Spatial correlation properties of focused partially coherent light

David G. Fischer

Microgravity Science Division, NASA Glenn Research Center, 21000 Brookpark Road, MS 110-3 Cleveland, Ohio 44135

Taco D. Visser

Dept. of Physics and Astronomy, Free University, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

We investigate the spatial coherence properties in the focal region of a converging, spatially partially coherent wave field. In particular, we find that, depending on the effective coherence length of the field in the aperture, the longitudinal and transverse coherence lengths in the focal region can be either larger or smaller than the corresponding width of the intensity distribution. Also, the correlation function is shown to exhibit phase singularities. © 2004 Optical Society of America

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1. INTRODUCTION

The focusing of partially coherent light has recently been the subject of several studies. Wang *et al.* generalized the classical Debye theory to include the focusing of partially coherent light by high-Fresnel-number systems.[1] Friberg *et al.* analyzed the axial intensity distribution of partially coherent wave fields focused by low-Fresnel-number systems.[2] Some related results were reported by Lü *et al.*[3] In addition, the effect of the state of coherence on the three-dimensional intensity distribution near focus was examined by Visser *et al.*[4]

The above-mentioned studies dealt exclusively with field intensities. To the best of our knowledge, the correlation properties of focused partially coherent fields have not been examined. Because, for example, light that is produced by a multimode laser or light that has traveled through the atmosphere or biological tissue is partially coherent, it is of prime importance to explore the correlation properties of such focused wave fields.

In the present paper an important class of partially coherent fields, namely, Gaussian-Schell model fields, is examined. In particular, the spectral degree of coherence of the field in the focal region is analyzed. It is shown that, depending on the effective coherence length of the field in the aperture, the coherence length in the focal region can be either greater or smaller than the width of the intensity distribution. Also, the spectral degree of coherence is found to possess phase singularities. This implies that the fields at certain pairs of points are completely uncorrelated.

2. PARTIALLY COHERENT FOCUSED FIELDS

Consider a converging, monochromatic field of frequency ω that is exiting a circular aperture with radius a in a plane screen (see Fig. 1). The origin O of the coordinate system coincides with the geometrical focus. The amplitude of the field is $U^{(0)}(\mathbf{r}', \omega)$, $\mathbf{r}'$ being the position vector of a point $Q(\mathbf{r}')$ in the aperture. The field at a point $P(\mathbf{r})$ in the focal region is, according to the Huygens–Fresnel principle (Ref. 5, Chap. 8.2), given by the expression

$$U(\mathbf{r}, \omega) = -\frac{\mathrm{i}}{\lambda} \int\int_S U^{(0)}(\mathbf{r}', \omega) \frac{\exp(\mathrm{i}ks)}{s} \mathrm{d}^2 r', \quad (1)$$

where the integration extends over the spherical wave front S that briefly fills the aperture, $s = |\mathbf{r} - \mathbf{r}'|$ denotes the distance QP, and we have suppressed a periodic time-dependent factor $\exp(-\mathrm{i}\omega t)$.

For a partially coherent wave field one must consider, instead of the field $U^{(0)}(\mathbf{r}', \omega)$, the cross-spectral density function (Ref. 6, Sec. 2.4.4) of the field at two points $Q_1(\mathbf{r}_1')$ and $Q_2(\mathbf{r}_2')$, namely,

$$W^{(0)}(\mathbf{r}_1', \mathbf{r}_2', \omega) = \langle U^{(0)*}(\mathbf{r}_1', \omega) U^{(0)}(\mathbf{r}_2', \omega) \rangle. \quad (2)$$

Here the angle brackets denote the average, taken over a statistical ensemble of monochromatic realizations $\{U^{(0)}(\mathbf{r}')\exp(-\mathrm{i}\omega t)\}$ (Ref. 6, Sec. 4.7), and the asterisk denotes the complex conjugate. The cross-spectral density of the focused field

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U^*(\mathbf{r}_1, \omega) U(\mathbf{r}_2, \omega) \rangle \quad (3)$$

is given by the formula

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \frac{1}{\lambda^2} \int\int_S \int\int_S W^{(0)}(\mathbf{r}', \mathbf{r}'', \omega) \times \frac{\exp[\mathrm{i}k(s_2 - s_1)]}{s_1 s_2} \mathrm{d}^2 r' \mathrm{d}^2 r'', \quad (4)$$

where we have used Eqs. (1) and (3), with

$$s_1 = |\mathbf{r}_1 - \mathbf{r}'|, \quad (5)$$

$$s_2 = |\mathbf{r}_2 - \mathbf{r}''|. \quad (6)$$

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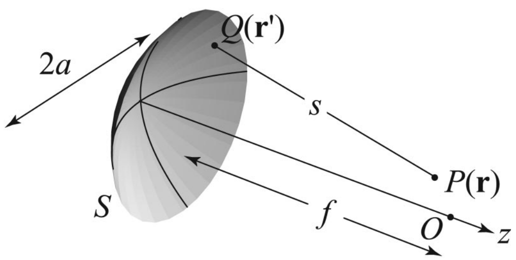

Fig. 1. Illustration of the notation.

From now on we omit the explicit dependence of the various quantities on the frequency ω.

We assume that the field in the aperture is a Gaussian Schell-model field with uniform intensity (Ref. 6, Sec. 5.3.2); i.e.,

$$W^{(0)}(\mathbf{r}', \mathbf{r}'') = W^{(0)}(\boldsymbol{\rho}', \boldsymbol{\rho}'') = \exp[-(\rho'' - \rho')^2/2\sigma_g^2], \tag{7}$$

where $\boldsymbol{\rho} = (x, y)$ is the two-dimensional transverse vector that specifies the position of a point Q on S, and σ_g is a positive constant that is a measure of the effective spectral coherence length of the field in the aperture. On substituting from Eq. (7) into Eq. (4) and approximating the factors $s_i\,(i = 1,2)$ in the denominator by the focal length f, we find that

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{(\lambda f)^2} \int\int_S \int\int_S \exp[-(\rho'' - \rho')^2/2\sigma_g^2] \times \exp[ik(s_2 - s_1)]\mathrm{d}^2r'\mathrm{d}^2r''. \tag{8}$$

The distances s_i appearing in the exponent may be approximated by the expressions

$$s_1 \approx f - \mathbf{q}'\cdot\mathbf{r}_1, \tag{9}$$

$$s_2 \approx f - \mathbf{q}''\cdot\mathbf{r}_2, \tag{10}$$

where $\mathbf{q}'$ and $\mathbf{q}''$ are unit vectors in the directions $O\mathbf{r}'$ and $O\mathbf{r}''$, respectively. Hence we obtain the expression

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{(\lambda f)^2} \int\int_S \int\int_S \exp[-(\rho'' - \rho')^2/2\sigma_g^2] \times \exp[ik(\mathbf{q}'\cdot\mathbf{r}_1 - \mathbf{q}''\cdot\mathbf{r}_2)]\mathrm{d}^2r'\mathrm{d}^2r''. \tag{11}$$

A quantitative measure of the strength of the field correlations at a pair of points $P_1(\mathbf{r}_1)$, $P_2(\mathbf{r}_2)$ in the focal region is given by the spectral degree of coherence (Ref. 6, Sec. 4.3.2), which is defined as

$$\mu(\mathbf{r}_1, \mathbf{r}_2) = \frac{W(\mathbf{r}_1, \mathbf{r}_2)}{[S(\mathbf{r}_1)S(\mathbf{r}_2)]^{1/2}}, \tag{12}$$

with the spectral density at position $\mathbf{r}_i$ given by the diagonal elements of the cross-spectral density; i.e.,

$$S(\mathbf{r}_i) = W(\mathbf{r}_i, \mathbf{r}_i). \tag{13}$$

Let us now examine the spectral degree of coherence of the focused fields for pairs of points on the z axis and for pairs of points in the focal plane.

3. AXIAL POINTS

We temporarily restrict ourselves to pairs of points on the z axis, i.e.,

$$\mathbf{r}_1 = (0, 0, z_1), \tag{14}$$

$$\mathbf{r}_2 = (0, 0, z_2). \tag{15}$$

On introducing cylindrical coordinates ρ and ϕ, while using the approximations

$$\mathbf{q}'\cdot\mathbf{r}_1 \approx -z_1(1 - \rho'^2/2f^2), \tag{16}$$

$$\mathbf{q}''\cdot\mathbf{r}_2 \approx -z_2(1 - \rho''^2/2f^2), \tag{17}$$

in Eq. (11), we obtain for the cross-spectral density the expression

$$\begin{aligned} W(0, 0, z_1; 0, 0, z_2) = {} & \left(\frac{1}{\lambda f}\right)^2 \int_0^{2\pi}\int_0^a\int_0^{2\pi}\int_0^a \\ & \times \exp\{-[\rho'^2 + \rho''^2 \\ & - 2\rho'\rho''\cos(\phi' - \phi'')]/2\sigma_g^2\} \\ & \times \exp\{ik[-z_1(1 - \rho'^2/2f^2) \\ & + z_2(1 - \rho''^2/2f^2)]\} \\ & \times \rho'\rho''\mathrm{d}\phi'\mathrm{d}\rho'\mathrm{d}\phi''\mathrm{d}\rho'', \end{aligned} \tag{18}$$

where we have used the relation $\mathrm{d}x\mathrm{d}y = \rho\mathrm{d}\rho\mathrm{d}\phi$. We note that in Eq. (18) only one factor depends on the variables ϕ' and ϕ''. Since[7]

$$\int_0^{2\pi}\int_0^{2\pi} \exp[\rho'\rho''\cos(\phi' - \phi'')/\sigma_g^2]\mathrm{d}\phi'\mathrm{d}\phi'' = 4\pi^2 I_0\left(\frac{\rho'\rho''}{\sigma_g^2}\right), \tag{19}$$

with I_0 denoting the modified Bessel function of order zero, we find for the cross-spectral density the formula

$$\begin{aligned} W(0, 0, z_1; 0, 0, z_2) = {} & \left(\frac{2\pi}{\lambda f}\right)^2 \int_0^a\int_0^a \exp[-(\rho'^2 \\ & + \rho''^2)/2\sigma_g^2] I_0\left(\frac{\rho'\rho''}{\sigma_g^2}\right) \\ & \times \exp\{ik[-z_1(1 - \rho'^2/2f^2) \\ & + z_2(1 - \rho''^2/2f^2)]\} \\ & \times \rho'\rho''\mathrm{d}\rho'\mathrm{d}\rho''. \end{aligned} \tag{20}$$

The axial spectral density distribution is given by the expression

$$S(0, 0, z) = W(0, 0, z; 0, 0, z) \tag{21}$$

$$\begin{aligned} = {} & \left(\frac{2\pi}{\lambda f}\right)^2 \int_0^a\int_0^a \exp[-(\rho'^2 \\ & + \rho''^2)/2\sigma_g^2] I_0\left(\frac{\rho'\rho''}{\sigma_g^2}\right) \\ & \times \exp\{ik[z(\rho'^2 - \rho''^2)/2f^2]\} \\ & \times \rho'\rho''\mathrm{d}\rho'\mathrm{d}\rho''. \end{aligned} \tag{22}$$

Since the spectral density is real-valued, it can be simplified to the form

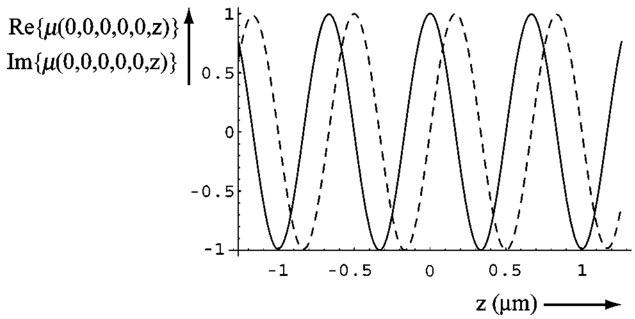

Fig. 2. Real part (solid curve) and imaginary part (dashed curve) of the spectral degree of coherence $\mu(0, 0, 0; 0, 0, z)$. In this example $a = 1$ cm, $f = 2$ cm, $\sigma_g = 0.5$ cm, and $\lambda = 0.6328\ \mu$m.

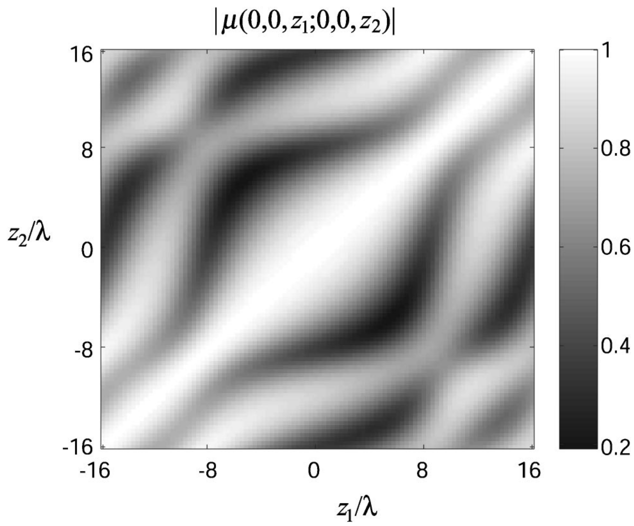

Fig. 3. Modulus of the spectral degree of coherence, $|\mu(0, 0, z_1; 0, 0, z_2)|$. In this example $a = 1$ cm, $f = 2$ cm, $\sigma_g = 0.4$ cm, and $\lambda = 0.6328\ \mu$m.

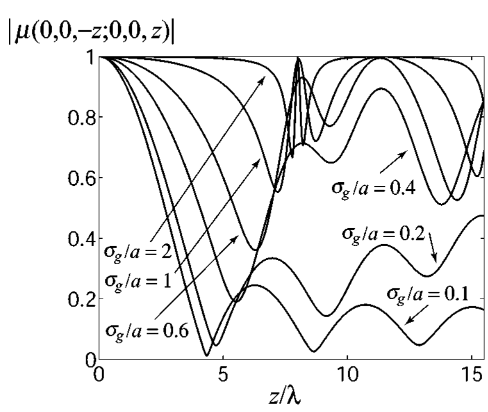

Fig. 4. Modulus of the spectral degree of coherence, $|\mu(0, 0, -z; 0, 0, z)|$ for several values of the scaled coherence length σ_g/a. In this example $a = 1$ cm, $f = 2$ cm, and $\lambda = 0.6328\ \mu$m.

$$S(0, 0, z) = \left(\frac{2\pi}{\lambda f}\right)^2 \int_0^a \int_0^a \exp[-(\rho'^2 + \rho''^2)/2\sigma_g^2] I_0\left(\frac{\rho'\rho''}{\sigma_g^2}\right) \times \cos[kz(\rho'^2 - \rho''^2)/2f^2]\rho'\rho''\mathrm{d}\rho'\mathrm{d}\rho''. \tag{23}$$

Several symmetry properties readily follow from Eqs. (20) and (23), viz.,

$$W(0, 0, -z_1; 0, 0, -z_2) = W(0, 0, z_1; 0, 0, z_2)^*, \tag{24}$$

$$S(0, 0, -z) = S(0, 0, z). \tag{25}$$

On using Eqs. (24) and (25) together with the definition of the spectral degree of coherence [Eq. (12)], we obtain the symmetry relation

$$\mu(0, 0, z_1; 0, 0, z_2) = \mu(0, 0, -z_1; 0, 0, -z_2)^*. \tag{26}$$

It follows from Eq. (20) that for z_1 kept fixed, $\mu(0, 0, z_1; 0, 0, z_2)$ is an oscillating function of z_2, with a period that is somewhat larger than the wavelength λ. An example is shown in Fig. 2, in which both the real part and the imaginary part of $\mu(0, 0, 0; 0, 0, z)$ are depicted.

For a converging field that is spatially fully coherent, the first and second axial zeros of intensity are located at (Ref. 5, Sec. 8.8.2)

$$z_{01} = \pm 2\lambda(f/a)^2, \tag{27}$$

$$z_{02} = \pm 4\lambda(f/a)^2, \tag{28}$$

respectively. For partially coherent fields, such zeros are absent. In Fig. 3 the modulus of the spectral degree of coherence, calculated by numerically integrating Eqs. (20) and (23), is shown for pairs of axial points z_1, z_2 up to the point z_{02}. It is seen that, contrary to what one might expect, $|\mu(0, 0, z_1; 0, 0, z_2)|$ is not a decreasing function of the distance $|z_1 - z_2|$ but rather has an oscillatory behavior.

The function $|\mu(0, 0, -z; 0, 0, z)|$ is shown for selected values of the normalized coherence length σ_g/a in Fig. 4. For comparison's sake the normalized axial intensity distribution, $S(0, 0, z)/S(0, 0, 0)$, is also plotted in Fig. 5. It is striking that for $\sigma_g/a \gtrsim 0.5$ [as in Figs. 5(a)–5(c)] the effective coherence length in the focal region is found to be greater than the width of the axial intensity distribution. For $\sigma_g/a \lesssim 0.5$ [as in Figs. 5(d)–5(f)] the effective coherence length in the focal region is seen to be less than the width of the intensity distribution.

4. FOCAL PLANE

We next study pairs of points that lie in the focal plane. One observation point is taken to be at the geometrical focus O and the other in the focal plane at a given distance from the z axis. Because of rotational symmetry we may, without loss of generality, assume that the latter point lies on the x axis, i.e.,

$$\mathbf{r}_1 = (0, 0, 0), \tag{29}$$

$$\mathbf{r}_2 = (x, 0, 0). \tag{30}$$

We then have

$$s_1 = f. \tag{31}$$

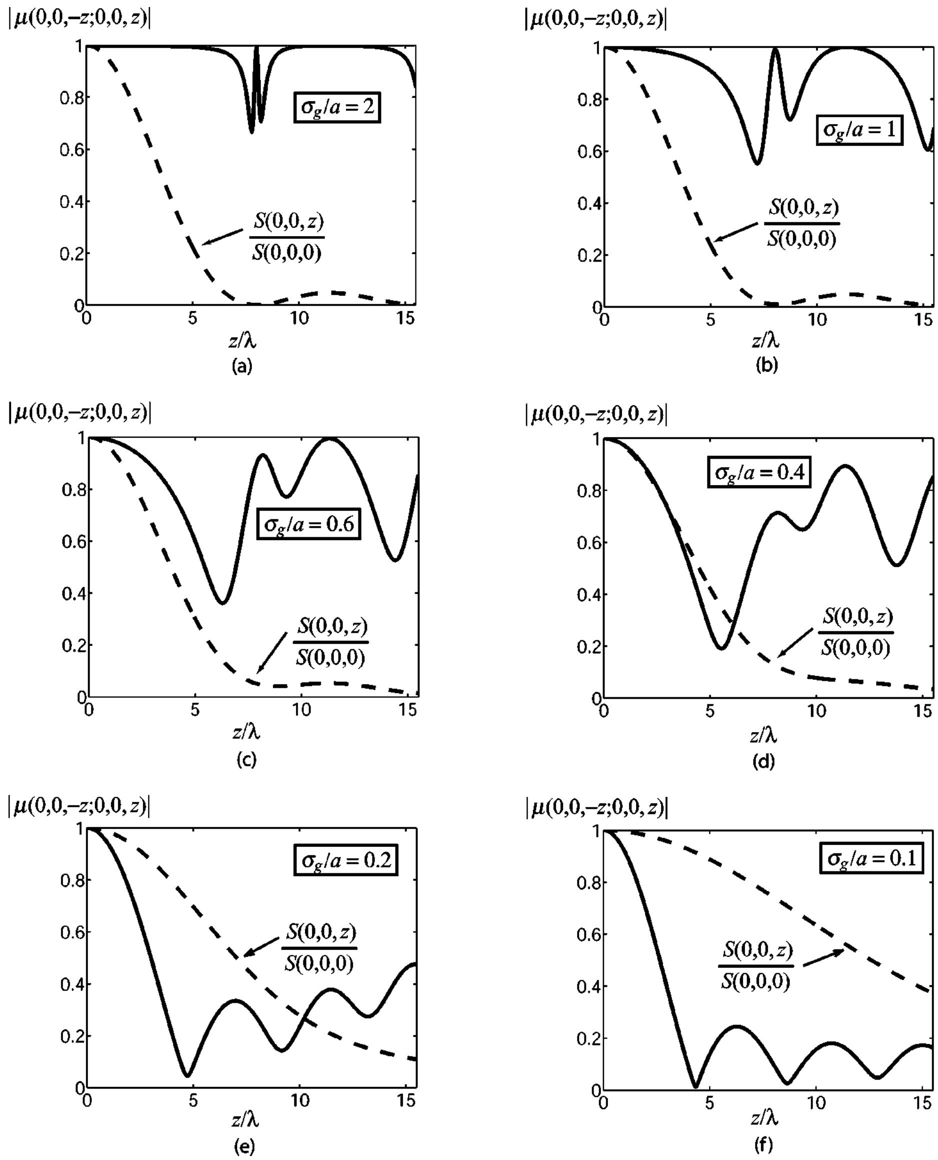

Fig. 5. Modulus of the spectral degree of coherence, $|\mu(0, 0, -z;\, 0, 0, z)|$ and the normalized axial spectral density $S(0, 0, z)/S(0, 0, 0)$ for different values of σ_g/a. (a) $\sigma_g/a = 2$, (b) $\sigma_g/a = 1$, (c) $\sigma_g/a = 0.6$, (d) $\sigma_g/a = 0.4$, (e) $\sigma_g/a = 0.2$, (f) $\sigma_g/a = 0.1$. In all examples $a = 1$ cm, $f = 2$ cm, and $\lambda = 0.6328\ \mu$m.

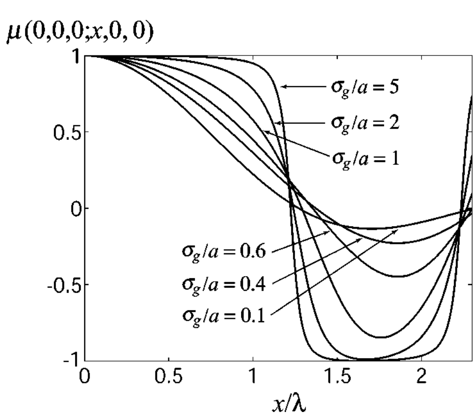

Fig. 6. Spectral degree of coherence, $\mu(0, 0, 0;\, x, 0, 0)$ for several values of the scaled coherence length σ_g/a. Notice that $\mu(0, 0, 0;\, x, 0, 0)$ is strictly real. In all examples $a = 1$ cm, $f = 2$ cm, and $\lambda = 0.6328\ \mu$m.

Also, since

$$\mathbf{q}'' = [\rho'' \cos \phi''/f,\ \rho'' \sin \phi''/f,\ -(1 - \rho''^2/f^2)^{1/2}], \tag{32}$$

we obtain from Eq. (10) that

$$s_2 \approx f - \rho'' x \cos \phi''/f. \tag{33}$$

On substituting from Eqs. (31) and (33) into Eq. (8) and again using cylindrical coordinates, we find that

$$\begin{aligned} W(0, 0, 0;\, x, 0, 0) &= \frac{1}{(\lambda f)^2} \int_0^{2\pi} \int_0^{a} \int_0^{2\pi} \int_0^{a} \exp\{-[\rho'^2 + \rho''^2 \\ &\quad - 2\rho'\rho'' \cos(\phi' - \phi'')]/2\sigma_g^2\} \\ &\quad \times \exp[-ik(\rho'' x \cos \phi'')/f]\rho'\rho'' \mathrm{d}\phi' \mathrm{d}\rho' \mathrm{d}\phi'' \mathrm{d}\rho'' \end{aligned} \tag{34}$$

$$= \frac{2\pi}{(\lambda f)^2} \int_0^a \int_0^{2\pi} \int_0^a \exp[-(\rho'^2 + \rho''^2)/2\sigma_g^2] I_0\left(\frac{\rho'\rho''}{\sigma_g^2}\right) \times \exp[-ik(\rho'' x \cos\phi'')/f]\rho'\rho'' d\rho' d\phi'' d\rho'', \tag{35}$$

$$= \left(\frac{2\pi}{\lambda f}\right)^2 \int_0^a \int_0^a \exp[-(\rho'^2 + \rho''^2)/2\sigma_g^2] I_0\left(\frac{\rho'\rho''}{\sigma_g^2}\right) \times J_0\left(\frac{k\rho'' x}{f}\right)\rho'\rho'' d\rho' d\rho'', \tag{36}$$

with J_0 the Bessel function of the first kind of order zero. We notice from Eq. (36) that $W(0, 0, 0; x, 0, 0)$, and hence also $\mu(0, 0, 0; x, 0, 0)$, is real-valued.

In order to obtain the spectral density, we note from Eq. (8) that

$$W(x, 0, 0; x, 0, 0)$$

$$= \frac{1}{(\lambda f)^2} \int_0^{2\pi} \int_0^a \int_0^{2\pi} \int_0^a \exp\{-[\rho'^2 + \rho''^2 - 2\rho'\rho'' \times \cos(\phi' - \phi'')]/2\sigma_g^2\} \times \exp[ikx(\rho' \cos\phi' - \rho'' \cos\phi'')/f]\rho'\rho'' d\phi' d\rho' d\phi'' d\rho'' \tag{37}$$

$$= \frac{1}{(\lambda f)^2} \int_0^{2\pi} \int_0^a \int_0^{2\pi} \int_0^a \exp\{-[\rho'^2 + \rho''^2 - 2\rho'\rho'' \cos(\phi' - \phi'')]/2\sigma_g^2\} \times \cos[kx(\rho' \cos\phi' - \rho'' \cos\phi'')/f] \times \rho'\rho'' d\phi' d\rho' d\phi'' d\rho'', \tag{38}$$

where we used the fact that the spectral density is real-valued.

Examples of the spectral degree of coherence $\mu(0, 0, 0; x, 0, 0)$ are shown in Fig. 6 for selected values of the normalized effective coherence length σ_g/a. An oscillatory behavior can be observed, with the damping increasing with decreasing σ_g. It can be seen that for the case $\sigma_g/a = 5$ two different regions can be discerned: regions where the fields at the points (0, 0, 0) and $(x, 0, 0)$ are essentially coherent and co-phasal and regions where the fields are essentially coherent and have opposite phases. In addition, all depicted curves intersect the line $\mu(0, 0, 0; x, 0, 0) = 0$. At these points the field is completely incoherent with the field at (0, 0, 0). Such phase singularities of the spectral degree of coherence have recently been discussed in different contexts.[8–10]

It is interesting to compare the spectral degree of coherence with the intensity (or "spectral density") $S(x, 0, 0)/S(0, 0, 0)$. In Figs. 7(a)–(f) several examples are presented. We notice that when $\sigma_g/a \gtrsim 0.5$, the width of the spectral density distribution is less than that

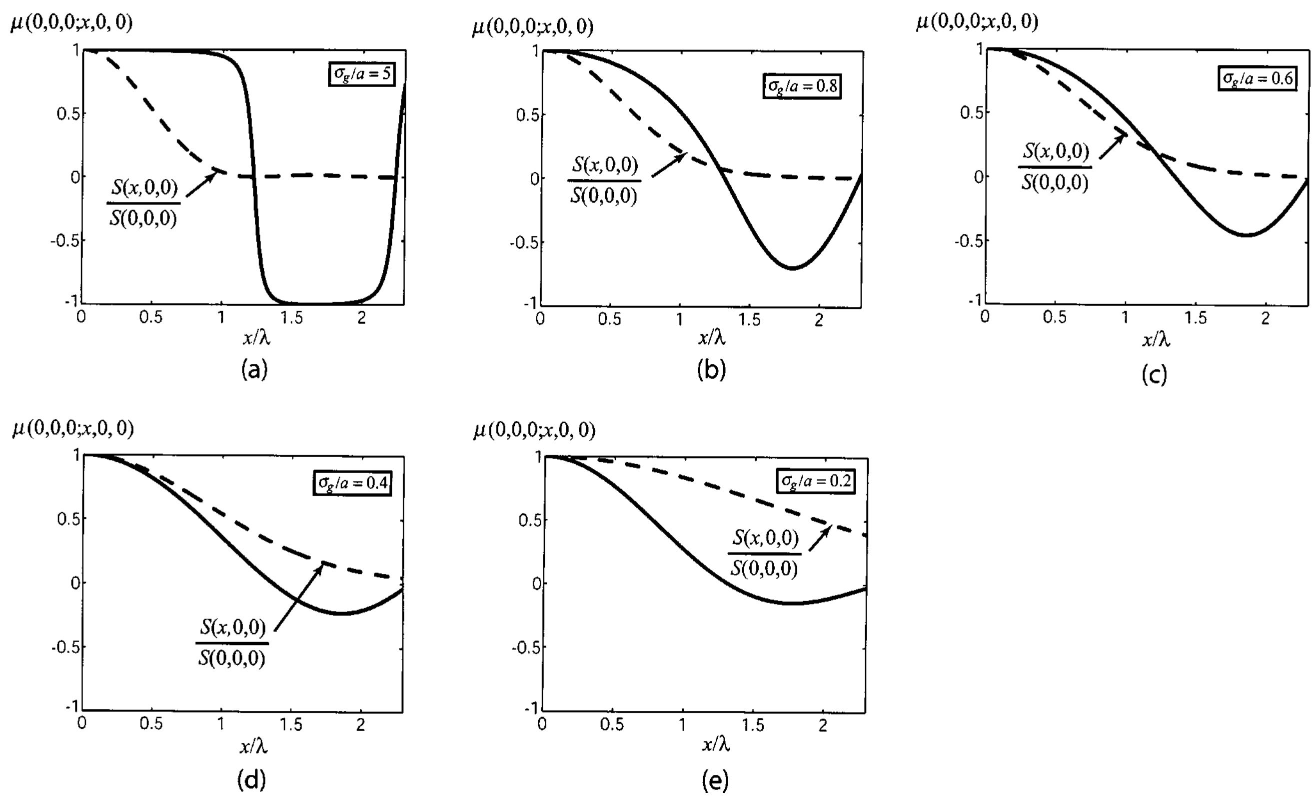

Fig. 7. Spectral degree of coherence, $\mu(0, 0, 0; x, 0, 0)$ and the normalized spectral density $S(x, 0, 0)/S(0, 0, 0)$ for different values of σ_g/a. (a) $\sigma_g/a = 5$, (b) $\sigma_g/a = 0.8$, (c) $\sigma_g/a = 0.6$, (d) $\sigma_g/a = 0.4$, (e) $\sigma_g/a = 0.2$. In all examples $a = 1$ cm, $f = 2$ cm, and $\lambda = 0.6328\ \mu$m.

of the spectral degree of coherence. However, when $\sigma_g/a \lesssim 0.5$, the width of the spectral density distribution exceeds that of the spectral degree of coherence. It can also be seen from Figs. 5(a) and 7(a) that as σ_g/a gets large, we recover the spectral density distribution for the fully coherent case. In Fig. 7(a), the zeros of the spectral density occur at $x/\lambda = 1.21$ and 2.23, as in the coherent case.

5. CONCLUSIONS

We have studied the spectral degree of coherence in the focal region of a converging, partially coherent wave field. Expressions were derived for pairs of points on the axis of symmetry and for pairs of points in the focal plane. For both cases it was found that, depending on the effective coherence length of the field in the aperture, the width of the spectral degree of coherence can be either larger or smaller than that of the spectral density distribution. In addition, the spectral degree of coherence was shown to possess phase singularities.

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Corresponding author T. D. Visser's e-mail address is tvisser@nat.vu.nl.

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