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A special irreducible matrix representation of the real Clifford algebra *C*(3,1)

K. Scharnhorst[a)]
Humboldt-Universität zu Berlin, Institut für Physik,
Invalidenstrasse 110, D-10115 Berlin, Germany

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4×4 Dirac (gamma) matrices [irreducible matrix representations of the Clifford algebras $C(3,1)$, $C(1,3)$, $C(4,0)$] are an essential part of many calculations in quantum physics. Although the final physical results do not depend on the applied representation of the Dirac matrices (e.g., due to the invariance of traces of products of Dirac matrices), the appropriate choice of the representation used may facilitate the analysis. The present paper introduces a particularly symmetric real representation of 4×4 Dirac matrices (Majorana representation) which may prove useful in the future. As a by-product, a compact formula for (transformed) Pauli matrices is found. The consideration is based on the role played by isoclinic 2-planes in the geometry of the real Clifford algebra $C(3,0)$ which provide an invariant geometric frame for it. It can be generalized to larger Clifford algebras. © *1999 American Institute of Physics.* [S0022-2488(99)04606-X]

I. INTRODUCTION

Dirac (gamma) matrices used within many calculations in quantum physics can be understood as representations of Clifford algebras. In four-dimensional Minkowski or Euclidean space they are representations of the Clifford algebras $C(3,1)$, $C(1,3)$ or $C(4,0)$, respectively. While there is no problem to write down sets of complex 4×4 Dirac matrices which form irreducible representations of these Clifford algebras, a set of real 4×4 Dirac matrices (Majorana representation), which we will be interested in, can only be obtained for the Clifford algebra $C(3,1)$[1–4] (further material on real Clifford algebras can be found in Ref. 5, Chap. 13, Refs. 6–11). These matrices obey the standard relation

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\,\eta_{\mu\nu}\mathbf{1}, \tag{1}$$

where $\eta_{\mu\nu}$, $\mu,\nu=1,...,4$ are the elements of the diagonal matrix η with $\mathrm{diag}(\eta)=(1,1,1,-1)$ and $\mathbf{1}$ is the 4×4 unit matrix. An explicit representation of real gamma matrices is provided by the following expressions (adapted from Ref. 4):

$$\gamma_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}, \tag{2}$$

$$\gamma_2=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}, \tag{3}$$

[a)]Electronic mail: scharnh@physik.hu-berlin.de

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$$\gamma_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\tag{4}$$

$$\gamma_4=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}.\tag{5}$$

But, Eq. (1) is invariant under orthogonal transformations O of the gamma matrices

$$\gamma'_\mu=O\gamma_\mu O^T\tag{6}$$

and any other set of congruent [by virtue of (6)] gamma matrices γ'_μ will also be equally appropriate as representation of $C(3,1)$. (The general situation is described by Pauli's fundamental theorem.[12,13]) Now, let us denote the real linear vector space $\mathbf{R}_4$ in which the elements of the Clifford algebra $C(3,1)$ act as operators by V (spinor space). Then, the matrices γ_μ can be understood as representations of the generators of $C(3,1)$ with respect to a certain orthonormal basis in V which defines in it a rectangular coordinate system. Any transformation (6) of the gamma matrices corresponds to an orthogonal transformation in V and consequently to a change of the coordinate system in V. The concrete shape of the gamma matrices changes in performing these transformations. In explicit calculations in which gamma matrices occur the required effort may depend on the explicit shape of the gamma matrices. Therefore, in dependence on the physical problem under consideration one may ask whether it is possible to find a coordinate system in which the gamma matrices assume a particularly convenient shape for some calculational purpose. The detailed requirements certainly may depend on the purpose. From such a problem, recently we have been led to ask ourselves whether it is possible to find an irreducible representation of the real Clifford algebra $C(3,1)$ which is particularly symmetric with respect to the index μ of the gamma matrices γ'_μ. Indeed, it is possible to find an orthogonal transformation which transforms the gamma matrices (2)–(5) into the following expressions which are obviously particularly symmetric with respect to the index of the gamma matrices $k=1,2,3$ ($\mathbf{1}$ and $\mathbf{0}$ are the 2×2 unit and null matrices, respectively; φ_0 is some arbitrary real constant; cf. Sec. V),

$$\gamma'_k=\frac{1}{\sqrt{3}}\begin{pmatrix}\mathbf{1}&\mathbf{F}_k\\\mathbf{F}_k&-\mathbf{1}\end{pmatrix},\quad \mathbf{F}_k=\begin{pmatrix}f(-\varphi_k)&f(\varphi_k)\\f(\varphi_k)&-f(-\varphi_k)\end{pmatrix},\tag{7}$$

$$f(\varphi)=\cos\varphi+\sin\varphi=\sqrt{2}\cos\left(\varphi-\frac{\pi}{4}\right),\tag{8}$$

$$\varphi_k=\varphi(k)=\varphi_0+\frac{2\pi}{3}k,\tag{9}$$

$$\gamma'_4=\begin{pmatrix}\mathbf{0}&-\mathbf{1}\\\mathbf{1}&\mathbf{0}\end{pmatrix}.\tag{10}$$

As a by-product, from the above expressions one obtains the following compact formula for transformed Pauli matrices [irreducible matrix representations of the complex Clifford algebra $C(3,0)$; cf. Appendix B].

$$\sigma_k' = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} e^{-i\varphi_k} \\ \sqrt{2} e^{i\varphi_k} & -1 \end{pmatrix}. \tag{11}$$

It is the purpose of the present article to systematically derive the above expressions relying on certain information not applied previously within the present context. The discussion is accompanied by references to the relevant but scattered literature.

Our considerations will be guided by the following idea. Related to the Clifford algebra $C(3,0)$, it should be possible to find an expression for the set of the gamma matrices γ_k', $k = 1,2,3$ which is particularly symmetric with respect to the index k. We approach the problem by noting that each gamma matrix γ_k has 2 two-dimensional eigenspaces related to the eigenvalues $\rho = 1$ and $\rho = -1$ (which are orthogonal to each other). Any coordinate system in V stands in a certain geometric relation to all the eigenspaces of the gamma matrices whose mutual relation is an invariant under any transformation (6). Now, the idea consists in finding such a coordinate system in V with respect to which all the eigenspaces of the gamma matrices lie in a particularly symmetric way. Then, one may expect that the explicit expressions for the gamma matrices γ_k' reflect this symmetry. Therefore, in Sec. II we start with some observations concerning the eigenspaces of the generators of the Clifford algebra $C(3,0)$ (more precisely, in using this term we always mean the generators of its irreducible representations).

II. ISOCLINIC 2-PLANES IN $\mathbf{R}_4$

To begin with, let us discuss some aspects of the geometry of 2-planes in the affine space $\mathbf{R}_4$ which we also denote by V for simplicity. We restrict our consideration to 2-planes containing the point $\mathbf{x} = (0,0,0,0)$ [i.e., to the Grassmann manifold $G(2,4)$, for a related review see Ref. 14]. We will rely here on the general multidimensional matrix formalism presented in Ref. 15, Chap. 3, Sec. 3 (also see Ref. 16, Chap. III, Sec. 3.3) which we specialize to $\mathbf{R}_4$. In the following we will start with some material which provides the necessary information on those aspect of the formalism of Refs. 15 and 16 which is relevant for the present paper.

For our purposes, a point $\mathbf{x}$ of a given 2-plane A can be described in terms of the equation

$$\mathbf{x} = \mathbf{A}\mathbf{t}, \tag{12}$$

where $\mathbf{A}$ is a 4×2 matrix whose two columns are given by the coordinates of two linearly independent vectors spanning the 2-plane A while $\mathbf{t}$ is the two-component vector of the coordinates of the point $\mathbf{x} \in A$. Two 2-planes A and B can intersect in V in various ways. In order to study their relation, to each pair of lines $X \subset A$, $Y \subset B$ the angle they enclose can be calculated. Once a line $X \subset A$ is fixed, for any arbitrary line $Y \subset B$ the angle enclosed assumes values between some $\alpha_0 \geqslant 0$ and $\pi/2$. In general, α_0 may lie between some minimal and some maximal value—the so-called *stationary angles (principal angles)* $\alpha_{\min}$, $\alpha_{\max}$—which are characteristic for the geometry of the pair of 2-planes A, B. Now, from an extremum principle a 2×2 matrix

$$\mathbf{W} = (\mathbf{A}^T\mathbf{A})^{-1}(\mathbf{A}^T\mathbf{B})(\mathbf{B}^T\mathbf{B})^{-1}(\mathbf{B}^T\mathbf{A}) \tag{13}$$

can be constructed[17] for whose eigenvalues w_1 and w_2 the equations

$$w_1 = \cos^2 \alpha_{\max}, \tag{14}$$

$$w_2 = \cos^2 \alpha_{\min} \tag{15}$$

apply. If the 2-planes A, B are given by means of Eq. (12) in terms of two orthonormal vectors each, Eq. (13) simplifies to the form[35]

$$\mathbf{W} = (\mathbf{A}^T\mathbf{B})(\mathbf{B}^T\mathbf{A}). \tag{16}$$

If the matrix $\mathbf{W}$ is proportional to the unit matrix (i.e., $w_1 = w_2 = w$)

$$\mathbf{W} = w\mathbf{1}, \tag{17}$$

the 2-planes A and B are said to be (mutually) *isoclinic*.[41] Then, to each vector $\mathbf{x} \in A$ a unique line in B exists (determined by the orthogonal projection of $\mathbf{x}$ onto B) which encloses with $\mathbf{x}$ the (stationary) angle $\alpha = \arccos\sqrt{w}$.[48] Finally, we would like to mention that under some natural bijection between $\mathbf{R}_4$ and $\mathbf{C}_2$ $((z_1,z_2) = (x_1 + ix_2, x_3 + ix_4) \in \mathbf{C}_2,\ (x_1,x_2,x_3,x_4) \in \mathbf{R}_4)$ two isoclinic 2-planes in $\mathbf{R}_4$ correspond to two lines through the origin in $\mathbf{C}_2$ (Ref. 23, Sec. 1-7, p. 51, theorem 1-7.4).

Now, the above formalism can be used to analyze the geometry of the set of 6 two-dimensional eigenspaces of the generators of the Clifford algebra $C(3,0)$ (i.e., more precisely, the generators of its irreducible representation). After some calculation using, e.g., the explicit representations of the gamma matrices (2)–(4) one finds that all their six eigenspaces are pairwise isoclinic 2-planes (some choice for the matrices $\mathbf{A}$ describing the eigenspaces is given in Appendix A). Of course, the two eigenspaces of a given gamma matrix γ_k are orthogonal to each other. But, any other two eigenspaces are pairwise isoclinic with an (stationary) angle $\alpha = \pi/4$. Consequently, we can find, at maximum, a set of three eigenspaces of the gamma matrices γ_k, $k = 1,2,3$, whose elements are pairwise isoclinic with the angle $\pi/4$.[54] Such a set of 2-planes is called an *equiangular frame* (Ref. 22, Pt. I, Sec. 5, p. 40). With respect to the aim of the present paper, in the following we will just be interested in such sets.

III. THE CLIFFORD ALGEBRA *C*(3,0) AND EQUIANGULAR FRAMES

We begin this section with some necessary information taken from Ref. 22[58] and specialized to the present needs (in the following the term ''adapted quote'' always means that the original text is quoted exactly except that any reference to the general multidimensional space $\mathbf{R}_{2n}$ has been specialized to $\mathbf{R}_4$). The following definition will be used: ''A *set of mutually isoclinic 2-planes* in $\mathbf{R}_4$ is characterized by the property that every two 2-planes of the set are isoclinic with each other. A set of mutually isoclinic 2-planes in $\mathbf{R}_4$ is called a *maximal* set if it is not subset of a larger set of mutually isoclinic 2-planes'' (this is an adapted quote from Ref. 22, Pt. I, Sec. 3, p. 19).[60]

In order to make contact with the formalism used in Ref. 22 which we will rely on in the further discussion we need to rewrite the defining equation (12) for a 2-plane A in one of the following two (alternative) ways:

$$\mathbf{x}_{(3,4)} = \tilde{\mathbf{A}}\mathbf{x}_{(1,2)}, \quad \tilde{\mathbf{A}} = \bar{\bar{\mathbf{A}}}(\bar{\mathbf{A}})^{-1}, \tag{18}$$

$$\mathbf{x}_{(1,2)} = \tilde{\tilde{\mathbf{A}}}\mathbf{x}_{(3,4)}, \quad \tilde{\tilde{\mathbf{A}}} = \bar{\mathbf{A}}(\bar{\bar{\mathbf{A}}})^{-1}. \tag{19}$$

Here, the notation $\mathbf{x}_{(1,2)} = (x_1,x_2)^T$, $\mathbf{x}_{(3,4)} = (x_3,x_4)^T$ is used and the 2×2 matrices $\bar{\mathbf{A}}$, $\bar{\bar{\mathbf{A}}}$ are related to the matrix $\mathbf{A}$ in the following way:

$$\mathbf{A} = \begin{pmatrix} \bar{\mathbf{A}} \\ \bar{\bar{\mathbf{A}}} \end{pmatrix}. \tag{20}$$

Equation (18) [(19)] is valid for any 2-plane which is isoclinic but not identical to the 2-plane $O_{(3,4)}$: $\mathbf{x}_{(1,2)} = 0$ [$O_{(1,2)}$: $\mathbf{x}_{(3,4)} = 0$] (this entails that the 2-plane A intersects the 2-plane $O_{(3,4)}$ [$O_{(1,2)}$] in the point $\mathbf{x} = (0,0,0,0)$ only and, therefore, ensures the invertibility of $\bar{\mathbf{A}}$ [$\bar{\bar{\mathbf{A}}}$]).

According to Wong (Ref. 22, Pt. I, Sec. 7, p. 54, theorem 7.2; also see Ref. 23, Sec. 1-7, p. 43), every maximal set of mutually isoclinic 2-planes in $\mathbf{R}_4$ is of dimension 2 and is congruent (i.e., related by an orthogonal transformation in $\mathbf{R}_4$) to the maximal set given by

$$\mathbf{x}_{(3,4)} = \tilde{\mathbf{B}}(\lambda_0,\lambda_1)\mathbf{x}_{(1,2)} = [\lambda_0\tilde{\mathbf{B}}_0 + \lambda_1\tilde{\mathbf{B}}_1]\mathbf{x}_{(1,2)}, \tag{21}$$

$$\tilde{\mathbf{B}}_0=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad \tilde{\mathbf{B}}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{22}$$

or

$$\mathbf{x}_{(1,2)}=\tilde{\tilde{\mathbf{B}}}(\lambda_0,\lambda_1)\mathbf{x}_{(3,4)}, \tag{23}$$

$$\begin{gathered}\tilde{\tilde{\mathbf{B}}}(\lambda_0,\lambda_1)=\tilde{\mathbf{B}}(\lambda_0,\lambda_1)^{-1}=\frac{1}{\lambda_0^2+\lambda_1^2}\tilde{\mathbf{B}}(\lambda_0,\lambda_1)=\tilde{\mathbf{B}}(\lambda_0',\lambda_1'),\\ \lambda_n'=\frac{\lambda_n}{\lambda_0^2+\lambda_1^2},\quad n=1,2,\end{gathered} \tag{24}$$

where λ_0, λ_1 are two real parameters.[64]

Both of the 2-planes $O_{(1,2)}:\mathbf{x}_{(3,4)}=0$ and $O_{(3,4)}:\mathbf{x}_{(1,2)}=0$ belong to this maximal set (Ref. 22, Pt. I, Sec. 2, p. 16, lemma 2.2). Equations (21) and (23) entail that the matrix **B** to be inserted in the corresponding Eq. (12) reads, e.g. (we have chosen particularly simple expressions),

$$\mathbf{B}(\lambda_0,\lambda_1)=\frac{1}{\sqrt{1+\lambda_0^2+\lambda_1^2}}\begin{pmatrix}\mathbf{1}\\ \tilde{\mathbf{B}}(\lambda_0,\lambda_1)\end{pmatrix}, \tag{25}$$

or

$$\mathbf{B}(\lambda_0',\lambda_1')=\frac{1}{\sqrt{1+\lambda_0'^2+\lambda_1'^2}}\begin{pmatrix}\tilde{\mathbf{B}}(\lambda_0',\lambda_1')\\ \mathbf{1}\end{pmatrix}. \tag{26}$$

Furthermore, Wong finds that (adapted quote) ''in $\mathbf{R}_4$, any maximal set of mutually isoclinic 2-planes which contains the 2-plane $O_{(1,2)}$ corresponds to a linear subspace of the linear space of all 2×2 matrices'' (Ref. 22, Pt. I, Sec. 3, p. 20, lemma 3.2). Now, in this two-dimensional subspace a matrix basis can be chosen in such a way that the 2-planes described by the elements of the basis and the 2-plane $O_{(1,2)}$ (or $O_{(3,4)}$) form an equiangular frame (Ref. 22, Pt. I, Sec. 3, p. 24, lemma 3.3 and p. 40). As one may convince oneself easily by means of the explicit expressions given in Appendix A, each equiangular frame built from the eigenspaces of the gamma matrices contains a basis of one and the same maximal set of mutually isoclinic 2-planes.

For the purpose of the present paper it appears to be useful to consider two disjoint equiangular frames Ω connected with the gamma matrices (2)–(4)—one (Ω_1) related to the three eigenspaces to the eigenvalue $\rho=1$, and the other one (Ω_{-1}) related to the three eigenspaces to the eigenvalue $\rho=-1$. The following theorem by Wong will be helpful then (Φ is any maximal set of mutually isoclinic 2-planes in $\mathbf{R}_4$; the following is an adapted quote; the indices have also been changed to conform to the notation used in the present article): ''If the angles between any 2-plane of Φ and the three 2-planes of an equiangular frame are θ_k $(1\leqslant k\leqslant 3)$, then

$$\cos^2 2\theta_1+\cos^2 2\theta_2+\cos^2 2\theta_3=1. \tag{27}$$

Conversely, given any set of three angles θ_k $(1\leqslant k\leqslant 3)$ such that $0\leqslant\theta_k\leqslant\pi$ and $\Sigma\cos^2 2\theta_k=1$, then there exists a unique 2-plane isoclinic to each of the three 2-planes of a given equiangular frame, making angles θ_k with them, and this 2-plane belongs to Φ'' [Ref. 22, Pt. I, Sec. 5, p. 41, theorem 5.3 (b)]. From this insight we conclude that, obviously, to each equiangular frame Ω_1 $[\Omega_{-1}]$ two uniquely determined 2-planes $A_{1\pm}$ $[A_{-1\pm}]$ exist which lie in a particularly symmetric way (isoclinic) relative to the elements of Ω_1 $[\Omega_{-1}]$. For $A_{1\pm}$, $A_{-1\pm}$ it holds

$$\theta_1=\theta_2=\theta_3=\theta_{\text{sym}}, \quad \cos 2\theta_{\text{sym}}=\pm\frac{1}{\sqrt{3}}. \tag{28}$$

For the corresponding eigenvalue of the matrix **W**, Eq. (17), one obtains

$$w=\cos^2\theta_{\text{sym}}=\frac{1}{2}(1+\cos 2\theta_{\text{sym}})=\frac{1}{2}\left(1\pm\frac{1}{\sqrt{3}}\right)=w_{\pm}. \tag{29}$$

The two different values of θ_{sym} (and w)) will not cause any major difference in the following considerations as both cases are related by a simple permutation of the indices of the gamma matrices.

IV. CHANGE OF THE COORDINATE SYSTEM

We may now set out to determine the position of the 2-planes $A_{1\pm}$, $A_{-1\pm}$ using the formulas given in Secs. II and III. For the 2-planes $A_{1\pm}$, $A_{-1\pm}$ we can apply a general ansatz according to Eqs. (21), (23), (25), (26) and calculate the eigenvalue of the matrix **W** for each of the three pairs given by one of the elements of the equiangular frame Ω_1 $[\Omega_{-1}]$ and $A_{1\pm}$ $[A_{-1\pm}]$. For each eigenvalue ρ of the gamma matrices (2)–(4), this leads to a set of three equations for the parameters λ_0, λ_1 which have to be solved simultaneously taking into account Eq. (29). These equations read for $\rho=1$ (in sequence for the indices $k=1$, $k=2$ and $k=3$ of the gamma matrices, respectively)

$$w_{\pm}=\frac{\lambda_0'^2+(1+\lambda_1')^2}{2(1+\lambda_0'^2+\lambda_1'^2)}, \tag{30}$$

$$w_{\pm}=\frac{(1-\lambda_0')^2+\lambda_1'^2}{2(1+\lambda_0'^2+\lambda_1'^2)}, \tag{31}$$

$$w_{\pm}=\frac{\lambda_0'^2+\lambda_1'^2}{1+\lambda_0'^2+\lambda_1'^2}, \tag{32}$$

and for $\rho=-1$,

$$w_{\pm}=\frac{\lambda_0^2+(1-\lambda_1)^2}{2(1+\lambda_0^2+\lambda_1^2)}, \tag{33}$$

$$w_{\pm}=\frac{(1+\lambda_0)^2+\lambda_1^2}{2(1+\lambda_0^2+\lambda_1^2)}, \tag{34}$$

$$w_{\pm}=\frac{\lambda_0^2+\lambda_1^2}{1+\lambda_0^2+\lambda_1^2}. \tag{35}$$

(Equations (30)–(32) [(33)–(35)] have been derived using the expressions given in Appendix A and Eq. (26) [(25)].) The solution of the above equations reads for $\rho=1$,

$$\lambda_0'=-\lambda_1'=-\lambda_{\pm}, \tag{36}$$

and for $\rho=-1$,

$$\lambda_0=-\lambda_1=\lambda_{\pm}. \tag{37}$$

Here,

$$\lambda_{\pm} = \pm\sqrt{3}w_{\pm}, \tag{38}$$

which entails

$$2\lambda_{\pm}\lambda_{\mp} = -1. \tag{39}$$

Now, we may assume that the explicit representations for the gamma matrices (2)–(5) are related to the natural basis in V from which two pairs of basis vectors can be selected which define the orthogonal 2-planes $O_{(1,2)}$, $O_{(3,4)}$. In order to obtain a particularly symmetric representation for the gamma matrices it appears to be advantageous now to go over to an orthonormal basis from which two pairs of basis vectors can be chosen which define the orthogonal 2-planes $A_{1\pm}$, $A_{-1\pm}$. This change of the basis in V is associated with an orthogonal transformation O in V which transforms the gamma matrices in accordance with Eq. (6). We start by choosing an appropriate orthonormal basis in V from which the matrices $\mathbf{A}_{1\pm}$, $\mathbf{A}_{-1\pm}$ describing the 2-planes $A_{1\pm}$, $A_{-1\pm}$ can be built [we simply insert the solutions (36) and (37) into Eqs. (26) and (25), respectively],

$$\mathbf{A}_{1\pm} = \frac{1}{\sqrt{1+2\lambda_{\pm}^2}} \begin{pmatrix} \lambda_{\pm} & \lambda_{\pm} \\ \lambda_{\pm} & -\lambda_{\pm} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{40}$$

$$\mathbf{A}_{-1\pm} = \frac{1}{\sqrt{1+2\lambda_{\pm}^2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\lambda_{\pm} & -\lambda_{\pm} \\ -\lambda_{\pm} & \lambda_{\pm} \end{pmatrix}. \tag{41}$$

One immediately recognizes that the 2-planes $A_{1\pm}$, $A_{-1\pm}$ are orthogonal to each other. Furthermore, by virtue of Eq. (39) it holds $A_{1\pm} = A_{-1\mp}$. Of course, the above choice for the matrices $\mathbf{A}_{1\pm}$, $\mathbf{A}_{-1\pm}$ is not unique and any orthonormal basis which is related to the basis used in the above equations by a rotation within the 2-planes $A_{1\pm}$, $A_{-1\pm}$ is equally well suited. In fact, further below we will use exactly this freedom to obtain our final result (7)–(10).

The transition from the natural basis in V which is related to the 2-planes $O_{(1,2)}$, $O_{(3,4)}$ to the basis which is given in terms of Eqs. (40) and (41) and which is related to the 2-planes $A_{1\pm}$, $A_{-1\pm}$ is described by the orthogonal transformation $O_{\pm}$,

$$O_{\pm} = \frac{1}{\sqrt{1+2\lambda_{\pm}^2}} \begin{pmatrix} \lambda_{\pm} & \lambda_{\pm} & 1 & 0 \\ \lambda_{\pm} & -\lambda_{\pm} & 0 & 1 \\ 1 & 0 & -\lambda_{\pm} & -\lambda_{\pm} \\ 0 & 1 & -\lambda_{\pm} & \lambda_{\pm} \end{pmatrix}, \tag{42}$$

which leads via $\gamma''_{\mu} = O_{\pm}\gamma_{\mu}O^{T}_{\pm}$ to the correspondingly transformed expressions for the gamma matrices γ_{μ} [of course, for our choice (42) it holds $O_{\pm} = O^{T}_{\pm}$]. After some algebra [taking into account Eq. (39)] one finds

$$\gamma''_{1\pm} = -\gamma''_{2\mp} = \pm\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & -\lambda_{\pm} & -\lambda_{\mp} \\ 0 & 1 & -\lambda_{\mp} & \lambda_{\pm} \\ -\lambda_{\pm} & -\lambda_{\mp} & -1 & 0 \\ -\lambda_{\mp} & \lambda_{\pm} & 0 & -1 \end{pmatrix}, \tag{43}$$

$$\gamma''_{3\pm}=\pm\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}, \tag{44}$$

$$\gamma''_{4}=\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{45}$$

From Eq. (43) one immediately recognizes that the two cases differing by the sign in Eq. (28) are related to each other by a permutation of the gamma matrices with the indices $k=1$ and $k=2$.

V. RESIDUAL ROTATIONS

Although in Sec. IV we have performed the transformation to a coordinate system which lies in a particularly symmetric way with respect to the equiangular frames Ω_1, Ω_{-1} built from the eigenspaces of the gamma matrices, at first glance the transformed expressions (43), (44) do not seem to exhibit any particular symmetry with respect to the index $k=1,2,3$ of the gamma matrices. However, the expected symmetry is there and we are going to reveal it now. Let us remind ourselves that the choice of the new basis (coordinate system) was not unique and we have disregarded for the moment the remaining freedom to perform rotations within the 2-planes $A_{1\pm}$, $A_{-1\pm}$. Any such rotation can be described by the orthogonal transformation

$$O(\beta_1,\beta_{-1})=\begin{pmatrix} \cos\beta_1 & -\sin\beta_1 & 0 & 0 \\ \sin\beta_1 & \cos\beta_1 & 0 & 0 \\ 0 & 0 & \cos\beta_{-1} & -\sin\beta_{-1} \\ 0 & 0 & \sin\beta_{-1} & \cos\beta_{-1} \end{pmatrix}, \tag{46}$$

where β_1 and β_{-1} are the independent rotation angles within the orthogonal 2-planes $A_{1\pm}$ and $A_{-1\pm}$, respectively (for the sake of completeness we mention that in addition to the above rotations an inversion within one of the 2-planes $A_{1\pm}$, $A_{-1\pm}$ may be considered). Again, we can write down the further transformed gamma matrices $\gamma'_\mu=O(\beta_1,\beta_{-1})\gamma''_\mu O(\beta_1,\beta_{-1})^T$. For brevity, we give the relatively simple expressions for $\gamma'_{3\pm}$ and γ'_4 only,

$$\gamma'_{3\pm}(\varphi)=\pm\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & f(-\varphi) & f(\varphi) \\ 0 & 1 & f(\varphi) & -f(-\varphi) \\ f(-\varphi) & f(\varphi) & -1 & 0 \\ f(\varphi) & -f(-\varphi) & 0 & -1 \end{pmatrix}, \tag{47}$$

$$\gamma'_4(\bar\varphi)=\begin{pmatrix} 0 & 0 & -\cos\bar\varphi & \sin\bar\varphi \\ 0 & 0 & -\sin\bar\varphi & -\cos\bar\varphi \\ \cos\bar\varphi & \sin\bar\varphi & 0 & 0 \\ -\sin\bar\varphi & \cos\bar\varphi & 0 & 0 \end{pmatrix}. \tag{48}$$

Here, $\varphi=\beta_1+\beta_{-1}$ and $\bar\varphi=\beta_1-\beta_{-1}$. The gamma matrices $\gamma'_{k\pm}$, $k=1,2,3$, do not depend on $\bar\varphi$ while γ'_4 does not depend on φ. The function f is given by

$$f(\varphi)=\cos\varphi+\sin\varphi=\sqrt{2}\cos\left(\varphi-\frac{\pi}{4}\right). \tag{49}$$

Symmetry considerations now suggest that any set of (three) rotations $O(\beta_1,\beta_{-1})$ among whose elements $\varphi=\beta_1+\beta_{-1}$ changes by a multiple of $2\pi/3$ (mod 2π) will lead to a set of three gamma matrices with the indices $k=1,2,3$. Consequently, in order to describe this set we can write

$$\varphi(k)=\varphi_0+\frac{2\pi}{3}k=\varphi_k, \tag{50}$$

where φ_0 is some real constant. Any three gamma matrices given by Eqs. (47), (49), and (50) can be chosen to serve as an irreducible representation of the real Clifford algebra $C(3,0)$. If we choose $\varphi_0=0$, Eqs. (47), (49), and (50) exactly reproduce the set of gamma matrices (43), (44), i.e.,

$$\gamma'_{3\pm}\left(\frac{2\pi}{3}\right)=\gamma''_{1\pm}, \quad \gamma'_{3\pm}\left(\frac{4\pi}{3}\right)=\gamma''_{2\pm}. \tag{51}$$

Furthermore, for the sake of simplicity it seems to be convenient to set $\bar{\varphi}=0$ and to vary φ exclusively. [Such an orthogonal transformation is called a *Clifford translation* (Ref. 23, Sec. 2-6, p. 102) and has special properties. In this context, also note Ref. 66.] This way the final result [Eqs. (7)–(10), also see Appendix B for some related consideration] quoted in Sec. I is obtained [where we have omitted, for simplicity, the ± sign on the right-hand side of Eq. (47) which relates to the two inequivalent irreducible representations of $C(3,0)$ (Ref. 2, p. 1657)]. The generators of the real Clifford algebra $C(3,0)$ are found from one of them by means of a discrete $\mathbf{Z}_6\sim\mathbf{Z}_2\times\mathbf{Z}_3$ subgroup of the orthogonal group $O(4)$ (in other words, the $\mathbf{Z}_6$ subgroup realizes a permutation among the gamma matrices). The Clifford translation in the spinor space V with $\beta_1=\beta_{-1}=\pi/3$ corresponds to a rotation by $2\pi/3$ around the axis (1,1,1) in the vector space $\mathbf{R}_{3,0}$ associated with the Clifford algebra $C(3,0)$ (it is an element of the group Spin(3)).

We want to extend our discussion now to the real Clifford algebra $C(3,2)$, which is the largest Clifford algebra admitting an irreducible representation by means of 4×4 matrices. From Eqs. (47), (51) we can calculate the product

$$\gamma'_{3\pm}(\varphi_1)\gamma'_{3\pm}(\varphi_2)\gamma'_{3\pm}(\varphi_3)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{52}$$

which is found to be independent of the choice of φ_0. Allowing an arbitrary value for $\bar{\varphi}$, γ'_5 can then be calculated and reads

$$\begin{aligned} \gamma'_5=\gamma'_5(\bar{\varphi}) &= \gamma'_{3\pm}(\varphi_1)\gamma'_{3\pm}(\varphi_2)\gamma'_{3\pm}(\varphi_3)\gamma'_4(\bar{\varphi}) \\ &= \gamma'_4\left(\bar{\varphi}-\frac{\pi}{2}\right) \\ &= \begin{pmatrix} 0 & 0 & -\sin\bar{\varphi} & -\cos\bar{\varphi} \\ 0 & 0 & \cos\bar{\varphi} & -\sin\bar{\varphi} \\ \sin\bar{\varphi} & -\cos\bar{\varphi} & 0 & 0 \\ \cos\bar{\varphi} & \sin\bar{\varphi} & 0 & 0 \end{pmatrix}, \\ \gamma_5'^2 &= -\mathbf{1}. \end{aligned} \tag{53}$$

Finally, the charge conjugation operator C ($C^T = -C$, $C\gamma'_\mu C^{-1} = -\gamma'^T_\mu$) can be given by $C = \gamma'_4(\bar{\varphi})$. In difference to the $C(3,0)$ subalgebra of the Clifford algebra $C(3,2)$, which is generated by relying on Eq. (50) [a variation of φ by 2π leads to just one copy of the generators of $C(3,0)$], the $C(0,2)$ subalgebra can be represented by $\gamma'_4 = \gamma'_4(\bar{\varphi})$, $\gamma'_5 = \gamma'_4(\bar{\varphi} \pm \pi/2)$ [a variation of $\bar{\varphi}$ by 2π leads to two copies of the generators of $C(0,2)$].[67] In this context, note

$$\gamma'_4(\bar{\varphi}) = -\gamma'_4(\bar{\varphi} + \pi). \tag{54}$$

For $\varphi = 0$, the second generator of the real Clifford algebra $C(0,2)$ is obtained from the first by means of a discrete $\mathbf{Z}_8 \sim (\mathbf{Z}_2)^3$ subgroup of the orthogonal group $O(4)$. A rotation (46) in the spinor space V with $\beta_1 = -\beta_{-1} = \pi/4$ corresponds to a rotation by $\pi/2$ in the vector space $\mathbf{R}_{0,2}$ associated with the Clifford algebra $C(0,2)$ [it is an element of the group Spin(2)].

VI. DISCUSSION

According to Pauli's fundamental theorem[12,13] any set of (in general, complex) 4×4 gamma matrices γ_μ, which represent the Clifford algebra $C(3,1)$, is related to our expressions for γ'_μ [Eqs. (7)–(10)] by means of a nonsingular transformation S ($\gamma_\mu = S\gamma'_\mu S^{-1}$). Therefore, any such set can, in principle, be written in a form analogous to Eqs. (7)–(10) (of course, in general such a representation may look fairly cumbersome). It is clear, that this consideration of the (complex) Clifford algebra $C(3,1)$ immediately carries over with little change to the Clifford algebra $C(1,3)$ and does not require any further special investigation. Furthermore, it seems natural to expect that the discussion of the real Clifford algebra $C(3,1)$ performed in the present paper can appropriately be generalized also to other Clifford algebras. Of course, the simpler and rather trivial case of the real Clifford algebra $C(2,1)$ which is presented in Appendix C carries the traces of the structures found for $C(3,0)$. On the other hand, one should expect that these structures themselves are also traces of more general structures of Clifford algebras which contain $C(3,0)$ as a subalgebra. Let us emphasize at this point that the mathematical tools we have relied on in Secs. II and III are not specific to the present case (although, we have specialized them to the present case, for simplicity) and they can also be used in more general situations. As interesting as this may be, it goes far beyond the purpose of the present study and, therefore, will not be investigated here.

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APPENDIX A: DESCRIPTION OF THE EIGENSPACES OF γ_k

In this Appendix we give some explicit expressions for the matrices $\mathbf{A}_{k,\rho}$ which define via Eq. (12) the eigenspace (i.e., the 2-plane $A_{k,\rho}$) of the gamma matrix γ_k, $k = 1,2,3$, to the eigenvalue $\rho = 1, -1$. We rely on orthonormal basis vectors for each eigenspace.

$$\mathbf{A}_{1,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{A1}$$

$$\mathbf{A}_{1,-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A2}$$

$$\mathbf{A}_{2,1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\\ 1 & 0\\ 0 & -1\end{pmatrix}, \tag{A3}$$

$$\mathbf{A}_{2,-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & 1\end{pmatrix}, \tag{A4}$$

$$\mathbf{A}_{3,1}=\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix}, \tag{A5}$$

$$\mathbf{A}_{3,-1}=\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}. \tag{A6}$$

From Eqs. (12), (18)–(20) one easily recognizes that for the 2-planes $A_{3,1}$, $A_{3,-1}$ holds $A_{3,1}=O_{(1,2)}$, $A_{3,-1}=O_{(3,4)}$ ($O_{(1,2)}:\mathbf{x}_{(3,4)}=0$, $O_{(3,4)}:\mathbf{x}_{(1,2)}=0$).

APPENDIX B: TRANSFORMED PAULI MATRICES

As Pauli matrices [irreducible matrix representations of the complex Clifford algebra $C(3,0)$] play a significant role in theoretical physics, in this Appendix we wish to comment on the derivation of a particularly symmetric expression for these 2×2 matrices by means of the approach discussed in the main part of the paper. The standard expressions for the Pauli matrices read

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{B1}$$

In order to make contact with the main part of the paper it turns out to be useful to represent the complex numbers which are entries of the matrices (B1) by means of 2×2 matrices using the rule

$$z=a+ib\rightarrow\begin{pmatrix}a & -b\\ b & a\end{pmatrix}. \tag{B2}$$

This leads to a set of three real 4×4 matrices which are congruent to the gamma matrices given by Eq. (7). In order to obtain the desired final result we have to subject the latter gamma matrices to a further orthogonal transformation—an inversion [mentioned below Eq. (46)]. Then the rule (B2) can be reversed yielding the following transformed Pauli matrices ($k=1,2,3$):

$$\sigma'_k=\frac{1}{\sqrt{3}}\begin{pmatrix}1 & \sqrt{2}e^{-i\varphi_k}\\ \sqrt{2}e^{i\varphi_k} & -1\end{pmatrix}, \tag{B3}$$

$$\varphi_k=\varphi(k)=\varphi_0+\frac{2\pi}{3}k. \tag{B4}$$

Here, φ_0 is some arbitrary real constant which, however, has been shifted with respect to Eq. (9).

APPENDIX C: THE CASE OF THE REAL CLIFFORD ALGEBRA *C*(2,1)

In the present Appendix we want to illustrate the formalism used in the main part of the paper in the rather trivial case of the real Clifford algebra $C(2,1)$. We display the equations (including the notation) in close analogy to the discussion performed in the main part of the paper. We start with some explicit expressions for the gamma matrices [σ_k are the standard Pauli matrices (B1)],

$$\gamma_1=\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{C1}$$

$$\gamma_2=\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{C2}$$

$$\gamma_3=i\sigma_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{C3}$$

The eigenspaces of the gamma matrices γ_1, γ_2 are described by the following matrices:

$$\mathbf{A}_{1,1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, \tag{C4}$$

$$\mathbf{A}_{1,-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}, \tag{C5}$$

$$\mathbf{A}_{2,1}=\begin{pmatrix}1\\ 0\end{pmatrix}, \tag{C6}$$

$$\mathbf{A}_{2,-1}=\begin{pmatrix}0\\ 1\end{pmatrix}. \tag{C7}$$

It is clear that the angle between the eigenspaces (lines, 1-planes) which relate to different gamma matrices γ_1, γ_2 is $\pi/4$ (cf. Fig. 1).[68] Each line through the origin $\mathbf{x}=(0,0)$ is (trivially) isoclinic to each other such line. Therefore, the analogs of Eqs. (21), (23) are

$$x_2=\lambda x_1, \tag{C8}$$

$$x_1=\lambda' x_2, \quad \lambda'=\lambda^{-1}. \tag{C9}$$

Equations (25), (26) are mirrored by

$$\mathbf{B}(\lambda)=\frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix}1\\ \lambda\end{pmatrix}, \tag{C10}$$

and

$$\mathbf{B}(\lambda')=\frac{1}{\sqrt{1+\lambda'^2}}\begin{pmatrix}\lambda'\\ 1\end{pmatrix}. \tag{C11}$$

Of course, to each set of the eigenspaces $\{A_{1,1}, A_{2,1}\}$, $\{A_{1,-1}, A_{2,-1}\}$ two lines $A_{1\pm}$, $A_{-1\pm}$ exist, respectively, which lie symmetric with respect to the elements of the set (cf. Fig. 1). The analog of Eq. (27) reads (θ_k are the angles between the two eigenspaces to the eigenvalue $\rho=1$ $[\rho=-1]$ and $A_{1\pm}$ $[A_{-1\pm}]$)

$$\cos^2 2\theta_1+\cos^2 2\theta_2=1. \tag{C12}$$

For $A_{1\pm}$, $A_{-1\pm}$ the relations [in analogy to Eqs. (28), (29)]

$$\theta_1=\theta_2=\theta_{\mathrm{sym}}, \quad \cos 2\theta_{\mathrm{sym}}=\pm\frac{1}{\sqrt{2}}, \tag{C13}$$

and

$$w=\cos^2\theta_{\mathrm{sym}}=\frac{1}{2}(1+\cos 2\theta_{\mathrm{sym}})=\frac{1}{2}\left(1\pm\frac{1}{\sqrt{2}}\right)=w_\pm \tag{C14}$$

are valid.

In analogy to Eqs. (30)–(35), in order to determine the lines $A_{1\pm}$, $A_{-1\pm}$ we have to solve the following equations for $\rho=1$ (in sequence for the indices $k=1$, $k=2$ of the gamma matrices, respectively)

$$w_\pm=\frac{(1+\lambda')^2}{2(1+\lambda'^2)}, \tag{C15}$$

$$w_\pm=\frac{\lambda'^2}{1+\lambda'^2}, \tag{C16}$$

and for $\rho=-1$,

$$w_\pm=\frac{(1-\lambda)^2}{2(1+\lambda^2)}, \tag{C17}$$

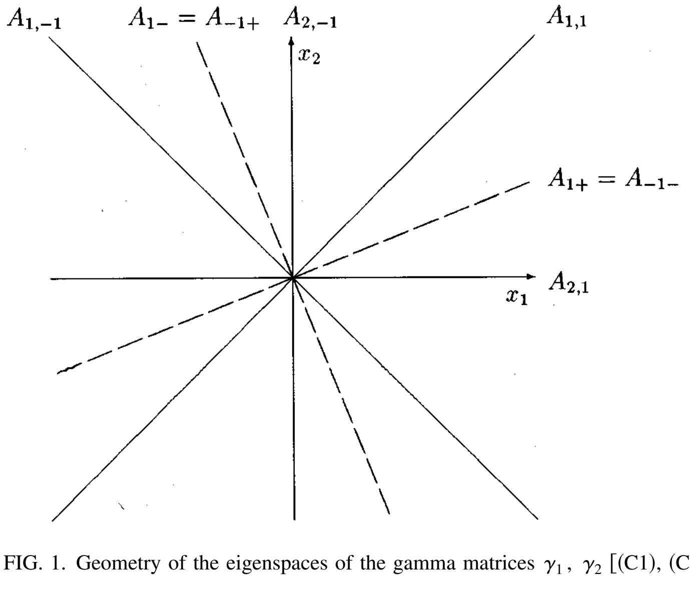

FIG. 1. Geometry of the eigenspaces of the gamma matrices γ_1, γ_2 [(C1), (C2)].

$$w_{\pm}=\frac{\lambda^2}{1+\lambda^2}. \tag{C18}$$

[Equations (C15), (C16) [(C17), (C18)] have been derived using Eq. (C11) [(C10)].] The solution of the above equations reads for $\rho=1$,

$$\lambda'=\lambda_{\pm}, \tag{C19}$$

and for $\rho=-1$,

$$\lambda=-\lambda_{\pm}. \tag{C20}$$

Here,

$$\lambda_{\pm}=\pm 2\sqrt{2}w_{\pm}, \tag{C21}$$

which entails

$$\lambda_{\pm}\lambda_{\mp}=-1. \tag{C22}$$

Inserting Eqs. (C20) and (C22) into Eqs. (C11) and (C10), respectively, one finds

$$\mathbf{A}_{1\pm}=\frac{1}{\sqrt{1+\lambda_{\pm}^2}}\begin{pmatrix}\lambda_{\pm}\\ 1\end{pmatrix}, \tag{C23}$$

$$\mathbf{A}_{-1\pm}=\frac{1}{\sqrt{1+\lambda_{\pm}^2}}\begin{pmatrix}1\\ -\lambda_{\pm}\end{pmatrix} \tag{C24}$$

(cf. Fig. 1; it holds $A_{1+}=A_{-1-}$, $A_{1-}=A_{-1+}$). The orthogonal transformation leading to the new coordinate system consequently reads

$$O_{\pm}=\frac{1}{\sqrt{1+\lambda_{\pm}^2}}\begin{pmatrix}\lambda_{\pm} & 1\\ 1 & -\lambda_{\pm}\end{pmatrix}. \tag{C25}$$

This way the following final result is obtained:

$$\gamma'_{1\pm}=\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \tag{C26}$$

$$\gamma'_{2\pm}=\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}, \tag{C27}$$

$$\gamma'_{3}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{C28}$$

It is clear that in the present case there is no residual continuous symmetry which has been exploited in Sec. V of the main part of the paper which is dealing with the real Clifford algebra $C(3,1)$.

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[67] This difference between the Clifford subalgebras $C(3,0)$ and $C(0,2)$ is caused by the fact that during a full 2π turn around the axis $(1,1,1)$ in the vector space $\mathbf{R}_{3,0}$ associated with $C(3,0)$ the rectangular coordinate system is mapped 3 times onto itself while during a full 2π turn in the vector space $\mathbf{R}_{0,2}$ associated with $C(0,2)$ the rectangular coordinate system is mapped 4 times onto itself (allowing reflections). Taking into account the different number of generators of $C(3,0)$ and $C(0,2)$, 3 and 2, respectively, this leads to a natural explanation for the difference.

[68] A related discussion can be found in Ref. 69, Vol. 2, *Mathematische Zusätze und Ergänzungen [Mathematical Supplements]*, no. 13: Zwei- und vierreihige Matrizen. Darstellung der hyperkomplexen γ-Einheiten durch Matrizen. Zu Kap. IV, § 5 [Two- and four-row matrices. Representation of the hypercomplex γ units by matrices. To Chap. IV, § 5], p. 780 of the cited German edition, also earlier editions of Ref. 69, Vol. 2, contain this material in some appendix.

[69] A. Sommerfeld, *Atombau und Spektrallinien*, 3rd ed. (Vieweg, Braunschweig, 1951); reprint (Verlag Harri Deutsch, Thun, 1978); English translation of an earlier edition: *Atomic Structure and Spectral Lines* (Methuen, London, 1934) (Dutton, New York, 1934), 3rd rev. ed.