## Phase singularities of the longitudinal field components in the focal region of a high-aperture optical system

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We study the behavior of the longitudinal components of strongly focused electromagnetic fields. These components possess phase singularities and phase saddles that can be annihilated when the aperture angle of the lens is changed. © 2004 Optical Society of America OCIS codes: 050.1960, 260.2110, 260.5430.

#### 1. INTRODUCTION

When a beam of light is focused by a high-angularaperture system, the vectorial character of the field can no longer be neglected.<sup>1</sup> In the germinal work of Richards and Wolf,<sup>2–4</sup> and also Boivin and Wolf,<sup>5,6</sup> expressions were derived for the field components in the focal region of such a system. For the case of a monochromatic, linearly polarized plane wave, they found that near focus, the field components were nonzero in the two directions perpendicular to the polarization of the incident field. This is of interest because a precise knowledge of the structure of the focused field is necessary for many applications. For example, it has been suggested that the longitudinal field component (i.e., the component along the direction of propagation of the incident field) might be used in a table-top accelerator for charged particles<sup>7</sup> or as a probe for examining the absorption dipole moments of individual molecules.<sup>8</sup> In recent experiments, Novotny et al. succeeded in measuring the longitudinal field components for an incident field that was radially polarized.<sup>9</sup>

In the present paper the longitudinal field components in the focal region of a linearly polarized incident field are studied. It is demonstrated that these components exhibit phase singularities and phase saddles.<sup>10–12</sup> In addition, it is shown that when the semiaperture angle of the focusing system is increased in a continuous manner, such singularities can annihilate each another via a process in which topological charge and index are conserved.

#### 2. BASIC EQUATIONS

Consider an aplanatic focusing system L, as illustrated in Fig. 1. The system has a focal length f and a semiaperture angle  $\alpha$ . The geometrical focus is indicated by O and is chosen to be the origin of the coordinate system. A monochromatic plane wave with angular frequency  $\omega$  is incident on the lens with the electric field polarized in the x direction. The notation for the problem is greatly simplified by introducing longitudinal and transverse optical coordinates that are defined as follows:

$$u = kz \sin^2 \alpha, \tag{1}$$

$$v = k(x^2 + y^2)^{1/2} \sin \alpha,$$
 (2)

where  $k = \omega/c = 2\pi/\lambda$ , with *c* and  $\lambda$  being, respectively, the speed and wavelength of light in vacuum. The socalled Lommel variables, *u* and *v*, together with the angle  $\phi$ , specify the position of an observation point *P* in the focal region. The electric and magnetic fields are given by

$$\mathbf{E}(u, v, \phi, t) = \operatorname{Re}[\mathbf{e}(u, v, \phi)\exp(-i\omega t)], \quad (3)$$

$$\mathbf{H}(u, v, \phi, t) = \operatorname{Re}[\mathbf{h}(u, v, \phi) \exp(-i\omega t)], \quad (4)$$

respectively, where Re denotes the real part. It has been shown by Richards and Wolf that the time-independent parts, **e** and **h**, of the electric and magnetic fields at the point  $P(u, v, \phi)$  are given by the following expressions<sup>4</sup>:

$$e_x(u, v, \phi) = -iA(I_0 + I_2 \cos 2\phi),$$
 (5a)

$$e_{y}(u, v, \phi) = -iAI_{2}\sin 2\phi, \qquad (5b)$$

$$e_z(u, v, \phi) = -2AI_1 \cos \phi, \qquad (5c)$$

$$h_x(u, v, \phi) = -iAI_2 \sin 2\phi, \qquad (6a)$$

$$h_{y}(u, v, \phi) = -iA(I_{0} - I_{2}\cos 2\phi),$$
 (6b)

$$h_z(u, v, \phi) = -2AI_1 \sin \phi, \qquad (6c)$$

where

$$I_{0}(u, v) = \int_{0}^{\alpha} \cos^{1/2} \theta \sin \theta (1 + \cos \theta) J_{0} \left( \frac{v \sin \theta}{\sin \alpha} \right)$$
$$\times \exp \left( \frac{iu \cos \theta}{\sin^{2} \alpha} \right) d\theta, \tag{7}$$

$$I_{1}(u, v) = \int_{0}^{\alpha} \cos^{1/2} \theta \sin^{2} \theta J_{1}\left(\frac{v \sin \theta}{\sin \alpha}\right) \\ \times \exp\left(\frac{iu \cos \theta}{\sin^{2} \alpha}\right) d\theta,$$
(8)

$$I_{2}(u, v) = \int_{0}^{\alpha} \cos^{1/2} \theta \sin \theta (1 - \cos \theta) J_{2} \left( \frac{v \sin \theta}{\sin \alpha} \right)$$
$$\times \exp \left( \frac{iu \cos \theta}{\sin^{2} \alpha} \right) d\theta. \tag{9}$$

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Here  $J_n(x)$  denotes the Bessel function of the first kind of order n, and the coefficient A is defined as

$$A = \pi f l / \lambda. \tag{10}$$



Fig. 1. Illustration of a high-numerical-aperture focusing system.



Fig. 2. Contours of  $e_z$  in the focal plane (u = 0) for a semiaperture angle of  $\alpha = 45^{\circ}$ . In this plane,  $e_z$  is strictly real.

For this paper we have chosen a value of A = 141.20 statV/cm. For this choice of A, the electricenergy density, defined as  $\langle w_e \rangle = \langle E^2 \rangle / 16\pi$ , will have a value of 100 statV<sup>2</sup>/cm<sup>2</sup> at the geometrical focus. It should be noted that, although not explicitly stated in the list of arguments, all of the functions in Eqs. (5)–(9) are dependent on the value of the semi-aperture angle  $\alpha$ . The following symmetry relations follow from Eqs. (5)– (9):

$$e_{x}(-u, v, \phi) = -e_{x}^{*}(u, v, \phi),$$

$$e_{y}(-u, v, \phi) = -e_{y}^{*}(u, v, \phi),$$

$$e_{z}(-u, v, \phi) = e_{z}^{*}(u, v, \phi),$$

$$h_{x}(-u, v, \phi) = -h_{x}^{*}(u, v, \phi),$$

$$h_{y}(-u, v, \phi) = -h_{y}^{*}(u, v, \phi),$$

$$h_{z}(-u, v, \phi) = h_{z}^{*}(u, v, \phi),$$
(11)

From the above symmetry relations, it follows that the phase behavior of the longitudinal components differs from that of the transverse components in the focal region. This behavior will be studied in more detail in the next section.

#### 3. LONGITUDINAL FIELD COMPONENTS

By comparing Eqs. (5c) and (6c) we can see clearly that the behavior of the longitudinal component of the electric field  $e_z$  in any meridional plane  $\phi = \text{constant}$  is identical to that of the longitudinal component of the magnetic field  $h_z$  in the meridional plane  $\phi = \text{constant} + \pi/2$ . It therefore suffices to restrict the analysis to  $e_z$ , as we will henceforth do.



Fig. 3. Contours of  $|e_z|$  in the meridional plane  $\phi = 0$  for a semiaperture angle of  $\alpha = 45^{\circ}$ .

We notice from Eqs. (5c) and (8) that in the focal plane  $(u = 0), e_z$  is strictly real-valued. In Fig. 2 contour lines of  $e_z$  in this plane are shown. This plot is in agreement with previously published data.<sup>5</sup> It is interesting to note that the values for  $e_z$  on the right-hand side of the plot are the negatives of the values on the left-hand side. This is due to the cosine dependence of  $e_z$  on  $\phi$ . Within the plane  $\phi = \pm \pi/2$ ,  $e_z$  is identically zero. Crossing this plane switches the sign of the field, which is equivalent to a  $\pi$  phase jump. This will be discussed further in Section 4. Another consequence of the dependence of  $e_z$  on the cosine of  $\phi$  is a series of circular contours of zero amplitude. In Fig. 2 these can be seen as rings with radii of v = 5.07, 8.37, 11.58, and 14.76.

It has been noted by Boivin and Wolf that there are points along the  $\phi = 0$  axis of the focal plane where the transverse components,  $e_x$  and  $e_y$ , vanish but  $e_z$  remains nonzero.<sup>5</sup> In Fig. 2 these points occur at v = 3.99, 7.13, 10.26, and 13.39. At these locations the electric field is strictly longitudinal.

In Fig. 3 contours of  $|e_z|$  are plotted in the meridional plane  $\phi = 0$ . In this plane  $e_z$  is, in general, complexvalued. From considering Eqs. (5c) and (8) it follows that  $e_z$  is strictly zero along the u axis (v = 0). In the previous discussion of Fig. 2, it was remarked that  $|e_z|$  vanishes for particular values of v along the v axis as well. It is worthwhile to note that, in addition to these zeros along the u axis where  $|e_z|$  appears to have minima. If  $|e_z| = 0$  in any of these regions, then the phase  $\Phi_{e_z}$ , as defined by the expression

$$e_z(u, v, \phi) = |e_z(u, v, \phi)| \exp[i\Phi_{e_z}(u, v)],$$
 (12)

is indeterminate, resulting in a phase singularity. We remark that the phase  $\Phi_{e_z}$  is independent of the angle  $\phi$ , because the  $\cos \phi$  term in Eq. (5c) is strictly real and thus affects only the amplitude of the field.

It is of interest to study how the phase of  $e_z$  changes as the point of observation moves along a ray through the geometrical focus. It is convenient to compare this phase with that of a converging wave in the half-space z < 0, namely, -kR, and with that of a diverging wave in the half space z > 0, namely, +kR. Here  $R = (x^2 + y^2 + z^2)^{1/2}$ . The phase anomaly of the longitudinal electric field component can be defined as follows<sup>13</sup>:

$$\delta(u, v) = \Phi_{e_z}(u, v) + kR \quad \text{when} \quad z < 0,$$
(13a)

$$\delta(u, v) = \Phi_{e_z}(u, v) - kR \qquad \text{when} \quad z > 0. \eqno(13b)$$

The plots in Fig. 4 show the phase anomaly for a selection of rays of various angles of inclination, each passing through the geometrical focus. When a ray passes through the focus, where the phase is indeterminate,  $e_z$  undergoes a  $\pi$  phase jump, as can be inferred from Eqs. (5c) and (8).



Fig. 4. Phase anomaly  $\delta(u, v)$  along different rays passing through the geometrical focus.  $\theta$  indicates the angle of inclination. For all cases  $\alpha = 45^{\circ}$ .

# 4. PHASE SINGULARITIES OF THE LONGITUDINAL FIELD COMPONENTS

To identify regions where  $|e_z|$  is identically zero, it is useful to study the zero contours of the real and imaginary parts of  $e_z$ . The amplitude of  $e_z$  vanishes if and only if the zero contours of the real and imaginary parts intersect at that point. Figure 5 is a color-coded plot of the phase  $\Phi_{e_z}$ . Overlaid on this plot are the zero contours of  $\operatorname{Re}(e_z)$  and  $\operatorname{Im}(e_z)$ .

As has been previously noted in this paper,  $|e_z|$  is identically zero along the *u* axis and for certain values of *v* in the focal plane. As Fig. 5 illustrates, all of these points correspond to regions where the zero contours of  $\text{Re}(e_z)$ and  $\text{Im}(e_z)$  coincide. Furthermore, in the upper-right quadrant, three intersections can be seen, one near (u, v) = (13, 3.25) and a pair near (u, v) = (17, 5). These intersections are independent of the angle  $\phi$ , and thus each intersection actually represents a ring-shaped singularity about the *u* axis. A corresponding set of singular points exist on the opposite side of the *u* axis, where the ring singularities again intersect the plane. Furthermore, because of the symmetry relations given in Eq. (11), a similar set of ring singularities exist for negative values of *u*.

In the study of singular optical phenomena, it is useful to consider the topological charge and index of the features. The topological charge s of a phase singularity is defined as

$$s \equiv \frac{1}{2\pi} \oint_C \nabla \Phi_{e_z} \cdot d\mathbf{r}, \qquad (14)$$

where C is a closed contour of winding number one that is traversed counterclockwise. The topological index t is defined as the topological charge of the field  $\nabla \Phi_{e_z}$ . It is well known from previous studies<sup>14–17</sup> that phase singularities can be created or annihilated when a system parameter is changed in a continuous manner. In the focal plane u = 0, all of the ring-shaped singularities have topological charge s = +1, reminiscent of the Airy rings in scalar focusing theory (see, for example, Ref. 13, Sec. 8.8). Saddle points lie between the singularities. Annihilation of these structures cannot occur because topological charge would not be conserved. This is not true, however, for the pair of singularities near (u, v) = (17, 5). The upper phase singularity is a negative vortex with topological charge s = -1 and topological index t = +1; the lower phase singularity is a positive vortex with topological charge s = +1 and topological index t = +1. For annihilation to occur, two phase saddles, each with s= 0 and t = -1, are also needed, (see Ref. 11, Sec. 5.5.2), so that topological index is also conserved. Figure 6 illustrates the existence of the phase singularities and phase saddles. As the semiaperture angle increases, the pair of singularities and the pair of saddle points move together until they annihilate each other. This progression is illustrated in Fig. 7, where the contours of  $\text{Im}(e_z) = 0$ (dotted curves) and  $\operatorname{Re}(e_z) = 0$  (solid lines) are shown for selected values of the semiaperture angle  $\alpha$ . As  $\alpha$  increases, the two phase singularities move closer to each other and eventually annihilate, meaning that the zero contours of the real and imaginary parts of  $e_z$  no longer intersect. This can be seen in Fig. 7(f). It is interesting to note that before annihilation occurs, the different contour lines in the vicinity of the phase singularities change shape, splitting and joining.



Fig. 5. Phase of the longitudinal electric field component, plotted as a color spectrum ranging from red for 0 through violet for  $2\pi$ . Overlaid onto the phase plots are lines corresponding to zero contours for the real and imaginary parts of  $e_z$ . The solid lines are zeros of the real part, and the dashed lines are zeros of the imaginary part. The intersection of these contours indicate phase singularities. For this example  $\alpha = 45^{\circ}$ .

#### 5. CONCLUSIONS

In this paper we have studied the longitudinal field components of a linearly polarized beam focused by a highangular-aperture system. In the focal plane, the longitudinal component is strictly real. In this plane, there exist rings where the amplitude of the longitudinal component is identically zero. In addition to these phase singularities, phase saddles are present in the focal plane. Ring singularities and saddle points also exist outside the focal plane. Of special interest are a pair of singularities with opposite topological charge, located near a pair of saddle points. Because these four elements have a total topological charge and topological index of zero, mutual annihilation is possible. Indeed, as was shown, the two singularities and two saddle points merge and annihilate as the semiaperture angle is increased.

Our results are relevant for all applications in which electromagnetic fields are strongly focused.

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Fig. 6. Phase of the longitudinal electric field component, plotted as a color spectrum ranging from red for 0 through violet for  $2\pi$ . Overlaid onto the plot are lines corresponding to contours of equal phase. The solid lines indicate contours of phase  $\Phi_{e_z} = 0.074\pi$ , and the dashed lines indicate contours of phase  $\Phi_{e_z} = 0.516\pi$ . The two X-shaped intersections are saddle points. The dashed and dotted contour lines intersect at the singular points, where the phase is undefined. In this example the semiaperture angle  $\alpha = 45^{\circ}$ .



Fig. 7. Demonstration of the annihilation of a pair of phase singularities with increasing semiaperture angle.

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