



Phase anomaly and phase singularities of the field in the focal region of high-numerical aperture systems

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ABSTRACT

The phase behavior of the three Cartesian components of the electric field in the focal region of a high-numerical aperture focusing system is studied. The Gouy phase anomaly and the occurrence of phase singularities are examined in detail. It is found that the three field components exhibit different behaviors.

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1. Introduction

In 1890 Gouy found that the phase in the region of focus of a diffracted converging wave, compared to that of a plane wave of the same frequency, undergoes a rapid change of 180° near the geometric focus [1,2]. Since then many observations of this so-called *phase anomaly* have been reported and many different explanations for its origin have been presented [3–29]. Because of the importance of the anomalous phase behavior in the focal region, for example, in mode conversion [9], in coherence tomography [17] and in the tuning of the resonance frequency of laser cavities [7,20], the Gouy phase continues to attract a good deal of attention.

In a recent publication [30] it was pointed out that the phase anomaly near focus can be understood by considering a converging wave of a more general form, namely a converging wave exhibiting astigmatism. As is well-known, a geometrical optics analysis of this situation shows that the wavefront of such a field has, at each point, two principal radii of curvature and two, mutually orthogonal, focal lines ([31], Section 4.6). Geometrical optics may be regarded as the asymptotic limit of physical optics as the wavenumber $k = 2\pi/\lambda$, (λ denoting the wavelength) tends to infinity ([31], Section 3.1). With the help of the method of stationary phase it can be shown that in this

limit the field exhibits a phase discontinuity of an amount $\pi/2$ at each focal line [32,33]. Geometrical optics is governed by the eikonal equation, the actual wave field however, satisfies the Helmholtz equation. The solutions of the latter are well known to be continuous. Hence, according to physical optics, the two phase discontinuities have to be “smoothed out”, and become continuous but rapid phase changes. When the astigmatic wave aberration decreases to zero, i.e., when the field in the aperture becomes a converging spherical wave, the two foci coincide and the sharp phase change in the focal region is the Gouy phase change of an amount π . Hence the phase anomaly can be understood from elementary properties of rays and from the relation between geometrical optics and physical optics.

When the focusing geometry is such that the focal length of the system is appreciably larger than the aperture size, treating the optical field as a scalar (as is done in Ref. [30]) is usually justified. However, when the system has a high angular aperture, the vector character of the field can no longer be neglected and a scalar description becomes inaccurate. Wolf et al. [34–38] derived expressions for the electric and magnetic field vectors in the focal region of such a system. In the present paper we use this formalism to analyze the phase behavior, in particular the occurrence of phase singularities and the Gouy phase anomaly. Restricting ourselves to the electric field, three phases—one for each Cartesian component—rather than a single phase have to be considered. As we will demonstrate, all the three phases exhibit singularities, and their associated phase anomalies are markedly different.

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2. Focusing systems with a high angular aperture

Let us consider an aplanatic focusing system L of focal length f and with a semi-aperture angle α (see Fig. 1). We take the origin O of a right-handed Cartesian coordinate system at the geometrical focus. A monochromatic plane wave of angular frequency ω is incident upon the system, with the electric field polarized along the x -direction. The position of an observation point P is indicated by the dimensionless Lommel variables u and v , together with the azimuthal angle ϕ , defined as

$$u = kz \sin^2 \alpha, \quad (1)$$

$$v = k(x^2 + y^2)^{1/2} \sin \alpha. \quad (2)$$

Here the wavenumber $k = \omega/c$, with c denoting the speed of light. The electric and magnetic fields are of the form

$$\mathbf{E}(u, v, \phi, t) = \text{Re} [\mathbf{e}(u, v, \phi) \exp(-i\omega t)], \quad (3)$$

$$\mathbf{H}(u, v, \phi, t) = \text{Re} [\mathbf{h}(u, v, \phi) \exp(-i\omega t)], \quad (4)$$

respectively, where Re denotes the real part and t the time. The time-independent parts, \mathbf{e} and \mathbf{h} , of the electric and magnetic fields at a point $P(u, v, \phi)$ have been shown to be given by the expressions [36]:

$$e_x(u, v, \phi) = -iA[I_0(u, v) + I_2(u, v) \cos 2\phi], \quad (5a)$$

$$e_y(u, v, \phi) = -iAI_2(u, v) \sin 2\phi, \quad (5b)$$

$$e_z(u, v, \phi) = -2AI_1(u, v) \cos \phi, \quad (5c)$$

$$h_x(u, v, \phi) = -iAI_2(u, v) \sin 2\phi, \quad (6a)$$

$$h_y(u, v, \phi) = -iA[I_0(u, v) - I_2(u, v) \cos 2\phi], \quad (6b)$$

$$h_z(u, v, \phi) = -2AI_1(u, v) \sin \phi, \quad (6c)$$

where

$$I_0(u, v) = \int_0^\alpha \cos^{1/2} \theta \sin \theta (1 + \cos \theta) J_0 \left(\frac{v \sin \theta}{\sin \alpha} \right) \exp \left(\frac{i u \cos \theta}{\sin^2 \alpha} \right) d\theta, \quad (7)$$

$$I_1(u, v) = \int_0^\alpha \cos^{1/2} \theta \sin^2 \theta J_1 \left(\frac{v \sin \theta}{\sin \alpha} \right) \exp \left(\frac{i u \cos \theta}{\sin^2 \alpha} \right) d\theta, \quad (8)$$

$$I_2(u, v) = \int_0^\alpha \cos^{1/2} \theta \sin \theta (1 - \cos \theta) J_2 \left(\frac{v \sin \theta}{\sin \alpha} \right) \exp \left(\frac{i u \cos \theta}{\sin^2 \alpha} \right) d\theta. \quad (9)$$

In these integrals $J_n(x)$ denotes the Bessel function of the first kind and of order n . The amplitude A will be taken to be unity from now on.

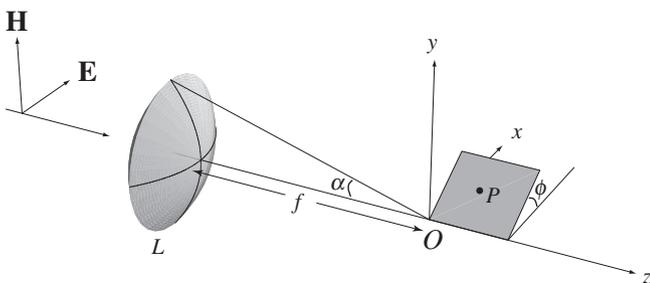


Fig. 1. Illustrating the geometry.

It is to be noted that all the functions in Eqs. (5)–(9) depend on the semi-aperture angle α (not explicitly shown).

The following symmetry relations follow immediately from Eqs. (5) and (7)–(9):

$$e_x(-u, v, \phi) = -e_x^*(u, v, \phi), \quad (10a)$$

$$e_y(-u, v, \phi) = -e_y^*(u, v, \phi), \quad (10b)$$

$$e_z(-u, v, \phi) = e_z^*(u, v, \phi). \quad (10c)$$

By comparing Eqs. (5) and (6) it is clear that the behavior of the magnetic field components is similar to that of the electric field components. In particular, the magnetic field component h_x is identical to the electric field component e_y ; h_y in a meridional plane $\phi = \text{constant}$ is identical to e_x in the plane $\phi \rightarrow \phi + \pi/2$; and h_z in the meridional plane ϕ is identical to e_z in the plane $\phi \rightarrow \phi + \pi/2$. In view of these relations we will restrict our analysis to the electric field only.

3. Phase singularities

The three components of the electric field, given by Eq. (5), are complex-valued. At points at which both the real and the imaginary parts of a component have the value zero, the amplitude is also zero and consequently the phase $\psi(\mathbf{r})$ is undetermined or “singular” at these points. The study of the topology of phase singularities is the subject of a relatively new subdiscipline, called *singular optics* [39–48]. Two of its key concepts are the topological charge and the topological index. The topological charge s of a phase singularity is defined by the expression

$$s \equiv \frac{1}{2\pi} \oint_C \nabla \psi(\mathbf{r}) \cdot d\mathbf{r}, \quad (11)$$

where C is a closed contour of winding number one that is traversed counter-clockwise. The topological index is defined as the topological charge of the field $\nabla \psi(\mathbf{r})$.

According to Eq. (5a) the electric field component e_x in the focal plane ($u = 0$) is purely imaginary. As noted by Richards and Wolf [36], this plane contains ring-shaped phase singularities of e_x , centered on the u -axis, at which $\text{Im}[e_x]$ changes sign. They also showed that in the low aperture limit ($\alpha \rightarrow 0$), e_x is the only non-vanishing component of the electric field, and these singularities form the well-known Airy rings of classical scalar diffraction theory.

The phase behavior of e_y is illustrated in Fig. 2. In this figure the phase is color-coded, with phase singularities indicated by the intersections of contour lines. A pair of singularities of opposite topological charge can be seen along the line $u = 23$. It follows from Eq. (5b) that the phase singularities of e_y form rings centered on the z -axis.

The phase of the longitudinal field component e_z is shown in Fig. 3. Again, several ring-shaped phase singularities can be observed. As shown in [49], a pair of these singularities merges with two phase saddle points when the semi-aperture angle α is changed. In such an annihilation process both the topological charge and the topological index are conserved.

4. The Gouy phase anomaly

The only component of the electric field which does not vanish along the optical axis ($v = 0$) is e_x . The wavefront spacing of that component is highly irregular (see for example [50,51] and the references therein). This behavior is seen from a plot of the real and the imaginary part, $\text{Re}[e_x(u, v, \phi)]$ and $\text{Im}[e_x(u, v, \phi)]$, with the longitudinal Lommel variable u as the parameter. An example is presented in Fig. 4.

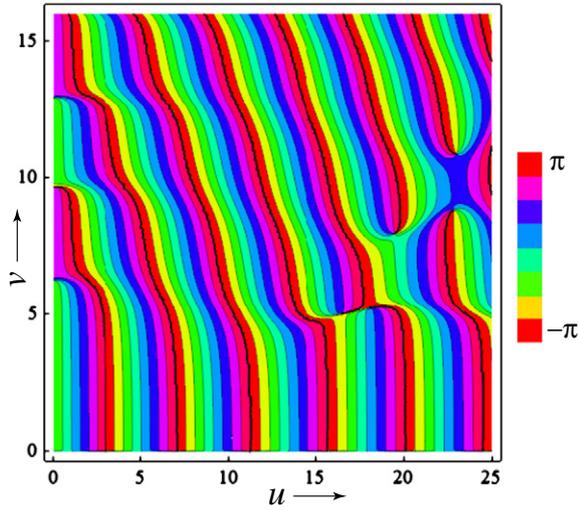


Fig. 2. Contours of the phase of the transverse electric field component $e_y(u, v, \phi)$ in the u, v -plane. Intersections of different contours (e.g. at $u=20, v=8$) indicate phase singularities. The semi-aperture angle α of the focusing system was taken to be 45° .

Alternatively, one can compare the phase $\psi[e_x(u, v, \phi)]$ of e_x , with that of a converging, non-diffracted spherical wave in the half-space $z < 0$, namely $-kR$, and with that of a diverging spherical wave in the half space $z \geq 0$, namely $+kR$, where $kR = k(x^2 + y^2 + z^2)^{1/2} = (v^2 + u^2/\sin^2\alpha)^{1/2}/\sin\alpha$. The Gouy phase anomaly for the x -component of the electric field, $\delta_x(u, v, \phi)$, is then defined as (see ([31], Section 8.8.4) or ([32], Ch. 8)):

$$\delta_x(u, v, \phi) = \begin{cases} \psi[e_x(u, v, \phi)] + kR & \text{when } z < 0, \\ \psi[e_x(u, v, \phi)] - kR & \text{when } z \geq 0. \end{cases} \quad (12)$$

From Eqs. (5a) and (12) one immediately finds that the phase anomaly at two points that are symmetrically located with respect to the geometrical focus, satisfies the relation

$$\delta_x(u, v, \phi) + \delta_x(-u, v, \phi + \pi) = -\pi. \quad (13)$$

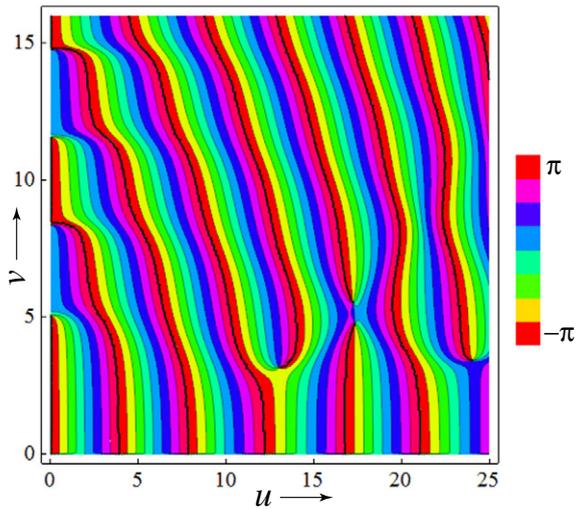


Fig. 3. Contours of the phase of the longitudinal electric field component $e_z(u, v, \phi)$ in the u, v -plane. Intersections of different contours (e.g. at $u=13, v=3$) indicate phase singularities. The semi-aperture angle α was taken to be 45° .

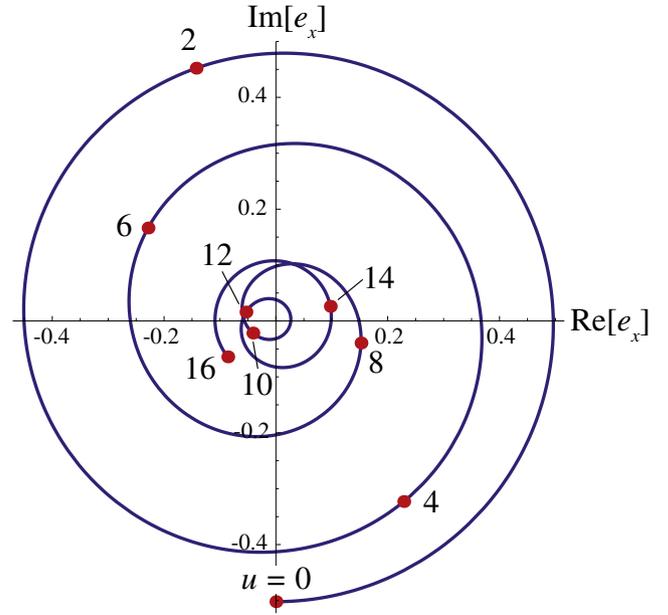


Fig. 4. Parametric plot of $\text{Re}[e_x]$ and $\text{Im}[e_x]$ along the optical axis. The dots correspond with the values $u = 0, 2, \dots, 16$. The semi-aperture angle α was taken to be 45° .

At the focus ($u = v = 0$) one has, according to Eq. (5a),

$$\delta_x(0, 0) = \psi[e_x(0, 0)] = -\pi/2. \quad (14)$$

The on-axis phase anomaly $\delta_x(u, v=0)$ is shown in Fig. 5 for selected values of the semi-aperture angle α of the focusing system. When α increases, the change in phase near focus is seen to become more gradual and to decrease. Scalar theory ([31], Section 8.8.4) predicts a linear behavior of the phase anomaly, with a discontinuity of π at each phase singularity (panel a). It is seen that for smaller values of the semi-aperture angle the phase behavior tends to that given by scalar theory. In connection with Fig. 5 it is important to bear in mind that the longitudinal coordinate u is, by virtue of Eq. (1), dependent on the value of the semi-aperture angle α .

In Fig. 6 the behavior of the phase anomaly of e_x is shown along several rays through the geometrical focus O . As an oblique ray passes through focus, the angle ϕ that defines the meridional plane in which the ray lies, changes by π . It is seen that when the angle of inclination θ of the ray (with $\theta = \tan^{-1}[v \sin \alpha/|u|]$) increases, the change in $\delta_x(u, v, \phi)$ near focus decreases.

According to Eq. (5b) the y -component of the electric field vanishes along the optical axis, and hence its phase $\psi[e_y(u, v)]$ is singular there. Along oblique rays through the geometric focus, however, this phase is defined. In analogy with Eq. (12) we define the phase anomaly $\delta_y(u, v, \phi)$ of e_y as

$$\delta_y(u, v, \phi) = \begin{cases} \psi[e_y(u, v, \phi)] + kR & \text{when } z < 0, \\ \psi[e_y(u, v, \phi)] - kR & \text{when } z > 0. \end{cases} \quad (15)$$

From Eqs. (5b) and (15) we find that the phase anomaly at two points that are symmetrically located with respect to the geometrical focus, satisfies the relation

$$\delta_y(u, v, \phi) + \delta_y(-u, v, \phi + \pi) = -\pi. \quad (16)$$

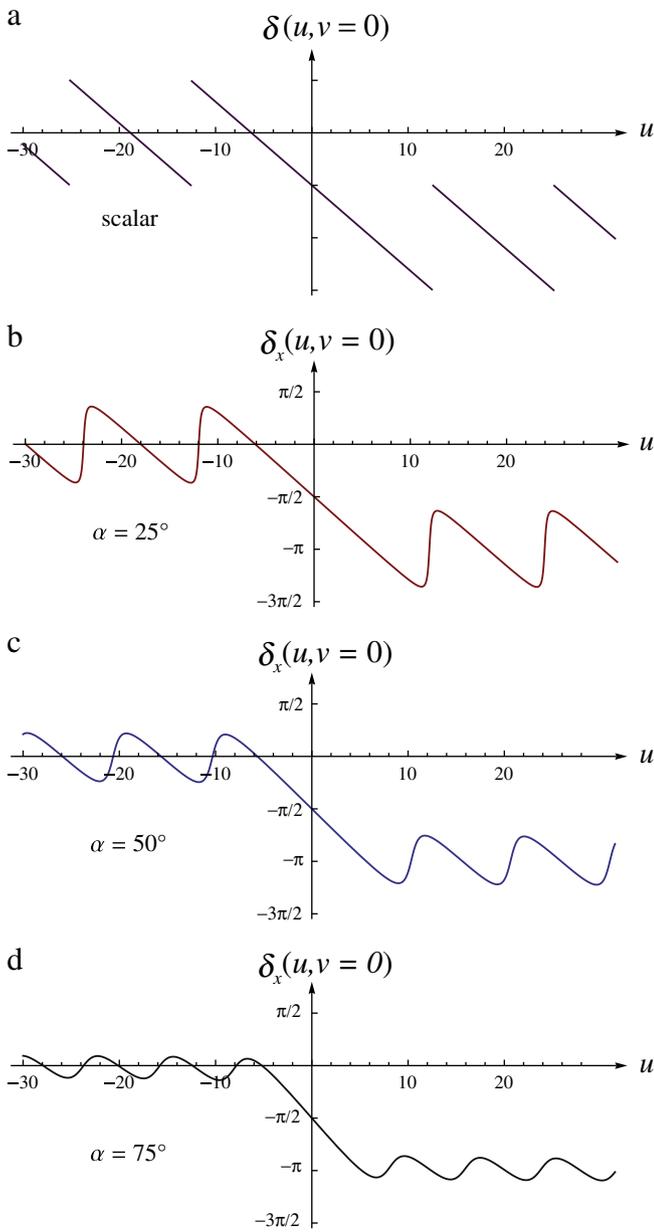


Fig. 5. The phase anomaly along the optical axis according to scalar theory (a), and the phase anomaly $\delta_x(u, v = 0)$ of the electric field component e_x for selected values of the semi-aperture angle α , (b) $\alpha = 25^\circ$, (c) $\alpha = 50^\circ$, and (d) $\alpha = 75^\circ$.

A ray with $v \propto |u|$ runs through the geometrical focus. On using the fact that for small arguments $J_n(x) \sim x^n$, we find from Eq. (5b) that along such a ray $e_y \sim -iu^2 \sin 2\phi$. Hence

$$\lim_{u \downarrow 0} \delta_y(u, v, \phi + \pi) = \lim_{u \uparrow 0} \delta_y(u, v, \phi) = -\frac{\pi}{2} \times \text{sign}[\sin 2\phi]. \quad (17)$$

Here the subscripts $u \downarrow 0$ and $u \uparrow 0$ indicate that the quantity u approaches the limiting value 0 from above and from below, respectively. Further, $\text{sign}(x)$ denotes the sign function

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \quad (18)$$

Although both limits in Eq. (17) are equal, the phase anomaly $\delta_y(u, v, \phi)$ is undefined at the geometrical focus because e_y vanishes there. An example of this behavior is shown in Fig. 7. The two discontinuities

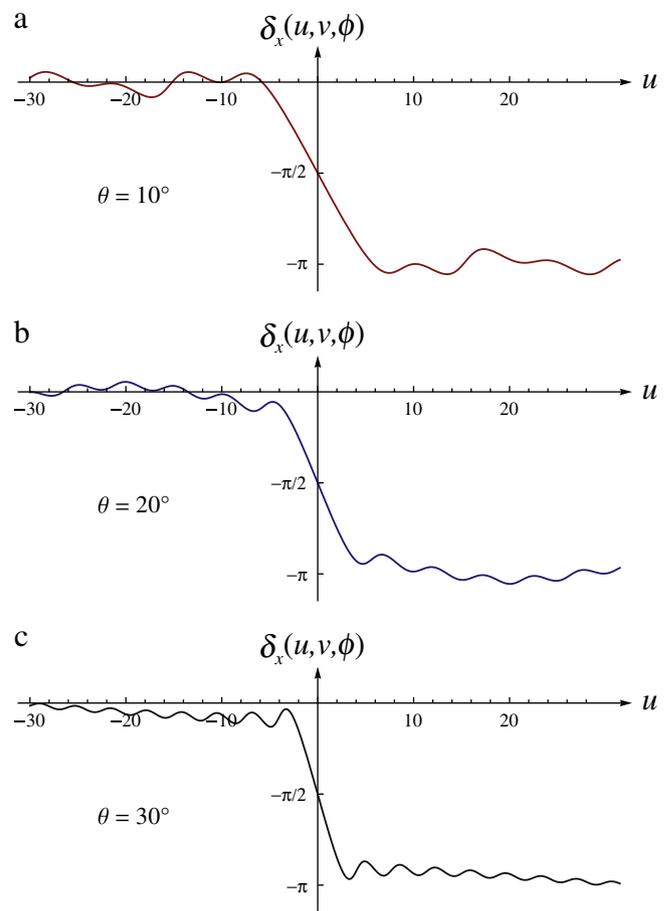


Fig. 6. The phase anomaly $\delta_x(u, v, \phi)$ of the electric field component e_x along several rays in the meridional plane $\phi = 0^\circ$ through the geometrical focus. The angle of inclination of each ray is denoted by θ , with (a) $\theta = 10^\circ$, (b) $\theta = 20^\circ$, and (c) $\theta = 30^\circ$. In this example the semi-aperture angle α was taken to be 45° .

near $u = 1$ and $u = 3$ are a consequence of the fact that the phase is defined up to an integral number of 2π .

Next we define, again in analogy with Eq. (12), the phase anomaly $\delta_z(u, v, \phi)$ of the longitudinal component of the electric field as

$$\delta_z(u, v, \phi) = \begin{cases} \psi[e_z(u, v, \phi)] + kR & \text{when } z < 0, \\ \psi[e_z(u, v, \phi)] - kR & \text{when } z > 0. \end{cases} \quad (19)$$

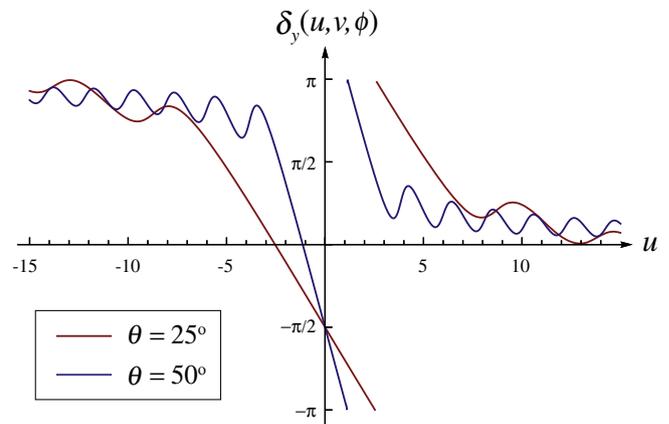


Fig. 7. The phase anomaly $\delta_y(u, v, \phi)$ of the electric field component e_y along two rays in the meridional plane $\phi = 45^\circ$ through the geometrical focus. The angle of inclination of each ray is denoted by θ . In this example the semi-aperture angle $\alpha = 50^\circ$.

It is seen from Eq. (10) that the phase behavior of the longitudinal component e_z of the electric field in the focal region differs from that of the two transverse components. From Eqs. (5c) and (19) it follows that the phase anomaly at two points that are symmetrically located with respect to the geometrical focus, satisfies the relation

$$\delta_z(u, v, \phi) + \delta_z(-u, v, \phi + \pi) = \pi. \quad (20)$$

Just as the y -component, the longitudinal component e_z equals zero along the optical axis. On using the small argument approximation for the Bessel function in Eq. (5c), one finds that along an oblique ray through the geometrical focus $e_z \sim -|u| \cos \phi$, and hence

$$\lim_{u \rightarrow 0} \delta_z(u, v, \phi) = \pi \times \Theta[\cos \phi], \quad (21a)$$

$$\lim_{u \rightarrow 0} \delta_z(u, v, \phi + \pi) = \pi \times \Theta[-\cos \phi], \quad (21b)$$

with $\Theta(x)$ being the Heaviside stepfunction

$$\Theta(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \quad (22)$$

The π phase discontinuity in e_z as the ray passes through focus is related to the fact that the angle ϕ , which defines the orientation of the meridional plane that contains the ray, has a discontinuity there of an amount π . (It is to be noted that the ϕ -dependence of e_x and e_y is such that this jump does not affect these two field components.) Examples of the phase anomaly of the longitudinal electric field are shown in Fig. 8. It is seen that when the angle that the ray makes with the axis becomes larger, the oscillations of the phase anomaly become more damped. A comparison of Eqs. (13), (16) and (20) shows that the phases of the three Cartesian components of the electric field have different symmetry relations. Furthermore, Eqs. (14), (17) and (21) show that their behavior at the geometrical focus is also different.

5. Conclusions

We have examined the phase behavior of the electric field in the vicinity of the geometric focus of an aplanatic, high-numerical aperture system. All three Cartesian components were found to

posses phase singularities. We also showed that the phase anomalies associated with each of the phases are markedly different. The x -component, along the direction of polarization of the incident field, shows the classical Gouy phase behavior expressed by Eqs. (13) and (14). Its precise behavior depends on the semi-aperture angle α of the focusing system. In contrast to the x -component of the electric field, the other transverse component, e_y , is singular at the geometric focus. Eq. (17) shows that its phase anomaly at the focus depends on the orientation of the meridional plane (i.e., on the angle ϕ), but behaves in a similar manner. The phase anomaly of the longitudinal component e_z is the only one which does not tend to $\pm\pi/2$ at the focus. Instead this phase undergoes a phase discontinuity there, by an amount π .

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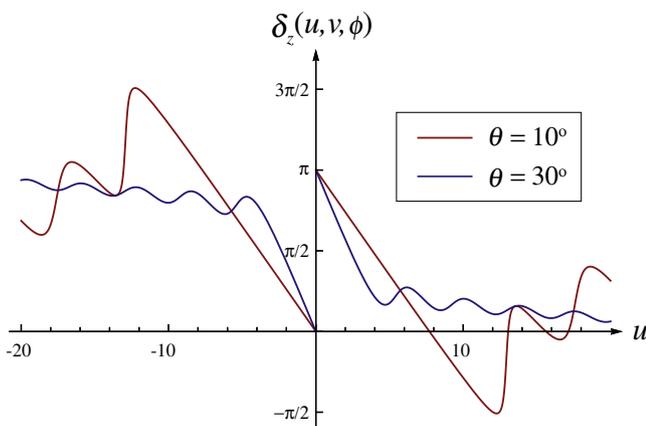


Fig. 8. The phase anomaly $\delta_z(u, v, \phi)$ of the electric field component e_z along two rays through the geometric focus in the meridional plane $\phi = 180^\circ$. The angle of inclination of each ray is denoted by θ . In this example the semi-aperture angle $\alpha = 45^\circ$.

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