

Conservation laws for partially coherent waves

Mengdi Li

*Shandong Provincial Key Laboratory of Light Field Manipulation Physics and Applications, School of Physics and Optoelectronics, Shandong Normal University, Jinan, China*Hugo F. Schouten *Department of Physics and Astronomy, VU University, Amsterdam, The Netherlands*Taco D. Visser **Shandong Provincial Key Laboratory of Light Field Manipulation Physics and Applications, School of Physics and Optoelectronics, Shandong Normal University, Jinan, China
and Department of Physics and Astronomy, VU University, Amsterdam, The Netherlands*

(Received 31 March 2026; accepted 28 July 2026; published 18 August 2026)

We examine the conservation law of coherence which, for partially coherent scalar wave fields, connects a coherence density to a coherence flux. A relation is established between this law and the law of energy conservation. In addition, the physical meaning of the coherence density and its associated flux is clarified. Our findings are illustrated by analyzing partially coherent fields in the focal region of a converging lens.

DOI: [10.1103/3qgq-1zjf](https://doi.org/10.1103/3qgq-1zjf)

I. INTRODUCTION

The notion that certain quantities are conserved underpins all major physical theories [1]. In particular, it could be argued that the conservation of energy is the most important principle in physics. Conservation laws relate the density of a certain quantity to its flux. For example, the law expressing the conservation of electric charge connects the charge density $\rho(\mathbf{r}, t)$ to the current density $\mathbf{J}(\mathbf{r}, t)$. In this example, as in others that come to mind, one deals with *one-point quantities*, i.e., quantities that are specified at one point in space $\mathbf{r}$ at one moment in time t . In contrast, correlation functions are *two-point quantities* that reside in a higher-dimensional space. They describe the correlation of a quantity at $(\mathbf{r}_1, t_1)$ with (often) the same quantity at $(\mathbf{r}_2, t_2)$. Conservation laws for correlation functions have been derived in several areas of physics, e.g., in statistical mechanics [2], quantum field theory [3], and condensed matter physics [4].

In the field of physical optics, Wang and Takeda [5] derived a conservation law for optical coherence functions. They called this law, the subject of the present study, the conservation law of coherence. It takes the form of a continuity equation in terms of the mutual coherence function and relates a coherence current (or flux) to a coherence density. This flux was shown to circulate around coherence singularities. These are pairs of points that are completely uncorrelated, i.e., the complex correlation function is 0 at such positions, implying that its phase is undefined or singular [6–10]. Although the mathematical form of this conservation law is clear and unambiguous, until now the physical interpretation of

the coherence flux and the coherence density, both one-point quantities associated with a two-point correlation function, remains somewhat vague.

Here we examine two conservation laws for partially coherent scalar wave fields, namely, coherence conservation and energy conservation. A physical interpretation of the former is presented. In addition, we establish that these two laws are interconnected. More specifically, we show that the coherence flux and coherence density are equal to the energy flux and energy density for fields that are spatially fully coherent. We illustrate our results by analyzing partially coherent fields in the focal region.

II. CONSERVATION LAWS

Many conservation laws for scalar densities can be expressed in the differential form

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) + \frac{\partial B(\mathbf{r}, t)}{\partial t} = 0. \quad (1)$$

Here $\mathbf{A}(\mathbf{r}, t)$ is the flux or current of the conserved quantity (per m^2 per s), and $B(\mathbf{r}, t)$ is its density (per m^3). On integrating this equation over a volume, V , that is bounded by a surface, S , and then using Gauss' theorem, one obtains

$$\int_S \mathbf{A}(\mathbf{r}, t) \cdot \hat{\mathbf{n}} dS = -\frac{\partial}{\partial t} \int_V B(\mathbf{r}, t) dV, \quad (2)$$

with $\hat{\mathbf{n}}$ being the outward unit normal on S . This integral form has a clear physical meaning. The left-hand side is the rate at which the conserved quantity enters (leaves) the volume V through the surface S . This equals the right-hand side, which represents the rate at which the total amount of the conserved quantity contained within V increases (decreases).

*Contact author: t.d.visser@vu.nl

Expressions like Eq. (1) or Eq. (2) are used, for example, to describe the conservation of mass in hydrodynamics and the conservation of charge in electrodynamics. In electrodynamics they are also used to describe the conservation of energy. In that case the role of $\mathbf{A}(\mathbf{r}, t)$ is played by the Poynting vector and $B(\mathbf{r}, t)$ is the sum of the electric and magnetic energy densities.

It is well-established that in a wide class of problems in optics, scalar wave theory provides remarkably accurate predictions for the observed intensity, being expressed as the squared modulus of a complex function called the *disturbance* or the *complex amplitude*. This has motivated, over the years, several generalizations of scalar theory to include an energy current. In Ref. [11] it was shown that, in a region free of currents and charges, any electromagnetic field may be represented by a single complex-valued wave function. Both an energy current and an energy density can be defined in terms of this wave function. In Ref. [12] it was established that such a scalar energy current cannot always reproduce the local Poynting vector. However, within the validity of the scalar approximation, an energy conservation law can be formulated [13, Appendix XI]. This law finds application, for example, in the derivation of the optical theorem [13, Sec. 13.3]. In the following, we assume that we are dealing with partially coherent fields for which the scalar approximation is valid, and we analyze their coherence flux, coherence density, energy flux, and energy density.

III. ENERGY FLUX AND ENERGY DENSITY

Consider a real, deterministic and monochromatic scalar wave field with frequency ω in free space:

$$V(\mathbf{r}, t) = \text{Re}[U(\mathbf{r}, \omega)e^{-i\omega t}]. \quad (3)$$

Associated with this field are a time-averaged energy flux vector ($\text{J/m}^2\text{s}$) and energy density (J/m^3) [13, Appendix XI], i.e.,

$$\mathbf{F}(\mathbf{r}) = -i\beta[U^*(\mathbf{r})\nabla U(\mathbf{r}) - U(\mathbf{r})\nabla U^*(\mathbf{r})] \quad (4)$$

and

$$H(\mathbf{r}) = \frac{\beta k^2}{\omega} \left[U^*(\mathbf{r})U(\mathbf{r}) + \frac{1}{k^2} \nabla U^*(\mathbf{r}) \cdot \nabla U(\mathbf{r}) \right], \quad (5)$$

where β is a positive constant, ensuring that the density $H(\mathbf{r})$ is positive, whose value depends on the choice of units, and k is the wave number corresponding to the frequency ω . The quantities $\mathbf{F}(\mathbf{r})$ and $H(\mathbf{r})$ are the scalar analogs of the Poynting vector and the total energy density of the electromagnetic theory [13, Chap. 1]. The two terms on the right-hand side of Eq. (5) are the scalar counterparts of the electric and magnetic energy density, respectively [11]. This formalism can be generalized to partially coherent scalar fields whose statistical properties are, in the space-frequency domain, characterized by a cross-spectral density (CSD) function [14, Sec. 4.3.2],

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle U^*(\mathbf{r}_1, \omega)U(\mathbf{r}_2, \omega) \rangle, \quad (6)$$

where the angular brackets indicate an average over an ensemble of strictly monochromatic-field realizations that all have the same temporal frequency [15]. For such fields, the energy

density is given by [14, Sec. 5.7.1]

$$\langle H(\mathbf{r}) \rangle = \frac{\beta k^2}{\omega} \left[W(\mathbf{r}, \mathbf{r}) + \frac{1}{k^2} \lim_{\mathbf{r}_1 \rightarrow \mathbf{r}} \lim_{\mathbf{r}_2 \rightarrow \mathbf{r}} \nabla_2 \cdot \nabla_1 W(\mathbf{r}_1, \mathbf{r}_2) \right] \quad (7)$$

$$\equiv \frac{\beta k^2}{\omega} [S(\mathbf{r}) + S_H(\mathbf{r})], \quad (8)$$

where ∇_i is the gradient with respect to $\mathbf{r}_i$ ($i = 1$ and 2), $S(\mathbf{r}) = W(\mathbf{r}, \mathbf{r})$ denotes the spectral density, and we introduced the “spectral magnetic density” $S_H(\mathbf{r})$. Note that the frequency dependence of all quantities is suppressed. The energy current can be expressed as

$$\langle \mathbf{F}(\mathbf{r}) \rangle = -i\beta \lim_{\mathbf{r}_1 \rightarrow \mathbf{r}} \lim_{\mathbf{r}_2 \rightarrow \mathbf{r}} [\nabla_2 W(\mathbf{r}_1, \mathbf{r}_2) - \nabla_1 W(\mathbf{r}_1, \mathbf{r}_2)]. \quad (9)$$

To simplify the notation, we henceforth omit the angular brackets around the energy density and the energy flux and simply refer to them as $H(\mathbf{r})$ and $\mathbf{F}(\mathbf{r})$.

IV. COHERENCE FLUX AND COHERENCE DENSITY

As mentioned above, Wang and Takeda [5] derived a conservation law for optical coherence. Using the Wolf equations for correlation functions [14] as a starting point, they found that

$$\nabla_1 \cdot \mathbf{T}_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) + \frac{\partial D_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)}{\partial t_1} = 0, \quad (10)$$

where $\mathbf{T}_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$ is the “coherence current” and $D_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$ represents the “coherence density.” These quantities are defined as

$$\mathbf{T}_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) = -\alpha \left[\frac{\partial \Gamma^*}{\partial t_1} \nabla_1 \Gamma + \frac{\partial \Gamma}{\partial t_1} \nabla_1 \Gamma^* \right] \quad (11)$$

and

$$D_\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) = \alpha \left[\frac{1}{c^2} \frac{\partial \Gamma^*}{\partial t_1} \frac{\partial \Gamma}{\partial t_1} + \nabla_1 \Gamma^* \cdot \nabla_1 \Gamma \right], \quad (12)$$

where α is an arbitrary positive constant, ensuring that the density D_Γ is positive, and c is the speed of light. Furthermore, $\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2)$ is the mutual coherence function that characterizes the cross-correlation of field vibrations at two points at two moments in time [16]. An analogous coherence conservation law can be obtained in terms of t_2 and ∇_2 . We restrict our discussion to the case of varying $\mathbf{r}_1$ and t_1 , while keeping $\mathbf{r}_2$ and t_2 fixed.

If we assume that the field is statistically stationary, then the mutual coherence function Γ will depend on the two time arguments t_1 and t_2 only through the difference $\tau = t_2 - t_1$, and differentiation with respect to t_1 may be replaced by differentiation with respect to minus τ . If, in addition, the field is quasimonochromatic, Γ can be written as [14, Eq. 4.3–31]

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \approx J(\mathbf{r}_1, \mathbf{r}_2) e^{-i\omega\tau} = |J(\mathbf{r}_1, \mathbf{r}_2)| e^{i[\theta_J(\mathbf{r}_1, \mathbf{r}_2) - \omega\tau]}, \quad (13)$$

where $J(\mathbf{r}_1, \mathbf{r}_2) \equiv \Gamma(\mathbf{r}_1, \mathbf{r}_2, 0)$ is the mutual intensity and ω is now the mean angular frequency of the narrow-band light. Furthermore, θ_J represents the phase angle of the mutual intensity. On substituting from Eq. (13) into the definitions

of the coherence density and the coherence flux, Wang and Takeda [5] obtained the expressions

$$D_J(\mathbf{r}_1, \mathbf{r}_2) = \alpha k^2 [J^* J + k^{-2} \nabla_1 J^* \cdot \nabla_1 J], \quad (14)$$

$$\mathbf{T}_J(\mathbf{r}_1, \mathbf{r}_2) = i\alpha k c [J^* \nabla_1 J - J \nabla_1 J^*] = -2\alpha k c |J|^2 \nabla_1 \theta_J. \quad (15)$$

The subscript J indicates that both quantities are expressed in terms of the mutual intensity, rather than the mutual coherence function Γ . As a result of the assumption of quasimonochromaticity, they are now independent of time.

For our purposes, it is convenient to consider the state of coherence in the space-frequency domain; i.e., we employ the CSD function $W(\mathbf{r}_1, \mathbf{r}_2, \omega)$ as defined in Eq. (6). According to the generalized Wiener-Khinchine theorem, the mutual coherence function and the cross-spectral density function form a Fourier transform pair [14, Eqs. 4.3–40], i.e.,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \int_0^\infty W(\mathbf{r}_1, \mathbf{r}_2, \omega') e^{-i\omega'\tau} d\omega'. \quad (16)$$

Due to the assumption of a quasimonochromatic field, the CSD is approximately proportional to $\delta(\omega - \omega')$, and hence

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = W(\mathbf{r}_1, \mathbf{r}_2, \omega) e^{-i\omega\tau}, \quad (17)$$

which is in agreement with Eq. (13), provided that we set $J(\mathbf{r}_1, \mathbf{r}_2) = W(\mathbf{r}_1, \mathbf{r}_2)$. We can thus express the coherence density and the coherence flux in terms of the CSD as

$$D(\mathbf{r}_1, \mathbf{r}_2) = \alpha k^2 [W^* W + k^{-2} \nabla_1 W^* \cdot \nabla_1 W], \quad (18)$$

$$\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2) = i\alpha k c [W^* \nabla_1 W - W \nabla_1 W^*] = -2\alpha k c |W|^2 \nabla_1 \theta_W, \quad (19)$$

where θ_W denotes the phase angle of the CSD function $W(\mathbf{r}_1, \mathbf{r}_2, \omega)$. The gradient in the last equation implies that $\mathbf{T}$ is perpendicular to the phase contours of the CSD and hence circulates around coherence singularities [10].

The coherence current in a field can be reconstructed from interferometric measurements, as is detailed in the experimental sections of Refs. [5,9]. There, it was also shown that the current indeed circulates around coherence singularities.

V. PHYSICAL MEANING OF COHERENCE FLUX AND COHERENCE DENSITY

We begin by considering a field that is spatially fully coherent. In that case, it only has a single statistical realization and hence the CSD function factorizes, i.e.,

$$W(\mathbf{r}_1, \mathbf{r}_2) = U^*(\mathbf{r}_1) U(\mathbf{r}_2). \quad (20)$$

On substituting this into Eq. (19) while using definition (4), we obtain

$$\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2) = i\alpha k c S(\mathbf{r}_2) [U(\mathbf{r}_1) \nabla_1 U^*(\mathbf{r}_1) - U^*(\mathbf{r}_1) \nabla_1 U(\mathbf{r}_1)] \quad (21)$$

$$= \frac{\alpha k c}{\beta} S(\mathbf{r}_2) \mathbf{F}(\mathbf{r}_1). \quad (22)$$

Since the choice of α is arbitrary, we can, for any fixed position $\mathbf{r}_2$, set

$$\alpha = \frac{\beta}{k c S(\mathbf{r}_2)}. \quad (23)$$

With this judicious choice of α , it follows that for fully coherent fields the coherence flux $\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)$ is equal to the scalar energy flux $\mathbf{F}(\mathbf{r}_1)$. Similarly, on using Eq. (20) in Eq. (18), we readily find, for the same value of α , that the coherence density $D(\mathbf{r}_1, \mathbf{r}_2)$ is equal to the energy density $H(\mathbf{r}_1)$.

To shed more light on the partially coherent case, we recall the definition of the spectral degree of coherence [16],

$$\mu(\mathbf{r}_1, \mathbf{r}_2) \equiv \frac{W(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{S(\mathbf{r}_1) S(\mathbf{r}_2)}}, \quad (24)$$

the modulus of which is bounded by 0 and unity, corresponding to complete incoherence and full coherence, respectively. Furthermore, we introduce a “spectral magnetic degree of coherence” as

$$\mu_H(\mathbf{r}_1, \mathbf{r}_2) \equiv \frac{|\nabla_1 W(\mathbf{r}_1, \mathbf{r}_2)|}{k \sqrt{S_H(\mathbf{r}_1) S(\mathbf{r}_2)}}, \quad (25)$$

where S_H is defined by Eq. (8). In contrast to the spectral degree of coherence, μ_H is real. One can show that it is bounded by 0 and 1 (see Appendix A). Furthermore, $\mu_H = 1$ if the field is fully coherent, as can be seen using Eq. (20). Note that, unlike the spectral degree of coherence, $\mu_H(\mathbf{r}, \mathbf{r})$ is not necessarily equal to 1 because it is a cross-correlation, rather than an autocorrelation. Using the above two definitions, Eq. (18) can be rewritten as

$$D(\mathbf{r}_1, \mathbf{r}_2) = \alpha k^2 S(\mathbf{r}_2) [|\mu(\mathbf{r}_1, \mathbf{r}_2)|^2 S(\mathbf{r}_1) + |\mu_H(\mathbf{r}_1, \mathbf{r}_2)|^2 S_H(\mathbf{r}_1)]. \quad (26)$$

With the choice of α given by Eq. (23), this becomes

$$D(\mathbf{r}_1, \mathbf{r}_2) = \frac{\beta k^2}{\omega} [|\mu(\mathbf{r}_1, \mathbf{r}_2)|^2 S(\mathbf{r}_1) + |\mu_H(\mathbf{r}_1, \mathbf{r}_2)|^2 S_H(\mathbf{r}_1)]. \quad (27)$$

On integrating $\mathbf{r}_1$ over some volume V , one obtains for the right-hand side

$$\frac{\beta k^2}{\omega} \int_V [|\mu(\mathbf{r}_1, \mathbf{r}_2)|^2 S(\mathbf{r}_1) + |\mu_H(\mathbf{r}_1, \mathbf{r}_2)|^2 S_H(\mathbf{r}_1)] d^3 r_1. \quad (28)$$

A comparison with Eq. (8) shows that this can be interpreted as the energy, contained in the volume V , which is “coherent” with the field at the reference point $\mathbf{r}_2$. The energy is thus the sum of a weighted “electric energy” and a weighted “magnetic energy.” The electric energy density is 0 at coherence singularities, where $\mu(\mathbf{r}_1, \mathbf{r}_2) = 0$. The magnetic energy density has no generic zeros. Clearly, $D(\mathbf{r}_1, \mathbf{r}_2) \leq H(\mathbf{r}_1)$. Because of the continuity equation (10), a similar interpretation holds for the coherence current: it is the current that is coherent with the field at $\mathbf{r}_2$.

In the next section, we further illustrate our results by analyzing focused wave fields. Although such fields have been researched before (for example, in Refs. [17,18]), those

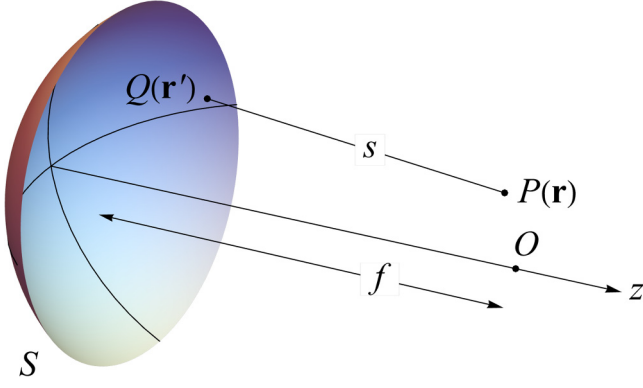


FIG. 1. A converging wave front S exits a circular aperture with radius a . The origin O is set at the geometrical focus, at a distance f along the z axis. $Q(\mathbf{r}')$ and $P(\mathbf{r})$ represent typical points of integration and observation, respectively.

studies were mainly concerned with the intensity distribution rather than the energy density.

VI. PARTIALLY COHERENT FIELDS NEAR FOCUS

We first consider a converging monochromatic scalar field, $U^{(0)}(\mathbf{r})$, of frequency ω , which emanates from a circular aperture with radius a in an opaque screen, as sketched in Fig. 1. In the paraxial approximation, it produces a field in the focal region of the form [13, Chap. 8.2]

$$U(\mathbf{r}) = -\frac{i}{\lambda} \int_S U^{(0)}(\mathbf{r}') \frac{e^{iks}}{s} d^2r', \quad (29)$$

where λ is the free-space wavelength, S is the wave front that briefly fills the aperture, and $s = |\mathbf{r}' - \mathbf{r}|$. If the field is spatially partially coherent we must take into account its CSD function. According to Eqs. (29) and (6), the CSD in the focal region becomes

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\lambda^2} \iint_S W^{(0)}(\mathbf{r}', \mathbf{r}'') \frac{e^{ik(s_2 - s_1)}}{s_1 s_2} d^2r' d^2r'', \quad (30)$$

where $W^{(0)}(\mathbf{r}', \mathbf{r}'') = \langle U^{(0)*}(\mathbf{r}') U^{(0)}(\mathbf{r}'') \rangle$, $s_1 = |\mathbf{r}_1 - \mathbf{r}'|$, and $s_2 = |\mathbf{r}_2 - \mathbf{r}''|$. We assume that $W^{(0)}(\mathbf{r}', \mathbf{r}'')$ describes a field with a Gaussian correlation and a uniform intensity, i.e.,

$$W^{(0)}(\mathbf{r}', \mathbf{r}'') = W^{(0)}(\boldsymbol{\rho}', \boldsymbol{\rho}'') = e^{-(\boldsymbol{\rho}'' - \boldsymbol{\rho}')^2 / (2\delta^2)}. \quad (31)$$

Here $\boldsymbol{\rho} = (x, y)$ is a two-dimensional transverse vector that specifies the position of an integration point Q on S , and δ is the field's transverse coherence width. For the two distances s we have, to a good approximation,

$$s_1 \approx f - \mathbf{q}' \cdot \mathbf{r}_1, \quad (32)$$

$$s_2 \approx f - \mathbf{q}'' \cdot \mathbf{r}_2, \quad (33)$$

with $\mathbf{q}'$ and $\mathbf{q}''$ being unit vectors in the directions $O\mathbf{r}'$ and $O\mathbf{r}''$, respectively. In addition, we use that $1/s_1 \approx 1/s_2 \approx 1/f$. On substituting this in Eq. (30), we obtain

$$W(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\lambda^2 f^2} \iint_S e^{-(\boldsymbol{\rho}'' - \boldsymbol{\rho}')^2 / (2\delta^2)} e^{ik(\mathbf{q}' \cdot \mathbf{r}_1 - \mathbf{q}'' \cdot \mathbf{r}_2)} d^2r' d^2r''. \quad (34)$$

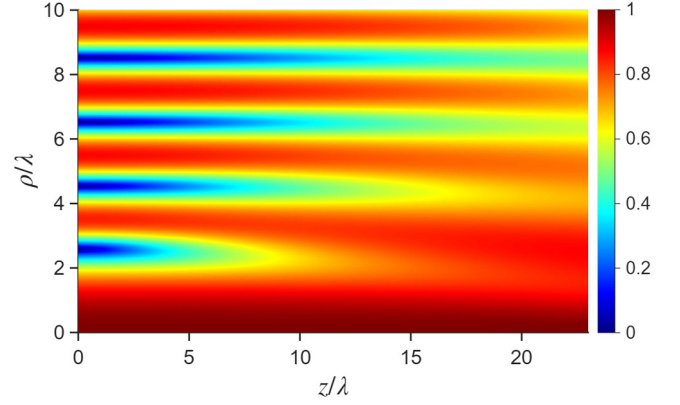


FIG. 2. Color-coded magnitude of the spectral degree of coherence $|\mu(0, z, \rho, z)|$. In this example $f = 4a = 4$ cm, $\sigma = a$, and $\lambda = 632.8$ nm. Note that $|\mu(0, z, \rho, z)|$ has rotational symmetry around the z axis and is mirror symmetric with respect to the focal plane.

We are interested in the strength of the correlation between the field on the axis and the field at a point in the same transverse z plane, separated by a distance ρ . The magnitude $|\mu(0, z, \rho, z)|$ can be computed from Eq. (24). An example is shown in Fig. 2. The isolated coherence singularities [8] in the focal plane ($z = 0$) at which $\mu(0, z, \rho, z) = 0$ are visible at $\rho/\lambda = 2.57, 4.53$, etc.

A useful measure of transverse spatial coherence in a cross-sectional plane, z , is the smallest value of ρ such that $|\mu(0, z, \rho, z)| = 0.5$, which we denote as $\rho_{0.5}$. Figure 3 illustrates how $\rho_{0.5}$ decreases towards the focus and then increases again behind the focal plane. One might loosely call this “the focusing of coherence.” We note that the steep vertical parts of the red curve are actually discontinuous jumps, as can be inferred from Fig. 2.

We next turn our attention to the coherence flux $\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)$ for the same situation. In all our examples, the reference point $\mathbf{r}_2$ is taken to be on the z axis. As shown in Appendix B, then only two-dimensional integrals need to be evaluated.

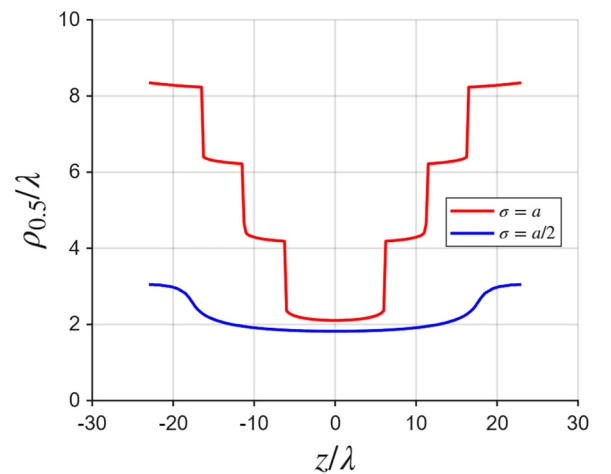


FIG. 3. Plot of $\rho_{0.5}$, the smallest value of ρ for which $|\mu(0, z, \rho, z)| = 0.5$, for two values of the coherence width σ . In this example $f = 4$ cm, $a = 1$ cm, $\sigma = a$ (red curve) or $a/2$ (blue curve), and $\lambda = 632.8$ nm.

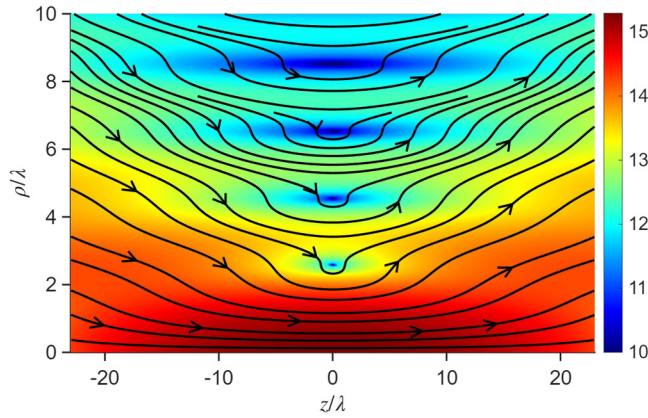


FIG. 4. Streamlines of the coherence flux $\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)$ in the focal region. Because of rotational symmetry the position $\mathbf{r}_1$ is represented as (ρ, z) . The colors represent $\log_{10} |\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)|$. In this example $\mathbf{r}_2 = \mathbf{0}$, $f = 4$ cm, $a = 1$ cm, $\sigma = a$, and $\lambda = 632.8$ nm.

Streamlines of the coherence flux $\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)$, for the same parameter values as in Fig. 2, are shown in Fig. 4. The reference point is initially set at the geometrical focus, i.e., $\mathbf{r}_2 = \mathbf{0}$. In agreement with the notion that coherence is being “focused,” the general behavior of the flux vectors for $z < 0$ is to converge towards focus, whereas for $z > 0$ they are seen to diverge. The background colors indicate $\log_{10} |\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)|$.

Figure 5 zooms in on a subwavelength region around the first correlation singularity at which $|\mu(0, z, \rho, z)| = 0$. The flux is indeed seen to circulate around the singularity. From the overall flow towards the right in Fig. 4, it follows that this vortex must be accompanied by a vector saddle point (see the white curve) at which again $|\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)| = 0$. All vortices of the coherence flux in the focal plane (see Fig. 4) rotate counterclockwise and are all accompanied by a saddle point.

It is instructive to compare our results for the coherence flux with those of the Poynting vector for fully coherent, focused fields. Streamlines, vortices, and saddle points similar

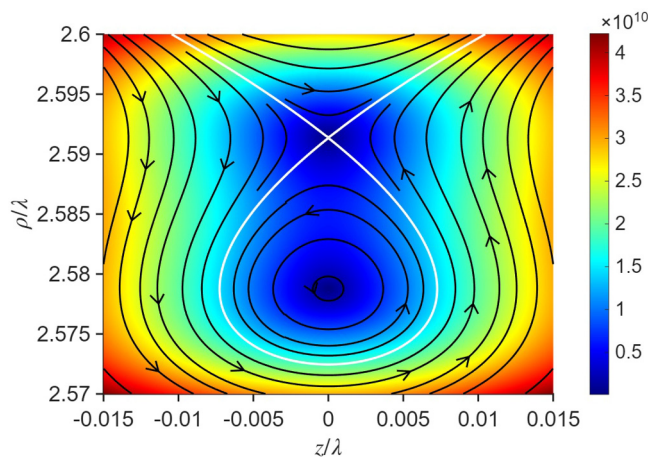


FIG. 5. Streamlines of the coherence flux $\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)$ near the first coherence singularity in the focal plane. The intersection of the white curve with itself is a saddle point. The parameter values are the same as those in Fig. 4. The colors represent $|\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)|$ in arbitrary units.

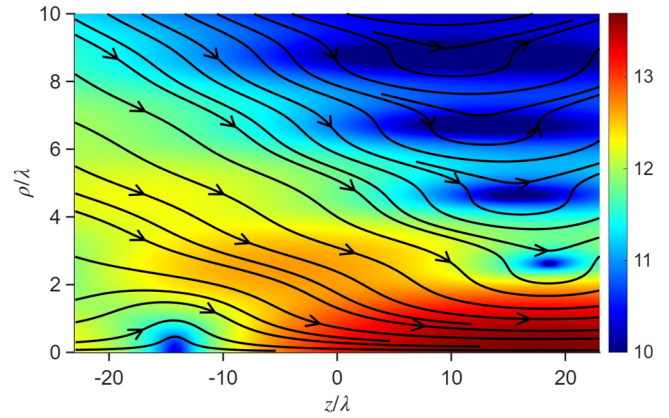


FIG. 6. Streamlines of the coherence flux $\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)$ in the focal region with the reference point chosen as $\mathbf{r}_2 = (0, 20\lambda)$ rather than at the geometrical focus. The coherence width $\sigma = 0.1a$. The other parameters are as in Fig. 4. The colors represent $\log_{10} |\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)|$.

to those shown in Figs. 4 and 5 have been reported in Ref. [19] for the Poynting vector in high-NA systems. In view of the relation between the coherence flux and the scalar energy flux discussed in Sec. V, such a correspondence is to be expected.

Also worth noting is the role of the Gouy phase, sometimes called the phase anomaly at focus. Its origin for scalar wave fields is discussed in Ref. [20]. It was found that, in high-NA systems, the phase behavior of the three Cartesian components of the electric field is different [21]. Although two-point correlation functions, like the spectral degree of coherence, undergo a generalized Gouy phase [22], the coherence flux $\mathbf{T}$, as defined in Eq. (9), is real valued and therefore has no such anomalous behavior in the focal region.

The physical meaning of the reference point $\mathbf{r}_2$ is discussed in Sec. V. It can be further illustrated by taking $\mathbf{r}_2$ not at the geometrical focus, as was done in creating Fig. 4, but at another position on the z axis. As expected, the resulting coherence flux pattern is now deformed towards the new position of $\mathbf{r}_2$, as shown in Fig. 6 for the case $\sigma = 0.1a$. The background colors indicate $\log_{10} |\mathbf{T}(\mathbf{r}_1, \mathbf{r}_2)|$. In agreement with the notion that the flux at $\mathbf{r}_1$ is coherent with respect to the field at $\mathbf{r}_2$, such a deformation does not occur when the field is highly coherent, i.e., when $\sigma \gg a$. In that case the flux pattern is independent of the choice of $\mathbf{r}_2$.

In the remainder, we examine the relation between the energy and coherence densities, as well as the energy and coherence flux. The axial distribution of the energy density $H(\mathbf{r})$ and the coherence density $D(\mathbf{r}_1, \mathbf{r}_2)$ is shown in Fig. 7. The latter is seen to be smaller than the former, in agreement with the analysis of Sec. V. The coherence density is plotted for three different axial positions of the reference point $\mathbf{r}_2 = (0, z_2)$. Just like the coherence flux depicted in Fig. 6, the peak of the coherence density shifts along with $\mathbf{r}_2$. The energy density H , being a one-point quantity, is of course independent of $\mathbf{r}_2$.

The dependence of the energy density $H(\mathbf{r})$ on the coherence radius σ is illustrated in Fig. 8. For the fully coherent case (red curve), it varies by several orders of magnitude (note the logarithmic scale on the vertical axis). When the

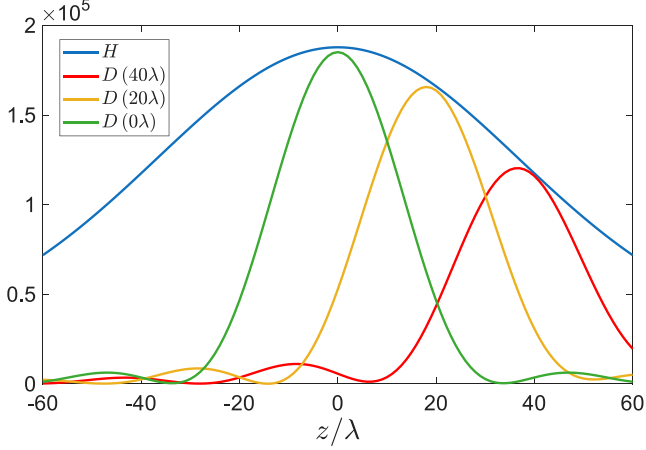


FIG. 7. The energy density H (blue curve), given by Eq. (7), and the coherence density D , given by Eq. (18), along the central axis (a.u.). The coherence density is shown for three values of z_2 : 0λ (green curve), 20λ (orange curve), and 40λ (red curve), with the other coordinates of $\mathbf{r}_2$ equal to 0. The coherence width $\sigma = 0.1a$. All other parameters are as in Fig. 4.

coherence width is comparable to the aperture radius a or smaller, as for the other three curves, the variations become much less pronounced.

Figure 9 illustrates a fundamental difference between the energy flux and the coherence flux, namely, the possible annihilation of a vortex-saddle pair. In this figure, F_z , the only nonzero component of the energy flux in the focal plane, is plotted for three values of σ . The zero crossings of the yellow curve correspond to a vortex and a saddle point, similar to the pattern shown in Fig. 5 for the coherence flux. When the coherence width σ decreases, the region in the focal plane where F_z is negative gradually becomes smaller. For $\sigma = 100a$ (blue curve) $F_z > 0$ everywhere, indicating the annihilation of the vortex and the saddle point. This event is confirmed by Fig. 10,

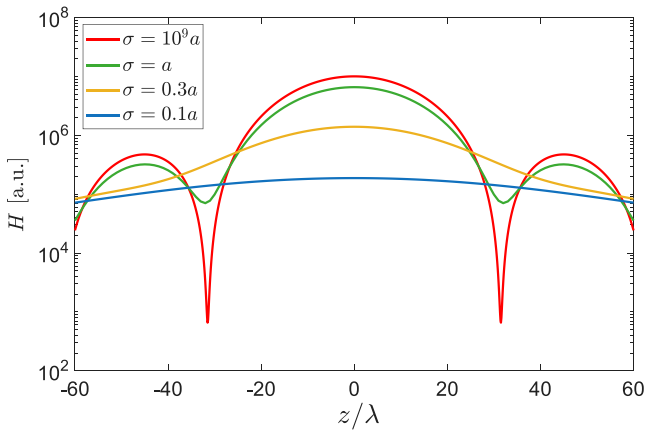


FIG. 8. The energy density H , given by Eq. (7), along the central axis (a.u.). The values of the coherence radius σ are $10^9 a$ (red curve), a (green curve), $0.3a$ (orange curve), and $0.1a$ (blue curve). All other parameters are as in Fig. 4. Note that the vertical axis uses a logarithmic scale.

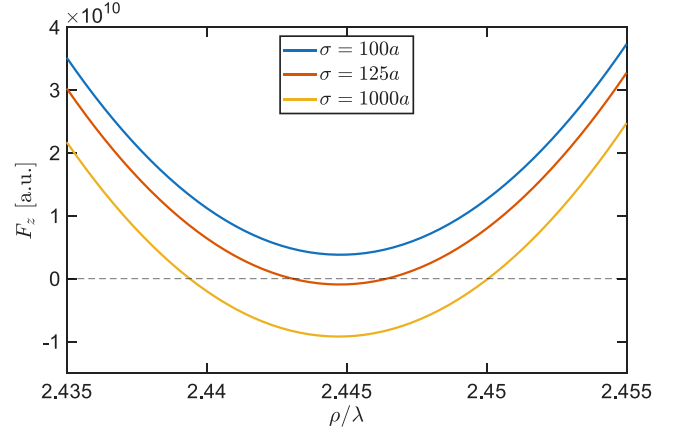


FIG. 9. The z component of the energy flux $\mathbf{F}$, as given by Eq. (9), in the focal plane. The coherence radius σ equals $100a$ (blue curve), $125a$ (red curve), and $1000a$ (yellow curve). All other parameters are as in Fig. 4.

where the resulting streamlines of the energy flux are shown. Clearly, no singular structures are present anymore.

A similar annihilation of the vortex-saddle pair of the coherence flux seen in Fig. 5 is *not* possible due to a restriction implied by Eq. (19). According to this expression, a vortex of $\mathbf{T}$ corresponds to a phase singularity of W , while a saddle point of $\mathbf{T}$ corresponds to a saddle point of θ_W (i.e., a point where $\nabla\theta_W = 0$, see Ref. [10]). Whereas the former has a topological charge, the latter does not. An annihilation would therefore violate topological charge conservation. Furthermore, we note that a vortex-saddle annihilation in $\mathbf{F}$ cannot occur in the fully coherent case. This is due to the relation (4), which can be cast in a form similar to that of Eq. (15).

In Fig. 11 the correlations μ [Eq. (24)] and μ_H [Eq. (25)] are plotted in the focal plane with the reference point $\mathbf{r}_2$ taken at the geometrical focus. In this case, both quantities are real valued [8]. The magnitude of both quantities is strikingly similar. Note that μ is necessarily unity at $\mathbf{r}_1 = 0$, but this is not the case for μ_H since it is a cross-correlation. Indeed, its value at $\rho = 0$ is slightly less than 1. Furthermore, while

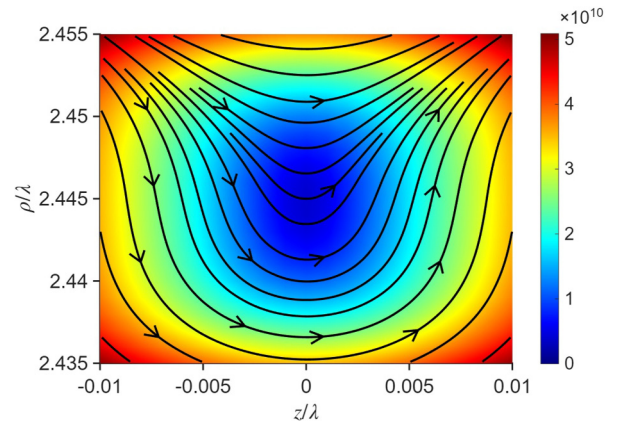


FIG. 10. Streamlines of the energy flux $\mathbf{F}$, given by Eq. (9), with the colors representing $|\mathbf{F}|$. The coherence radius $\sigma = 100a$. All other parameters are as in Fig. 4.

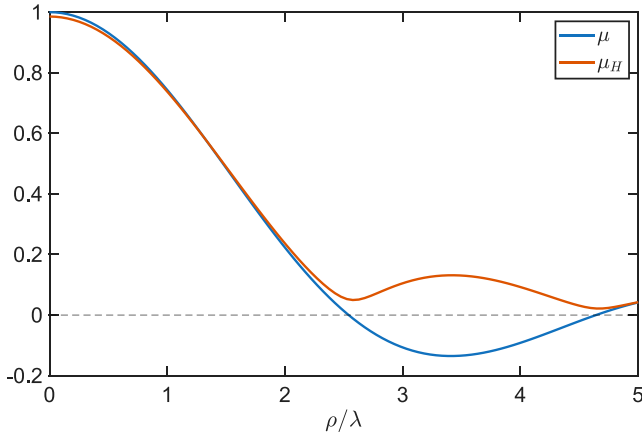


FIG. 11. μ and μ_H for $\mathbf{r}_1 = (\rho, 0)$ and $\mathbf{r}_2 = (0, 0)$. The coherence radius σ equals $0.1a$. All other parameters are as in Fig. 4.

μ has two zeros, μ_H has no zeros and is always non-negative. The zeros of μ correspond to a vortex and a saddle of $\mathbf{T}$.

VII. CONCLUSIONS

The state of coherence of an optical wave field is a fundamental property that determines how it diffracts [23], propagates [24], and scatters [25]. When characterizing a partially coherent field, one needs quantities such as its energy flux, energy density, spectral degree of coherence, coherence flux, and coherence density. The latter two can be reconstructed from interferometric measurements. Examining the laws of energy conservation and coherence conservation, we found that these quantities are interrelated. This provides a deeper physical insight into two-point correlation functions and corresponding one-point quantities such as flux and density. In particular, the coherence density and the coherence flux at the point $\mathbf{r}_1$ were shown to be weighted by the coherence strength between that point and the reference point $\mathbf{r}_2$. Also, the coherence density and flux turn out to be equivalent to the energy density and energy flux for the special case when the field is spatially fully coherent. However, for partially coherent fields, these quantities differ significantly. For example, the topology of the coherence flux differs from that of the energy flux. Whereas a vortex-saddle pair of the energy flux can annihilate, for the coherence flux such a process cannot occur as it would violate topological charge conservation. We illustrated the behavior of the coherence flux and density by calculating them in the focal region.

The notions of coherence flux and density have been shown to be useful in coherence holography [26] and surface profilometry [27]. Our results offer foundational insight into the relation between these two concepts and energy conservation; they may also prove to be helpful for the many applications for which partially coherent fields are preferred over fully coherent fields [28].

Finally, according to Noether's theorem [29] one expects, just like for energy conservation, a connection between the Wang-Takeda conservation law and the time invariance of the pertinent Lagrangian for optical coherence functions. This could be the subject of future study.

ACKNOWLEDGMENT

This research is supported by the National Natural Science Foundation of China (Grant No. W2441005).

The authors contributed equally to this work.

The authors declare no conflicts of interest.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A

In this Appendix, we proof the claim made below Eq. (25), namely that $\mu_H(\mathbf{r}_1, \mathbf{r}_2)$, has an upper bound equal to 1. Consider the cross-correlation of two arbitrary fields $V(\mathbf{r}_1)$ and $U(\mathbf{r}_2)$, i.e., $\langle V^*(\mathbf{r}_1)U(\mathbf{r}_2) \rangle$. One can show [14, Sec. 2.4.4] that the following inequality holds:

$$|\langle V^*(\mathbf{r}_1)U(\mathbf{r}_2) \rangle| \leq \sqrt{\langle |V(\mathbf{r}_1)|^2 \rangle \langle |U(\mathbf{r}_2)|^2 \rangle}. \quad (\text{A1})$$

For the case that $V = \partial_i U$, with ∂_i denoting partial differentiation with respect to x_i , y_i , or z_i , with $i = 1$ and 2 , we thus get

$$|\langle \partial_i U^*(\mathbf{r}_1)U(\mathbf{r}_2) \rangle| \leq \sqrt{\langle |\partial_i U(\mathbf{r}_1)|^2 \rangle \langle |U(\mathbf{r}_2)|^2 \rangle}. \quad (\text{A2})$$

Therefore,

$$|\langle \nabla_1 U^*(\mathbf{r}_1)U(\mathbf{r}_2) \rangle| \leq \sqrt{\langle |\nabla_1 U(\mathbf{r}_1)|^2 \rangle \langle |U(\mathbf{r}_2)|^2 \rangle}. \quad (\text{A3})$$

Using this result together with the definitions for μ_H , S , and S_H [Eqs. (8) and (25)], we obtain the upper bound $[\mu_H(\mathbf{r}_1, \mathbf{r}_2)]^2 \leq 1$.

APPENDIX B

In this Appendix, we proof the assertion made in Sec. VI, namely that in certain cases only two-dimensional integrals need to be evaluated. On introducing cylindrical coordinates

$$\boldsymbol{\rho}' = (\rho' \cos \phi', \rho' \sin \phi'), \text{ with } 0 \leq \rho' \leq a, \quad (\text{B1})$$

$$\mathbf{q}' = [\rho' \cos \phi', \rho' \sin \phi', -(f^2 - \rho'^2)^{1/2}]/f, \quad (\text{B2})$$

$$\mathbf{r}_1 = (\rho_1 \cos \phi_1, \rho_1 \sin \phi_1, z_1), \quad (\text{B3})$$

with similar expressions for $\boldsymbol{\rho}''$, $\mathbf{q}''$, and $\mathbf{r}_2$, Eq. (34) for the CSD in the focal region can be written as

$W(\mathbf{r}_1, \mathbf{r}_2)$

$$\begin{aligned} &= \frac{1}{(\lambda f)^2} \int_0^{2\pi} \int_0^a \int_0^{2\pi} \int_0^a e^{-(\rho'^2 + \rho''^2 - 2\rho'\rho'' \cos(\phi' - \phi''))/(2\sigma^2)} \\ &\quad \times e^{ik[\rho_1 \rho' \cos(\phi_1 - \phi')/f - z_1(1 - \rho'^2/f^2)^{1/2}]} \\ &\quad \times e^{-ik[\rho_2 \rho'' \cos(\phi_2 - \phi'')/f - z_2(1 - \rho''^2/f^2)^{1/2}]} \rho' \rho'' d\rho' d\phi' d\rho'' d\phi''. \end{aligned} \quad (\text{B4})$$

If we take the second reference point to be on the z axis, i.e., $\mathbf{r}_2 = (0, 0, z_2)$, the integrations over ϕ' and ϕ'' can be

carried out analytically, with the result

$$W(\rho_1, z_1; 0, z_2) = \left(\frac{2\pi}{\lambda f}\right)^2 \int_0^a \int_0^a I_0\left(\frac{\rho' \rho''}{\sigma^2}\right) J_0\left(\frac{k \rho_1 \rho'}{f}\right) e^{-(\rho'^2 + \rho''^2)/(2\sigma^2)} e^{-ikz_1 \sqrt{1 - \rho'^2/f^2}} e^{ikz_2 \sqrt{1 - \rho''^2/f^2}} \rho' \rho'' d\rho' d\rho'', \quad (\text{B5})$$

with J_0 being a Bessel function of the first kind and I_0 a modified Bessel function of the first kind. The two partial derivatives needed for calculating the coherence flux and the coherence density are

$$\frac{\partial W(\rho_1, z_1; 0, z_2)}{\partial \rho_1} = -\left(\frac{2\pi}{\lambda f}\right)^2 \int_0^a \int_0^a I_0\left(\frac{\rho' \rho''}{\sigma^2}\right) \frac{k \rho'}{f} J_1\left(\frac{k \rho_1 \rho'}{f}\right) e^{-(\rho'^2 + \rho''^2)/(2\sigma^2)} e^{-ikz_1 \sqrt{1 - \rho'^2/f^2}} \times e^{ikz_2 \sqrt{1 - \rho''^2/f^2}} \rho' \rho'' d\rho' d\rho'' \quad (\text{B6})$$

and

$$\frac{\partial W(\rho_1, z_1; 0, z_2)}{\partial z_1} = \left(\frac{2\pi}{\lambda f}\right)^2 \int_0^a \int_0^a I_0\left(\frac{\rho' \rho''}{\sigma^2}\right) J_0\left(\frac{k \rho_1 \rho'}{f}\right) e^{-(\rho'^2 + \rho''^2)/(2\sigma^2)} -ik \sqrt{1 - \rho'^2/f^2} e^{-ikz_1 \sqrt{1 - \rho'^2/f^2}} \times e^{ikz_2 \sqrt{1 - \rho''^2/f^2}} \rho' \rho'' d\rho' d\rho''. \quad (\text{B7})$$

For the energy flux and the energy density $\mathbf{r}_1 = \mathbf{r}_2$ and hence, with the exception of the geometrical focus, four-dimensional expressions like Eq. (B4) need to be evaluated.

-
- [1] J. Hanc, S. Tuleja, and M. Hancova, Symmetries and conservation laws: Consequences of Noether's theorem, *Am J. Phys.* **72**, 428 (2004).
 - [2] K. Huang, *Statistical Mechanics*, 2nd ed. (Wiley, New York, 1987), Sec. 3.5.
 - [3] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus, Reading, Massachusetts, 1995), Sec. 7.4.
 - [4] G. Baym and L. P. Kadanoff, Conservation laws and correlation functions, *Phys. Rev.* **124**, 287 (1961).
 - [5] W. Wang and M. Takeda, Coherence current, coherence vortex, and the conservation law of coherence, *Phys. Rev. Lett.* **96**, 223904 (2006). There is a minus sign error in Eq. (8). This is corrected in our corresponding Eq. (15).
 - [6] H. F. Schouten, G. Gbur, T. D. Visser, and E. Wolf, Phase singularities of the coherence functions in Young's interference pattern, *Opt. Lett.* **28**, 968 (2003).
 - [7] G. Gbur and T. D. Visser, Coherence vortices in partially coherent beams, *Opt. Commun.* **222**, 117 (2003).
 - [8] D. G. Fischer and T. D. Visser, Spatial correlation properties of focused partially coherent light, *J. Opt. Soc. Am. A* **21**, 2097 (2004).
 - [9] W. Wang, Z. Duan, S. G. Hanson, Y. Miyamoto, and M. Takeda, Experimental study of coherence vortices: Local properties of phase singularities in a spatial coherence function, *Phys. Rev. Lett.* **96**, 073902 (2006).
 - [10] G. J. Gbur, *Singular Optics* (CRC, Boca Raton, FL, 2017).
 - [11] H. S. Green and E. Wolf, A scalar representation of electromagnetic fields, *Proc. Phys. Soc. A* **66**, 1129 (1953).
 - [12] M. V. Berry, Optical currents, *J. Opt. A* **11**, 094001 (2009).
 - [13] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University, Cambridge, England, 1999).
 - [14] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).
 - [15] E. Wolf, New theory of partial coherence in the space-frequency domain. Part I: Spectra and cross spectra of steady-state sources, *J. Opt. Soc. Am.* **72**, 343 (1982).
 - [16] E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light* (Cambridge University, Cambridge, England, 2007).
 - [17] T. D. Visser, G. Gbur, and E. Wolf, Effect of the state of coherence on the three-dimensional spectral intensity distribution near focus, *Opt. Commun.* **213**, 13 (2002).
 - [18] T. van Dijk, G. Gbur, and T. D. Visser, Shaping the focal intensity distribution using spatial coherence, *J. Opt. Soc. Am. A* **25**, 575 (2008).
 - [19] A. Boivin, J. Dow, and E. Wolf, Energy flow in the neighborhood of the focus of a coherent beam, *J. Opt. Soc. Am.* **57**, 1171 (1967).
 - [20] T. D. Visser and E. Wolf, The origin of the Gouy phase anomaly and its generalization to astigmatic wavefields, *Opt. Commun.* **283**, 3371 (2010).
 - [21] X. Pang, T. D. Visser, and E. Wolf, Phase anomaly and phase singularities of the field in the focal region of a high-numerical aperture system, *Opt. Commun.* **284**, 5517 (2011).
 - [22] X. Pang, D. G. Fischer, and T. D. Visser, Generalized Gouy phase for focused partially coherent light and its implications for interferometry, *J. Opt. Soc. Am. A* **29**, 989 (2012).
 - [23] G. Gbur and T. D. Visser, Young's interference experiment: Past, present, and future, in *Progress in Optics* (Elsevier, Amsterdam, 2022), Vol. 67, pp. 275–343.
 - [24] O. Korotkova and E. Wolf, Changes in the state of polarization of a random electromagnetic beam on propagation, *Opt. Commun.* **246**, 35 (2005).
 - [25] H. F. Schouten and T. D. Visser, Scattering of partially coherent electromagnetic beams by a sphere, *Opt. Express* **32**, 10690 (2024).
 - [26] M. Takeda, W. Wang, Z. Duan, and Y. Miyamoto, Coherence holography, *Opt. Express* **13**, 9629 (2005).
 - [27] J. Rosen and M. Takeda, Longitudinal spatial coherence applied for surface profilometry, *Appl. Opt.* **39**, 4107 (2000).
 - [28] O. Korotkova and G. Gbur, Applications of optical coherence theory, in *Progress in Optics* (Elsevier, Amsterdam, 2020), Vol. 65, pp. 43–104.
 - [29] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, Reading, MA, 2001), Sec. 13.7.