde{S}_x^{(0)}(0)\tilde{\mu}_{xx}^{(0)}[k(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})/2]+\tilde{S}_y^{(0)}(0)\tilde{\mu}_{yy}^{(0)}[k(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})/2]^{-1}\right\}\\&\times e^{ik(r_2-r_1)}.\end{aligned} \tag{24}$$

Eq. (24) is another reciprocity relation. It shows that the far-field spectral degree of coherence of an electromagnetic beam which is generated by a planar, secondary, quasi-homogeneous source, is related to the Fourier transforms of both the spectral densities and of the correlation coefficients of the field in the source plane.

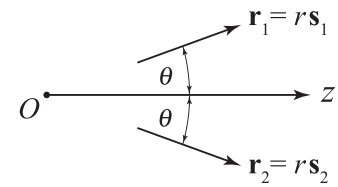

Fig. 2. Two symmetrically located observation points, with $\mathbf{s}_{1_\perp}=-\mathbf{s}_{2_\perp}$.

If we choose two observation points that are located symmetrically opposite to each other with respect to the z-axis (i.e., $r_1=r_2=r$; $\mathbf{s}_{1_\perp}=-\mathbf{s}_{2_\perp}$), as is illustrated in Fig. 2, this relation simplifies to the form

$$\eta^{(\infty)}(r\mathbf{s}_1,r\mathbf{s}_2)=[\tilde{S}_x^{(0)}(2k\mathbf{s}_{2_\perp})\tilde{\mu}_{xx}^{(0)}(0)+\tilde{S}_y^{(0)}(2k\mathbf{s}_{2_\perp})\tilde{\mu}_{yy}^{(0)}(0)]\times[\tilde{S}_x^{(0)}(0)\tilde{\mu}_{xx}^{(0)}(0)+\tilde{S}_y^{(0)}(0)\tilde{\mu}_{yy}^{(0)}(0)]^{-1}. \tag{25}$$

The two reciprocity relations (20) and (24) are generalizations of well-known results for scalar fields, derived by Carter and Wolf [12].

5. Off-diagonal matrix elements

In order to study the degree of polarization [given by Eq. (6)], we must also consider the two off-diagonal elements of the cross-spectral density matrix. The first one in the source plane reads

$$W_{xy}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2')=\sqrt{S_x^{(0)}(\boldsymbol{\rho}_1')\,S_y^{(0)}(\boldsymbol{\rho}_2')}\;\mu_{xy}^{(0)}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1'). \tag{26}$$

In writing Eq. (26) the homogeneity of the source has been used. Next we assume that both $S_x^{(0)}(\boldsymbol{\rho}_1')$ and $S_y^{(0)}(\boldsymbol{\rho}_2')$ vary much more slowly with their arguments than $\mu_{xy}^{(0)}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1')$ varies with $\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1'$. We then have, to a good approximation, that

$$S_i^{(0)}(\boldsymbol{\rho}_1')\approx S_i^{(0)}(\boldsymbol{\rho}_2')\approx S_i^{(0)}\left(\frac{\boldsymbol{\rho}_1'+\boldsymbol{\rho}_2'}{2}\right). \tag{27}$$

In such a case we may introduce a new function

$$S_{xy}^{(0)}\left(\frac{\boldsymbol{\rho}_1'+\boldsymbol{\rho}_2'}{2}\right)\equiv\sqrt{S_x^{(0)}\left(\frac{\boldsymbol{\rho}_1'+\boldsymbol{\rho}_2'}{2}\right)}\sqrt{S_y^{(0)}\left(\frac{\boldsymbol{\rho}_1'+\boldsymbol{\rho}_2'}{2}\right)}, \tag{28}$$

$$\approx\sqrt{S_x^{(0)}(\boldsymbol{\rho}_1')}\sqrt{S_y^{(0)}(\boldsymbol{\rho}_2')}. \tag{29}$$

In terms of $S_{xy}^{(0)}$ the matrix element of Eq. (26) may be expressed in the form

$$W_{xy}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2')=S_{xy}^{(0)}[\boldsymbol{\rho}^{(+)}]\;\mu_{xy}^{(0)}[\boldsymbol{\rho}^{(-)}], \tag{30}$$

where the sum and difference variables defined by Eqs. (10) and (11) have been used. In strict analogy with the derivation of Eq. (15) we find that the Fourier transform of this matrix element equals

$$\tilde{W}_{xy}^{(0)}(\mathbf{f}_1,\mathbf{f}_2)=\tilde{S}_{xy}^{(0)}(\mathbf{f}_1+\mathbf{f}_2)\tilde{\mu}_{xy}^{(0)}\left(\frac{\mathbf{f}_2-\mathbf{f}_1}{2}\right). \tag{31}$$

On substituting Eq. (31) into Eq. (18) we obtain the formula

$$W_{xy}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2)=(2\pi k)^2\cos\theta_1\cos\theta_2\,\tilde{S}_{xy}^{(0)}[k(\mathbf{s}_{2_\perp}-\mathbf{s}_{1_\perp})]\times\tilde{\mu}_{xy}^{(0)}[k(\mathbf{s}_{1_\perp}+\mathbf{s}_{2_\perp})/2]\frac{e^{ik(r_2-r_1)}}{r_1r_2}. \tag{32}$$

The second off-diagonal matrix element is given by the expression

$$W_{yx}^{(0)}(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2')=\sqrt{S_y^{(0)}(\boldsymbol{\rho}_1')\,S_x^{(0)}(\boldsymbol{\rho}_2')}\;\mu_{yx}^{(0)}(\boldsymbol{\rho}_2'-\boldsymbol{\rho}_1'). \tag{33}$$

It follows from the definition of the cross-spectral density matrix that

$$W^{(0)}_{yx}(\boldsymbol{\rho}'_1,\boldsymbol{\rho}'_2)=[W^{(0)}_{xy}(\boldsymbol{\rho}'_2,\boldsymbol{\rho}'_1)]^*, \tag{34}$$

$$=S^{(0)}_{xy}[\boldsymbol{\rho}^{(+)}]\mu^{(0)*}_{xy}[-\boldsymbol{\rho}^{(-)}], \tag{35}$$

where Eq. (30) has been used. Since

$$\frac{1}{(2\pi)^2}\int \mu^{(0)*}_{xy}[-\boldsymbol{\rho}^{(-)}]e^{-i\mathbf{f}\cdot\boldsymbol{\rho}^{(-)}}d^2\rho^{(-)}=[\tilde{\mu}^{(0)}_{xy}(\mathbf{f})]^*, \tag{36}$$

we find that

$$W^{(\infty)}_{yx}(r_1\mathbf{s}_1,r_2\mathbf{s}_2)=(2\pi k)^2\cos\theta_1\cos\theta_2\,\tilde{S}^{(0)}_{xy}[k(\mathbf{s}_{2\perp}-\mathbf{s}_{1\perp})]\times\tilde{\mu}^{(0)*}_{xy}[k(\mathbf{s}_{1\perp}+\mathbf{s}_{2\perp})/2]\frac{e^{ik(r_2-r_1)}}{r_1r_2}. \tag{37}$$

Expressions for all the four elements of the cross-spectral density matrix of the far-zone beam have now been derived. On substituting Eqs. (19), (32), and (37) into Eq. (6) we obtain for the degree of polarization of the beam at axial points ($\mathbf{s}_\perp=0$) in the far-zone the expression

$$P^{(\infty)}(0,0,z)=\{[\tilde{S}^{(0)}_x(0)\tilde{\mu}^{(0)}_{xx}(0)-\tilde{S}^{(0)}_y(0)\tilde{\mu}^{(0)}_{yy}(0)]^2+4[\tilde{S}^{(0)}_{xy}(0)|\tilde{\mu}^{(0)}_{xy}(0)|]^2\}^{1/2}\times|\tilde{S}^{(0)}_x(0)\tilde{\mu}^{(0)}_{xx}(0)+\tilde{S}^{(0)}_y(0)\tilde{\mu}^{(0)}_{yy}(0)|^{-1}. \tag{38}$$

We see from this formula that in this case the degree of polarization does *not* depend on the specific forms of the spectral densities or of the correlation coefficients, but rather on their Fourier transform at zero frequency, i.e., on their spatial integrals.

6. Examples

We will now make use of the two reciprocity relations, given by Eqs. (21) and (25), and of Eq. (38) to investigate changes in the spectrum, in the degree of coherence, and in the degree of polarization that occur on propagation from the source to the far-zone.

6.1. The far-field spectrum

Coherence-induced spectral changes have been examined for several years now. A review of this subject was given by Wolf and James [22]. As mentioned before, in contrast to the present work, most studies have dealt with scalar fields. To see how the vectorial nature of the beam influences the far-zone spectrum, we first recall Eq. (21),

$$S^{(\infty)}(0,0,z;\omega)=\left(\frac{2\pi k}{z}\right)^2[\tilde{S}^{(0)}_x(0;\omega)\tilde{\mu}^{(0)}_{xx}(0;\omega)+\tilde{S}^{(0)}_y(0;\omega)\tilde{\mu}^{(0)}_{yy}(0;\omega)], \tag{39}$$

where, for clarity, we again display the frequency-dependence of the various quantities.

Let us now investigate the incoherent superposition of two laser beams, each with constant intensity A, and with an identical Gaussian spectrum, with central frequency ω_0. One beam is assumed to be x-polarized and to have a radius a, whereas the other beam is assumed to be y-polarized and to have a radius b. In that case the two spectral densities are given by the expressions

$$S^{(0)}_x(\boldsymbol{\rho}';\omega)=\begin{cases}Ae^{-(\omega-\omega_0)^2/\Delta^2} & \text{if } |\boldsymbol{\rho}'|\le a,\\ 0 & \text{if } |\boldsymbol{\rho}'|>a,\end{cases} \tag{40}$$

$$S^{(0)}_y(\boldsymbol{\rho}';\omega)=\begin{cases}Ae^{-(\omega-\omega_0)^2/\Delta^2} & \text{if } |\boldsymbol{\rho}'|\le b,\\ 0 & \text{if } |\boldsymbol{\rho}'|>b,\end{cases} \tag{41}$$

with Δ the effective width of the two spectra. The two-dimensional spatial Fourier transforms of these spectra are given by the expressions

$$\tilde{S}^{(0)}_x(\mathbf{f};\omega)=\frac{a^2A}{2\pi}e^{-(\omega-\omega_0)^2/\Delta^2}\frac{J_1(fa)}{fa}, \tag{42}$$

$$\tilde{S}^{(0)}_y(\mathbf{f};\omega)=\frac{b^2A}{2\pi}e^{-(\omega-\omega_0)^2/\Delta^2}\frac{J_1(fb)}{fb}, \tag{43}$$

where J_1 denotes the first order Bessel function of the first kind, and $f=|\mathbf{f}|$. It follows that

$$\tilde{S}^{(0)}_x(0;\omega)=\frac{a^2A}{4\pi}e^{-(\omega-\omega_0)^2/\Delta^2}, \tag{44}$$

$$\tilde{S}^{(0)}_y(0;\omega)=\frac{b^2A}{4\pi}e^{-(\omega-\omega_0)^2/\Delta^2}. \tag{45}$$

We also assume that the correlation coefficients $\mu^{(0)}_{xx}$ and $\mu^{(0)}_{yy}$ are both represented by Gaussian functions, but with different spatial and different spectral widths, i.e.,

$$\mu^{(0)}_{xx}(\boldsymbol{\rho}';\omega)=e^{-\boldsymbol{\rho}'/2\delta^2_{xx}}e^{-(\omega-\omega_0)^2/\Delta^2_{xx}}, \tag{46}$$

$$\mu^{(0)}_{yy}(\boldsymbol{\rho}';\omega)=e^{-\boldsymbol{\rho}'/2\delta^2_{yy}}e^{-(\omega-\omega_0)^2/\Delta^2_{yy}}. \tag{47}$$

It thus follows that

$$\tilde{\mu}^{(0)}_{xx}(0;\omega)=\frac{\delta^2_{xx}}{2\pi}e^{-(\omega-\omega_0)^2/\Delta^2_{xx}}, \tag{48}$$

$$\tilde{\mu}^{(0)}_{yy}(0;\omega)=\frac{\delta^2_{yy}}{2\pi}e^{-(\omega-\omega_0)^2/\Delta^2_{yy}}. \tag{49}$$

On substituting Eqs. (44), (45), (48) and (49) into Eq. (39), we obtain for the on-axis spectral density in the far-zone the formula

$$S^{(\infty)}(0,0,z;\omega)=\frac{A}{2}\left(\frac{k}{z}\right)^2e^{-(\omega-\omega_0)^2/\Delta^2}\times\{a^2\delta^2_{xx}e^{-(\omega-\omega_0)^2/\Delta^2_{xx}}+b^2\delta^2_{yy}e^{-(\omega-\omega_0)^2/\Delta^2_{yy}}\}. \tag{50}$$

Using the fact that the on-axis spectral density in the source plane is given by the expression

$$S^{(0)}(0,0,0;\omega)=S^{(0)}_x(0,0,0;\omega)+S^{(0)}_y(0,0,0;\omega), \tag{51}$$

$$=2Ae^{-(\omega-\omega_0)^2/\Delta^2}, \tag{52}$$

we can write the on-axis far-zone spectral density in the form

$$S^{(\infty)}(0,0,z;\omega)=M(\omega)S^{(0)}(0,0,0;\omega), \tag{53}$$

where the *spectral modifier function M* is given by the expression

$$M(\omega)=\frac{1}{4}\left(\frac{\omega}{zc}\right)^2\{a^2\delta^2_{xx}e^{-(\omega-\omega_0)^2/\Delta^2_{xx}}+b^2\delta^2_{yy}e^{-(\omega-\omega_0)^2/\Delta^2_{yy}}\}. \tag{54}$$

Eq. (53) shows that the on-axis spectrum in the far-zone equals the on-axis spectrum in the source plane times the modifier function $M(\omega)$. We note that the function $M(\omega)$ contains several parameters: the beam sizes a and b, the coherence lengths δ_{xx} and δ_{yy} and the spectral widths Δ_{xx} and Δ_{yy}. Each of these parameters can give rise to changes of the spectrum on propagation. An example of the far-zone spectrum is shown in Fig. 3. It is seen the far-zone spectrum can be significantly narrower than that in the source plane (case a). Also, the maximum of the far-zone spectrum can be shifted to higher frequencies (case b).

6.2. The far-field spectral degree of coherence

Let us next consider a source with two equal diagonal correlation coefficients, i.e., $\mu^{(0)}_{xx}(\boldsymbol{\rho})=\mu^{(0)}_{yy}(\boldsymbol{\rho})$. We assume that the two spectral densities are Gaussian functions with the same

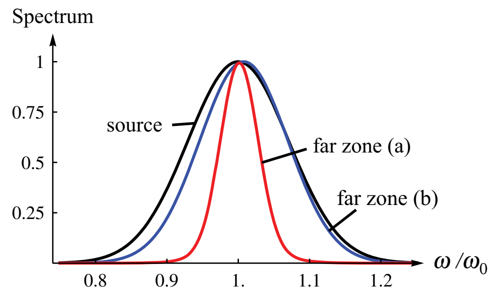

Fig. 3. The on-axis spectrum in the source plane (black curve) and two on-axis spectra in the far-zone. All spectra are normalized to have unity height. In this example $a=1$, $b=2a$, $\delta_{xx}=1$, $\delta_{yy}=2$, $\Delta=\omega_0/10$, $\Delta_{xx}=0.1$ and $\Delta_{yy}=0.04$ [far-zone spectrum (a)] or 0.2 [far-zone spectrum (b)].

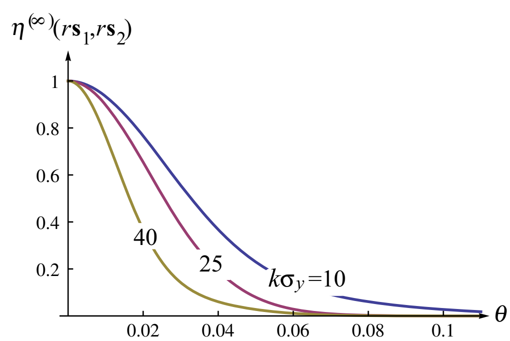

Fig. 4. The spectral degree of coherence in the far field of a beam generated by a quasi-homogeneous source. The two symmetrically located observation points are each making an angle θ with the beam axis (see Fig. 2). The normalized widths of the two spectral densities are taken to be $k\sigma_x=20$ and $k\sigma_y=10$, 25 and 40.

maxima, but with different widths, viz.,

$$S_x^{(0)}(\boldsymbol{\rho}')=Ae^{-\rho'^2/2\sigma_x^2}, \tag{55}$$

$$S_y^{(0)}(\boldsymbol{\rho}')=Ae^{-\rho'^2/2\sigma_y^2}. \tag{56}$$

From Eq. (25) it follows that in this case the spectral degree of coherence of the field at two far field points located symmetrically with respect to the beam axis is given by the formula

$$\eta^{(\infty)}(r\mathbf{s}_1,r\mathbf{s}_2)=\frac{\sigma_x^2e^{-2(k\sigma_x\,\sin\,\theta)^2}+\sigma_y^2e^{-2(k\sigma_y\,\sin\,\theta)^2}}{\sigma_x^2+\sigma_y^2}\qquad(\mathbf{s}_{1_\perp}=-\mathbf{s}_{2_\perp}). \tag{57}$$

An example of the angular dependence of $\eta^{(\infty)}(r\mathbf{s}_1,r\mathbf{s}_2)$ for this case is shown in Fig. 4 for various values of the scaled transverse coherence length $k\sigma_y$. It is seen that the width of the spectral degree of coherence decreases when the width of the spectral density $k\sigma_y$ increases.

6.3. The far-field spectral degree of polarization

As our last example, we consider a source in which the two components of the electric field have an identical spectral density, but are uncorrelated, i.e.,

$$S_x^{(0)}(\boldsymbol{\rho}')=S_y^{(0)}(\boldsymbol{\rho}'),$$
$$\mu_{xy}^{(0)}(\boldsymbol{\rho}')=\mu_{yx}^{(0)}(\boldsymbol{\rho}')=0. \tag{58}$$

We also assume that both the non-zero correlation coefficients have a Gaussian form

$$\mu_{ii}^{(0)}(\boldsymbol{\rho}')=e^{-\rho'^2/2\delta_{ii}^2},\qquad(i=x,y). \tag{59}$$

It immediately follows from Eq. (6) that everywhere in the source plane the field is completely unpolarized, i.e., the degree of polarization $P^{(0)}(\boldsymbol{\rho}')=0$. However, in the far-zone that is generally not the case (see also [7]). We have from Eq. (59) that

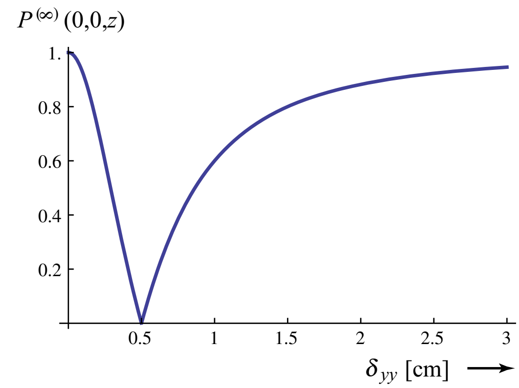

Fig. 5. The spectral degree of polarization of a beam generated by a quasi-homogeneous source at a far-field axial point. In this example one effective coherence length is taken to be $\delta_{xx}=0.5$ cm, whereas δ_{yy} varies from 0 to 3 cm. The two spectral densities, $S_x^{(0)}$ and $S_y^{(0)}$, are assumed to be equal.

$$\tilde{\mu}_{ii}^{(0)}(0)=\frac{1}{2\pi}\delta_{ii}^2. \tag{60}$$

Under these circumstances, the expression for the degree of polarization at points in the far-zone on the axis, Eq. (38), reduces to a function of the two effective correlation lengths only, namely

$$P^{(\infty)}(0,0,z)=\frac{|\delta_{xx}^2-\delta_{yy}^2|}{\delta_{xx}^2+\delta_{yy}^2}. \tag{61}$$

An example of the behavior of $P^{(\infty)}(0,0,z)$ is shown in Fig. 5. It is seen that the degree of polarization in the far-zone varies strongly with the ratio between the two correlation lengths δ_{xx} and δ_{yy}, and can take on any value between zero and unity.

7. Conclusions

We have studied the far-zone properties of electromagnetic beams that are generated by planar, secondary quasi-homogeneous sources. No assumptions regarding the shape of the source, it symmetries or its state of polarization were made. Two reciprocity relations were derived. The first one relates the spectral density in the far-zone to the Fourier transforms of the correlation coefficients in the source plane. The second one relates the spectral degree of coherence in the far-zone to the Fourier transforms of both the spectral densities and of the correlation coefficients of the source field. We applied these two relations to demonstrate that the spectral density, the coherence properties and the state of polarization of a beam that originates from a quasi-homogeneous source can all drastically change on propagation.

While this paper was being finalized, two papers [23,24] appeared in which some related results were reported.

Acknowledgments

T.D.V.'s research is supported by the Plasmonics program of the Foundation for Fundamental Research on Matter (FOM). E.W.'s research is supported by the Air Force Office of Scientific Research under Grant FA 9550-12-1-0284.

Appendix A. Derivation of Eq. (18)

For a beam-like field generated by a planar, secondary source, we have, according to the first Rayleigh diffraction formula

[1, Section 3.2.5]

$$E_i(\mathbf{r}) = \frac{-1}{2\pi}\int_{z=0} E_i^{(0)}(\boldsymbol{\rho}')\frac{\partial}{\partial z}\left[\frac{e^{ikR}}{R}\right] \mathrm{d}^2\rho', \tag{A.1}$$

where $R=|(\boldsymbol{\rho}',0)-\mathbf{r}|$. If $\mathbf{r}=r\mathbf{s}$ with $|\mathbf{s}|=1$ represents a point in the far-zone, we have, to a good approximation, that

$$R \approx r-\boldsymbol{\rho}'\cdot\mathbf{s}_{\perp}, \tag{A.2}$$

where $\mathbf{s}_{i_\perp}$ is the two-dimensional projection, considered as a vector, of $\mathbf{s}_i$ onto the xy-plane. Hence,

$$e^{\mathrm{i}kR} \approx e^{\mathrm{i}kr}e^{-\mathrm{i}k\boldsymbol{\rho}'\cdot\mathbf{s}_\perp}. \tag{A.3}$$

It then follows that

$$\frac{\partial}{\partial z}\left[\frac{e^{ikR}}{R}\right] = \frac{\partial r}{\partial z}\frac{\partial}{\partial r}\left[\frac{e^{ikR}}{R}\right], \tag{A.4}$$

$$\approx \frac{\mathrm{i}k}{r}\cos\theta\, e^{\mathrm{i}kr}e^{-\mathrm{i}k\boldsymbol{\rho}'\cdot\mathbf{s}_\perp}, \tag{A.5}$$

where we have made use of the facts that in the far-zone $r \gg \lambda$, together with $z=r\cos\theta$. On making use of Eq. (A.5) in Eq. (A.1) we find that

$$E_i^{(\infty)}(\mathbf{r}) = \frac{-\mathrm{i}k}{2\pi}\cos\theta\,\frac{e^{\mathrm{i}kr}}{r}\int_{z=0} E_i^{(0)}(\boldsymbol{\rho}')\, e^{-\mathrm{i}k\boldsymbol{\rho}'\cdot\mathbf{s}_\perp}\mathrm{d}^2\rho', \tag{A.6}$$

$$= -2\pi\mathrm{i}k\cos\theta\,\frac{e^{\mathrm{i}kr}}{r}\tilde{E}_i^{(0)}(k\mathbf{s}_\perp), \tag{A.7}$$

where we used the definition of the Fourier transform, Eq. (16). On substituting Eq. (A.7) into Eqs. (1) and (14) we obtain the result

$$W_{ij}^{(\infty)}(r_1\mathbf{s}_1,r_2\mathbf{s}_2) = \langle E_i^{(\infty)*}(r_1\mathbf{s}_1)E_j^{(\infty)}(r_2\mathbf{s}_2)\rangle, \tag{A.8}$$

$$= (2\pi k)^2\cos\theta_1\cos\theta_2\langle\tilde{E}_i^{(0)*}(k\mathbf{s}_{1_\perp})\tilde{E}_j^{(0)}(k\mathbf{s}_{2_\perp})\rangle\frac{e^{\mathrm{i}k(r_2-r_1)}}{r_1r_2}, \tag{A.9}$$

$$= (2\pi k)^2\cos\theta_1\cos\theta_2\,\tilde{W}_{ij}^{(0)}(-k\mathbf{s}_{1_\perp},k\mathbf{s}_{2_\perp})\frac{e^{\mathrm{i}k(r_2-r_1)}}{r_1r_2}, \tag{A.10}$$

which is Eq. (18).

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