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Reconstruction of an electromagnetic Gaussian Schell-model source from far-zone intensity measurements

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An electromagnetic Gaussian Schell-model source that produces a random beam may be characterized by eight independent quantities. We show how far-zone measurements of the Stokes parameters, together with the Hanbury Brown–Twiss coefficient, allow one to determine all the source parameters. This method provides, to the best of our knowledge, a new tool to identify distant sources.

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The inverse problem of determining the properties of a source or a scatterer from field or intensity measurements rarely has a unique solution [1]. One typically has to assume some *a priori* knowledge. Reconstruction has been attempted, for example, for delta-correlated thermal sources [2], illuminated apertures [3], illuminated object transparencies [4], sources in the Fresnel regime [5], and scalar quasi-homogeneous sources [6].

In the present study, we assume that we are dealing with a planar, secondary source of the Gaussian Schell-model (GSM) type that generates an electromagnetic beam [7]. Such sources, together with their scalar counterparts, have been used in numerous studies in the field of optical coherence [8]. Because of their widespread use as a representation of a general source, the question of how information can be obtained from beam measurements is of practical interest.

In the space–frequency domain, GSM sources are characterized by a spectral density distribution and a homogeneous correlation function that are both of a Gaussian form. As we will demonstrate, far-zone measurements of the Stokes parameters, together with the Hanbury Brown–Twiss (HBT) coefficient, allow one to recover the relevant source parameters. These parameters describe not only the source shape but also its coherence properties. Our approach uses intensity measurements, rather than field measurements, which are generally less robust.

The second-order statistical properties of a partially coherent electromagnetic beam that propagates along the z axis are described by its cross-spectral density (CSD) matrix, which is defined as [Ch. 9, 7]

$$\mathbf{W}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \begin{pmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \end{pmatrix}. \tag{1}$$

The matrix elements are functions of three variables, and given by the expression

$$W_{ij}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle E_i^*(\mathbf{r}_1, \omega) E_j(\mathbf{r}_2, \omega)\rangle, \quad (i, j = x, y), \tag{2}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are two points of observation, ω is the angular frequency, E_i is a Cartesian component of the electric field, and the angled brackets indicate an average taken over an ensemble of realizations. From now on, we now longer display the frequency dependence in our notation. The CSD matrix elements of an electromagnetic GSM source occupying the plane $z = 0$, indicated by the superscript (0), are [Ch. 9, 7]

$$W_{ij}^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = A_i A_j B_{ij} \exp\left[-\frac{\rho_1^2}{4\sigma_i^2} - \frac{\rho_2^2}{4\sigma_j^2} - \frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2}{2\delta_{ij}^2}\right], \tag{3}$$

where $\boldsymbol{\rho} = (x, y)$ denotes a transverse position vector. Furthermore, A_i denotes the amplitude of E_i, B_{ij} is the correlation between E_i and E_j, and σ_i is the effective width of E_i. The factors δ_{ij} are transverse coherence radii. All parameters in Eq. (3) are independent of position, but may depend on frequency. They cannot be chosen freely, but have to satisfy several constraints, i.e.,

$$B_{xx} = B_{yy} = 1, \tag{4}$$

$$B_{xy} = B_{yx}^*, \tag{5}$$

$$B_{xy} = |B_{xy}| e^{i\phi}, \text{ with } |B_{xy}| \leq 1, \text{ and } \phi \in \mathbb{R}, \tag{6}$$

$$\delta_{xy} = \delta_{yx}. \tag{7}$$

Furthermore, the so-called realizability conditions are [9]

$$\sqrt{\frac{\delta_{xx}^2 + \delta_{yy}^2}{2}} \leq \delta_{xy} \leq \sqrt{\frac{\delta_{xx}\delta_{yy}}{|B_{xy}|}}. \tag{8}$$

For the case $\sigma_x = \sigma_y = \sigma$, the source will generate a beam-like field if [10]

$$\frac{1}{4\sigma^2} + \frac{1}{\delta_{xx}^2} \ll \frac{2\pi^2}{\lambda^2} \quad \text{and} \quad \frac{1}{4\sigma^2} + \frac{1}{\delta_{yy}^2} \ll \frac{2\pi^2}{\lambda^2}, \tag{9}$$

where λ denotes the wavelength. Under these constraints, the source is described by eight independent parameters, namely, $A_x, A_y, |B_{xy}|, \phi, \delta_{xx}, \delta_{yy}, \delta_{xy}$, and σ, which we will attempt to recover from far-zone intensity measurements. It is seen from Eq. (3) that if A_x, A_y, or B_{xy} is zero, then the number of source parameters that describe the source is reduced. We will discuss these special cases later on.

On propagation to a transverse plane z, the beam matrix elements evolve into ([Ch. 9, 7], notice that the next to last minus sign in Eq. (10) on p. 184 should in fact be a plus sign)

$$W_{ij}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{A_i A_j B_{ij}}{\Delta_{ij}^2(z)} \exp\left[-\frac{(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)^2}{8\sigma^2 \Delta_{ij}^2(z)}\right] \times \exp\left[-\frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2}{2\Omega_{ij}^2 \Delta_{ij}^2(z)} + \frac{ik(\rho_2^2 - \rho_1^2)}{2R_{ij}(z)}\right], \tag{10}$$

where $k = 2\pi/\lambda$, and

$$\Delta_{ij}^2(z) = 1 + (z/\sigma k \Omega_{ij})^2, \tag{11}$$

$$\frac{1}{\Omega_{ij}^2} = \frac{1}{4\sigma^2} + \frac{1}{\delta_{ij}^2}, \tag{12}$$

$$R_{ij}(z) = \left[1 + (\sigma k \Omega_{ij}/z)^2\right] z. \tag{13}$$

When z tends to infinity, we have that

$$\Delta_{ij}^2(z) \sim \frac{z^2}{(\sigma k \Omega_{ij})^2}, \tag{14}$$

$$R_{ij}(z) \sim z. \tag{15}$$

We thus get for the four far-zone elements, denoted by the superscript (∞), the formulas

$$W_{ij}^{(\infty)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \frac{A_i A_j B_{ij} (k\sigma\Omega_{ij})^2}{z^2} \exp\left[-\frac{(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)^2 (k\Omega_{ij})^2}{8z^2}\right] \times \exp\left[-\frac{(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2 (k\sigma)^2}{2z^2} + \frac{ik(\rho_2^2 - \rho_1^2)}{2z}\right]. \tag{16}$$

In terms of the polar angle $\theta \approx \rho/z$, these can be expressed for the two cases $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_1 = -\boldsymbol{\rho}_2$ as

$$W_{ij}^{(\infty)}(\theta, \theta) = K^2 A_i A_j B_{ij} \Omega_{ij}^2 \exp\left(-\frac{\theta^2 k^2 \Omega_{ij}^2}{2}\right) \tag{17}$$

and

$$W_{ij}^{(\infty)}(\theta, -\theta) = K^2 A_i A_j B_{ij} \Omega_{ij}^2 \exp\left(-2\theta^2 k^2 \sigma^2\right), \tag{18}$$

respectively, where the z dependence is contained in the factor

$$K = k\sigma/z. \tag{19}$$

In far-zone measurements, the distance z is usually unknown, and therefore the precise value of K cannot be established.

The expectation values of the four spectral Stokes parameters can be expressed in terms of the far-zone CSD matrix elements as [7]

$$\langle S_0(\theta)\rangle = W_{xx}(\theta, \theta) + W_{yy}(\theta, \theta), \tag{20}$$

$$\langle S_1(\theta)\rangle = W_{xx}(\theta, \theta) - W_{yy}(\theta, \theta), \tag{21}$$

$$\langle S_2(\theta)\rangle = W_{xy}(\theta, \theta) + W_{yx}(\theta, \theta), \tag{22}$$

$$\langle S_3(\theta)\rangle = i\left[W_{yx}(\theta, \theta) - W_{xy}(\theta, \theta)\right]. \tag{23}$$

On substituting from Eq. (17), we obtain

$$\langle S_0(\theta)\rangle = K^2\left[A_x^2\Omega_{xx}^2 \exp\left(-\frac{\theta^2 k^2 \Omega_{xx}^2}{2}\right) + A_y^2\Omega_{yy}^2 \exp\left(-\frac{\theta^2 k^2 \Omega_{yy}^2}{2}\right)\right], \tag{24}$$

$$\langle S_1(\theta)\rangle = K^2\left[A_x^2\Omega_{xx}^2 \exp\left(-\frac{\theta^2 k^2 \Omega_{xx}^2}{2}\right) - A_y^2\Omega_{yy}^2 \exp\left(-\frac{\theta^2 k^2 \Omega_{yy}^2}{2}\right)\right], \tag{25}$$

$$\langle S_2(\theta)\rangle = 2K^2 A_x A_y |B_{xy}| \Omega_{xy}^2 \exp\left[-\frac{\theta^2 k^2 \Omega_{xy}^2}{2}\right] \cos\phi, \tag{26}$$

$$\langle S_3(\theta)\rangle = 2K^2 A_x A_y |B_{xy}| \Omega_{xy}^2 \exp\left[-\frac{\theta^2 k^2 \Omega_{xy}^2}{2}\right] \sin\phi. \tag{27}$$

On the beam axis ($\theta = 0$), these expressions simplify to

$$\langle S_0(0)\rangle = K^2\left(A_x^2\Omega_{xx}^2 + A_y^2\Omega_{yy}^2\right), \tag{28}$$

$$\langle S_1(0)\rangle = K^2\left(A_x^2\Omega_{xx}^2 - A_y^2\Omega_{yy}^2\right), \tag{29}$$

$$\langle S_2(0)\rangle = 2K^2 A_x A_y \Omega_{xy}^2 |B_{xy}| \cos\phi, \tag{30}$$

$$\langle S_3(0)\rangle = 2K^2 A_x A_y \Omega_{xy}^2 |B_{xy}| \sin\phi. \tag{31}$$

Measuring the Stokes parameters alone is not sufficient for our purpose. We also need the correlation of intensity fluctuations as described by the HBT coefficient. It turns out to be useful to consider this correlation for a pair of angles $(\theta, -\theta)$. Under the assumption of Gaussian statistics, the HBT coefficient can be expressed in terms of the CSD matrix as (Eq. 8, [11])

$$C(\theta,-\theta)=\sum_{i,j=\{x,y\}}\left|W_{ij}(\theta,-\theta)\right|^2$$

$$=\Big[(KA_x)^4\Omega_{xx}^4+(KA_y)^4\Omega_{yy}^4 \tag{32}$$

$$+\,2(KA_x)^2(KA_y)^2|B_{xy}|^2\Omega_{xy}^4\Big]\exp(-4\theta^2k^2\sigma^2), \tag{33}$$

where we made use of Eq. (18). The merit of Eq. (33), as we will see, lies in the fact that the source width σ appears explicitly in the last factor.

The above results, if applied in a specific order, can be used to establish the values of the eight source parameters as we now show.

1. Adding the two formulas for $\langle S_0\rangle$ and $\langle S_1\rangle$ yields

$$\langle S_0(\theta)\rangle+\langle S_1(\theta)\rangle=2(KA_x)^2\Omega_{xx}^2\exp\left(-\theta^2k^2\Omega_{xx}^2/2\right), \tag{34}$$

$$\langle S_0(0)\rangle+\langle S_1(0)\rangle=2(KA_x)^2\Omega_{xx}^2. \tag{35}$$

The only way the left-hand side of Eq. (34) or Eq. (35) can be zero is when $A_x=0$. In that case, the only non-zero matrix element is W_{yy}. This implies there are only three parameters that describe the source, namely, A_y, σ, and δ_{yy}. We return to this particular case later, and for now assume that both A_x and A_y are non-zero. From the two above equations, we find

$$\exp(-\theta^2k^2\Omega_{xx}^2/2)=\frac{\langle S_0(\theta)\rangle+\langle S_1(\theta)\rangle}{\langle S_0(0)\rangle+\langle S_1(0)\rangle}, \tag{36}$$

and hence,

$$\Omega_{xx}^2=-\frac{2}{k^2\theta^2}\ln\left[\frac{\langle S_0(\theta)\rangle+\langle S_1(\theta)\rangle}{\langle S_0(0)\rangle+\langle S_1(0)\rangle}\right]. \tag{37}$$

Clearly, the experimental uncertainty in the value of Ω_{xx} can be reduced by repeating the measurements for different values of the angle θ. This also pertains, as we will see, to Ω_{yy}, Ω_{xy}, and σ. Having thus established the value of Ω_{xx}^2, we can use it in Eq. (35) to obtain

$$KA_x=\frac{1}{\sqrt{2}\Omega_{xx}}\sqrt{[\langle S_0(0)\rangle+\langle S_1(0)\rangle]}. \tag{38}$$

2. A strictly similar procedure results in the values of Ω_{yy} and KA_y. Subtracting the expressions for $\langle S_0\rangle$ and $\langle S_1\rangle$ leads to

$$\Omega_{yy}^2=-\frac{2}{k^2\theta^2}\ln\left[\frac{\langle S_0(\theta)\rangle-\langle S_1(\theta)\rangle}{\langle S_0(0)\rangle-\langle S_1(0)\rangle}\right] \tag{39}$$

and

$$KA_y=\frac{1}{\sqrt{2}\Omega_{yy}}\sqrt{[\langle S_0(0)\rangle-\langle S_1(0)\rangle]}. \tag{40}$$

Notice that, due to the presence of the factor K, Eqs. (38) and (40) allow us to determine the ratio of the amplitudes A_x and A_y, but not their individual values.

3. The values of the off-diagonal quantities Ω_{xy} and $|B_{xy}|$ can be found as follows. First the trigonometry factors in the expressions for $\langle S_2(\theta)\rangle$ and $\langle S_3(\theta)\rangle$ are eliminated by squaring and summing, leading to the results

$$\sqrt{\langle S_2(\theta)\rangle^2+\langle S_3(\theta)\rangle^2}=2(KA_x)(KA_y)|B_{xy}|\Omega_{xy}^2\times\exp\left(-\theta^2k^2\Omega_{xy}^2/2\right) \tag{41}$$

and

$$\sqrt{\langle S_2(0)\rangle^2+\langle S_3(0)\rangle^2}=2(KA_x)(KA_y)|B_{xy}|\Omega_{xy}^2. \tag{42}$$

Because of our previous assumption of A_x and A_y both being non-zero, the only way the left-hand sides of Eqs. (41) and (42) can vanish is for $|B_{xy}|$ to be zero. Under that condition, $W_{xy}=W_{yx}=0$, and δ_{xy} and Ω_{xy} are undefined. We will return to this case later. When $B_{xy}\neq 0$, we get

$$\exp\left(-\theta^2k^2\Omega_{xy}^2/2\right)=\sqrt{\frac{\langle S_2(\theta)\rangle^2+\langle S_3(\theta)\rangle^2}{\langle S_2(0)\rangle^2+\langle S_3(0)\rangle^2}}, \tag{43}$$

from which

$$\Omega_{xy}^2=-\frac{1}{k^2\theta^2}\ln\left[\frac{\langle S_2(\theta)\rangle^2+\langle S_3(\theta)\rangle^2}{\langle S_2(0)\rangle^2+\langle S_3(0)\rangle^2}\right]. \tag{44}$$

With the factors KA_x, KA_y, and Ω_{xy}^2 now known, we can substitute their values into Eq. (42) to obtain for $|B_{xy}|$ the expression

$$|B_{xy}|=\frac{1}{2(KA_x)(KA_y)\Omega_{xy}^2}\sqrt{\langle S_2(0)\rangle^2+\langle S_3(0)\rangle^2}. \tag{45}$$

4. It is readily seen that the angle ϕ, which represents the average phase difference between E_x and E_y, can be obtained by dividing Eq. (27) by Eq. (26), i.e.,

$$\tan\phi=\frac{\langle S_3(\theta)\rangle}{\langle S_2(\theta)\rangle}, \tag{46}$$

for any observation angle θ. The signs of the Stokes parameters in Eq. (46) determine in which quadrant ϕ is located.

5. The parameters KA_x, KA_y, $|B_{xy}|$, ϕ, Ω_{xx}, Ω_{yy}, and Ω_{xy} are now all determined, and we can next use the expression for the HBT coefficient (33) to calculate the value of the effective source width σ, namely,

$$\sigma^2=-\frac{1}{4k^2\theta^2}\ln\left[\frac{C(\theta,-\theta)}{(KA_x)^4\Omega_{xx}^4+(KA_y)^4\Omega_{yy}^4+2(KA_x)^2(KA_y)^2|B_{xy}|^2\Omega_{xy}^4}\right]. \tag{47}$$

6. The three remaining source parameters are the coherence lengths δ_{xx}, δ_{yy}, and $\delta_{xy}=\delta_{yx}$. Using the previously

established values of σ^2 and Ω_{ij}^2, these lengths can all be determined by rewriting Eq. (12) as

$$\delta_{ij}^2 = \frac{4\sigma^2\Omega_{ij}^2}{4\sigma^2 - \Omega_{ij}^2}. \quad (48)$$

We note that in the procedure outlined above, the parameters are recovered in a certain order, the value of established parameters being used to obtain subsequent ones.

When one of the Cartesian components is zero, e.g., $A_x = 0$, the only non-zero matrix element is W_{yy}. This contains three parameters, namely, A_y, σ, and δ_{yy}. These can be determined using Eqs. (39), (40), (47), and (48). The case in which $A_y = 0$ is completely analogous.

If $B_{xy} = 0$, as happens when the beam is unpolarized, the angle ϕ is undetermined. The remaining parameters are then A_x, δ_{xx}, A_y, δ_{yy} and, σ. These can be determined using Eqs. (37)–(40), (47), and (48).

In conclusion, we have presented a procedure by which the source parameters of an electromagnetic GSM source can be determined from far-zone measurements. The observed quantities involve intensities that are less sensitive to perturbations than the amplitude and phase.

The expression for the HBT coefficient relies on the assumption of Gaussian statistics. Two further assumptions are that the source produces a beam-like field, and that the effective widths of E_x and E_y are equal. This implies that the source is specified by eight independent parameters. We find that all of them can be recovered, with the exception of the two perpendicular amplitudes of which only the ratio can be obtained.

Our approach involves measuring the Stokes parameters, both on-axis and for non-zero values of the polar angle θ. For this, there are well-established methods [12–14]. Techniques to determine the HBT coefficient are described in [15,16].

The results presented here may be applied to the characterization and identification of distant sources.

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