Transverse-momentum distributions and beyond: setting up the nucleon tomography

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In these lecture notes I describe a diagrammatic approach in high energy scattering processeses. Using in particular production processes initiated by a lepton-hadron or a hadron-hadron intitial state we identify the correlators that describe the *partons in the hadrons*. In this way one can generalize more rigorous approaches such as the operator product expansion techniques. Generalizations include the treatment of transverse momenta of partons. The latter allows a general treatment that includes all possible correlations between momenta and spins of partons and parent hadrons both in polarized and unpolarized cases. The effects of transverse momenta show up as azimuthal asymmetries in the inclusive production of jets or specific hadrons. Although correlators describe in general squared amplitudes, links can be made to amplitudes in other processes. Examples are form factors and generalized parton distributions. One can also look at extensions to multi-parton scattering phenomena. The parametrization in terms of *universal functions*, such as distribution and fragmentation functions are useful to optimally profit from the kinematic and spin-related degrees of freedom in high-energy processes but the correlators actually also encode interesting hadronic structure that can be studied in lattice approaches or specific models for hadron structure.

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Contents

I. Tomography of Composite Systems 3
A. Introduction 3
B. Quantum Mechanics 3
C. Quantum Field Theory 5
D. Appendix: kinematics, frames, phase space, ... 7

II. High energy processes with hadrons 10
A. Three basic processes 10
B. Cross sections for (semi-)inclusive leptoproduction 11
C. The cross section for the Drell-Yan process 12
D. Cross sections in terms of form factors and structure functions 13
Virtual photon cross sections 15
E. Form factors 16
F. Structure functions for 1PI leptoproduction 17

III. Parton correlators in hadrons 19
A. Deep inelastic scattering 19
The operator in coordinate space 21
Interpretation as densities 21
Relation to forward amplitudes 21
B. Distribution and fragmentation correlators in 1PI leptoproduction 21
C. Inclusion of spin in distribution and fragmentation correlators 24
D. Tree-level treatment of Drell-Yan scattering 25
E. Multi-parton distribution functions 27

IV. Properties and parametrizations of the correlation functions 30
A. Collinear parton distributions 30
B. Bounds on the distribution functions 30
C. Transverse momentum dependent correlation functions 32
D. Fragmentation functions 33
E. Examples of azimuthal asymmetries 33
F. Inclusion of subleading contributions 35
G. Working in impact parameter space 35

V. Color gauge invariance 35
A. Wilson lines in correlators 35
Gluonic pole matrix elements 37

VI. Gluon TMDs 39

VII. Miscellaneous material 41
A. Inclusion of color 41
Regions of importance in parton kinematics 42
B. Distribution correlators in high-energy processes 43
Twist expansion 43
Color gauge invariance 44
Sum rules 45
C. Fragmentation correlators in high-energy processes 45
Sum rules for fragmentation functions 46
D. Large transverse momenta 47
Appendix A: The gauge link 49
Appendix B: Calculation of gauge link 54

Acknowledgments 62

References 62

Version with exercises and answers

I. TOMOGRAPHY OF COMPOSITE SYSTEMS

A. Introduction

In these lectures the focus is on the proton being the groundstate of Quantum Chromodynamics (QCD) in the baryon number $B = 1$ sector. In particular two aspects will be considered, the first being the proton's structure in terms of the quark and gluon constituents, the second the proton's role as a basic tool in experiments that probe the fundamental forces in nature. Besides the global properties like energy, momentum and spin we then want to understand and employ details such as spatial distribution of charge or momentum distribution of quarks and gluons (then often referred to as partons). It is not at all trivial, however, to talk about spatial distributions or wave functions for a relativistic composite system. Already at the level of atomic nuclei there remain many open issues in the proper treatment of nucleons as constituents of nuclei. Part of the problems in those treatments of course could be attributed to the absence of a complete microscopic theory for the interactions between nucleons. Even if QCD provides such a microscopic theory at the level of quarks and gluons in the nucleons, other problems come in. Perturbative techniques in QCD cannot be used to understand the proton's structure and the quark and gluon fields in the theory do not correspond to asymptotically available states. In that sense the situation is very different for protons and electrons. For the lattere the quantum state, $|\text{electron}\rangle = |m_e, \boldsymbol{p}; s, m_s; \ldots\rangle = b_e^\dagger(\boldsymbol{p}, m_s)|0\rangle$, is created from the vacuum by a creation operator, satisfying elementary (anti-commutation) relations. Properties remaining in the $\ldots$ are charges determining the coupling of the electron to electroweak gauge bosons. Creation and annihilation operators appear in the free field expansion of the electron field. They are multiplied with plane-wave solutions including a spinor structure in Dirac space. Using these fields a nice description of electrons and their interactions through the coupling to photons is achieved within the framework of the U(1) gauge theory of Quantum Electrodynamics and its embedding in the larger framework of the Standard Model of Particle Physics, even if some calculations are cumbersome; think of $g - 2$. In comparison with the electron the proton's quantum state, $|\text{proton}\rangle = |M, \boldsymbol{P}; S, M_s; \ldots\rangle$, has lots of information hidden in the $\ldots$ that can be accessed in a variety of scattering processes, but which requires completely different ab initio calculational methods, such as lattice gauge theories. Nevertheless one likes to identify quantities that can be used and stored; think of quantities like magnetic moments, form factors (FFs) and parton momentum distribution functions (PDFs). In particular the latter quantities have in recent years been complemented with many novel observables like transverse momentum dependent (TMD) distribution functions, in short often referred to as TMDs, Generalized Parton Distributions (GPDs) and multi-parton distributions. These observables are part of the modern tomography of the nucleon, providing structural information and basic tools for scattering processes involving nucleons.

B. Quantum Mechanics

In order to illustrate some basics we start with simple wave function for a constituent in a particular 'orbit' n given by $\phi_n(x)$ or its equivalent $\phi_n(p)$ in momentum representation.

$$\phi_n(p) = \int dx\; e^{-i\,px}\,\phi_n(x), \tag{1}$$

$$\phi_n(x) = \int \frac{dp}{2\pi}\; e^{i\,px}\,\phi_n(\boldsymbol{p}). \tag{2}$$

To look at the various quantities that could be accessed as 'observables', we will continue with just writing down everything as 1-dimensional positions and momenta. A composite system may be probed in an electromagnetic process, where the kinematic essential is the absorption of a momentum q transferred to the system without breaking up the system. The interaction is described with the (charge) operator

$$\hat{Q}(q) = \sum_n \int \frac{dp_n}{2\pi}\; |p_n + \tfrac{1}{2}q\rangle\, e_n\, \langle p_n - \tfrac{1}{2}q| = \sum_n \int dx_n\; |x_n\rangle\, e_n\, e^{iqx_n}\, \langle x_n|. \tag{3}$$

One measures an elastic response $\sigma(q) \propto |G(q)|^2$, where $F(q)$ is the expectation value of the operator $\hat{Q}(q)$, known as the *form factor*, $F(q) = \sum_n e_n\, F_n(q)$

$$F_n(q) = \langle\phi_n|\hat{Q}(q)|\phi_n\rangle = \int \frac{dp}{2\pi}\; \phi_i^*(p + \tfrac{1}{2}q)\,\phi_n(p - \tfrac{1}{2}q) = \int dx\; e^{iqx}\,|\phi_i(x)|^2, \tag{4}$$

which is the Fourier transform of the charge density $\rho_n(x) = |\phi_n(x)|^2$ and satisfies $F_n(0) = 1$. The form factor at $q = 0$ is just a number, in this case the total charge of the system $Q = \sum e_n$.

Since the probing operator is a single-particle operator, it is sufficient to work with the single-particle wave functions. If the constituents are not elementary, one just replaces e_n by a constituent form factor $F_n^{(c)}(q)$ with $F_n^{(c)}(0) = e_n$. This form factor reflects the internal charge distribution. The total form factor is then a product of body form factor and constituent form factor(s); examples are nuclear form factors where nucleons are constituents.

Other ways to study the system, in particular 'at higher energies' is to knock out a particle from the system, such as for a nucleus $\gamma^*(q) + \mathrm{A} \to \mathrm{N}(k) + \text{remnant}$. The absorbed momentum q produces a free (plane-wave) final state $|k\rangle$. The inelastic response is now $\sigma(q) \propto W(q) = |\langle k\, X|\hat{Q}(q)|\Psi\rangle|^2 = |\phi_n(k-q)|^2$, which reflects the momentum distribution $f_n(p) = |\phi_n(p)|^2/2\pi$ at $p = k-q$. Interchanging the roles of x and p in the above equation, this momentum distribution can be written as

$$f_n(p) = \frac{1}{2\pi}\,|\phi_n(p)|^2 = \int \frac{dx\,db}{2\pi}\; e^{ipb}\,\phi_n^*(x+\tfrac{1}{2}b)\,\phi_n(x-\tfrac{1}{2}b), \tag{5}$$

satisfying $\int dp\, f_n(p) = 1$. It is important to note that the form factors or space densities are non-local in momentum space, while the momentum distribution or momentum desities are non-local in coordinate space. Furthermore a form factor appears as an amplitude, a momentum distribution as a cross section.

A convenient way to incorporate both is the concept of quantum phase-space (Wigner) distributions. For a nice introduction in the field of hadron physics we refer to Ref. [10]. The Wigner functions in quantum mechanics are defined as 'very general amplitudes',

$$W(x,p) \;=\; \int db\; e^{ipb}\,\phi^*(x+\tfrac{1}{2}b)\,\phi(x-\tfrac{1}{2}b) \tag{6}$$

$$\;=\; \int \frac{dq}{2\pi}\; e^{-iqx}\,\phi^*(p+\tfrac{1}{2}q)\,\phi(p-\tfrac{1}{2}q). \tag{7}$$

The observable (real and positive definite) densities discussed above are obtained after integration over positions or momenta, respectively

$$f(p) \;=\; \frac{1}{2\pi}\,\phi^*(p)\,\phi(p) = \int \frac{dx}{2\pi}\; W(x,p), \tag{8}$$

$$\rho(x) \;=\; \phi^*(x)\,\phi(x) = \int \frac{dp}{2\pi}\; W(x,p), \tag{9}$$

with the common norm $\int dxdp/2\pi\; W(x,p) = 1$. The Wigner distributions, generalized to appropriate 3-dimensional variables, are useful tools in analyzing the 3-dimensional structure of nucleons. In exclusive processes other combinations may play a role, for instance the socalled *generalized parton distributions* (GPDs).

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Exercise: Show that the Fourier transform of $f(p)$ can be written as

$$\mathscr{F}(b) = \int dp\; e^{ipb}\, f(p) = \int \frac{dx}{2\pi}\; \phi^*(x+\tfrac{1}{2}b)\phi(x-\tfrac{1}{2}b) \tag{10}$$

Construct Fourier transforms in $x \leftrightarrow q$ and $p \leftrightarrow b$ for $W(x,p)$ and show that their structure is that of $\phi^*(\ldots)\phi(\ldots)$. Express $\mathscr{F}(b)$ and $F(q)$ as integrals over $\mathscr{W}(x,b)$ and $\widetilde{\mathscr{W}}(q,p)$.

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Answer: Finding the Fourier transform of $f(p)$ in impact space is straightforward. In order to find the expressions for $F(q)$ and $F(b)$ in terms of the Wigner functions, we note that Fourier transforms $W(q,p)$ and $W(x,b)$ can be defined as

$$\widetilde{\mathscr{W}}(q,p) = \int \frac{dx}{2\pi}\; e^{iqx}\, W(x,p) = \phi^*(p+\tfrac{1}{2}q)\phi(p-\tfrac{1}{2}q),$$

$$\mathscr{W}(x,b) = \int dp\; e^{ipb}\, W(x,p) = \phi^*(x+\tfrac{1}{2}b)\phi(x-\tfrac{1}{2}b),$$

and they satisfy

$$F(q) = \int dp\; \widetilde{\mathscr{W}}(q,p) \quad \text{with} \quad F(0) = 1,$$

$$\mathscr{F}(b) = \int \frac{dx}{2\pi}\; \mathscr{W}(x,b) \quad \text{with} \quad \mathscr{F}(0) = 1.$$

Furthermore $\rho(x) = \mathscr{W}(x,0)$ and $f(p) = \widetilde{\mathscr{W}}(0,p)$.

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Exercise: Take a (normalized) Gaussian wave function,

$$\phi(x) = (1/\pi a^2)^{1/4}\, e^{-x^2/2a^2} \quad \text{with} \quad \langle x^2\rangle = a^2/2,$$

and calculate $\phi(p)$, $\rho(x)$, $f(p)$, $W(x,p)$, $F(q)$ and also the impact parameter space representation of the momentum distribution $\mathscr{F}(b) \equiv \int dp\; e^{ipb}\, f(p)$.

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Answer: The momentum space wave function is given by:

$$\phi(p) = (4\pi\, a^2)^{1/4}\, e^{-a^2p^2/2} \quad \text{with} \quad \langle p^2\rangle = 1/2a^2.$$

We have

$$\rho(x) = (1/\pi a^2)^{1/2}\, e^{-x^2/a^2} \quad \text{and} \quad F(q) = e^{-a^2q^2/4},$$
$$f(p) = (a^2/\pi)^{1/2}\, e^{-a^2p^2} \quad \text{and} \quad \mathscr{F}(b) = e^{-b^2/4a^2}.$$

The result for $W(x,p)$ is

$$W(x,p) = \left(\frac{8}{\pi a^2}\right)^{1/4} e^{-x^2/a^2}\, e^{-a^2p^2}.$$

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Note that a 'single particle' wave functions needs in a real calculation some extra explanation, even in the case that a composite system $|\Psi\rangle$ would be a product wave function of single particle wave functions (if necessary appropriately symmetrized) such as the waves $\phi_n(x)$ or $\phi_n(p)$ in momentum representation used above. More formal and general, one works with a spectral function depending on the energy and momentum of the relevant constituent particle with mass m (say an electron in an atom or a nucleon in a nucleus). It involves

$$(2\pi)^3\, \delta^3(\boldsymbol{p}-\boldsymbol{p}')\, S(E_m;\boldsymbol{p}) = \frac{1}{(2\pi)^3} \sum_X \int \frac{d^3P_X}{(2\pi)^3} \langle M;\boldsymbol{P}|a^\dagger(\boldsymbol{p}')\,|M_X;\boldsymbol{P}_X\rangle\langle M_X;\boldsymbol{P}_X|a(\boldsymbol{p})|M;\boldsymbol{P}\rangle, \tag{11}$$

with plane wave annihilation and creation operators,

$$a^\dagger(\boldsymbol{p})|0\rangle = |\boldsymbol{p}\rangle \quad \text{with} \quad [a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}'), \tag{12}$$

taking care of the appropriate normalization in coordinate and momentum representation. The missing energy is actually not an energy but stands for $E_m = M_X + m - M$ and fixes the mass of the remnant after *knocking out* a constituent. It is actually related to the constituents energy corrected for the recoil of the remnant (see below for some kinematic details). Playing a bit with a product wave function ansatz shows that $S(E_m;\boldsymbol{p})$ represents the momentum space wave function squared $|\phi_n(\boldsymbol{p})|^2/(2\pi)^3$ of the knocked out constituent with n being the specific orbit identified from the missing energy. In this way, one can associate it with the 'single particle' wave function and its Fourier transform, the coordinate space 'single particle' wave function via

$$(2\pi)^3\, \delta^3(\boldsymbol{P}_X + \boldsymbol{p} - \boldsymbol{P})\, \phi_n(\boldsymbol{p}) = \langle M_X;\boldsymbol{P}_X|a(\boldsymbol{p})|M;\boldsymbol{P}\rangle, \tag{13}$$

which together with its Fourier transform $\phi_n(\boldsymbol{r})$ can be used as discussed schematically above.

C. Quantum Field Theory

In field theory, the (complex-valued) wave functions show up as the space-time dependent coefficients in a field expansion of the quantum fields for scalar (spin 0) particles, $\phi(x)$, fermions $\psi(x)$ or vector/gauge fields $A_\mu(x)$, e.g. for fermions the Dirac field

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3\, 2E_k} \left(u(k,s)e^{-ik\cdot x}\, b^\dagger(k,s) + v(k,s)e^{ik\cdot x}\, d^\dagger(k,s)\right). \tag{14}$$

This is a free field expansions in which creation and annihilation operators multiply *wave functions*, which are plane waves multiplied with Dirac spinors,

$$\psi_{p,s}(x) = u(p,s)e^{-ip\cdot x} = \langle 0|\psi(x)|p,s\rangle = \langle 0|\psi(x)b^\dagger(p,s)|0\rangle, \tag{15}$$

with $p^0 = +E_p = +\sqrt{\boldsymbol{p}^2+m^2}$. The fields satisfy equal time commutation or anti-commutation relations, which are equivalent to the commutation or anti-commutation relations of the creation and annihilation operators. In the case of scalar fields the wave functions are just plane waves, in the case of vector/gauge fields $A_\mu(x)$ the wave functions are multiplied with the polarization vectors $\epsilon_\mu(k,\lambda)$. For these lectures, familiarity with basic field theoretical methods is assumed.

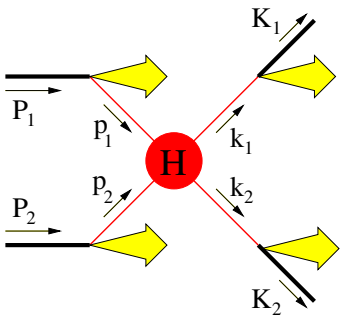


FIG. 1: Schematic illustration of the contribution of a hard subprocess, parton (p_1) + parton (p_2) $\to$ parton (k_1) + parton (k_2), to the (2-particle inclusive) scattering process hadron (P_1) + hadron (P_2) $\to$ hadron (K_1) + hadron (K_2) + X, at the level of the amplitude. The process being hard implies for the hadronic momenta $s = (P_1+P_2)^2 \approx 2P_1\cdot P_2 \gg M^2$ and is of the same order as other invariants involving different hadrons (at least if they belong to different jets) such as $t_i \approx 2K_i\cdot P_1$. The momenta of the outgoing partons are basically the 'jet' momenta. Also the partonic momenta in the subprocess involve invariants like $\hat{s} = (p_1+p_2)^2 \approx 2p_1\cdot p_2 \gg M^2$ and $\hat{t} = (k_1-p_1)^2 \approx 2k_1\cdot p_1$, etc. Clearly the picture suggests a *factorization* of the process, which will be discussed later.

In doing Feynman calculations in quantum electrodynamics (QED) one needs these wave functions to account for the particles in initial and final states, employing well-known relations like $\sum_s u(k,s)\overline{u}(k,s) = \not{k}+m$, etc. The basic framework for the strong interactions is Quantum Chromodynamics (QCD), but the basic constituents, quarks and gluons, are confined into hadrons. Hadrons, however, are not simply created via the quark and gluon operators in QCD. In fact the matrix element in Eq. 15 needs to be replaced,

$$u(p,s)\,e^{-ip\cdot x} \Longrightarrow \langle P_X|\psi(x)|P\rangle = \langle P_X|\psi(0)|P\rangle\, e^{-i(P-P_X)\cdot x}. \tag{16}$$

and (including Dirac indices for the fields and summation and integration over remnant X)

$$\begin{aligned}\sum_s u_i(k,s)\overline{u}_j(k,s) \Longrightarrow \Phi_{ij}(p;P) &= \sum_X \int \frac{d^3P_X}{(2\pi)^3\,2E_X}\,\langle P|\overline{\psi}_j(0)|X\rangle\,\langle X|\psi_i(0)|P\rangle\,\delta^4(p+P_X-P)\\ &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle.\end{aligned} \tag{17}$$

Details on spin and color (a summation over color is understood) will be discussed later. First we focus on momentum and spatial structure. Taking for $\psi(x)$ a plane wave expansion further illustrates the similarity to the quantum mechanical matrix elements used in Eq. 11. Taking the Fourier transform of $\psi(x)$ and defining momentum space fields, $\psi(p) \equiv \int d^4x\; e^{i\,p\cdot x}\,\psi(x)$ we have

$$\psi(p) = \sum_s \Big(u(\boldsymbol{p},s)\,b(\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(p^0) + v(-\boldsymbol{p},s)\,d^\dagger(-\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(-p^0)\Big). \tag{18}$$

We have for their matrix elements

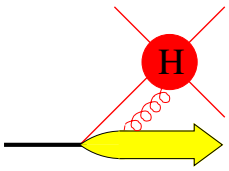
$$\langle X|\psi(x)|P\rangle = \sum_s \int \frac{d^3k}{(2\pi)^3\,2E_k}\langle P_X|b(\boldsymbol{k},s)|P\rangle\,u(\boldsymbol{k},s)\,e^{-ik\cdot x} + \langle P_X|d^\dagger(\boldsymbol{k},s)|P\rangle\,v(\boldsymbol{k},s)\,e^{ik\cdot x}, \tag{19}$$

$$\begin{aligned}\langle X|\psi(p)|P\rangle &= \sum_s \langle P_X|b(\boldsymbol{p},s)|P\rangle\,u(\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(p^0)\\ &\quad + \langle P_X|d^\dagger(-\boldsymbol{p},s)|P\rangle\,v(-\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(-p^0).\end{aligned} \tag{20}$$

In this matrix element $P = p + P_X$ and the first term contributes when $P^0 > P_X^0$ (emission of a quark), the second term contributes when $P_X^0 > P^0$ (absorption of an antiquark). Note that in front form one uses lightcone coordinates with integrations over p^+ and $\boldsymbol{p}_T$ (see section I D). It has the advantage that signs of p^+ and p^- are the same. For the correlator we now get

$$(2\pi)^4\delta^4(p-p')\,\Phi_{ij}(p;P) = \frac{1}{(2\pi)^4}\,\langle P|\overline{\psi}_j(p')\,\psi_i(p)|P\rangle. \tag{21}$$

Of course, the treatment assumes inclusive processes, in which (colored) remnants collinear with the incoming hadron (along a lightlike direction) are integrated over using completeness for the relevant sector.



In a process involving a composite hadronic state $|P\rangle$, contractions with one or several of the quark and gluon fields may be involved, leading not only to nonzero matrix elements for a quark between the hadron state and a remainder, but also to nonzero matrix elements involving multi-parton field combinations, like the quark-gluon combination shown here.

$$\langle X|\psi_i(\xi)|P\rangle,\ \langle X|A^\mu(\eta)\,\psi(\xi)|P\rangle, \ldots\ .$$

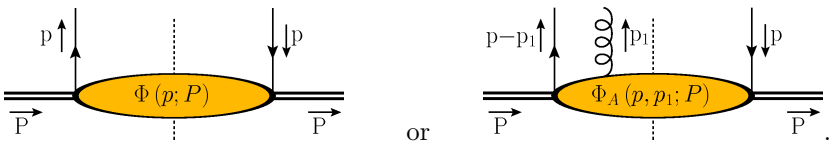
As a consequence one must include additional correlators, such as quark-gluon-quark correlators involving matrix elements of the form

$$\Phi^\mu_{A\,ij}(p,p_1;P) = \frac{1}{(2\pi)^8}\int d^4\xi\, d^4\eta\; e^{i\,(p-p_1)\cdot\xi}\; e^{i\,p_1\cdot\eta}\;\langle P|\overline{\psi}_j(0)\,A^\mu(\eta)\,\psi_i(\xi)|P\rangle, \tag{22}$$

or with momentum space operators (for gluons $A_\mu(p) \equiv \int d^4x\; e^{i\,p\cdot x}\,A_\mu(x)$) one has

$$(2\pi)^4\delta^4(p-p')\,\Phi^\mu_{A\,ij}(p,p_1;P) = \frac{1}{(2\pi)^8}\,\langle P|\overline{\psi}_j(p')\,A^\mu(p_1)\,\psi_i(p-p_1)|P\rangle. \tag{23}$$

Pictorially one has for the correlators,



In principle all of such correlators need to be included or, as we will see, we will need to resum some of them into a single correlator (this will be needed for instance to ensure color gauge invariance). To calculate the correlators ultimately requires nonperturbative methods in QCD. For instance all of the matrix elements involved are 'untruncated' as seen e.g. from the free field matrix element which includes the external propagator,

$$\langle 0|\psi_i(\xi)|p,s\rangle\,\theta(t) = \theta(t)\int\frac{d^4k}{(2\pi)^4}\;e^{-ik\cdot\xi}\;\frac{i(\not k+m)}{k^2-m^2+i\epsilon}\;\frac{u_i(\boldsymbol{p},s)}{2m}\,(2\pi)^3\,2E_p\,\delta^3(\boldsymbol{k}-\boldsymbol{p}). \tag{24}$$

But the full matrix elements involving states $|P\rangle$ will no longer exhibit poles corresponding to free quarks. Nevertheless, in general the correlators remain quite problematic quantities. For example, they are by themselves not even color gauge-invariant, an issue to be discussed below. We will later also discuss similar correlators for final state hadrons. When more hadrons are involved, one needs to consider two-hadron correlators, involving two-hadron states (or correlators involving hadronic states in initial and final state), etc. In multi-parton interactions one needs correlators for two 'independent' quarks. We'll try to get some feeling for these objects in these lectures.

D. Appendix: kinematics, frames, phase space, ...

Four vectors, such as x or p, generally are denoted by giving their coordinates p^μ. In fact this is an expansion in a basis, $p = p^\mu n_\mu$, where $n_\mu{\cdot}n_\nu = g_{\mu\nu}$ determines the metric. Working in Minkowski space, we will use either a basis with one time-like basisvector and three orthogonal space-like basisvectors with nonzero values for $g_{00} = -g_{ii} = 1$. or instead of n_0 and n_3 two light-like basisvectors $n_\pm = (n_0 \pm n_3)/\sqrt{2}$ with nonzero values for $g_{+-} = g_{-+} = 1$, thus we have $n_+{\cdot}n_- = 1$ and $n_+^2 = n_-^2 = 0$. The covariant coordinates are $p_\mu = p{\cdot}n_\mu = g_{\mu\nu}p^\nu$, with $p_\mp = p^\pm$. We denote the coordinates,

$$p = (p^0,p^1,p^2,p^3) = (p^0,\boldsymbol{p}), \quad \text{or} \quad p = [p^-,p^+,p^1,p^2] = [p^-,p^+,\boldsymbol{p}_T], \tag{25}$$

with $p^2 = p{\cdot}p = (p^0)^2 - \boldsymbol{p}^2 = 2p^-p^+ - \boldsymbol{p}_T^2$. We will also use the fourvector $p_T = [0,0,\boldsymbol{p}_T]$ with $p_T^2 = -\boldsymbol{p}_T^2$. The symmetric and antisymmetric 'transverse' projectors are defined as

$$g_T^{\mu\nu} = g^{\mu\nu} - n_0^\mu n_0^\nu + n_3^\mu n_3^\nu = g^{\mu\nu} - n_+^{\{\mu}n_-^{\nu\}} \tag{26}$$

$$\epsilon_T^{\mu\nu} = \epsilon^{n_0n_3\mu\nu} = \epsilon^{03\mu\nu} = \epsilon^{n_+n_-\mu\nu} = \epsilon^{-+\mu\nu}, \tag{27}$$

where $\epsilon^{0123} = \epsilon^{-+12} = 1$. In a given process, it is often convenient to use external vectors as the ones defining our basis vectors, e.g. the proton momentum P as the time-like vector $n_0 \equiv P/M$. In high energy processes where the proton mass squared is negligible, it is more convenient to use the proton as an approximately light-like vector. While a boost in the z-direction mixes the coordinates p^0 and p^3, it is actually very simple in lightcone coordinates, namely a scaling $p^+ \to \alpha p^+$ and $p^- \to p^-/\alpha$.

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Exercise: For an on-shell particle one has $p^2 = m^2$ and

$$p = p^+ n_+ + p^- n_- + \boldsymbol{p}_T \quad \text{with} \quad p^- = (m^2 + \boldsymbol{p}_T^2)/2p^+,$$
$$p = p^0 n_0 + \boldsymbol{p} \quad \text{with} \quad p^0 = \pm\sqrt{\boldsymbol{p}^2 + m^2}.$$

As mentioned, a boost corresponds to rescaling plus and minus components, $a^+ \to \alpha\, a^+$ and $a^- \to a^-/\alpha$. It corresponds to the boost generated by M^{-+}. It leaves the transverse momenta unchanged. Another very useful Lorentz transformation is the one that leaves a^+ invariant, using parameters b^+ and b_T acting as

$$\Big[a^-, a^+, \boldsymbol{a}_T\Big] \;\longrightarrow\; \Big[a^- - \frac{\boldsymbol{a}_T\cdot\boldsymbol{b}_T}{b^+} + \frac{\boldsymbol{b}_T^2\, a^+}{2(b^+)^2}, a^+, \boldsymbol{a}_T - \frac{a^+}{b^+}\,\boldsymbol{a}_T\Big]. \tag{28}$$

Show that this transformation, indeed, does not change a^2. Similarly there is a transformation that leaves a^+ invariant. Show that a (rescaling) boost with parameter $\alpha = p^+\sqrt{2}/M$ brings the rest-frame vector $p = [m/\sqrt{2}, m/\sqrt{2}, \boldsymbol{0}_T]$ into $[p^-, p^+, \boldsymbol{0}_T]$ with $p^- = m^2/2p^+$. Then apply a transformation that leaves a^+ invariant with $b^+ = a^+$ and $\boldsymbol{b}_T = \boldsymbol{p}_T$ to get the lightcone components for an on-shell particle.

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Exercise: Show that for an off-shell momentum $p = P - P_X$ with on-shell momenta $P^2 = M^2$ and $P_X^2 = M_X^2$ one can write in terms of light-cone components

$$P = P^+ n_+ + (M^2/2P^+) n_- \quad \text{and} \quad p = xP^+ n_+ + p^-\, n_- + \boldsymbol{p}_T$$

(x is the light-cone momentum fraction) with

$$2P^+p^- = 2\,p\cdot P - x\,M^2 = p^2 - M_X^2 + (1-x)M^2 \tag{29}$$
$$(1-x)p^2 = x(1-x)\,M^2 - x\,M_X^2 - \boldsymbol{p}_T^2, \tag{30}$$

which is useful for applications to quarks in nucleons (note that we did on purpose not put $p^2 = m^2$). In terms of the usual four vector components one can write

$$P = M n_0 \quad \text{and} \quad p = E n_0 + \boldsymbol{p}$$

with

$$M\,E = p\cdot P = (M^2 + m^2 - M_X^2)/2 \approx Mm - M_X B \tag{31}$$
$$p^2 = (M^2 + M_X^2) - 2M\sqrt{\boldsymbol{p}^2 + M_X^2} \approx m^2 - 2mB - (M/M_X)\,\boldsymbol{p}^2. \tag{32}$$

where we have shown the results also for small binding energies, i.e. for $M_X = M - m + B$ with $B \ll \{m, M, M_X\}$, e.g. applicable to electrons in atoms or nucleons in nuclei.

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The well-known measure or phase space for momentum states (with $p^2 = m^2$) and positive energy ($p^0 > 0$ or $p^+ > 0$) is given by

$$\int d\tilde{p} \;=\; \int \frac{d^4p}{(2\pi)^4}\,(2\pi)\,\delta(p^2 - m^2)\,\theta(p^0) = \int \frac{d^4p}{(2\pi)^4}\,(2\pi)\,\delta(p^2 - m^2)\,\theta(p^+) \tag{33}$$
$$= \int \frac{d^3p}{(2\pi)^3\,2p^0} = \int \frac{dp^0\,d^2p_T}{(2\pi)^3\,2p^3} = \int \frac{dp^+\,d^2p_T}{(2\pi)^3\,2p^+} = \int \frac{dp^-\,d^2p_T}{(2\pi)^3\,2p^-}. \tag{34}$$

In high energy (hard) scattering processes, usually many particles are produced. In inclusive measurements no particles are detected in the final state, in exclusive measurements all particles are detected. Consider the 1-particle inclusive case, in which one particle is detected, $H_1 + H_2 \to h + X$. At high energies, there is usually a preferred direction, for

instance the momenta of incoming (colliding) hadrons and it is useful to use for the produced particle rapidity as a variable. Writing

$$E = m_T \cosh y, \quad p_z = m_T \sinh y, \quad p_T = (p_x, p_y), \tag{35}$$

with $m_T^2 = m^2 - p_T^2 = m^2 + p_x^2 + p_y^2$, the rapidity y is defined

$$y = \frac{1}{2} \ln\left(\frac{E+p_z}{E-p_z}\right) = \frac{1}{2} \ln\left(\frac{p^+}{p^-}\right) = \ln\left(\frac{E+p_z}{m_T}\right) = \tanh^{-1}\left(\frac{p_z}{E}\right).$$

It is convenient because under a boost (along z) with velocity β it changes as $y \to y - \tanh^{-1}\beta$, which means that rapidity distributions dN/dy maintain their shape. For large energies and not too small scattering angles ($\theta \gg 1/\gamma$) y is approximately equal to the pseudo-rapidity η,

$$\eta = -\ln(\tan(\theta/2)) = \tanh^{-1}(\cos\theta), \tag{36}$$

which (only involving angles) is easier to determine. The one-particle phase space in terms of these variables becomes

$$d\tilde{p} = \frac{dy\, d^2p_T}{2\,(2\pi)^3} = \frac{dy\, d|p_T|^2}{16\pi^2}\frac{d\phi}{2\pi} = \frac{|\boldsymbol{p}|}{E}\frac{d\eta\, d|p_T|^2}{16\pi^2}\frac{d\phi}{2\pi}, \tag{37}$$

with at high energies the factor $|\boldsymbol{p}|/E \simeq 1$.

==========

Exercise: The phenomenon of scaling occurs when a constituent with an energy-momentum distribution $F(p)$ is put on-shell after absorption of a momentum q while there is no further dependence on the constituents momentum
Show that for nucleons in a nucleus with approximation that $\boldsymbol{p}^2 \ll m_N^2$ and $E \approx m_N$ one gets y-scaling (don't confuse this y with the rapidity above),

$$\int d\tilde{p}\; F_{N/A}(p)\, \delta\left((p+q)^2 - m_N^2\right) \ldots \;\propto\; \int dy\; f_{N/A}(y) \ldots, \tag{38}$$

where

$$y = \frac{2m_N\nu - Q^2}{|\boldsymbol{q}|},$$

is a function of the energy transfer ν and the momentum transfer $\boldsymbol{q}$ (or $Q^2 = \boldsymbol{q}^2 - \nu^2$). The function $f_{N/A}$ represent the p_z nucleon momentum distribution along the direction of $\boldsymbol{q}$, i.e. $f_{N/A}(p_z)$. Take q to be $(q^0, \boldsymbol{q}_\perp, q^3) = (\nu, \boldsymbol{0}_\perp, |\boldsymbol{q}|)$ in the nuclear rest-frame.
Show that for quarks in a nucleon with approximation that $m_N^2 \gg p^2 \approx m_q^2 \approx 0$, one gets x-scaling,

$$\int d\tilde{p}\; F_{q/N}(p)\, \delta\left((p+q)^2 - m_q^2\right) \ldots \;\propto\; \int dx\; f_{q/N}(x) \ldots, \tag{39}$$

where

$$x = \frac{Q^2}{2M_N\nu},$$

is a function of energy and momentum transfer, which in the rest-frame is equal to $\nu = P{\cdot}q/M_N$ and the momentum transfer $Q^2 - -q^2 = \boldsymbol{q}^2 - \nu^2$. The function $f_{q/N}$ represents the p^+ (fractional) lightcone momentum distribution, i.e. $f_{q/N}(p^+/P^+)$. Take q to be $[q^-, q^+, \boldsymbol{q}_\perp]$ with $q^- = Q^2/2q^+ \gg q^+$ (that is, as is convential in this situation, now taking q along minus z-direction).

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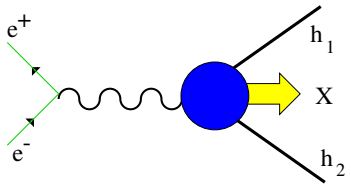
II. HIGH ENERGY PROCESSES WITH HADRONS

A. Three basic processes

Electroweak scattering processes are suitable to study properties of hadrons, The leptons, electroweak gauge bosons (photon, $W^{\pm}$, Z or Higgs particle) do not feel strong interactions. Characteristic processes are the following three types, the *annihilation* process, $\ell\bar{\ell} \to X$, the *lepton-hadron* scattering process, $\ell H \to \ell' X$, and the *lepton-pair production* process, $AB \to \ell\bar{\ell}X$ (Drell-Yan process) or generalizations thereof. The basic processes mentioned here involve electroweak currents, coupling to the leptons in a known way. In this way one easily recognizes the subprocesses, such as $\gamma^* \to q\bar{q}$, $\gamma^* + q \to q$ or $q + \bar{q} \to \gamma^*$. The advantage of electromagnetic processes further lies in the fact that the process is accurately described in terms of the exchange of one photon, as the coupling, $\alpha = e^2/4\pi \approx 1/137$ is weak. The same is true for the processes involving production or exchange of electroweak gauge bosons. Generalizations of these processes include subprocesses as gluon fusion into a Higgs particle, $g + g \to H$ or even hard subprocesses involving more than just one hard vertex, but a full hard subprocess, like $\gamma^* + q \to Z + q$ or $q + \bar{q} \to q + \bar{q}$, although these processes may depend on the recognizability of the final state involving jets originating from the produced quarks or gluons. At that point, it is of course often possible to look at production of heavy quark - antiquark ($Q\overline{Q}$) pairs because they have easily recognizable decay chains. The structure of hadrons in terms of quarks and gluons is the unknown part. The fact that the electroweak coupling to the quarks is known, however, enables the study of quarks and (usually indirectly) gluons.

First let us consider the variables that are used to describe these basic scattering processes.

(1) Electron-positron annihilation: $\ell\bar{\ell} \to X$ or $\ell\bar{\ell} \to h(P_h)X$ or $\ell\bar{\ell} \to h_1(P_1)h_2(P_2)X$



$$q^2 = (\ell + \ell')^2 = s \equiv Q^2 \geq 0 \tag{40}$$
$$2\,P_i \cdot q \equiv z_i\,Q^2 \tag{41}$$
$$P_1 \cdot \ell \equiv y\,P_1 \cdot q \tag{42}$$

In the case of production of hadrons with a timelike (virtual) photon one can consider the rest-frame of the virtual photon. This shows that Q is a measure of the excitation energy.

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Exercise: Show that $0 \leq z_i - M_i^2/Q^2 \leq 1$, thus in the limit that $Q^2 \to \infty$ one has $0 \leq z_i \leq 1$ and $P_1 \cdot \ell' = (1-y)P_1 \cdot q$.

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Answer: Use that $0 \leq (q - P_i)^2 \leq s = Q^2$.

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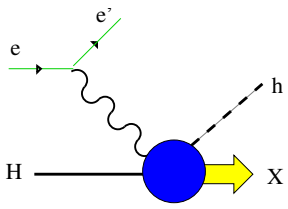
(2) Lepton-hadron scattering: $\ell H(P) \to \ell' X$ or $\ell H(P) \to \ell' h(P_h)X$



$$q^2 = (\ell - \ell')^2 \equiv -Q^2 \leq 0 \tag{43}$$
$$2P \cdot q \equiv 2M\,\nu \equiv \frac{Q^2}{x_B} \tag{44}$$
$$2P_h \cdot q \equiv -z_h\,Q^2 \tag{45}$$
$$P \cdot \ell \equiv \frac{P \cdot q}{y} = \frac{Q^2}{2x_B y} \tag{46}$$

The variable x_B is the Bjorken scaling variable. In this scattering process a hadron is probed with a spacelike (virtual) photon, for which one can consider a frame in which the momentum only has a spatial component, from which it is clear that the resolving power of the probing photon is of the order $\lambda \approx 1/Q$. Roughly spoken one probes a nucleus (1 - 10 fm) with $Q \approx 10 - 100$ MeV, baryon or meson structure (with sizes in the order of 1 fm) with $Q \approx 0.1 - 1$ MeV and one probes deep into the nucleon (< 0.1 fm) with $Q > 2$ GeV.

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Exercise: Rewrite the invariant mass squared of the hadronic final state, W^2, in terms of invariants and use that $W^2 \geq M^2$ to show that $0 \leq x_B \leq 1$, with $x_B = 1$ corresponding to elastic scattering.

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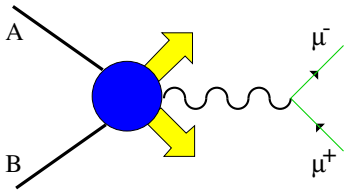
(3) Lepton-pair production or Drell-Yan scattering: $A(P_A)B(P_B) \to \ell\bar{\ell}X$



$$q^2 = (\ell + \ell')^2 \equiv Q^2 \geq 0 \tag{47}$$

$$2P_A \cdot q \equiv \frac{Q^2}{x_A} \quad \text{and} \quad 2P_B \cdot q \equiv \frac{Q^2}{x_B} \tag{48}$$

$$2\ell \cdot q \equiv \frac{P_A \cdot q}{y} \tag{49}$$

In Drell-Yan scattering one already has two hadrons to start with. Here Q^2 is the invariant mass of the lepton pair. The above processes are characteristic for a large number of other processes involving particles for which the interactions are fully known on the one side and hadrons on the other side.

B. Cross sections for (semi-)inclusive leptoproduction

Taking inclusive lepton-hadron scattering as an example, the cross section is given by

$$E' \frac{d\sigma}{d^3\ell'} = \frac{1}{s - M^2} \frac{\alpha^2}{Q^4} L_{\mu\nu}\, 2MW^{\mu\nu}, \tag{50}$$

where $L_{\mu\nu}$ is the lepton tensor,

$$L_{\mu\nu}(\ell, \ell'; \lambda_e) = 2\ell_\mu \ell'_\nu + 2\ell_\nu \ell'_\mu - Q^2 g_{\mu\nu} + 2i\lambda_e\, \epsilon_{\mu\nu\rho\sigma} q^\rho \ell^\sigma. \tag{51}$$

and $W_{\mu\nu}$ is the hadron tensor, which depends on exchanged momentum q and target momentum P. It contains the information on the hadronic part of the scattering process,

$$2M\, W_{\mu\nu}(P, S; q) = \frac{1}{2\pi} \sum_n \int \frac{d^3P_n}{(2\pi)^3\, 2E_n} \langle P, S|J^\dagger_\mu(0)|P_n\rangle \langle P_n|J_\nu(0)|P, S\rangle\, (2\pi)^4\, \delta^4(P + q - P_n). \tag{52}$$

We have included a spin vector S satisfying $S^2 = -1$ and $P{\cdot}S = 0$ to characterize the spin of the target, of which the details will be discussed later.

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Exercise: Show, using $\delta^4(P + q - P_n) = \int d^4x\ \exp(iP \cdot x + iq \cdot x - iP_n \cdot x)$, shifting the argument of the current, $J_\mu(x) = \exp(iP_{op} \cdot x) J_\mu(0) \exp(-iP_{op} \cdot x)$ and using completeness for the intermediate states that the hadron tensor for inclusive deep inelastic scattering can be written as the expectation value of the product of currents $J_\mu(x)J_\nu(0)$. Then, adding a second term $\propto J_\nu(0)J^\dagger_\mu(0)\delta^4(P - q - P_n)$, which in the physical region ($\nu > 0$) is zero because of the spectral conditions of the intermediate states n ($P^0_n > M$) one can after a similar procedure combine the terms to

$$2M\, W_{\mu\nu} = \frac{1}{2\pi} \int d^4x\ e^{i\,q\cdot x}\, \langle P|[J^\dagger_\mu(x), J_\nu(0)]|P\rangle, \tag{53}$$

where summation and averaging over spins is understood.

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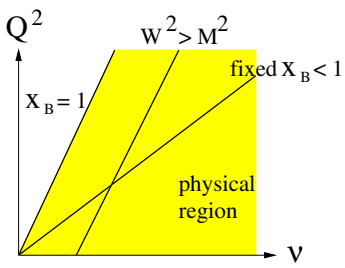

FIG. 2: *The physical region in deep inelastic scattering in the Q^2-ν plane. The variable $\nu = P \cdot q/M$ is the photon energy in the target (momentum P) rest frame. Indicated are lines of constant invariant mass squared for the hadronic system, $W^2 = (P + q)^2$. The elastic limit $W^2 = M^2$ corresponds with $\nu = Q^2/2M$ or $x_B = 1$.*

In the leptoproduction $e + H \to e' + X$ the (unobserved) final state X can be the target (elastic scattering) or an excitation thereof. In Fig. 2) a plot of the two independent variables ν and Q^2 is given as well as lines for fixed W^2 or fixed x_B. Elastic scattering corresponds to $W^2 = M^2$ or $x_B = 1$. For an elementary fermion that cannot be broken up or excited, this line relates ν and Q^2. For nucleons the elastic scattering cross section (on top of the $1/Q^4$ of the Mott cross section) is proportional to a form factor squared, measuring the expectation value of the electromagnetic current $\langle P'|J^\mu(x)|P\rangle$. It is also possible to excite the system. This gives rise to inelastic contributions in the cross section at $\nu > Q^2/2M$, starting at the threshold $W = M + M_\pi$. When Q^2 and the energy transfer ν are high enough the cross section will reflect elastic scattering off the pointlike constituents of the nucleon and the cross section will become equal to an incoherent sum of the electron-quark cross section. This is known as the deep inelastic scattering region, in which one finds Bjorken scaling. The cross section, more precisely the structure functions, become functions of one (kinematic) variable x_B, which is identified with the momentum fraction of the struck quark in the nucleon, enabling measurement of quark distributions. In zeroth order the cross section thus becomes constant along lines of fixed x_B (scaling).

Depending on the details of the final state, we can distinguish semi-inclusive or exclusive processes. If the final state is the same system as the initial state, a nucleon (or a specific nucleon excitation), it is a part of the inclusive cross section for which the phase space is rewritten as $d\tilde{P}'\,(2\pi)^4\,\delta^4(P+q-P') = (2\pi)\,\delta((P+q)^2 - P'^2) = 2\pi\,\delta(2P{\cdot}q - Q^2)$, giving

$$2M\,W_{\mu\nu}(P,S;q) = \underbrace{\langle P,S|J_\mu(0)|P'\rangle\langle P'|J_\nu(0)|P,S\rangle}_{H_{\mu\nu}(P,S;P')}\,\frac{1}{Q^2}\,\delta(1-x_B). \tag{54}$$

One needs the hadronic current matrix elements, instead of those of the commutator or the time-ordered product. These current matrix elements are expressed in form factors, which are Fourier transforms of charge and current distributions. This will be discussed in more detail in the next section.

Although one has local matrix elements between hadronic state, which in general only give access to global properties, it is useful to realize that we still know that that the currents are basically the vector or axial vector currents of the quarks probed via the exchange of the electroweak bosons, in particular

$$J_\mu^{(\gamma)} = :\overline{\psi}(x)\,Q\,\gamma_\mu\,\psi(x):, \tag{55}$$

$$\begin{aligned} J_\mu^{(Z)} &= :\overline{\psi}(x)\,(I_W^3 - Q\sin^2\theta_W)\,\gamma_{\mu L}\,\psi(x): - :\overline{\psi}(x)\,Q\sin^2\theta_W\,\gamma_{\mu R}\,\psi(x): \\ &= :\overline{\psi}(x)\,(I_W^3 - 2\,Q\sin^2\theta_W)\,\gamma_\mu\,\psi(x): - :\overline{\psi}(x)\,I_W^3\,\gamma_\mu\gamma_5\,\psi(x):, \end{aligned} \tag{56}$$

$$J_\mu^{(W)} = :\overline{\psi}(x)\,I_W^\pm\,\gamma_{\mu L}\,\psi(x):, \tag{57}$$

where $\gamma_{\mu R/L} = \gamma_\mu(1 \pm \gamma_5)$. Note that the factors e, $e/2\,\sin\theta_W\,\cos\theta_W$ and $e/2\sqrt{2}\,\sin\,\theta_W$ that would be expected for the γ, Z^0 and W-couplings to the quarks are omitted here. In writing cross sections, they will be included as explicit factors of the fine structure constant α in Eq. 50) or the Fermi-coupling $G_F/\sqrt{2} = e^2/8M_W^2\sin^2\theta_W$, As mentioned above, one has known currents of which even the flavor decomposition is known, but their expectation values are not known and need to be parametrized in terms of form factors.

Other processes that can be considered are the semi-inclusive processes in which out of the final state one hadron is measured in coincidence with the scattered electron. The cross section is also differential in $d\tilde{P}_h$, the phase space of the observed hadron and the hadronic tensor becomes

$$\begin{aligned} 2M\mathscr{W}_{\mu\nu}(q;PS;P_hS_h) &= \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q+P-P_X-P_h)\,\langle PS|J_\mu(0)|P_X;P_hS_h\rangle\langle P_X;P_hS_h|J_\nu(0)|PS\rangle \\ &= \frac{1}{(2\pi)^4}\int d^4x\;e^{iq\cdot x}\,\langle PS|J_\mu(x)\sum_X|X;P_h,S_h\rangle\langle X;P_hS_h|J_\nu(0)|PS\rangle, \end{aligned}$$

C. The cross section for the Drell-Yan process

Consider the process $A + B \to \ell + \bar{\ell} + X$, where two spin-$\frac{1}{2}$ hadrons with momenta P_A^μ and P_B^μ interact and two outgoing leptons (considered massless) are measured with momenta ℓ^μ and ℓ'^μ. The leptons are assumed to originate from massive (virtual) photon with momentum $q = \ell + \ell'$ with $Q^2 \equiv q^2 > 0$. The unobserved outstate will be denoted by $|P_X\rangle$. We will consider the case of pure incoming spin states, characterized by the spin vectors S_A and S_B, respectively, and observed lepton helicities λ and λ'. In the deep inelastic limit $Q^2, s \to \infty$, with the ratio $\tau = Q^2/s$

fixed. The invariant amplitude for the process is given by

$$\mathscr{M} = \bar{u}(k,\lambda)\gamma^\mu v(k',\lambda')\frac{e^2}{Q^2}\langle P_X|J_\mu(0)|P_A S_A; P_B S_B\rangle. \tag{58}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathscr{M}|^2 = \frac{e^4}{Q^4} L^{(DY)}_{\mu\nu} H^{\mu\nu(DY)}, \tag{59}$$

with the lepton tensor (neglecting the lepton masses) being

$$L^{(DY)}_{\mu\nu}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma\right). \tag{60}$$

The product of hadronic current matrix elements is written as

$$H^{(DY)}_{\mu\nu}(P_X; P_A S_A; P_B S_B) = \langle P_A S_A; P_B S_B|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|P_A S_A; P_B S_B\rangle, \tag{61}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathscr{M}|^2 d\mathscr{R}, \tag{62}$$

with the flux factor

$$F = 4\sqrt{(P_A\cdot P_B)^2 - M_A^2 M_B^2} \;\approx\; 2s \tag{63}$$

and the Lorentz invariant phase space

$$d\mathscr{R} = (2\pi)^4\delta^4(P_A + P_B - P_X - k - k')\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3k'}{(2\pi)^3 2k'^0}. \tag{64}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$\mathscr{W}^{(DY)}_{\mu\nu}(q; P_A S_A; P_B S_B) \;=\; \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(P_A + P_B - P_X - q)\,H^{(DY)}_{\mu\nu}(P_X; P_A S_A; P_B S_B). \tag{65}$$

The remaining phase space is conveniently written as

$$\frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3k'}{(2\pi)^3 2k'^0} = \frac{d^4q}{(2\pi)^4}\frac{d\Omega}{32\pi^2}, \tag{66}$$

where the angles are those of the lepton axis in the rest frame of the two leptons. Including the fine-structure constant $\alpha \equiv e^2/4\pi$, we then obtain the Drell-Yan cross section (including a factor 2 from the summation over the lepton polarizations)

$$\frac{d\sigma^{(DY)}}{d^4q\,d\Omega} = \frac{\alpha^2}{2s\,Q^4}L^{(DY)}_{\mu\nu}\mathscr{W}^{\mu\nu(DY)}, \tag{67}$$

where the lepton tensor is given by the symmetric part in Eq. (60) and the hadron tensor by Eq. (65).

D. Cross sections in terms of form factors and structure functions

The simplest thing one can do with the hadron tensor is to express it in standard tensors and functions depending on the invariants, the *structure functions*. Instead of the traditional choice for lepton-hadron scattering using tensors, $g_{\mu\nu}$, $P_\mu P_\nu$ and $\epsilon_{\mu\nu\rho\sigma}\,q^\rho P^\sigma$ multiplying structure functions W_1, W_2 and W_3 depending on ν and Q^2, we immediately go to a dimensionless representation. First we define a Cartesian basis of vectors [46], starting with the natural space-like momentum (defined by q). Using the target hadron momentum P^μ one can construct an orthogonal four vector $\tilde{P}^\mu = P^\mu - (P\cdot q/q^2)\,q^\mu$, which is timelike with length $\tilde{P}^2 = \kappa\,P\cdot q$ with in the limit of large Q^2 the result $\kappa \to 1$

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Exercise: Show that the correction for finite Q^2 is given by

$$\kappa = 1 + \frac{M^2\,Q^2}{(P\cdot q)^2} = 1 + \frac{4\,M^2 x_B^2}{Q^2}. \tag{68}$$

taking into account the so-called target mass corrections $\propto M^2/Q^2$.

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The Cartesian set is defined as

$$Z^\mu \equiv -q^\mu, \tag{69}$$

$$T^\mu \equiv -\frac{q^2}{P\cdot q}\,\tilde{P}^\mu = q^\mu + 2x_B\,P^\mu. \tag{70}$$

For these vectors we have $Z^2 = -Q^2$ and $T^2 = \kappa\,Q^2$ and we will often use the normalized vectors $\hat{z}^\mu = -\hat{q}^\mu = Z^\mu/Q$ and $\hat{t}^\mu = T^\mu/Q\sqrt{\kappa}$. With respect to these vectors one can use the transverse tensors, $g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu$ and $\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{q}_\sigma$. To get the parametrization of hadronic tensors, such as the one in Eq. 53, including for generality also an (axial) spin vector S, we use the general symmetry property,

$$W_{\mu\nu}(q,P,S) = W_{\nu\mu}(-q,P,S) \tag{71}$$

as well as properties following from hermiticity, parity and time-reversal invariance,

$$W^*_{\mu\nu}(q,P,S) = W_{\nu\mu}(q,P,S), \tag{72}$$

$$W_{\mu\nu}(q,P,S) = \overline{W}_{\nu\mu}(\bar{q},\bar{P},-\bar{S}) \qquad \text{[Parity]}, \tag{73}$$

$$W^*_{\mu\nu}(q,P,S) = \overline{W}_{\mu\nu}(\bar{q},\bar{P},\bar{S}) \qquad \text{[Time reversal]}, \tag{74}$$

where $\bar{p} = (p^0, -\boldsymbol{p})$. Finally we use current conservation implying $q^\mu W_{\mu\nu} = W_{\mu\nu}q^\nu = 0$. Note that depending on the situation not all constraints can be applied. For inclusive unpolarized leptoproduction one obtains as the most general form for the symmetric tensor,

$$\begin{aligned} M\,W^{\mu\nu\,(S)}(P,q) &= \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) F_1(x_B,Q^2) + \frac{\tilde{P}^\mu\tilde{P}^\nu}{P\cdot q}\,F_2(x_B,Q^2) \\ &= -g_\perp^{\mu\nu}\,\underbrace{F_1(x_B,Q^2)}_{F_T} + \hat{t}^\mu\hat{t}^\nu\,\underbrace{\left(-F_1 + \frac{\kappa}{2x_B}\,F_2\right)}_{F_L}, \end{aligned} \tag{75}$$

where the structure functions F_1, F_2 or the transverse and longitudinal structure functions, $F_T = F_1$ and F_L, depend only on the for the hadron part relevant invariants Q^2 and x_B. This is the structure for the electromagnic (photon exchange) part of the electroweak interaction. For the weak (W- or Z-exchange) part both vector and axial vector currents with different parity behavior come in. In that case also the following antisymmetric tensor is allowed,

$$M\,W^{\mu\nu\,(A)}(q,P) = \frac{i\epsilon^{\mu\nu\rho\sigma}P_\rho q_\sigma}{(P\cdot q)}\,F_3(x_B,Q^2) = i\,\kappa\,\epsilon_\perp^{\mu\nu}\,F_3(x_B,Q^2). \tag{76}$$

It appears in the part of the tensor in which one of the currents in the product is a vector current and the other an axial vector current.

The cross section is obtained from the contraction of lepton and hadron tensors. It is convenient to expand also the lepton momenta ℓ and $\ell' = \ell - q$ in other vectors. Examples using ℓ and P (and the scaling variables for leptoproduction) are

$$\ell' = x_B y\,P + (1-y)\,\ell + Q\sqrt{1-y}\,\hat{\ell}_{\perp(\ell,P)}, \tag{77}$$

or using P and q (or linear combinations like P and n or $\hat{t}$ and $\hat{z}$)

$$\begin{aligned} \ell' &= \frac{x_B}{y}\,P + \frac{1-y}{y}\,\frac{Q^2}{2x_B}\,n + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}_{\perp(q,P)} \\ &= \frac{(2-y)Q}{2y}\,\hat{t}^\mu + \frac{Q}{2}\,\hat{q}^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}_{\perp(q,P)}. \end{aligned} \tag{78}$$

where $n = (q + x_B P)/P{\cdot}q = (\hat{t} + \hat{z})/\sqrt{2}$, which is the (approximately) light-like vector $n^2 \approx 0$, conjugate to P, $P{\cdot}n = 1$. The unit vectors $\hat{\ell}_\perp$ define the transverse directions with between brackets following the $\perp$ the momenta with respect to which the perpendicular directions are being considered. We note that in particular when one considers azimuthal asymmetries one should be careful with the phase space,

$$d\tilde{\ell}' = \frac{d(\ell'{\cdot}P)\, d|\ell_{\perp(\ell,P)}|^2\, d\varphi_{\ell P}}{(2\pi)^3\, 2(\ell'{\cdot}P)} = \frac{y\, s}{8\pi^2}\, dx_B\, dy\, \frac{d\varphi_{\ell P}}{2\pi} \tag{79}$$

$$= \frac{d(\ell'{\cdot}n)\, d|\ell_{\perp(q,P)}|^2\, d\varphi_{qP}}{(2\pi)^3\, 2(\ell'{\cdot}n)} = \frac{s}{8\pi^2\, y}\, dx_B\, dy\, \frac{d\varphi_{qP}}{2\pi}. \tag{80}$$

The kinematics in the frame where virtual photon and target are collinear (including target rest frame) will become relevant in cases where additional directions play a role, such as 1-particle inclusive (1PI) leptoproduction. With the definition of $\hat{\ell}_\perp$, we obtain neglecting mass corrections ($\kappa = 1$) for unpolarized leptons the symmetric and antisymmetric parts of the leptonic tensor in Eq. 51,

$$L^{\mu\nu\,(S)} = \frac{Q^2}{y^2}\left[-2\left(1 - y + \tfrac{1}{2}\, y^2\right) g_\perp^{\mu\nu} + 4(1-y)\hat{t}^\mu \hat{t}^\nu + 4(1-y)\left(\hat{\ell}_\perp^\mu\, \hat{\ell}_\perp^\nu + \tfrac{1}{2}\, g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y}\; \hat{t}^{\{\mu}\hat{\ell}_\perp^{\nu\}}\right], \tag{81}$$

$$L^{\mu\nu\,(A)} = \frac{Q^2}{y^2}\left[-i\, y(2-y)\, \epsilon_\perp^{\mu\nu} - 2i\, y\sqrt{1-y}\; \hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\right]. \tag{82}$$

We will need this form when we look at explicit production of particles h in the final state. The explicit contraction of lepton and hadron tensors for inclusive scattering gives for electromagnetic scattering (only symmetric tensor and $\kappa = 1$) the result

$$\frac{d\sigma^{ep}}{dx_B dy} = \frac{4\pi\, \alpha^2\, x_B\, s}{Q^4}\left[\left(1 - y + \tfrac{1}{2}\, y^2\right) F_T(x_B, Q^2) + (1-y)\, F_L(x_B, Q^2)\right] \tag{83}$$

$$= \frac{2\pi\, \alpha^2\, s}{Q^4}\left[(1-y)\, F_2(x_B, Q^2) + x_B\, y^2\, F_1(x_B, Q^2)\right].$$

We have now used the known photon coupling to the lepton and parametrized our ignorance for what happened with the hadron in a hadronic tensor. The fact that we know how the photon interacts with the (quark) constituents of the hadrons will be used later to relate the structure functions to quark properties. In the same way one also knows for the weak interaction processes leading to antisymmetric part containing F_3 in the tensor for unpolarized hadrons, how the Z^0 and W couple to quarks. To describe weak interactions also the antisymmetric part of the lepton tensor is needed, which is also encountered when one looks at polarization.

Virtual photon cross sections

Dressing the hadronic tensor $W_{\mu\nu}$ with photon polarization vectors one obtains the total cross section for $\gamma^* H \to$ everything, where γ^* indicates a virtual photon. For a given virtuality Q^2 of the photon this cross section depends on only one variable, $W^2 = (P+q)^2$, or equivalently on the variable $\nu = P \cdot q/M$,

$$\sigma^{\gamma^* H}(\nu) = \frac{4\pi^2\alpha}{K}\, \epsilon^{\mu *} W_{\mu\nu}\epsilon^\nu, \tag{84}$$

where $4M\,K$ is the photon flux factor. This flux factor only is physical for real photons ($Q^2 = 0$). One convention is to take $4M\,K = 4\sqrt{(P\cdot q)^2 - P^2 q^2}$, i.e. $K = \sqrt{\nu^2 + Q^2}$. Other possibilities are to take the real photon result $4M\,K = 4\,P\cdot q$ or $K = \nu$ (Hand convention). Another convention that has been used is to equate the final state invariant mass squared $W^2 = (P+q)^2$, i.e. take the result $4M\,K = 2(W^2 - M^2)$ for a massless photon and equate W^2 to the invariant mass in the case of a virtual photon, $W^2 = 2\,P\cdot q + M^2 - Q^2$ or $K = \nu - Q^2/2M$.

Being a (total) cross section for (virtual) photoabsorption, the hadronic tensor is related to the forward (virtual) Compton amplitude through the optical theorem,

$$W_{\mu\nu} = \frac{1}{\pi}\, \mathrm{Im}\; T_{\mu\nu}, \tag{85}$$

where

$$2M\, T_{\mu\nu}(P,q) = i\int d^4x\; e^{iq\cdot x}\langle P|\mathcal{T} J_\mu(x) J_\nu(0)|P\rangle, \tag{86}$$

For a virtual photon we need three polarization vectors ϵ^μ_α, where α indicates the polarization directions (perpendicular to q^μ),

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}\,(0,\mp 1,-i,0) = \mp\frac{1}{\sqrt{2}}(\epsilon_x \pm i\,\epsilon_y) \quad \text{and} \quad \epsilon^\mu_L = \frac{1}{\sqrt{Q^2}}\,(q^3,0,0,q^0). \tag{87}$$

One gets two transverse structure functions and one longitudinal structure function, $F_\alpha = \epsilon^{\mu *}_\alpha\, M W_{\mu\nu}\epsilon^\nu_\alpha$. Because of the fact that they are cross sections for (virtual) photons, the structure functions, F_+, F_- and F_L are positive. We have (as already indicated)

$$F_T = \frac{1}{2}\,(F_+ + F_-) = F_1, \quad F_L = \frac{F_2}{2x_B} - F_1, \quad \text{and} \quad F_3 = F_+ - F_-. \tag{88}$$

E. Form factors

In the previous sections we have also seen matrix elements, which in particular for elastic scattering contained local operators such as the currents. These contributions are part of the inclusive scattering where $P' = P + q$ and is fixed to be $(P+q)^2 = M^2$, i.e. $x_B = 1$. We can still use the formalism developed so far, but the hadronic tensor becomes becomes

$$2M\,W_{\mu\nu}(q,P) = \underbrace{\langle P|J_\mu(0)|P'\rangle\langle P'|J_\nu(0)|P\rangle}_{H_{\mu\nu}(P;P')}\,\frac{1}{Q^2}\,\delta(1-x_B). \tag{89}$$

One needs the (local) hadronic current matrix elements, instead of the nonlocal combination in the commutator or the time-ordered product. Like the structure functions, the local matrix elements for hadrons cannot be calculated ab initio and one usually parametrizes them. The current matrix elements are expressed in form factors, which are Fourier transforms of charge and current densities. We will consider in more detail the vector and axial vector currents, V_μ and A_μ. The most general form consistent with the requirements of Poincaré invariance, gauge invariance and invariance under parity and time reversal yields for a spin 0 system (e.g. a pion or a ^{12}C nucleus) the form

$$\langle P',S'|V_\mu(x)|P,S\rangle = (P_\mu + P'_\mu)\,F(Q^2)e^{i(P'-P)\cdot x}, \tag{90}$$

involving one invariant function (*form factor*) depending on the (only) invariant, $q^2 \equiv -Q^2$, where $q = P' - P$. For spin 1/2 systems (such as a nucleon, ^{3}H or ^{3}He) the most general expression for the expectation value of the current can be written using free Dirac spinors $U(P,S)$,

$$\langle P',S'|V_\mu(0)|P,S\rangle = \overline{U}(P',S')\left[\gamma_\mu\,F_1(Q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}\,F_2(Q^2)\right]U(P,S), \tag{91}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu]$. The functions $F_1(Q^2)$ and $F_2(Q^2)$ are the Dirac and Pauli form factors.

The elastic cross section then is found by inserting the parametrizations, for instance in the case of electromagnetic scattering one obtains the tensor $H_{\mu\nu}$

$$\begin{aligned} H_{\mu\nu}(P,P') &= \frac{1}{2}\sum_{S,S'}\langle P,S|J_\mu(0)|P',S'\rangle\langle P',S'|J_\nu(0)|P,S\rangle \\ &= \frac{1}{2}\,\mathrm{Tr}\;[\Gamma_\mu(\not{P}' + M)\Gamma_\nu(\not{P} + M)] \\ &= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)\,Q^2(F_1+F_2)^2 + 4\,\tilde{P}_\mu\tilde{P}_\nu\,\left(F_1^2 + \frac{Q^2}{4M^2}\,F_2^2\right) \\ &= -g_{\perp\,\mu\nu}\,Q^2G_M^2 + \hat{t}_\mu\hat{t}_\nu\,4M^2\,G_E^2 \end{aligned} \tag{92}$$

One thus sees the following *elastic* contribution in the transverse and longitudinal structure functions

$$F_T(x_B,Q^2) = G_M^2(Q^2)\,\delta(1-x_B), \tag{93}$$

$$F_L(x_B,Q^2) = \frac{4M^2}{Q^2}\,G_E^2(Q^2)\,\delta(1-x_B). \tag{94}$$

In order to interpret the space-like form factors above, with $q = P' - P$ and $q^2 = -Q^2$, we recall that there is a time-like vector $\hat{t} = n_0$, which in this case is proportional to $T^\mu = P^\mu + P'^\mu$ satisfying thus $q{\cdot}n_0 = 0$. Any four-vector can be expanded as $a = (a \cdot \hat{t})\,\hat{t} + \boldsymbol{a}$, specifically $q = \boldsymbol{q}$. A further specification of the spacelike direction (z-direction or transverse) is at this stage not needed, but may for specific applications be useful. The frame in which $n_0 = \hat{t} = (1,0,0,0)$ is known as the Breit frame or the Brick-Wall frame.

Working out the current expression for the nucleon with the above definition of zero and space-like components (or if one likes in the Breit frame) with $P'^0 = P^0$, $\boldsymbol{P} = -\boldsymbol{q}/2$, $\boldsymbol{P'} = +\boldsymbol{q}/2$, gives

$$\langle P', S'|J_0^{\rm em}(0)|P, S\rangle \;=\; 2M\,\left[F_1 - \frac{Q^2}{4M^2}\,F_2\right] \equiv 2M\,G_E(Q^2), \tag{95}$$

$$\langle P', S'|\boldsymbol{J}^{\rm em}(0)|P, S\rangle \;=\; [F_1 + F_2]\;(i\boldsymbol{\sigma}_N\,\times \boldsymbol{q}) \equiv G_M(Q^2)\,(i\boldsymbol{\sigma}_N \times \boldsymbol{q}), \tag{96}$$

where $\chi^\dagger_{S'}\,\boldsymbol{\sigma}\,\chi_S \equiv \boldsymbol{\sigma}_N$. These expressions show the relevance of the Sachs form factors G_E and G_M. The quantity $e\,G_E(0)$ is the charge Q of the nucleon, $e\,G_M(0)/M$ is the magnetic moment of the nucleon. The quantity $\kappa = G_M(0) - Q = G_M(0) - G_E(0)$ is the anomalous magnetic moment. For an elementary fermion one has $F_1(Q^2) = 1$ and $F_2(Q^2) = 0$. For the nucleon one has for the proton $F_1^p(0) = Q_p = 1$ and $F_2^p(0) = \kappa_p = 1.793$ and for the neutron $F_1^n(0) = Q_n = 1$ and $F_2^n(0) = \kappa_n =$ -1.913. Roughly one has for nucleons the dipole-like behavior

$$G_E^p(Q^2) \approx \frac{G_M^p(Q^2)}{\mu_p} \approx \frac{G_M^n(Q^2)}{\mu_n} \approx \frac{1}{(1 + Q^2/\Lambda^2)^2}, \tag{97}$$

with $\Lambda^2 \approx 0.71$ GeV^2 and $\mu_N = G_M^N(0)$ the nucleon magnetic moment. We note that the term proportional to q^μ in the parametrization can only be measured off-forward, i.e. if $P \neq P'$. Generalizing the term linear in the momentum transfer into

$$\langle P'|O(x)|P\rangle = e^{i\,(P'-P)\cdot x}\;\langle P'|O(0)|P\rangle = e^{i\,q\cdot x}\big[G_1(Q^2) - i\,q_\mu\,G_2^\mu(Q^2)\big], \tag{98}$$

we may write

$$G_1(Q^2) - i\,q_\mu\,G_2^\mu(Q^2) \;=\; \langle P'|\,O(0)\,|P\rangle = (2\pi)^3\,\delta^3(\boldsymbol{P'} - \boldsymbol{P} - \boldsymbol{q})\frac{\langle P'|\,O(0)\,|P\rangle}{V}\int d^3x\;\exp\,(i\,\boldsymbol{q}\cdot\boldsymbol{x})\,\underbrace{\frac{\langle P'|\,O(x)\,|P\rangle}{V}}_{\rho_O(x)}$$

where the density $\rho_O(x) = \rho_O(\boldsymbol{x})$ (which is independent of component $x{\cdot}n_0$ as follows from Eq. 98). Next we consider the matrix element $\langle P'|\,x^\mu\,O(x)\,|P\rangle$, of which the component along n_0 vanishes and use this to obtain

$$\int d^3x\;\exp\,(i\,\boldsymbol{q}\cdot\boldsymbol{x})\;\frac{\langle P'|\,\boldsymbol{x}\,O(x)\,|P\rangle}{V} \;=\; -i\boldsymbol{\nabla}_q\int d^3x\;\exp\,(i\,\boldsymbol{q}\cdot\boldsymbol{x})\;\frac{\langle P'|\,O(x)\,|P\rangle}{V} = \boldsymbol{G}_2(Q^2) - 2i\,\boldsymbol{q}\,\frac{\partial G_1}{\partial Q^2} - 2\,\boldsymbol{q}\,q_\mu\,\frac{\partial G_2^\mu}{\partial Q^2}.$$

In this way we obtain for the forward matrix elements

$$\lim_{P'\to P}\langle P'|\,O(x)\,|P\rangle = G_1(0), \tag{99}$$

$$\lim_{P'\to P}\langle P'|\,x^\mu\,O(x)\,|P\rangle = G_2^\mu(0). \tag{100}$$

Applying this to the nucleon form factors one has the forward matrix elements

$$\langle P, S'|J_0^{\rm em}(0)|P, S\rangle = 2M\,F_1(0) = 2M\,G_E(0), \tag{101}$$

$$\langle P, S'|(\boldsymbol{r}\times\boldsymbol{J}^{\rm em})(0)|P, S\rangle = [F_1(0) + F_2(0)]\;\boldsymbol{\sigma}_N = G_M(0)\,\boldsymbol{\sigma}_N, \tag{102}$$

where $\chi^\dagger_{S'}\,\boldsymbol{\sigma}\,\chi_S \equiv \boldsymbol{\sigma}_N$.

We can also look at the quark currents that make up the charge current and note that for the (diagonal) vector currents of the quarks, $V_\mu^q = \overline{\psi}^q\gamma_\mu\psi^q$, the quantities $F_1^q(0)$ are (by definition) the quark numbers $F_1^q(0) = n_q$. The matrix element of the electromagnetic current J_μ^{em} in Eq. 55 then indeed satisfying $F_1^{em}(0) = Q$ (the charge of the hadron state).

F. Structure functions for 1PI leptoproduction

In cases like Drell-Yan scattering or 1-particle inclusive (1PI) leptoproduction one has two hadrons and a momentum characterizing the hard process. Taking leptoproduction as an example. We have momenta P, P_h and q. This enables us to define

$$q_T^\mu = q^\mu + x_B\,P^\mu - \frac{P_h^\mu}{z_h} \equiv -Q_T\,\hat{h}^\mu. \tag{103}$$

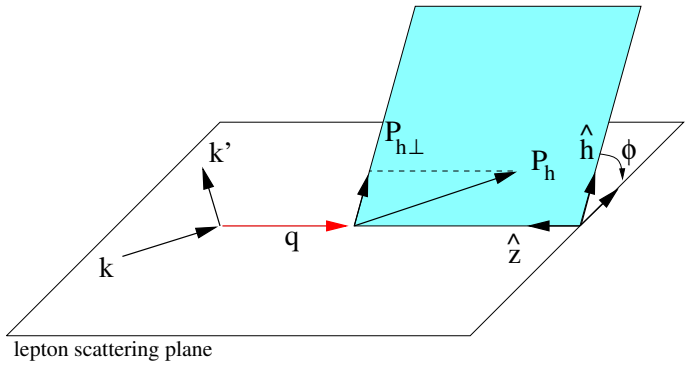

FIG. 3: *Kinematics for lepton-hadron scattering. Transverse directions indicated with a $\perp$ index are orthogonal to P and q, e.g. the orthogonal component of the momentum of a produced hadron has been indicated as example. Similarly one can consider the orthogonal component of the spin vector of the target.*

In this equation the hadronic momenta serve as the (approximate) light-like directions $n_\pm$, with respect to which q has a transverse component. But of course one also could view at q_T in a frame where q and P define the collinear directions. In that case the vector q_T (or actually $-z_h\, q_T$ is the perpendicular component of P_h, justifying the righthandside renaming of this vector. Important is that this provides covariant formulations of *transverse component of* q as well as *perpendicular component of* $-z_h\, P_h$ in terms of the external vectors q, P and P_h which are in general not collinear. The vector $\hat{h}$ defines the orientation of the hadronic plane in Fig. 3.

For an unpolarized (or spin 0) hadron in the final state the symmetric part of the tensor is given by

$$M\mathcal{W}^{\mu\nu(S)}(q,P,P_h) \;=\; -g_\perp^{\mu\nu}\,\mathcal{H}_T + \hat{t}^\mu\hat{t}^\nu\,\mathcal{H}_L + \hat{t}^{\,\{\mu}\hat{h}^{\nu\}}\,\mathcal{H}_{LT} + \Big(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\Big)\,\mathcal{H}_{TT}.$$

Noteworthy is that also an antisymmetric term in the tensor is allowed,

$$M\mathcal{W}^{\mu\nu(A)}(q,P,P_h) = -i\hat{t}^{\,[\mu}\hat{h}^{\nu]}\,\mathcal{H}'_{LT}. \tag{104}$$

Clearly the lepton tensor in Eq. 81 or 82 is able to distinguish all the structures in the semi-inclusive hadron tensor.

The symmetric part gives the cross section for unpolarized leptons,

$$\frac{d\sigma_{OO}}{dx_B dy\, dz_h d^2q_T} \;=\; \frac{4\pi\,\alpha^2\,s}{Q^4}\,x_B z_h\Big\{\big(1-y+\tfrac{1}{2}\,y^2\big)\,\mathcal{H}_T + (1-y)\,\mathcal{H}_L \tag{105}$$
$$-\,(2-y)\sqrt{1-y}\,\cos\phi_h^\ell\;\mathcal{H}_{LT} + (1-y)\,\cos 2\phi_h^\ell\;\mathcal{H}_{TT}\Big\}$$

while the antisymmetric part gives the cross section for a polarized lepton (note the target is not polarized!)

$$\frac{d\sigma_{LO}}{dx_B dy\, dz_h d^2q_T} = \lambda_e\,\frac{4\pi\,\alpha^2}{Q^2}\,z_h\,\sqrt{1-y}\,\sin\phi_h^\ell\;\mathcal{H}'_{LT}. \tag{106}$$

III. PARTON CORRELATORS IN HADRONS

A. Deep inelastic scattering

The basic idea in the diagrammatic approach is to realize that a diagram involves hadronic states and quark and gluon operators. Therefore we use correlators as discussed in our introductory section. Although there may be many of those, let us make a start and solve problems along the way. The correlator is the Fourier transform in the space-time arguments of the quark and gluon fields. In the correlators, all momenta of hadrons *and* quarks and gluons (partons) inside the hadrons are soft which means that $p^2 \sim p \cdot P \sim P^2 = M_N^2 \ll Q^2 \sim s$. The *off-shellness* being of hadronic order implies that in the hard process partons are in essence *on-shell*. Consistency of this has of course to be checked by using QCD interactions to give partons a large off-shellness of $\mathscr{O}(Q)$ and check the behavior as a function of the momenta. In these considerations one must also realize that beyond tree-level one has to distinguish bare and renormalized fields.

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Exercise: Derive the following expansion for the (external) vectors P and q,

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} q = \frac{Q}{\sqrt{2}}\,n_- \;-\; \frac{Q}{\sqrt{2}}\,n_+ \\ \\ P = \frac{x_B M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right. ,$$

or in light-cone components ($n_+ \equiv [0, A, 0_\perp]$ and $n_- \equiv [1/A, 0, 0_\perp]$)

$$q = \Big[\frac{Q}{\sqrt{2}}, -\frac{Q}{\sqrt{2}}, \mathbf{0}_\perp\Big], \quad \text{and} \quad P = \Big[\frac{x_B M^2}{Q\sqrt{2}}, \frac{Q}{x_B\sqrt{2}}, \mathbf{0}_\perp\Big] \;\approx\; \Big[0, \frac{Q}{x_B\sqrt{2}}, \mathbf{0}_\perp\Big].$$

What is the effect of a boost? Which value of A represents the nucleon rest-frame. Note that $A \to \infty$ is referred to as the *infinite momentum frame*. The representation with light-like vectors exhibits that when Q^2 becomes large, the nucleon momentum is 'on the scale Q in essence light-like. While the hard momentum has both components proportional to Q, this is not the case for P and one has $P^- \ll q^-$. In deep-inelastic scattering the plus components are of the same order with ratio being the scaling variable $x_B = -q^+/P^+$.

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The simplest expression that is needed in scattering a photon from the target, is the correlator and the scattering of the photon from a quark. Restricting us to the quark part we write the contribution

$$\begin{aligned} 2M\,W^{\mu\nu}(P,q) \;&=\; \sum_q e_q^2 \int dp^- dp^+ d^2p_\perp \;\mathrm{Tr}\,(\Phi(p)\,\gamma^\mu(\not p + \not q + m)\gamma^\nu)\;\delta((p+q)^2 - m^2) \\ &\approx\; \sum_q e_q^2 \int dp^- dp^+ d^2p_\perp \;\mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,\frac{\not q}{2q^-}\,\gamma^\nu\right)\;\delta(p^+ + q^+) \\ &\approx\; -g_\perp^{\mu\nu}\,\frac{1}{2}\int dp^- d^2p_\perp \;\mathrm{Tr}\left(\gamma^+\,\Phi(p)\right)\Big|_{p^+ = x_B\,P^+} + \ldots, \end{aligned} \tag{107}$$

where $\Phi(p)$ is a projection of the (Dirac-space) correlator discussed in section I. The relevant soft part then is a particular Dirac trace of the quantity

$$\Phi_{ij}(x) = \int dp^- d^2p_T\;\Phi_{ij}(p,P,S) = \int \frac{d\xi^-}{2\pi}\;e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{108}$$

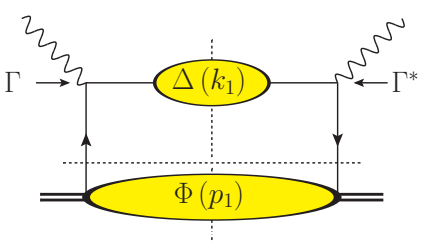


FIG. 4: *The simplest (parton-level) diagrams for inclusive or semi-inclusive lepton-hadron scattering. Note that also the diagram with opposite fermion flow has to be added. For DIS the outgoing quark (i.e. $\Delta(k)$ can be calculated involving $\sum_s u(k)\overline{u}(k) = \not k + m$). The vertices Γ are electromagnetic vertices $i\,e_q\gamma^\mu$. For SIDIS the part $\Delta(k)$ will be expressed in terms of parton fields like $\Phi(p)$.*

depending on the lightcone fraction $x = p^+/P^+$, which is the fractional momentum when we take the lightcone parametrization

$$p = [p^-, p^+, p_T] = \Big[\frac{1}{xQ\sqrt{2}}, \frac{x\,Q}{x_B\sqrt{2}}, \boldsymbol{p}_\perp\Big],$$

and also appears in the (covariantly formulated) Sudakov expansion for the parton momentum. Choosing $P{\cdot}n = 1$ and $n^2 = 0$ we write

$$p = x\,P + p_T + \underbrace{(p{\cdot}P - x\,M^2)}_{\sigma}\,n, \tag{109}$$

where the term $x\,P \sim Q$, $p_T \sim M$ and $\sigma\,n \sim M^2/Q$. We have the exact relations $p{\cdot}p_T = p_T^2 = (p - x\,P)^2$. Note that if we rather keep $P{\cdot}n$ general, this is achieved by replacing $n \to n/P{\cdot}n$. With $P{\cdot}n = 1$, the momentum fraction is $x = p{\cdot}n$ and it is $\mathscr{O}(1)$. Note that one can construct *two* exactly conjugate null-vectors,

$$n_+ = P - \tfrac{1}{2}\,M^2\,n \quad \text{and} \quad n_- = n, \tag{110}$$

satifying $n_+{\cdot}n_- = 1$ and $n_+^2 = n_-^2 = 0$. These can be used to define light-cone components, $p^\pm = p{\cdot}n_\mp$ and allow also for inclusion of finite mass corrections. The symmetric and antisymmetric tensors in Eqs 26 and 27 become

$$g_T^{\mu\nu} \approx g^{\mu\nu} - P^{\{\mu}n^{\nu\}} \quad \text{and} \quad \epsilon_T^{\mu\nu} = \epsilon^{Pn\mu\nu}. \tag{111}$$

Using the Sudakov decomposition for the parton momentum p he delta function in Eq. 107 becomes proportional to $\delta(x - x_B)$. Comparing with the general form of the hadronic tensor, we read off (including now also the antiquark part)

$$2\,F_1(x_B) = 2M\,W_1(x_B, Q^2) = \sum_q e_q^2\,\left[f_1^q(x_B) + \bar{f}_1^q(x_B)\right], \tag{112}$$

with

$$f_1^q(x) \;=\; \int \frac{d\xi{\cdot}P}{4\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\not{n}\psi(\xi)|P,S\rangle\Big|_{\xi\cdot n=\xi_\perp=0} = \frac{1}{2}\,\mathrm{Tr}\,(\Phi(x)\not{n})\,, \tag{113}$$

$$\bar{f}_1^q(x) \;=\; \int \frac{d\xi{\cdot}P}{4\pi}\; e^{-ip\cdot\xi}\langle P,S|\overline{\psi}(0)\not{n}\psi(\xi)|P,S\rangle\Big|_{\xi\cdot n=\xi_\perp=0}\,, \tag{114}$$

often simply denoted as $q(x) = f_1^q(x)$, satisfying $\overline{q}(x) = -q(-x)$ (Exercise!). The result is (as expected) a light-cone correlation function of quark fields.

==========

Exercise: A choice of a different null-vector n', in principle leads to different x', σ' and $p_{T'}$ as well as different transverse projectors. With $P{\cdot}n' = 1$, implying $P{\cdot}\Delta n = 0$. Show that the differences vanish at order Q^0,

$$\mathscr{O}(Q^0): \qquad \Delta x = \Delta p_T^2 = \Delta n = 0. \tag{115}$$

With $\Delta n = n' - n$ arbitrary (of order $1/Q$) one easily shows that the changes Δx and Δp_T are related,

$$\mathscr{O}(Q^0): \qquad \Delta p_T = -\Delta x\,P \tag{116}$$

(corresponding to the validity of $xP + p_T \approx x'P + p_{T'}$ up to $\mathscr{O}(Q^0)$). Given two (hard) hadronic momenta P and P' one thus can (disregarding $1/Q^2$ mass corrections) use $P'/(P{\cdot}P')$ as the null-vector n for the hadron with momentum P. At $\mathscr{O}(Q^0)$ the momentum fractions $x \approx (p{\cdot}P')/(P{\cdot}P')$ thus is the same for any of the hard (hadronic) momenta P' involved in the process. Finally, show that the integration

$$\int d^4p = \int dx\,d^2p_T\,d\sigma = \int d(p{\cdot}n)\,d^2p_T\,d(p{\cdot}P), \tag{117}$$

is insensitive to the choice of n-vector.

==========

The operator in coordinate space

The parton result for the structure functions can also be derived by inserting free currents in the hadronic tensor for the current commutator and using the expression for the free field commutator.

==========

Exercise: Use the anticommutation relations for free quark fields, $\{\psi(\xi), \overline{\psi}(0)\} = \frac{1}{2\pi}\,\not{\partial}\,\delta(\xi^2)\,\epsilon(\xi^0)$ to derive for the $g_{\mu\nu}$ contribution in the current-current commutator for quarks

$$\begin{aligned}[J_\mu(\xi), J_\nu(0)] &= [:\overline{\psi}(\xi)\gamma_\mu\psi(\xi):, :\overline{\psi}(0)\gamma_\nu\psi(0):] \\ &= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(\xi^2)\,\epsilon(\xi^0)\right] : \overline{\psi}(\xi)\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\psi(\xi) : \,.\end{aligned} \tag{118}$$

An important feature, evident in the free-current commutator, is the light-cone dominance. By sandwiching the commutator between physical states and taking the Fourier transform, it is a straightforward calculation to obtain again the hadron tensor and the same result as in the diagrammatic approach above. Details can be found in Ref. [38].

==========

Interpretation as densities

To convince oneself that the above expressions for $f_1(x)$ and $\bar{f}_1(x)$ actually can be interpreted as quark momentum density one needs to realize that $\overline{\psi}(\xi)\gamma^+\psi(0) = \sqrt{2}\,\psi_+^\dagger(\xi)\psi_+(0)$ where $\psi_\pm = P_\pm\,\psi$ are projections obtained with projection operators $P_\pm = \frac{1}{2}\gamma^\mp\gamma^\pm$. One then can insert a complete set of states and obtain

$$\begin{aligned} f_1(x) &= \int \frac{d\xi^-}{2\pi\sqrt{2}}\; e^{ip\cdot\xi}\,\langle P,S|\psi_+^\dagger(0)\psi_+(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0} \\ &= \frac{1}{\sqrt{2}}\sum_n |\langle P_n|\psi_+|P\rangle|^2\;\delta\left(P_n^+ - (1-x)P^+\right), \end{aligned} \tag{119}$$

which represents the probability that a quark is annihilated from $|P\rangle$ giving a state $|n\rangle$ with $P_n^+ = (1-x)P^+$. As a further note, we mention that ψ_+ fields are actually the good fields of the light-cone quantized field theory. For these good fields one can write down the expansion in plane wave states with specified p^+ and p_T components, even in the interacting theory

Relation to forward amplitudes

The full hadronic tensor as a squared amplitude is also the imaginary part of a forward amplitude (optical theorem). The same relation holds for the hard (QCD) amplitude. A similar property also holds for the soft part after integration over the light-cone momentum $p^- = p\cdot P$. All operators in the correlator now are evaluated at light-cone time $\xi^+ = \xi\cdot n = 0$. This implies that time-ordering has become automatic. Thus not only the full hadronic tensor, but also the correlators $\Phi(x, p_T)$ and $\Phi(x)$, besides being densities, can be seen as time-ordered products of fields, which means related to the forward antiquark-hadron scattering amplitudes [27, 37]. This is important because amplitudes in a field theoretical setting have various analytic properties, e.g. related to unitarity. That the time-ordering is only evident in light-cone time is not a problem, because the ordering as such is covariant.

B. Distribution and fragmentation correlators in 1PI leptoproduction

We now consider the case in which one particle is detected in coincidence with the scattered lepton, *one-particle-inclusive* or 1PI leptoproduction. The kinematics of this process is already in the picture given before (Fig. 3). With a target hadron (momentum P) and a detected hadron h in the final state (momentum K_h) one has a situtation in which two hadrons are involved and the operator product expansion cannot be used. Within the framework of QCD and knowing that the photon or Z^0 current couples to the quarks, it is possible to write down a diagrammatic

expansion for leptoproduction, with in the deep inelastic limit ($Q^2 \to \infty$) as relevant diagrams only the ones given in Fig. 4 for 1-particle inclusive scattering (modulo collinear gluons to be discussed later).

In analogy with the case of inclusive scattering, we also in 1-particle inclusive scattering parametrize the momenta with the help of two lightlike vectors, which are choosen now along the hadron momenta,

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ K_h^2 = M_h^2 \\ 2\,P{\cdot}q = \frac{Q^2}{x_B} \\ 2\,K_h{\cdot}q = -z_h\,Q^2 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} K_h = \frac{z_h Q}{\sqrt{2}}\,n_- + \frac{M_h^2}{z_h\,Q\sqrt{2}}\,n_+ \\ \\ q = \frac{Q}{\sqrt{2}}\,n_- \;-\; \frac{Q}{\sqrt{2}}\,n_+ \;+\; q_T \\ \\ P = \frac{x_B M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right.$$

An additional invariants z_h comes in. Note that the expansion is appropriate for the socalled current fragmentation, in which case the produced hadron is *hard* with respect to the target momentum, i.e. $P \cdot K_h \sim Q^2$. The minus component p^- is irrelevant in the lower soft part, while the plus component k^+ is irrelevant in the upper soft part. Note that after the choice of P and K_h one can no longer omit a transverse component in the other vector, in the consideration above put in the momentum transfer q. One sees that one has (up to mass effects) a nonzero q_T,

$$q_T^\mu = q^\mu + x_B\,P^\mu - \frac{P_h^\mu}{z_h} \equiv -Q_T\,\hat{h}^\mu. \tag{120}$$

that is a new small (transverse) scale (small at least compared to Q^2), which can be determined experimentally from the external vectors q, P and K_h which are in general not collinear. The vector $\hat{h}$ defines the orientation of the hadronic plane (see Fig. 3). There are new structures in the cross section as discussed in the previous section. In the hard process final state partons decay into a jet, in the discussion of which we limit ourselves to the consideration of an identified hadronic state (which could in principle also be a multi-particle, e.g. two-pion, state).

An important consequence in the theoretical approach (Fig. 4) is that one can no longer simply integrate over the transverse components of the quark momenta. Parametrizing the quark momenta (omitting the 'small' components p^- and k^+, we have

$$p = x\,P + p_T \quad \text{and} \quad k = K_h/z_h - k_T, \tag{121}$$

which shows that the partonic energy-momentum conservation becomes

$$\delta^4(p+q-k) = \delta(p^+ + q^+)\delta(q^- - k^-)\delta^2(p_T - k_T + q_T) \approx (1/P{\cdot}P_h)\delta(x - x_B)\delta(z^{-1} - z_h^{-1})\delta^2(q_T + p_T - k_T),$$

in particular $q_T = k_T - p_T$, i.e. one can obtain information on quark transverse momenta (even if it is convoluted) from a kinematically accessible observable q_T, measuring the noncollinearity in the process.

==========

Exercise: Show that there are various ways to determine the scaling variables if Q^2 is large. One has $x_B \equiv Q^2/2P{\cdot}q \approx -q{\cdot}K_h/P{\cdot}K_h$ and $z_h = P{\cdot}K_h/P{\cdot}q \approx -2K_h{\cdot}q/Q^2$.

==========

We can use our kinematics to evaluate the tree level result for the hadron tensor $\mathcal{W}_{\mu\nu}$ by inserting the free currents $j_\mu(x) =: \overline{\psi}(x)\gamma_\mu\psi(x) :$. The nonlocal product of these currents then can be rewritten in the following way,

$$\begin{aligned} &2M\mathcal{W}_{\mu\nu}(q;PS;P_hS_h) = \\ &\quad= \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle PS| : \overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x) : \sum_X |X;P_hS_h\rangle\,\langle X;P_hS_h| : \overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0) : |PS\rangle \\ &\quad= \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle(\gamma_\mu)_{jk}\langle 0|\psi_k(x)\sum_X |X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li} \\ &\qquad+ \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle PS|\psi_k(x)\overline{\psi}_l(0)|PS\rangle(\gamma_\nu)_{li}\langle 0|\overline{\psi}_j(x)\sum_X |X;P_hS_h\rangle\langle X;P_hS_h|\psi_i(0)|0\rangle(\gamma_\mu)_{jk} + \ldots \\ &\quad= \int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\,(\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu) + \left\{\begin{array}{c} q \leftrightarrow -q \\ \mu \leftrightarrow \nu \end{array}\right\}, \end{aligned} \tag{122}$$

where we encounter not only a correlator describing quark distributions but also a correlator describing the fragmentation process,

$$\Phi_{ij}(p) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ip\cdot\xi} \, \langle PS|\overline{\psi}_j(0)\psi_i(\xi)|PS\rangle, \tag{123}$$

$$\Delta_{kl}(k) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ik\cdot\xi} \, \langle 0|\psi_k(\xi) \sum_X |X; P_h S_h\rangle\langle X; P_h S_h|\overline{\psi}_l(0)|0\rangle. \tag{124}$$

Note that in Φ (quark production) a summation over colors is assumed, while in Δ (quark decay) an averaging over colors is assumed. The quantities Φ and Δ correspond to the blobs in Fig. 4 and parametrize the 'soft collinear' physics, leading to the definitions of distribution and fragmentation functions [25, 56, 57]. 'Soft collinear' refers to all invariants of momenta being small as compared to the hard scale, i.e. for $\Phi(p)$ one has $p^2 \sim p\cdot P \sim P^2 = M^2 \ll Q^2$.

Using the kinematic approximations in the momentum conservation relation $p + q = k$ then gives

$$2M\mathcal{W}_{\mu\nu}(q; PS; P_h S_h) = \int d^2p_T \, d^2k_T \, \delta^2(p_T + q_T - k_T) \; \mathrm{Tr}\left(\Phi(x, p_T)\gamma_\mu \Delta(z, k_T)\gamma_\nu\right), \tag{125}$$

with convoluted transverse momentum dependent (TMD) correlators for distributions and fragmentation [25],

$$\Phi_{ij}(x, p_T; n) = \int \frac{d\xi^- \, d^2\xi_T}{(2\pi)^3} \; e^{i\,p\cdot\xi} \; \langle P|\overline{\psi}_j(0)\, \psi_i(\xi)|P\rangle\Big|_{\xi^+=0}, \tag{126}$$

$$\Delta_{ij}(z, k_T; n) = \sum_X \int \frac{d\xi^+ d^2\xi_T}{(2\pi)^3} \, e^{ik\cdot\xi} \, \mathrm{Tr}\langle 0|\psi_i(\xi)|P_h, X\rangle\langle P_h, X|\overline{\psi}_j(0)|0\rangle\Big|_{\xi^-=0}, \tag{127}$$

where we have suppressed the hadron momenta.

We first of all notice that also in these matrix elements time ordering is not relevant, since they are still at equal lightcone time, although the nonlocality only is at a light-front rather than at the lightcone. A second point to note is that in general many more diagrams have to be considered in evaluating the hadron tensors, but in the deep inelastic limit they can be neglected or considered as corrections to the soft blobs. We return to this later. One specific contribution that we want to mention here are target fragmentation parts involving matrix elements of the form $\langle PS|\psi_j(x)\sum_X |X; P_h S_h\rangle \; \langle X; P_h S_h|\psi_i(0)|PS\rangle$, known as *fracture functions*. They are relevant in the situation where $P\cdot P_h \sim M^2$ (target fragmentation region), which at sufficiently high energies can be distinguished experimentally from the region that we are interested in, $P\cdot P_h \sim Q^2$ (current fragmentation region).

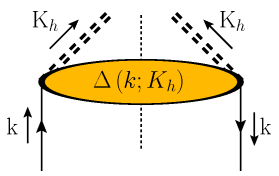
Returning to the language that we used in our introductory section for the distribution functions, we note that in the fragmentation process of a parton (with momentum k) into hadrons (with momentum K_h) we combine the squared decay matrix elements in a (fragmentation) correlator, for quarks

$$\begin{aligned}\Delta_{ij}(k; K_h) &= \sum_X \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ik\cdot\xi} \, \langle 0|\psi_i(\xi)|K_h, X\rangle\langle K_h, X|\overline{\psi}_j(0)|0\rangle \\ &= \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ik\cdot\xi} \, \langle 0|\psi_i(\xi) a_h^\dagger a_h \overline{\psi}_j(0)|0\rangle,\end{aligned} \tag{128}$$

where an averaging over color indices is implicit. In a momentum space representation for the operators, we have

$$(2\pi)^4\delta^4(k - k') \, \Delta_{ij}(k; K_h) = \frac{1}{(2\pi)^4} \sum_X \langle 0|\psi_i(k)|K_h, X\rangle\langle K_h, X|\overline{\psi}_j(k')|0\rangle. \tag{129}$$

Pictorially we have



In particular, we note that in fragmentation correlators, one no longer deals with plane wave hadronic states, but with out states $|K_h, X\rangle$. There are a a number of other subtleties in these definitions. The use of intermediate states

X and in addition one specified state with momentum K_h needs some explanation. First note that the unit operator can be written as

$$\mathbb{I} \equiv \sum_X |X\rangle\langle X| = \sum_{n=0}^{\infty} \mathbb{I}_n \tag{130}$$

with (assuming at this stage one type/flavor of hadrons)

$$\mathbb{I}_n = \frac{1}{n!} \int d\widetilde{K}_1 \ldots d\widetilde{K}_n \; a^\dagger(K_1) \ldots a^\dagger(K_n)|0\rangle\langle 0|a(K_1)\ldots a(K_n) \tag{131}$$

containing the n-particle states (with $d\widetilde{K}$ being the invariant one-particle phase-space). Thus the summation appearing in the definition of the fragmentation correlator is for a given hadron sector

$$\begin{aligned} \sum_X |P_h, X\rangle\langle P_h, X| &= |P_h\rangle\langle P_h| + \int d\widetilde{K}_1 \; |P_h, K_1\rangle\langle P_h, K_1| + \frac{1}{2!}\int d\widetilde{K}_1 d\widetilde{K}_2 \; |P_h, K_1, K_2\rangle\langle P_h, K_1, K_2| + \ldots \\ &= a_h^\dagger \mathbb{I}\, a_h = a_h^\dagger a_h. \end{aligned} \tag{132}$$

After integrating over P_h one obtains

$$\int d\tilde{P}_h \sum_X |P_h, X\rangle\langle P_h, X| = \sum_{n=0}^{\infty} n\, \mathbb{I}_n = \mathbb{N}_h, \tag{133}$$

which is the number operator for hadrons h. This will become relevant when one integrates over the phase-space of particles in the final state to go from 1-particle inclusive to inclusive scattering processes. Another useful operator is the momentum operator

$$\begin{aligned} \hat{P}^\mu &= \int d\widetilde{K}_1 \; |K_1\rangle K_1^\mu \langle K_1| + \frac{1}{2!}\int d\widetilde{K}_1 d\widetilde{K}_2 \; |K_1, K_2\rangle (K_1^\mu + K_2^\mu)\langle K_1, K_2| + \ldots \\ &= \sum_{h,X} \int d\widetilde{K}_h |K_h, X\rangle K_h^\mu \langle K_h, X|. \end{aligned} \tag{134}$$

C. Inclusion of spin in distribution and fragmentation correlators

In principle hadrons could be polarized, having additional degrees of freedom, $|P, \alpha\rangle$, etc. In order to treat the spin of initial states, one then can explicitly work with distribution correlators in the hadron spin-space,

$$\Phi_{ij,\beta\alpha}(p; P) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{i\,p\cdot\xi} \; \langle P, \beta|\overline{\psi}_j(0)\, \psi_i(\xi)|P, \alpha\rangle. \tag{135}$$

It is convenient to include the off-diagonal elements in the definition. Having a non-pure initial state described by a spin density matrix $\rho(P, S) = \sum_\alpha |P, \alpha\rangle \mathrm{Prob}_\alpha \langle P, \alpha|$ one then finds the spin-dependent correlator

$$\Phi_{ij}(p; P, S) \equiv \rho_{\alpha\beta}(P, S)\, \Phi_{ij,\beta\alpha}(p; P). \tag{136}$$

A single spin vector S is sufficient to parameterize the density matrix for a spin 1/2 hadron. For hadrons with higher spins one needs additional parameters (e.g. a spin vector *and* a symmetric traceless tensor to describe spin 1).

For fragmentation correlators, the role of spin is different. In that case not only the specific kind of hadron in the final state, but also its spin state is fixed. This means that besides having Dirac structure, we also include spin states (off-diagonal to remain as general as possible)

$$\Delta_{ij,\beta\alpha}(k; P_h) = \sum_X \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ik\cdot\xi} \; \langle 0|\psi_i(\xi)|P_h, \alpha, X\rangle\langle P_h, \beta, X|\overline{\psi}_j(0)|0\rangle. \tag{137}$$

In many applications we will use a spin-dependent fragmentation correlator $\Delta_{ij}(k; P_h, S_h)$ by defining

$$\Delta_{ij}(k; P_h, S_h) \equiv (2s_h + 1)\, \Delta_{ij,\beta\alpha}(k; P_h)\, \rho_{\alpha\beta}(P_h, S_h), \tag{138}$$

where $\rho(P_h, S_h)$ is the usual spin density matrix. The factor $(2s_h + 1)$ assures that for a spin vector $S_h = 0$ one ends up with a sum over spins for the produced hadron. Depending on the spin the parametrization of density matrix may require beside the spin vector polarization tensors of higher rank. Note that in most applications S_h (or other tensors) will be replaced by analyzing power $A_h(P_h, f)$ of the decay channel (with f representing the final state variables) of the produced particle, e.g. in the case of production of Λ's or ρ's, rather than the tunable polarization for initial states.

In order to find a covariant parametrization of the spin vector in analogy with the decomposition of the parton momentum in terms of x and p_T, one must go back to its definition. The hadron spin vector is used to parametrize the spin density matrix and satisfies $S^2 = -1$ and $P{\cdot}S = 0$. Thus one can write

$$P = P^+ n_+ + \frac{M^2}{2P^+}, \tag{139}$$

$$S = S_L \frac{P^+}{M} n_+ + S_T + S_L \frac{M}{2P^+} n_-, \tag{140}$$

satisfying $S_L^2 + \boldsymbol{S}_T^2 = 1$. The quantity S_L is called the *light-cone helicity*, S_T the *transverse spin* vector.Note that this is a covariant way of defining the components. The rest-frame spin vector $(0, \boldsymbol{S})$ has $\boldsymbol{S} = (\boldsymbol{S}_T, S_L)$. In the infinite momentum frame, S_L is the helicity. Using just a single lightlike vector n one can write

$$S = S_L \frac{P}{M} + S_T - S_L\, M\, n \;\approx\; S_L \frac{P}{M} + S_T, \tag{141}$$

with

$$S_L = M\, S{\cdot}n \qquad \text{and} \qquad S_T^\mu = g_T^{\mu\nu} S_\nu. \tag{142}$$

The spin vector is used in the parametrization of the density matrix.

D. Tree-level treatment of Drell-Yan scattering

The hadronic tensor for DY is rewritten as

$$\begin{aligned}
\mathscr{W}_{\mu\nu}(q; P_A S_A; P_B S_B) &= \frac{1}{(2\pi)^4} \int \frac{d^3 P_X}{(2\pi)^3 2P_X^0} (2\pi)^4 \delta^4(P_A + P_B - P_X - q) \\
&\qquad \times \langle P_A S_A; P_B S_B | J_\mu(0) | P_X \rangle \langle P_X | J_\nu(0) | P_A S_A; P_B S_B \rangle, \\
&= \frac{1}{(2\pi)^4} \int d^4x\; e^{-iq\cdot x}\, \langle P_A S_A; P_B S_B | J_\mu(x)\; J_\nu(0) | P_A S_A; P_B S_B \rangle,
\end{aligned}$$

which in tree approximation (Born terms) becomes

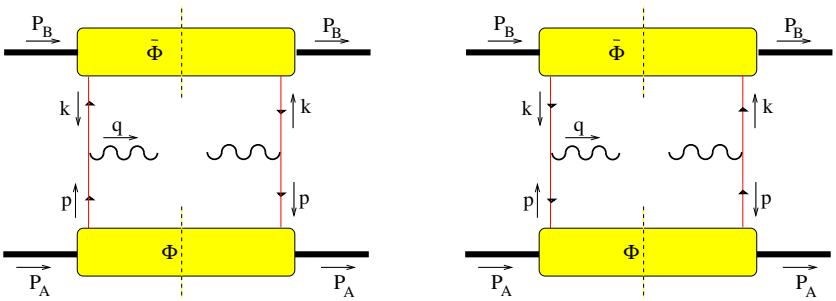


FIG. 5: Born diagrams for Drell-Yan scattering

$$
\begin{aligned}
\mathscr{W}_{\mu\nu} &= \frac{1}{(2\pi)^4}\int d^4x\; e^{-iq\cdot x}\,\langle P_A S_A; P_B S_B| : \overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x) : : \overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0) : |P_A S_A; P_B S_B\rangle \\
&= \frac{1}{(2\pi)^4}\,\frac{1}{3}\int d^4x\; e^{-iq\cdot x}\,\langle P_A S_A|\overline{\psi}_j(x)\psi_i(0)|P_A S_A\rangle(\gamma_\mu)_{jk}\langle P_B S_B|\psi_k(x)\overline{\psi}_l(0)|P_B S_B\rangle(\gamma_\nu)_{li} \\
&\quad +\frac{1}{(2\pi)^4}\,\frac{1}{3}\int d^4x\; e^{-iq\cdot x}\,\langle P_A S_A|\psi_k(x)\overline{\psi}_l(0)|P_A S_A\rangle(\gamma_\nu)_{li}\langle P_B S_B|\overline{\psi}_j(x)\psi_i(0)|P_B S_B\rangle(\gamma_\mu)_{jk}, \\
&= \frac{1}{3}\int d^4p\, d^4k\, \delta^4(p+k-q)\;\mathrm{Tr}\left(\Phi(p)\gamma_\mu\overline{\Phi}(k)\gamma_\nu\right) + \left\{\begin{array}{c} q\leftrightarrow -q \\ \mu\leftrightarrow\nu \end{array}\right\},
\end{aligned}
\tag{143}
$$

where we have used

$$
\begin{aligned}
\Phi_{ij}(p) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-ip\cdot\xi}\,\langle P_A S_A|\overline{\psi}_j(\xi)\psi_i(0)|P_B S_B\rangle, \\
\overline{\Phi}_{kl}(k) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-ik\cdot\xi}\,\langle P_B S_B|\psi_k(\xi)\overline{\psi}_l(0)|P_B S_B\rangle,
\end{aligned}
$$

and its symmetry properties. Note that since in both Φ (quark production) and $\overline{\Phi}$ (antiquark production) summations over colors are assumed, a factor $1/N_c = 1/3$ appears in the result in Eq. 143

Using the lightcone representation of the momenta it is easy to see that if the quark momenta in the matrix elements Φ are limited, i.e. p^2, $p\cdot P_A$ are of hadronic scale and similarly in the matrix elements $\overline{\Phi}$ for k^2 and $k\cdot P_B$, one can write the delta function up to $\mathscr{O}(1/Q^2)$ as

$$
\delta^4(p+k-q) \approx \delta(p^+ - q^+)\,\delta(k^- - q^-)\,\delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T),
\tag{144}
$$

where P_A and P_B define (up to mass corrections) the directions n_+ and n_- and $q_T = q - x_A\,P_A - x_B\,P_B$ with $x_A = q\cdot P_B/P_A\cdot P_B$ and $x_B = q\cdot P_A/P_A\cdot P_B$ (see previous section).

The result in leading order is then

$$
\begin{aligned}
\mathscr{W}_{\mu\nu} &= \frac{1}{3}\int d^2p_T\, d^2k_T \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\mathrm{Tr}\left([\textstyle\int dp^-\Phi(p)]\gamma_\mu[\int dk^+\overline{\Phi}(k)]\gamma_\nu\right)\Big|_{\substack{p^+ = x_A P_A^+ \\ k^- = x_B P_B^-}} \\
&= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} + \frac{Z_\mu Z_\nu}{Z^2}\right)\frac{1}{3}\,I[f_1\,\overline{f}_1],
\end{aligned}
\tag{145}
$$

where

$$
I[f_1\,\overline{f}_1] = \int d^2p_T\, d^2k_T\, \delta^2(\boldsymbol{p}_T + \boldsymbol{k}_T - \boldsymbol{q}_T)\, f_1(x_A, p_T)\overline{f}_1(x_B, k_T).
\tag{146}
$$

Using the contraction with the leptonic tensor,

$$
\begin{aligned}
L^{\mu\nu}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} + \frac{Z_\mu Z_\nu}{Z^2}\right) &= 4Q^2\left(\frac{1}{2} - y + y^2\right) \\
&= Q^2\left(1 + \cos^2\theta_{\mu\mu}\right),
\end{aligned}
\tag{147}
$$

the cross section becomes

$$
\frac{d\sigma(AB\to\mu^+\mu^- X)}{dx_A\,dx_B\,d^2q_T\,d\Omega_{\mu\mu}} = \frac{\alpha^2}{12\,Q^2}\,\left(1+\cos^2\theta_{\mu\mu}\right)\,I[f_1\,\overline{f}_1].
\tag{148}
$$

Integrated over the transverse momenta of the produced mu pair,

$$
\frac{d\sigma(AB\to\mu^+\mu^- X)}{dx_A\,dx_B\,d\Omega_{\mu\mu}} = \frac{\alpha^2}{12\,Q^2}\,\left(1+\cos^2\theta_{\mu\mu}\right)\,f_1(x_A)\,\overline{f}_1(x_B),
\tag{149}
$$

and integrated over the muon angular distribution,

$$
\frac{d\sigma}{dx_A\,dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\,f_1(x_A)\,\overline{f}_1(x_B),
\tag{150}
$$

or including the summation over quarks and antiquarks,

$$\frac{d\sigma(AB\to\mu^+\mu^-X)}{dx_A\,dx_B} = \frac{4\pi\,\alpha^2}{9\,Q^2}\sum_a e_a^2\,f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)$$
$$= \frac{1}{3}\sum_a f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\hat{\sigma}(a\overline{a}\to\mu^-\mu^+), \tag{151}$$

where the quark-antiquark annihilation cross section is given by

$$\hat{\sigma}(a\overline{a}\to\mu^-\mu^+) = \frac{4\pi\alpha^2}{3Q^2}\,e_a^2. \tag{152}$$

and the factor 1/3 multiplying the summation is the color factor that can be naturally understood because only quarks of the same color can annihilate and we have seen that the definitions of the quark distribution functions included a summation over colors.

Introducing the virtuality of the photon (i.e. the invariant mass $\hat{s}$ of the produced mu pair) as a variable one can consider the Drell-Yan cross section as a function of s. Writing

$$\frac{d\hat{\sigma}}{dQ^2}\,(a\overline{a}\to\mu^-\mu^+) = \frac{4\pi\,\alpha^2}{3\,Q^2}\,e_a^2\,\delta(\hat{s}-Q^2), \tag{153}$$

one has

$$\frac{d\sigma(AB\to\mu^-\mu^+X)}{dQ^2\,dx_A\,dx_B} = \frac{1}{3}\sum_a f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\frac{d\hat{\sigma}}{dQ^2}(a\overline{a}\to\mu^-\mu^+)$$
$$= \frac{4\pi\alpha^2}{9Q^4}\sum_a e_a^2\,f_{1\,a/A}(x_A)\,\overline{f}_{1\,\overline{a}/B}(x_B)\,\delta\left(\frac{s}{Q^2}\,x_Ax_B-1\right), \tag{154}$$

which exhibits explicitly the scaling in $\tau = s/Q^2$ for the cross section $Q^4\,d\sigma/dQ^2$.

==========

Exercise: Show that by introducing

$$\Phi(x;\boldsymbol{b}_T) = \int d^2p_T\;\exp(i\boldsymbol{p}_T\cdot\boldsymbol{b}_T)\,\Phi(x,\boldsymbol{p}_T) = \int\frac{d\xi^-}{2\pi}\;e^{i\,p^+\xi^-}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi^-,b_T)|P\rangle\Big|_{\xi^+=0}, \tag{155}$$

one can rewrite

$$I[\Phi_A\,\overline{\Phi}_B] = \int d^2p_T\,d^2k_T\,\delta^2(\boldsymbol{p}_T+\boldsymbol{k}_T-\boldsymbol{q}_T)\,\Phi_A(x_A,\boldsymbol{p}_T)\overline{\Phi}_B(x_B,\boldsymbol{k}_T)$$
$$= \int d^2b_T\;\exp(-i\boldsymbol{q}_T\cdot\boldsymbol{b}_T)\,\Phi_A(x_A;\boldsymbol{b}_T)\,\overline{\Phi}(x_B;\boldsymbol{b}_T). \tag{156}$$

The convolution in transverse momenta becomes a product in impact parameter space. Note that $\Phi(x) = \Phi(x;\mathbf{0}_T)$ and $\int d^2q_T\,I[\Phi_A\overline{\Phi}_B] = \Phi_A(x_A)\overline{\Phi}_B(x_B)$.

==========

E. Multi-parton distribution functions

In order to study this phenomenon, we look at *Double Drell-Yan scattering*, in which two independent hard scatterings occur among the partons of just two colliding hadrons. The subprocesses in the hard amplitude are parton_1$(p_1-r/2)$ + antiparton_1$(k_1+r/2)\to\gamma_1^*(q_1)$ and parton_2$(p_2+r/2)$ + antiparton_2$(k_2-r/2)\to\gamma_2^*(q_2)$, where the momentum r is an internal momentum in the amplitude. Only the momenta P_A, P_B, $P_A-p_1-p_2$, $P_B-k_1-k_2$, $q_1=p_1+k_1$ and $q_2=p_2+k_2$ are external momenta on the amplitude. Therefore one can allow a

difference in the partonic momenta of partons in amplitude and conjugate amplitude, i.e. in the conjugate amplitude one uses the opposite sign for $r/2$ (make a picture). The soft part to be considered is the double parton distribution

$$\begin{aligned}
&(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')\Phi_{i_1i_2,i_1'i_2'}(p_1,p_2,r;P_A)\\
&\quad=\frac{1}{(2\pi)^8}\langle P_A|\overline{\psi}_{i_1'}(p_1')\overline{\psi}_{i_2'}(p_2')\psi_{i_2}(p_2)\psi_{i_1}(p_1)|P_A\rangle\\
&\quad=\frac{1}{(2\pi)^8}\langle P_A|\overline{\psi}_{i_1'}(p_1+\tfrac{1}{2}r)\overline{\psi}_{i_2'}(p_2-\tfrac{1}{2}r)\psi_{i_2}(p_2+\tfrac{1}{2}r)\psi_{i_1}(p_1-\tfrac{1}{2}r)|P_A\rangle.
\end{aligned}\tag{157}$$

This has to be combined with a second correlator

$$\begin{aligned}
&(2\pi)^4\delta^4(k_1+k_2-k_1'-k_2')\overline{\Phi}_{j_1j_2,j_1'j_2'}(k_1,k_2,-r;P_B)\\
&\quad=\frac{1}{(2\pi)^8}\langle P_B|\overline{\psi}_{j_1'}(k_1-\tfrac{1}{2}r)\overline{\psi}_{j_2'}(k_2+\tfrac{1}{2}r)\psi_{j_2}(k_2-\tfrac{1}{2}r)\psi_{j_1}(k_1+\tfrac{1}{2}r)|P_B\rangle.
\end{aligned}\tag{158}$$

a hard cross section $\hat\sigma_1(q_1)\hat\sigma_2(q_2)$ and an integration

$$\begin{aligned}
&\int d^4p_1\,d^4p_2\,d^4k_1\,d^4k_2\,\frac{d^4r}{(2\pi)^4}\,\delta^4(p_1+k_1-q_1)\,\delta^4(p_2+k_2-q_2)\\
&\quad=\int d^4p_1\,d^4p_2\,d^4k_1\,d^4k_2\,\frac{d^4r}{(2\pi)^4}\\
&\qquad\times\delta(p_1^+-q_1^+)\delta(p_2^+-q_2^+)\,\delta(k_1^--q_1^-)\delta(k_2^--q_2^-)\,\delta^2(p_{1T}+k_{1T}-q_{1T})\,\delta^2(p_{2T}+k_{2T}-q_{2T}),
\end{aligned}$$

where we have used that

$$p_1^-\sim p_2^-\sim r^-\sim k_1^+\sim k_2^+\sim r^+\ll Q_1\equiv\sqrt{q_1^2}\sim Q_2\equiv\sqrt{q_2^2}.$$

We write

$$q_1^+=x_{A1}\,P_A^+\quad\text{and}\quad q_2^+=x_{A2}\,P_A^+,\quad q_1^-=x_{B1}\,P_B^-\quad\text{and}\quad q_2^-=x_{B2}\,P_B^-,\tag{159}$$

which implies

$$x_{A1}=\frac{q_1\cdot P_B}{P_A\cdot P_B},\quad x_{A2}=\frac{q_2\cdot P_B}{P_A\cdot P_B},\quad x_{B1}=\frac{q_1\cdot P_A}{P_A\cdot P_B},\quad x_{B2}=\frac{q_2\cdot P_A}{P_A\cdot P_B},\tag{160}$$

$$\hat s_1=q_1^2=x_{A1}\,x_{A2}\,s\quad\text{and}\quad\hat s_2=q_2^2=x_{A2}\,x_{B2}\,s.\tag{161}$$

Omitting Dirac indices, we get for the cross section at measured q_{1T} and q_{2T},

$$\begin{aligned}
\frac{d\sigma(P_A,P_B,q_1,q_2)}{d^4q_1\,d^4q_2}\;&\propto\;\int d^2p_{1T}d^2k_{1T}\,\delta^2(p_{1T}+k_{1T}-q_{1T})\int d^2p_{2T}d^2k_{2T}\,\delta^2(p_{2T}+k_{2T}-q_{2T})\int\frac{d^2r_T}{(2\pi)^2}\\
&\quad\times\left(\int dp_1^-dp_2^-dr^-\,\Phi(p_1,p_2,r;P_A)\right)\frac{d\hat\sigma_1}{d^4q_1}\times\left(\int dk_1^+dk_2^+dr^+\,\overline{\Phi}(k_1,k_2,-r;P_B)\right)\frac{d\hat\sigma_2}{d^4q_2}\\
&\propto\;\int d^2p_{1T}d^2k_{1T}\,\delta^2(p_{1T}+k_{1T}-q_{1T})\int d^2p_{2T}d^2k_{2T}\,\delta^2(p_{2T}+k_{2T}-q_{2T})\int\frac{d^2r_T}{(2\pi)^2}\\
&\quad\times\Phi(x_{A1},x_{A2},p_{1T},p_{2T},r_T;P_A)\frac{d\hat\sigma_1}{d^4q_1}\times\overline{\Phi}(x_{A1},x_{A2},p_{1T},p_{2T},-r_T;P_B))\frac{d\hat\sigma_2}{d^4q_2},
\end{aligned}\tag{162}$$

with de TMD double parton distribution function (only given for one hadron)

$$\begin{aligned}
\Phi(x_1,x_2,p_{1T},p_{2T},r_T)\;=\;&\int\frac{d\xi_1^-d\xi_2^-d^2\xi_{1T}d^2\xi_{2T}}{(2\pi)^6}\,\frac{d^2\eta_T}{(2\pi)^2}\;e^{i(p_1\cdot\xi_1+p_2\cdot\xi_2+r\cdot\eta)}\\
&\times\langle P_A|\overline{\psi}_{i_1'}(-\tfrac{1}{2}\xi_1)\psi_{i_1}(\tfrac{1}{2}\xi_1)\,\overline{\psi}_{i_2'}(\eta-\tfrac{1}{2}\xi_2)\psi_{i_2}(\eta+\tfrac{1}{2}\xi_2)|P_A\rangle\Big|_{\xi_1^+=\xi_2^+=\eta^+=\eta^-=0}.
\end{aligned}\tag{163}$$

Integrated over transverse momenta q_{1T} and q_{2T}, the cross sections are

$$\frac{d\sigma(P_A,P_B,q_1,q_2)}{dx_{A1}dx_{A2}dx_{B1}dx_{B2}}\;\propto\;\int\frac{d^2r_T}{(2\pi)^2}\;\Phi_A(x_{A1},x_{A2},r_T)\frac{d\hat\sigma_1}{dx_{A1}dx_{B1}}\,\overline{\Phi}_B(x_{B1},x_{B2},-r_T)\frac{d\hat\sigma_2}{dx_{A2}dx_{B2}},\tag{164}$$

with collinear double parton distribution function (only given for one hadron)

$$\Phi(x_1,x_2,r_T) \;=\; \int \frac{d\xi_1^- d\xi_2^-}{(2\pi)^2}\,\frac{d^2\eta_T}{(2\pi)^2}\; e^{i(p_1\cdot\xi_1+p_2\cdot\xi_2+r\cdot\eta)} \times \langle P_A|\overline{\psi}_{i_1'}(-\tfrac{1}{2}\xi_1)\psi_{i_1}(\tfrac{1}{2}\xi_1)\,\overline{\psi}_{i_2'}(\eta-\tfrac{1}{2}\xi_2)\psi_{i_2}(\eta+\tfrac{1}{2}\xi_2)|P_A\rangle\Big|_{\xi_1^+=\xi_{1T}=\xi_2^+=\xi_{2T}=\eta^+=\eta^-=0}. \tag{165}$$

==========

Exercise: The appropriate double parton TMD functions in impact parameter space are

$$\Phi(x_1,x_2;\boldsymbol{b}_{1T},\boldsymbol{b}_{2T};\boldsymbol{r}_T) \;=\; \int \frac{d\xi_1^- d\xi_2^-}{(2\pi)^2}\,\frac{d^2\eta_T}{(2\pi)^2}\; e^{i(p_1^+\xi_1^-+p_2^+\xi_2^-+r\cdot\eta)} \times \langle P_A|\overline{\psi}_{i_1'}(-\tfrac{1}{2}\xi_1)\psi_{i_1}(\tfrac{1}{2}\xi_1)\,\overline{\psi}_{i_2'}(\eta-\tfrac{1}{2}\xi_2)\psi_{i_2}(\eta+\tfrac{1}{2}\xi_2)|P_A\rangle\Big|_{\substack{\xi_1^+=\xi_2^+=\eta^+=\eta^-=0,\\ \xi_{1T}=b_{1T},\xi_{2T}=b_{2T}}}.$$

==========

Exercise: Including also the Fourier transform in $\boldsymbol{r}_T$,

$$\begin{aligned}\Phi(x_1,x_2;\boldsymbol{b}_{1T},\boldsymbol{b}_{2T};\boldsymbol{y}_T) \;&=\; \int d^2r_T\;\exp(i\boldsymbol{r}_T\cdot\boldsymbol{y}_T)\,\Phi(x_1,x_2;\boldsymbol{b}_{1T},\boldsymbol{b}_{2T};\boldsymbol{r}_T) \qquad (166)\\ &=\; \int \frac{d\xi_1^- d\xi_2^-}{(2\pi)^2}\,\frac{d^2\eta_T}{(2\pi)^2}\; e^{i(p_1^+\xi_1^-+p_2^+\xi_2^-)} \times \langle P_A|\overline{\psi}_{i_1'}(-\tfrac{1}{2}\xi_1)\psi_{i_1}(\tfrac{1}{2}\xi_1)\,\overline{\psi}_{i_2'}(\eta-\tfrac{1}{2}\xi_2)\psi_{i_2}(\eta+\tfrac{1}{2}\xi_2)|P_A\rangle\Big|_{\substack{\xi_1^+=\xi_2^+=\eta^+=\eta^-=0,\\ \xi_{1T}=b_{1T},\xi_{2T}=b_{2T},\eta_T=y_T}}.\end{aligned}$$

one gets

$$\frac{d\sigma(P_A,P_B,q_1,q_2)}{dx_{A1}dx_{A2}dx_{B1}dx_{B2}} \;\propto\; \int d^2y_T\;\Phi_A(x_{A1},x_{A2};\boldsymbol{0}_{1T},\boldsymbol{0}_{2T};\boldsymbol{y}_T)\frac{d\hat{\sigma}_1}{dx_{A1}dx_{B1}}\,\overline{\Phi}_B(x_{B1},x_{B2};\boldsymbol{0}_{1T},\boldsymbol{0}_{2T};\boldsymbol{y}_T)\frac{d\hat{\sigma}_2}{dx_{A2}dx_{B2}}, \tag{167}$$

One has impact parameter distributions with relative impact parameter being zero and common impact parameter being integrated over.

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IV. PROPERTIES AND PARAMETRIZATIONS OF THE CORRELATION FUNCTIONS

A. Collinear parton distributions

The form of Φ is constrained by hermiticity, parity and time-reversal invariance. The quantity depends besides the quark momentum p on the target momentum P and the spin vector S and one must have

$$[\text{Hermiticity}] \quad \Rightarrow \quad \Phi^\dagger(p,P,S) = \gamma_0\,\Phi(p,P,S)\,\gamma_0, \tag{168}$$

$$[\text{Parity}] \quad \Rightarrow \quad \Phi(p,P,S) = \gamma_0\,\Phi(\bar{p},\bar{P},-\bar{S})\,\gamma_0, \tag{169}$$

$$[\text{Time reversal}] \;\Rightarrow\; \Phi^*(p,P,S) = (-i\gamma_5 C)\,\Phi(\bar{p},\bar{P},\bar{S})\,(-i\gamma_5 C), \tag{170}$$

where $C = i\gamma^2\gamma_0$, $-i\gamma_5 C = i\gamma^1\gamma^3$ and $\bar{p} = (p^0, -\boldsymbol{p})$.

To obtain the leading contribution in inclusive deep inelastic scattering one can integrate over the component p^- and the transverse momenta (see discussion in the section where the parton model has been derived). This integration restricts the nonlocality in $\Phi(p)$. When one wants to calculate the leading order in $1/Q$ for a hard process, one looks for leading parts in M/P^+ because $P^+ \propto Q$. The leading contribution [39] in the integrated part turns out to be proportional to $(M/P^+)^0$, given by

$$\Phi(x) = \frac{1}{2}\left\{ f_1(x)\,\not{n}_+ + S_L\,g_1(x)\,\gamma_5\,\not{n}_+ + h_1(x)\,\frac{\gamma_5\,[\not{S}_\perp, \not{n}_+]}{2} \right\}. \tag{171}$$

where one can also write $\not{n}_+ = \not{P}$. The precise expression of the functions $f_1(x)$, etc. as integrals over the amplitudes can be easily written down after tracing with the appropriate Dirac matrix,

$$f_1(x) \;=\; \int \frac{d\xi^-}{4\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{172}$$

$$S_L\,g_1(x) \;=\; \int \frac{d\xi^-}{4\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{173}$$

$$S_T^i\,h_1(x) \;=\; \int \frac{d\xi^-}{4\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\,i\sigma^{i+}\gamma_5\,\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{174}$$

where one can also write $\xi^- = \xi\cdot P$. Including flavor indices, the functions $f_1^q(x) = q(x)$ and $g_1^q(x) = \Delta q(x)$ are precisely the functions that we encountered before.

The third function in the above parametrization is known as *transversity* or *transverse spin distribution* [2, 26]. Including flavor indices one also denotes $h_1^q(x) = \delta q(x)$. In the same way as we have seen for $f_1(x)$ and $g_1(x)$, the function h_1 can be interpreted as a density, but one needs instead of the projectors on quark chirality states, $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$, those on quark transverse spin states, $P_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^i\gamma_5)$. One has

$$f_1(x) = f_{1R}(x) + f_{1L}(x) = f_{1\uparrow}(x) + f_{1\downarrow}(x), \tag{175}$$

$$g_1(x) = f_{1R}(x) - f_{1L}(x), \tag{176}$$

$$h_1(x) = f_{1\uparrow}(x) - f_{1\downarrow}(x). \tag{177}$$

This results in some trivial bounds such as $f_1(x) \geq 0$ and $|g_1(x)| \leq f_1(x)$. We already did discuss the support and charge conjugation properties of $f_1(x)$. The analysis for all these functions shows that the support is in all cases $-1 \leq x \leq 1$, while the charge conjugation properties of the functions are $\overline{f}(x) = -f(-x)$ (C-even) for f_1 and h_1 and $\overline{f}(x) = +f(-x)$ (C-odd) for g_1.

==========

Exercise: **Exercise**: Show that the Dirac structure for h_1 in terms of chirality states is $\overline{\psi}_R\psi_L$ and $\overline{\psi}_L\psi_R$. Such functions are called *chiral-odd*. Explain why chiral-odd functions cannot be measured in inclusive deep inelastic scattering.

==========

B. Bounds on the distribution functions

The trivial bounds on the distribution functions ($|h_1(x)| \leq f_1(x)$ and $|g_1(x)| \leq f_1(x)$) can be sharpened. For instance one can look explicitly at the structure in Dirac space of the correlation function Φ_{ij}. Actually, we will look

at the correlation functions $(\Phi\,\gamma_0)_{ij}$, which involves at leading order matrix elements $\psi^\dagger_{+j}(0)\psi_{+i}(\xi)$. One has in Weyl representation ($\gamma^0=\rho^1$, $\gamma^i=-i\rho^2\sigma^i$, $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3=\rho^3$) the matrices

$$P_+ \;=\; \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix}, \quad P_+\gamma_5 = \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1 \end{pmatrix}, \quad P_+\gamma^1\gamma_5 = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix}.$$

The good projector only leaves two (independent) Dirac spinors, one righthanded (R), one lefthanded (L). On this basis of good R and L spinors the for hard scattering processes relevant matrix $(\Phi\not{n}_-)$ is given by

$$(\Phi\,\not{n}_-)_{ij}(x) = \begin{pmatrix} f_1 + S_L\,g_1 & (S_T^1 + i\,S_T^2)\,h_1 \\ (S_T^1 - i\,S_T^2)\,h_1 & f_1 - S_L\,g_1 \end{pmatrix} \tag{178}$$

One can also turn the S-dependent correlation function Φ into a matrix in the nucleon spin space via the standard spin 1/2 density matrix $\rho(P,S)$. The relation is $\Phi(x;P,S) = \mathrm{Tr}\,[\Phi(x;P)\,\rho(P,S)]$. Writing

$$\Phi(x;P,S) \;=\; \Phi_O + S_L\,\Phi_L + S_T^1\,\Phi_T^1 + S_T^2\,\Phi_T^2, \tag{179}$$

one has on the basis of spin 1/2 target states with $S_L=+1$ and $S_L=-1$ respectively

$$\Phi_{ss'}(x) = \begin{pmatrix} \Phi_O + \Phi_L & \Phi_T^1 - i\,\Phi_T^2 \\ \Phi_T^1 + i\,\Phi_T^2 & \Phi_O - \Phi_L \end{pmatrix} \tag{180}$$

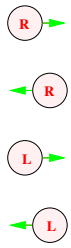
==========

Exercise: Show by generalizing $\Phi(p)$ to a matrix elements between states $\langle P,s|$ and $|P,s'\rangle$ that for the matrix $M = (\Phi\not{n}_-)^T$ (transposed in Dirac space) one has $v^\dagger M v \geq 0$ for any direction v in Dirac space.

==========

On the basis $+R$, $-R$, $+L$ and $-L$ the matrix in quark $\otimes$ nucleon spin-space becomes

$$(\Phi(x)\,\not{n}_-)^T \;=\; \begin{pmatrix} f_1+g_1 & 0 & 0 & 2\,h_1 \\ 0 & f_1-g_1 & 0 & 0 \\ 0 & 0 & f_1-g_1 & 0 \\ 2\,h_1 & 0 & 0 & f_1+g_1 \end{pmatrix} \tag{181}$$



Of this matrix any diagonal matrix element must always be positive, hence the eigenvalues must be positive, which gives a bound on the distribution functions stronger than the trivial bounds, namely

$$|h_1(x)| \leq \frac{1}{2}\,(f_1(x) + g_1(x)) \tag{182}$$

known as the Soffer bound [55].

==========

Exercise: Show that a change to the transverse quark spin basis gives the quark production matrix

$$(\Phi(x)\,\not{n}_-)^T \;=\; \begin{pmatrix} f_1+h_1 & 0 & 0 & g_1+h_1 \\ 0 & f_1-h_1 & g_1-h_1 & 0 \\ 0 & g_1-h_1 & f_1-h_1 & 0 \\ g_1+h_1 & 0 & 0 & f_1+h_1 \end{pmatrix} \tag{183}$$

==========

C. Transverse momentum dependent correlation functions

For the TMD correlators for distributions in Eq. 126 one can write down parametrizations which for the parts involving unpolarized targets (O), longitudinally polarized targets (L) and transversely polarized targets (T) up to parts proportional to M/P^+ take the form [53, 58]

$$\Phi_O(x,\boldsymbol{p}_T) = \frac{1}{2}\left\{ f_1(x,\boldsymbol{p}_T)\,\not{n}_+ + h_1^\perp(x,\boldsymbol{p}_T)\,\frac{i\,[\not{p}_T,\not{n}_+]}{2M}\right\} \tag{184}$$

$$\Phi_L(x,\boldsymbol{p}_T) = \frac{1}{2}\left\{ S_L\, g_{1L}(x,\boldsymbol{p}_T)\,\gamma_5\,\not{n}_+ + S_L\, h_{1L}^\perp(x,\boldsymbol{p}_T)\frac{\gamma_5\,[\not{p}_T,\not{n}_+]}{2M}\right\} \tag{185}$$

$$\begin{aligned}\Phi_T(x,\boldsymbol{p}_T) = \frac{1}{2}\Bigg\{ & f_{1T}^\perp(x,\boldsymbol{p}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_+^\nu p_T^\rho S_T^\sigma}{M} + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\,g_{1T}(x,\boldsymbol{p}_T)\,\gamma_5\,\not{n}_+ \\ & + h_{1T}(x,\boldsymbol{p}_T)\,\frac{\gamma_5\,[\not{S}_T,\not{n}_+]}{2} + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\,h_{1T}^\perp(x,\boldsymbol{p}_T)\,\frac{\gamma_5\,[\not{p}_T,\not{n}_+]}{2M}\Bigg\}.\end{aligned} \tag{186}$$

All functions appearing here have a natural interpretation as densities. This is seen as discussed before for the $\boldsymbol{p}_T$-integrated functions. Now it includes densities such as the density of longitudinally polarized quarks in a transversely polarized nucleon (g_{1T}) and the density of transversely polarized quarks in a longitudinally polarized nucleon ($h_{1L}^\perp$).

Upon integration over p_T not all functions survive. We are then left with the collinear correlator in Eq. 171 with $f_1(x) = \int d^2p_T\; f_1(x,p_T)$, $g_1(x) = \int d^2p_T\; g_{1L}(x,p_T)$ and $h_1(x) = \int d^2p_T\,\left[h_{1T}(x) + \frac{\boldsymbol{p}_T^2}{2M^2}\,h_{1T}^\perp(x,p_T)\right]$. The explicit treatment of transverse momenta also provides a way to include the evolution of quark distribution and fragmentation functions. The assumption that soft parts vanish sufficiently fast as a function of the invariants $p\cdot P$ and p^2, which at constant x implies a sufficiently fast vanishing as a function of $\boldsymbol{p}_T^2$, simply turns out not to be true. Assuming that the result for $\boldsymbol{p}_T^2 \geq \mu^2$ is given by the emission of an additional gluon one finds that the extra distribution written in terms of p_T becomes

$$f_1(x,\boldsymbol{p}_T^2) \overset{\boldsymbol{p}_T^2\geq\mu^2}{\Longrightarrow} \frac{1}{\pi\,\boldsymbol{p}_T^2}\,\frac{\alpha_s(\mu^2)}{2\pi}\int_x^1 \frac{dy}{y}\,P_{qq}\left(\frac{x}{y}\right)\,f_1(y;\mu^2), \tag{187}$$

which gives $f_1(x;\mu^2) \equiv \pi\int_0^{\mu^2} d\boldsymbol{p}_T^2\; f_1(x,\boldsymbol{p}_T^2)$ a logarithmic scale dependence. A more detailed discussion of problems in matching small and large p_T is given in Ref. [5].

Actually we find that different functions survive when one integrates over $\boldsymbol{p}_T$ weighting with p_T^α, e.g.

$$\Phi_\partial^\alpha(x) \equiv \int d^2p_T\,\frac{p_T^\alpha}{M}\,\Phi(x,\boldsymbol{p}_T) \tag{188}$$

$$= \frac{1}{2}\left\{ -g_{1T}^{(1)}(x)\,S_T^\alpha\,\not{n}_+\gamma_5 - S_L\,h_{1L}^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2} - f_{1T}^{\perp(1)}\,\epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_-^\nu S_T^\rho - h_1^{\perp(1)}\,\frac{i[\gamma^\alpha,\not{n}_+]}{2}\right\}, \tag{189}$$

involving *transverse moments* defined as

$$g_{1T}^{(1)}(x) = \int d^2p_T\;\frac{\boldsymbol{p}_T^2}{2M^2}\,g_{1T}(x,\boldsymbol{p}_T), \tag{190}$$

and similarly for the other functions. The functions $h_1^\perp$ and $f_{1T}^\perp$ are *T-odd*. As we will explain in the section on color gauge invariance they do not to vanish because time reversal invariance cannot be used for the transverse moments. Also for fragmentation functions they will not vanish. The T-odd functions correspond to unpolarized quarks in a transversely polarized nucleon ($f_{1T}^\perp$) or transversely polarized quarks in an unpolarized hadron ($h_1^\perp$). The easiest way to interpret the functions is by considering their place in the quark production matrix $(\Phi(x,p_T)\,\not{n}_-)^T$, which

becomes [4]

$$\begin{array}{cccc} \text{R}\rightarrow & \leftarrow\text{R} & \text{L}\rightarrow & \leftarrow\text{L} \end{array}$$

$$\begin{pmatrix} f_1 + g_{1L} & \frac{|p_T|}{M}\, e^{i\phi}\, g_{1T} & \frac{|p_T|}{M}\, e^{-i\phi}\, h_{1L}^{\perp} & 2\, h_1 \\ \frac{|p_T|}{M}\, e^{-i\phi}\, g_{1T}^{*} & f_1 - g_{1L} & \frac{|p_T|^2}{M^2}\, e^{-2i\phi}\, h_{1T}^{\perp} & -\frac{|p_T|}{M}\, e^{-i\phi}\, h_{1L}^{\perp *} \\ \frac{|p_T|}{M}\, e^{i\phi}\, h_{1L}^{\perp *} & \frac{|p_T|^2}{M^2}\, e^{2i\phi}\, h_{1T}^{\perp} & f_1 - g_{1L} & -\frac{|p_T|}{M}\, e^{i\phi}\, g_{1T}^{*} \\ 2\, h_1 & -\frac{|p_T|}{M}\, e^{i\phi}\, h_{1L}^{\perp} & -\frac{|p_T|}{M}\, e^{-i\phi}\, g_{1T} & f_1 + g_{1L} \end{pmatrix}.$$

In this representation T-odd functions can be incorporated as imaginary parts, $f_{1T}^{\perp} = -\mathrm{Im}\, g_{1T}$ and $h_1^{\perp} = \mathrm{Im}\, h_{1L}^{\perp}$.

D. Fragmentation functions

Just as for the distribution functions one can perform an analysis of the soft part describing the quark fragmentation in Eq. 127. For the production of unpolarized (or spin 0) hadrons h in hard processes one needs to leading order in $1/Q$ the $(M_h/P_h^-)^0$ part of the correlation function,

$$\Delta_O(z, \boldsymbol{k}_T) = z\, D_1(z, \boldsymbol{k}'_T)\, \not{n}_- + z\, H_1^{\perp}(z, \boldsymbol{k}'_T)\, \frac{i\, [\not{k}_T, \not{n}_-]}{2M_h}. \tag{191}$$

The arguments of the fragmentation functions D_1 and $H_1^{\perp}$ are $z = P_h^-/k^-$ and $\boldsymbol{k}'_T = -z\boldsymbol{k}_T$. The first is the (lightcone) momentum fraction of the produced hadron, the second is the transverse momentum of the produced hadron with respect to the quark. The fragmentation function D_1 is the equivalent of the distribution function f_1. It can be interpreted as a quark decay function, giving the probability of finding a hadron h in a quark. The quantity $n_h = \int dz\; D_1(z)$ is the number of hadrons.

==========

Exercise: Show that the normalization of the fragmentation functions is given by $\sum_h \int dz\; z\, D_1^{q\to h}(z)$ by relating this quantity to a local operator. One needs to eliminate the hadrons in the intermediate state via the momentum operator

$$P^{\mu} = \sum_{h,X} |P_h, X\rangle\, P_h^{\mu}\, \langle P_h, X|.$$

==========

The function $H_1^{\perp}$, interpretable as the difference between the numbers of unpolarized hadrons produced from a transversely polarized quark depending on the hadron's transverse momentum, is allowed because of the non-applicability of time reversal invariance [22]. This is natural for the fragmentation functions [36, 40] because of the appearance of out-states $|P_h, X\rangle$ in the definition of Δ, in contrast to the plane wave states appearing in Φ. The function $H_1^{\perp}$ is of interest because it is chiral-odd. This means that it can be used to probe the chiral-odd quark distribution function h_1, which can be achieved e.g. by measuring a particular azimuthal asymmetry of produced pions in the current fragmentation region.

The spin structure of fragmentation functions is also conveniently summarized by explicitly giving it on a R and L chiral quark basis, for which we find for decay into spin zero hadrons,

$$(\Delta(z, k_T)\not{n}_+)^T = \begin{array}{cc} \begin{pmatrix} D_1 & i\frac{|k_T|\, e^{-i\phi}}{M_h}\, H_1^{\perp} \\ -i\frac{|k_T|\, e^{+i\phi}}{M_h}\, H_1^{\perp} & D_1 \end{pmatrix} & \begin{array}{c} \text{R} \\ \text{L} \end{array} \\ \begin{array}{cc} \text{R} & \text{L} \end{array} & \end{array} \tag{192}$$

E. Examples of azimuthal asymmetries

Transverse momentum dependence shows up in the azimuthal dependence in the SIDIS cross section (via $\hat{h}$ or transverse spin vectors), in most cases requiring polarization of beam and/or target or requiring polarimetry [6, 13, 49].

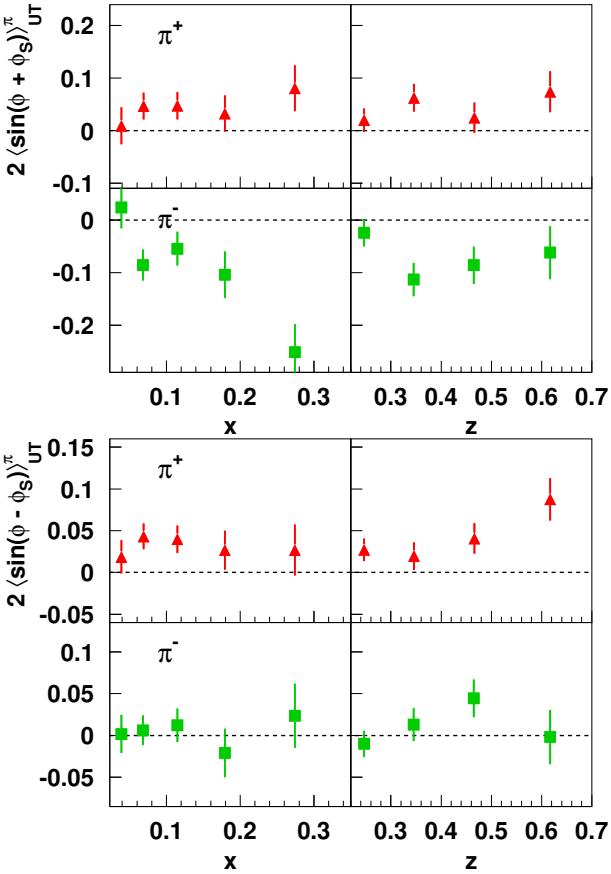


FIG. 6: Weighted asymmetries for the Collins and Sivers angles (see Eqs 193 and 194) obtained in semi-inclusive single spin asymmetries measured on a transversely polarized Hydogen target by the HERMES collaboration at DESY [1]. The error bars represent the statistical uncertainties.

Examples of leading azimuthal asymmetries, appearing for polarized leptoproduction are

$$\left\langle \frac{Q_T}{M} \sin(\phi_h^\ell - \phi_S^\ell) \right\rangle_{OT} = \frac{2\pi\alpha^2\, s}{Q^4} |\boldsymbol{S}_T| \left(1 - y + \frac{1}{2}\, y^2\right) \sum_{a,\bar{a}} e_a^2\, x_B\, f_{1T}^{\perp(1)a}(x_B) D_1^a(z_h). \tag{193}$$

$$\left\langle \frac{Q_T}{M_h} \sin(\phi_h^\ell + \phi_S^\ell) \right\rangle_{OT} = \frac{4\pi\alpha^2\, s}{Q^4} |\boldsymbol{S}_T| \,(1 - y) \sum_{a,\bar{a}} e_a^2\, x_B\, h_1^a(x_B) H_1^{\perp(1)a}(z_h). \tag{194}$$

The notation $\langle W \rangle$ is the q_T-integrated cross section including a weight W. The factor Q_T is included, because it together with the direction $\hat{h}$ combines to q_T, allowing a defolding of the cross section in distribution and fragmentation parts (one of them weighted with transverse momentum). Note that both of these asymmetries involve T-odd functions, which can only appear in single spin asymmetries. The latter can easily be checked from the conditions on the hadronic tensor, which are the same as those in Eq. 72 to 74. They require an odd number of spins vectors entering in the symmetric part and an even number of spins entering in the antisymmetric part of the hadron tensor. The results of single spin asymmetries in SIDIS measurements on a transversely polarized target from HERMES [1] are shown in Fig. 6. An extended review of transverse momentum dependent functions and transversity can be found in Ref. [7, 8]

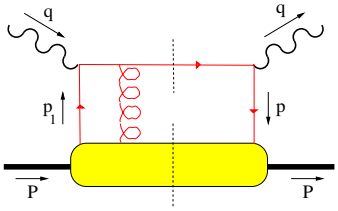


FIG. 7: Example of a contribution involving a quark-gluon matrix element that must be included at subleading order in lepton hadron inclusive scattering.

F. Inclusion of subleading contributions

If one proceeds up to order $1/Q$ one also needs terms in the parametrization of the soft part proportional to M/P^+. Limiting ourselves to the $\boldsymbol{p}_T$-integrated correlations one needs [39]

$$\Phi(x) = \frac{1}{2}\left\{ f_1(x)\,\not{n}_+ + S_L\, g_1(x)\,\gamma_5\,\not{n}_+ + h_1(x)\,\frac{\gamma_5\,[\not{S}_T, \not{n}_+]}{2} \right\} + \frac{M}{2P^+}\left\{ e(x) + g_T(x)\,\gamma_5\,\not{S}_T + S_L\, h_L(x)\,\frac{\gamma_5\,[\not{n}_+, \not{n}_-]}{2} \right\} \tag{195}$$

We will use inclusive scattering off a transversely polarized nucleon ($|\boldsymbol{S}_\perp| = 1$) as an example to show how higher twist effects can be incorporated in the cross section. The hadronic tensor for a transversely polarized nucleon is zero in leading order in $1/Q$. At order $1/Q$ one obtains a contribution from the handbag diagram, which turns out to involve the transverse moments in Φ^α_∂ in Eq. 189. There is a second contribution at order $1/Q$, however, coming from diagrams as the one shown in Fig. 7. For these gluon diagrams one needs bilocal matrix elements containing $1/Q$ one only needs the matrix element of the bilocal combinations $\overline{\psi}(0)\,gA^\alpha_T(\xi)\,\psi(\xi)$ and $\overline{\psi}(0)\,gA^\alpha_T(0)\,\psi(\xi)$. The $\Phi^\alpha_A(x)$ and $\Phi^\alpha_\partial(x)$ contributions sum to $\Phi^\alpha_D(x)$ involving matrix elements of bilocal combinations $\overline{\psi}(0)\,iD^\alpha_T\,\psi(\xi)$ for which one can Use the QCD equations of motion to relate them to the functions appearing in Φ [15, 39],

$$\Phi^\alpha_D(x) = \frac{M}{2}\left\{ -\left(x\,g_T - \frac{m}{M}\,h_1\right)\,S^\alpha_T\,\not{n}_+\gamma_5 - S_L\left(x\,h_L - \frac{m}{M}\,g_1\right)\,\frac{[\gamma^\alpha, \not{n}_+]\gamma_5}{2} \right\}. \tag{196}$$

The distribution function g_T e.g. shows up in the corresponding structure function of polarized inclusive deep inelastic scattering

$$2M\,W^{\mu\nu}_A(q,P,S_T) = i\,\frac{2Mx_B}{Q}\,\hat{t}^{\,[\mu}\epsilon^{\nu]\rho}_\perp S_{\perp\rho}\,g_T(x_B), \tag{197}$$

leading for the structure function $g_T(x_B, Q^2)$ in the antisymmetric part of the tensor to the result

$$g_T(x_B, Q^2) = \frac{1}{2}\sum_q e_q^2\left(g^q_T(x_B) + g^{\bar{q}}_T(x_B)\right). \tag{198}$$

G. Working in impact parameter space

V. COLOR GAUGE INVARIANCE

A. Wilson lines in correlators

We have sofar disregarded two issues. The first issue is that the correlation function Φ discussed in previous sections involve two quark fields at different space-time points and hence are not color gauge invariant. The second issue are the gluonic diagrams similar as the ones we have discussed in the previous section (see Fig. 7), among which also correlation functions appear involving matrix elements with longitudinal (A^+) gluon fields,

$$\overline{\psi}_j(0)\,gA^+(\eta)\,\psi_i(\xi).$$

These do not lead to any suppression. The reason is that because of the +-index in the gluon field the matrix element is proportional to P^+, p^+ or $M\,S^+$ rather than the proportionality to $M\,S^\alpha_T$ or p^α_T that one gets for a gluonic matrix element with transverse gluons.

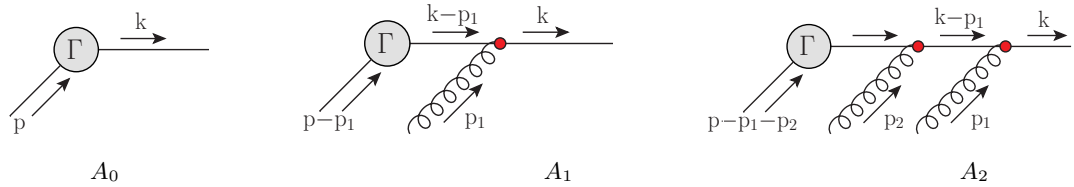


FIG. 8: Inclusion of collinear gluons from $\Phi_{A\ldots A}(p-p_1\ldots-p_N,p_1,\ldots,p_N)$ coupling to an outgoing (colored) quark line with momentum k results in a Wilson line running from the point ξ in field $\psi(\xi)$ to the point where $\xi^-=\infty$.

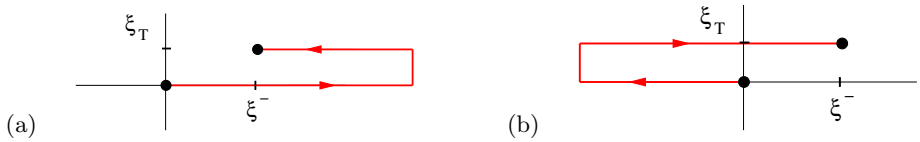


FIG. 9: The gauge link structure in the quark-quark correlator Φ in SIDIS (a) and DY (b) respectively

A straightforward calculation, however, shows that the gluonic diagrams with one or more longitudinal gluons involve matrix elements (soft parts) of operators $\overline{\psi}\psi$, $\overline{\psi}\,A^+\,\psi$, $\overline{\psi}\,A^+A^+\,\psi$, etc. (see Fig. 8) can be resummed into a correlation function

$$\Phi_{ij}(x)=\int\frac{d\xi^-}{2\pi}\;e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,U_{[0,\xi]}\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{199}$$

where U is a gauge link operator [9, 15, 17–19]

$$U_{[0,\xi]}=\mathcal{P}\exp\left(-i\int_0^{\xi^-}d\zeta^-\,A^+(\zeta)\right) \tag{200}$$

(path-ordered exponential with path along $-$-direction). Et voila, the unsuppressed gluonic diagrams combine into a color gauge invariant correlation function [29, 30]. We note that at the level of operators, one expands

$$\overline{\psi}(0)\psi(\xi)=\sum_n\frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(0), \tag{201}$$

in a set of local operators, but only the expansion of the non-local combination with a gauge link

$$\overline{\psi}(0)\,U_{[0,\xi]}\,\psi(\xi)=\sum_n\frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)D_{\mu_1}\ldots D_{\mu_n}\psi(0), \tag{202}$$

is an expansion in terms of local gauge invariant operators. The latter operators are precisely the local (quark) operators that appear in the operator product expansion applied to inclusive deep inelastic scattering.

For the p_T-dependent functions, one finds that inclusion of A^+ gluonic diagrams leads to a color gauge invariant matrix element with links running via $\xi^=\pm\infty$ [14, 23] where the direction depends on the coupling of the gluons to an final state or initial state color line. For instance in lepton-hadron scattering one has a situation like the one in Fig. 8 and one finds

$$\Phi^{[+]}(x,\boldsymbol{p}_T)=\int\frac{d\xi^-d^2\boldsymbol{\xi}_T}{(2\pi)^3}\;e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\,U^{[+]}_{[0,\xi]}\,\psi(\xi)|P,S\rangle\Big|_{\xi^+=0}, \tag{203}$$

where the link $U^{[+]}$ is shown in Fig. 9a. In contrast, for Drell-Yan scattering one finds a gauge link that runs via minus infinity, involving the link in Fig. 9b. We note that the gauge link involves transverse gluons [9, 15] showing that one in processes involving more hadrons the effects of transverse gluons are not necessarily suppressed, as has also been shown in explicit model calculations [20]. Integration over transverse momenta implies in the correlator, which is a Fourier transform that $\xi_T=0_T$, in which case the $U^{[+]}$ and $U^{[-]}$ links reduce to a unique collinear link connecting 0 and ξ^-.

The transverse momentum dependent distribution functions, however, do depend on the gauge link structure, which is still tractable in simple processes like DIS or DY with a simple color flow, but the color structure of various correlators become entangled if the color flow is more complicated [17–19].

Even if the color flow is simple, like SIDIS or DY, there are effects. Most notable is a sign change in single spin asymmetries going from SIDIS to DY. To understand this sign change, one notes that TMDs are no longer constrained by time-reversal, as the time reversal operation interchanges the $U^{[+]}$ and $U^{[-]}$ links, leading to the appearance of T-odd functions in Eqs 184 - 186 and also occuring in Eq. 189. One has e.g.

$$\Phi_O^{[\pm]}(x,\boldsymbol{p}_T) = \frac{1}{2}\left\{ f_1(x,\boldsymbol{p}_T)\,\not{n}_+ \pm h_1^\perp(x,\boldsymbol{p}_T)\,\frac{i\,[\not{p}_T,\not{n}_+]}{2M}\right\}. \tag{204}$$

Looking back at our example of an azimuthal asymmetry, we have seen the $\sin(\phi_h^\ell - \phi_S^\ell)$ asymmetry proportional to $f_{1T}^\perp\, D_1$. The corresponding asymmetry in Drell-Yan proportional to $f_{1T}^\perp\, \overline{f}_1$ would get an additional minus sign. Actually for the fragmentation correlator, such as Δ_O in Eq. 191 there is no such dependence on the gauge link [34, 35, 44]. The function $H_1^\perp$, however, is nonzero because the states $|P_h, X\rangle$ in the case of fragmentation are out-states and time reversal (changing out- into in-states) simply cannot be used as a constraint. The asymmetry in Eq. 194 thus will not change sign going from SIDIS to the corresponding DY asymmetry. In general situations in which one has a convolution of TMD distribution functions of two hadrons in the initial state, factorization is, already at tree-level hampered by the entanglement of Wilson lines [21, 54]. entanglement of Wilson lines. At the level of the weighted asymmetries, it implies more complicated factors than just a sign change in the appearance of the weighted functions.

Gluonic pole matrix elements

The inclusion of gauge links also allows us to study in some more detail the operator structure of the T-odd parts in the correlators. Given the full operator structure for a TMD correlator including a gauge links $U^{[\pm]}$ one can explicitly calculate the transverse moments $\Phi_\partial^\alpha(x)$ of which we have given the parametrization in Eq. 189. One finds that they, depending on the gauge links, can be related to color gauge invariant quark-quark-gluon matrix elements Φ_G^α and Φ_D^α,

$$\begin{aligned}
\left(\Phi_\partial^{[\pm]\alpha}\right)_{ij}(x) &= \int \mathrm{d}^2 p_T \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{(2\pi)^3}\; e^{i\,p\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\pm\infty]}U^T_{[0_T,\pm\infty_T]} \\
&\qquad\qquad i\partial_\xi^\alpha\, U^T_{[\pm\infty_T,\xi_T]}U^-_{[\pm\infty,\xi]}\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0} \\
&= \int \frac{d\xi^-}{(2\pi)}\; e^{ip\cdot\xi}\Big\{\langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\xi]}\,iD_T^\alpha\psi_i(\xi)|P,S\rangle\Big|_{LC} \\
&- \langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\pm\infty]}\int_{\pm\infty}^{\xi^-} d\eta^-\,U^-_{[\pm\infty,\eta]}\,g\,G^{+\alpha}(\eta)\,U^-_{[\eta,\xi]}\,\psi_i(\xi)|P,S\rangle\Big|_{LC}\Big\},
\end{aligned}$$

or

$$\Phi_\partial^{[\pm]\,\alpha}(x) = \Phi_D^\alpha(x) - \int_{-\infty}^{\infty}\mathrm{d}p_1^+\;\frac{i}{p_1^+\mp i\epsilon}\,\Phi_G^\alpha(p^+,p^+-p_1^+), \tag{205}$$

where

$$\Phi_{D\,ij}^\alpha(x) = \int\frac{\mathrm{d}\xi^-}{2\pi}\;e^{i\,p\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\xi]}\,iD^\alpha(\xi)\psi_i(\xi)|P,S\rangle\Big|_{LC}. \tag{206}$$

The difference between correlation functions with links running to $\pm\infty$, respectively, is related to a collinear quark-gluon correlator,

$$\Phi_\partial^{[+]\alpha}(x) - \Phi_\partial^{[-]\alpha}(x) = 2\pi\,\Phi_G^\alpha(x,x), \tag{207}$$

the latter being given the name *gluonic-pole* matrix element since it corresponds to the soft-gluon point $p_1^+ = 0$. It is a collinear matrix element which has been extensively studied [31–33, 42, 43, 50–52] as a collinear mechanism to generate single spin asymmetries. With the definition

$$\Phi_A^\alpha(x) = \mathrm{PV}\int_{-\infty}^{\infty}\mathrm{d}p_1^+\;\frac{i}{p_1^+}\,\Phi_G^\alpha(p^+,p^+-p_1^+), \tag{208}$$

one gets a separation

$$\Phi_{\partial}^{[\pm]\,\alpha}(x) = \underbrace{\Phi_D^{\alpha}(x) - \Phi_A^{\alpha}(x)}_{\widetilde{\Phi}_{\partial}^{\alpha}} \pm\pi\, \Phi_G^{\alpha}(x,x), \tag{209}$$

into T-even and T-odd parts.

VI. GLUON TMDS

Except in higher twist matrix elements and in gauge links, gluon fields also appear in the gluon distribution functions. To better understand the treatment of gluon fields, we also make for gluon fiels $A^\mu(\eta)$ a Sudakov expansion,

$$A^\mu(\eta) = A^n(\eta)\, P^\mu + A_T^\mu(\eta) + \big(A^P(\eta) - A^n(\eta)\, M^2\big)\, n^\mu. \tag{210}$$

A similar expansion can be written down for $A^\mu(p)$. It will be convenient to look at the (collinear) gluon field component along parton momentum p^μ, hence we write

$$\begin{aligned} A^\mu(p) &= \int d^4\eta\; e^{i\,p\cdot\eta}\, A^\mu(\eta) \\ &= \int d^4\eta\; e^{ip\cdot\eta} \left[\frac{A^n(\eta)}{p\cdot n}\, p^\mu + \frac{(p\cdot n)\, A_T^\mu(\eta) - p_T^\mu\, A^n(\eta)}{p\cdot n} + \frac{(p\cdot n)\, A^P(\eta) - (p\cdot P)\, A^n(\eta)}{p\cdot n}\, n^\mu \right]. \end{aligned} \tag{211}$$

In the correlator the momentum $p^\mu \longrightarrow i\partial^\mu(\eta)$, so

$$\begin{aligned} A^\mu(p) &= \frac{1}{p\cdot n} \int d^4\eta\; e^{i\,p\cdot\eta} \left[A^n(\eta)\, p^\mu + i\partial^n(\eta)\, A_T^\mu(\eta) - i\partial_T^\mu(\eta)\, A^n(\eta) + \big(i\partial^n(\eta) A^P(\eta) - i\partial^P(\eta) A^n(\eta)\big)\, n^\mu \right] \\ &= \frac{1}{p\cdot n} \left[A^n(p)\, p^\mu + i\, G_T^{n\mu}(p) + i\, G^{nP}(p)\, n^\mu \right]. \end{aligned} \tag{212}$$

Although the latter appears to be only true for the Abelian case, we will find the same result in the non-abelian case, but to complete that proof, we first need to incorporate the collinear gluons into the matrix elements as gauge links. For that we need the $A^n(p)$ gluons and actually also some boundary terms (shown very explicitly in Ref. [21]). Using the expansion in Eq. 212 rather than the one in Eq. 210 streamlines the inclusion of collinear gluons, because one can immediately make use of Ward identities. It circumvents the explicit treatment of transverse momentum dependent parts (as done in Ref. [15]). The results are of course identical.

Rather than transverse gluon fields one thus encounters $F^{n\alpha}$ field strengths in the matrix elements, as we saw explicitly already in the previous section. Transverse momentum dependent gluon distribution functions are projections of the TMD correlator Γ (often also referred to as Φ^g)

$$\Gamma^{[U,U']\,\mu\nu}(x,p_T) = \frac{1}{(p\cdot n)^2} \int \frac{d(\xi\cdot P)\; d^2\xi_T}{(2\pi)^3}\; e^{ip\cdot\xi}\; \mathrm{Tr}\, \langle P,S|\, F^{n\mu}(0)\, U_{[0,\xi]}\, F^{n\nu}(\xi)\, U'_{[\xi,0]}\, |P,S\rangle\, \big]_{\mathrm{LF}}\,.$$

Here the field-operators are written in the color-triplet representation requires the inclusion of *two* Wilson lines $U_{[0,\xi]}$ and $U'_{[\xi,0]}$. They again arise from the resummation of gluon initial and final-state interactions. In general this will lead to two unrelated Wilson lines U and U'. In the particular case that $U' = U^\dagger$, the gluon correlator can also be written as the product of two gluon fields with the Wilson line U in the adjoint representation of $SU(N)$. This is for instance the case for the gluon correlators which acquire gaugelinks as in Figs. 10a and b, but not for the gluon correlators in Figures. 10c and d.

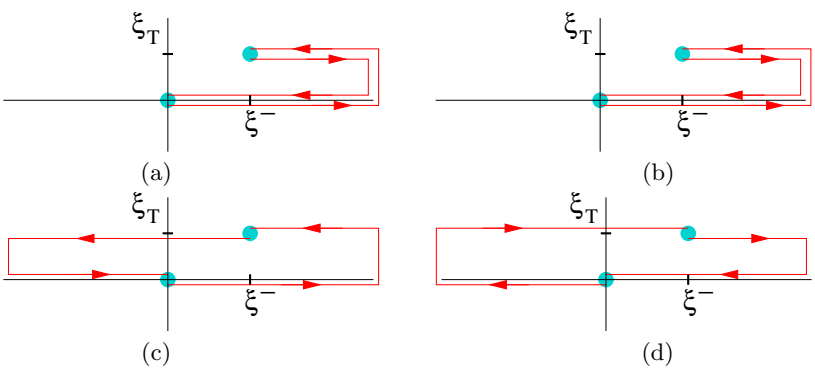


FIG. 10: The gauge links for gluon TMDs

In the p_T-integrated correlator on the lightcone the process dependence of the TMD gluon correlator disappears,

$$\begin{aligned}\Gamma^{\mu\nu}(x) &= \int d^2p_T \;\; \Gamma^{[U,U']\,\mu\nu}(x,p_T) \\ &= \frac{1}{(p\cdot n)^2}\int \frac{d(\xi\cdot P)}{2\pi}\; e^{ix(\xi\cdot P)}\; \mathrm{Tr}\,\langle P,S|\, F^{n\mu}(0)\, U^n_{[0,\xi]}\, F^{n\nu}(\xi)\, U^n_{[\xi,0]}|P,S\rangle\,\rfloor_{\rm LC}\;.\end{aligned} \tag{213}$$

However, as for the quark correlator, a subprocess-dependence due to the Wilson lines in the TMD gluon correlators remains in the transverse moments. The analogue of the process-dependent decomposition in the case of the gluon correlator is (with for each diagrammatic contribution d a TMD correlator $\Gamma^{[\mathcal{U},\mathcal{U}']}(x,p_T) = \Gamma^{[\mathcal{U}(D),\mathcal{U}'(D)]}(x,p_T) \equiv \Gamma^{[D]}(x,p_T)$ and omitting the gluon field indices μ and ν) [16],

$$\Gamma^{[D]\,\alpha}_{\partial}(x) = \widetilde{\Gamma}^{\alpha}_{\partial}(x) + C_G^{(f)\,[D]}\,\pi\Gamma^{\alpha}_{G_f}(x,x) + C_G^{(d)\,[D]}\,\pi\Gamma^{\alpha}_{G_d}(x,x)\;. \tag{214}$$

The matrix elements Γ_{G_f} and Γ_{G_d} are the two gluonic pole matrix elements that correspond to the two possible ways to construct color-singlets from three gluon fields [16, 41]. They involve the antisymmetric f and symmetric d structure constants of $SU(3)$, respectively. The only process dependence coming from the Wilson lines in the TMD correlators is contained in the gluonic pole strengths $C_G^{(f/d)\,[D]} \equiv C_G^{(f/d)\,[\mathcal{U}(D),\mathcal{U}'(D)]}$. The collinear correlators are

$$\begin{aligned}\Gamma^{\mu\nu;\alpha}_{D}(x) &= \frac{1}{(p\cdot n)^2}\int \frac{d(\xi\cdot P)}{2\pi}\; e^{ix(\xi\cdot P)} \\ &\qquad\times \mathrm{Tr}\,\langle P,S|\, F^{n\mu}(0)\, U^n_{[0;\xi]}\,\big[iD^{\alpha}(\xi), F^{n\nu}(\xi)\big]\, U^n_{[\xi;0]}\,|P,S\rangle\,\rfloor_{\rm LC}\;, \\ \Gamma^{\mu\nu;\alpha}_{G_f}(x,x-x') &= \frac{1}{(p\cdot n)^2}\int \frac{d(\xi\cdot P)}{2\pi}\frac{d(\eta\cdot P)}{2\pi}\; e^{ix'(\eta\cdot P)}e^{i(x-x')(\xi\cdot P)} \\ &\qquad\times \mathrm{Tr}\,\langle P,S|\, F^{n\mu}(0)\,\big[U^n_{[0,\eta]}gF^{n\alpha}(\eta)U^n_{[\eta,0]}, U^n_{[0,\xi]}F^{n\nu}(\xi)U^n_{[\xi,0]}\big]\,|P,S\rangle\,\rfloor_{\rm LC}\;, \\ \Gamma^{\mu\nu;\alpha}_{G_d}(x,x-x') &= \frac{1}{(p\cdot n)^2}\int \frac{d(\xi\cdot P)}{2\pi}\frac{d(\eta\cdot P)}{2\pi}\; e^{ix'(\eta\cdot P)}e^{i(x-x')(\xi\cdot P)} \\ &\qquad\times \mathrm{Tr}\,\langle P,S|\, F^{n\mu}(0)\,\big\{U^n_{[0,\eta]}gF^{n\alpha}(\eta)U^n_{[\eta,0]}, U^n_{[0,\xi]}F^{n\nu}(\xi)U^n_{[\xi,0]}\big\}\,|P,S\rangle\,\rfloor_{\rm LC}\;,\end{aligned}$$

and

$$\widetilde{\Gamma}^{\alpha}_{\partial}(x) = \Gamma^{\alpha}_{D}(x) - \int dx'\; P\frac{i}{x'}\;\Gamma^{\alpha}_{G_f}(x,x-x')\;. \tag{215}$$

For gluon distribution functions we follow the naming convention discussed in Ref. [45] and use the parameterizations of the TMD gluon correlators in Ref. [47]

$$\begin{aligned}\Gamma^{(T\text{-even})\,\mu\nu}(x,p_T) &= \frac{1}{2x}\Big\{ -g_T^{\mu\nu}\, f_1^g(x,p_T^2) + \Big(\frac{p_T^{\mu}p_T^{\nu}}{M^2} + g_T^{\mu\nu}\,\frac{\boldsymbol{p}_T^2}{2M^2}\Big)\, h_1^{\perp g}(x,p_T^2) \\ &\qquad + i\epsilon_T^{\mu\nu}\, S_L\; g_{1L}^g(x,p_T^2) + i\epsilon_T^{\mu\nu}\,\frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\; g_{1T}^g(x,p_T^2)\Big\}\;, \\ \Gamma^{(T\text{-odd})\,\mu\nu}_{(f/d)}(x,p_T) &= \frac{1}{2x}\Big\{ g_T^{\mu\nu}\,\frac{\epsilon_T^{p_T S_T}}{M}\; f_{1T}^{\perp g\,(f/d)}(x,p_T^2) - \frac{\epsilon_T^{p_T\{\mu}S_T^{\nu\}} + \epsilon_T^{S_T\{\mu}p_T^{\nu\}}}{4M}\; h_{1T}^{g\,(f/d)}(x,p_T^2) \\ &\qquad - \frac{\epsilon_T^{p_T\{\mu}p_T^{\nu\}}}{2M^2}S_L\; h_{1L}^{\perp g\,(f/d)}(x,p_T^2) - \frac{\epsilon_T^{p_T\{\mu}p_T^{\nu\}}}{2M^2}\frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\; h_{1T}^{\perp g\,(f/d)}(x,p_T^2)\Big\}\;.\end{aligned}$$

In particular one has two distinct gluon-Sivers distribution functions $f_{1T}^{\perp g\,(f)}(x,p_T^2)$ and $f_{1T}^{\perp g\,(d)}(x,p_T^2)$ corresponding to the two ways to construct T-odd gluon correlators.

$$\begin{pmatrix} f_1 + g_{1L} & \frac{k_T e^{-i\phi}}{M} g_{1T}^* & e^{-2i\phi} h_1^{\perp(1)} & i\frac{k_T e^{-3i\phi}}{M} h_{1T}^{\perp(1)} \\ \frac{k_T e^{+i\phi}}{M} g_{1T} & f_1 - g_{1L} & \frac{ik_T e^{-i\phi}}{M} h_1 & e^{-2i\phi} h_1^{\perp(1)\,*} \\ e^{+2i\phi} h_1^{\perp(1)\,*} & \frac{-ik_T e^{+i\phi}}{M} h_1 & f_1 - g_{1L} & \frac{k_T e^{-i\phi}}{M} g_{1T}^* \\ -i\frac{k_T e^{+3i\phi}}{M} h_{1T}^{\perp(1)} & e^{+2i\phi} h_1^{\perp(1)} & \frac{k_T e^{+i\phi}}{M} g_{1T} & f_1 + g_{1L} \end{pmatrix}$$

FIG. 11: Illustration of the rich structure when one includes spin and transverse momentum dependence for gluon distribution (or fragmentation) functions. Given are matrix elements involving gluonic fields with different helicities (two possibilities) between proton states with different helicities (two possibilities). On top the basis entities are given, with inside the circle the gluon helicity. In this representation T-odd functions appear as imaginary entries, which are only allowed in off-diagonal elements, in particular $f_{1T}^{\perp} = -\mathrm{Im} g_{1T}$ and $h_{1L}^{\perp} = -\mathrm{Im} h_1^{\perp}$. All functions depend on longitudinal momentum fraction (x) and absolute value of transverse momentum (k_T). The angle ϕ is the azimuthal angle of the transverse momentum vector. The notation $h_1^{\perp(1)}(x, k_T^2) \equiv (k_T^2/2M^2)\, h_1^{\perp}(x, k_T^2)$ is used and $h_1 \equiv h_{1T} + h_{1T}^{\perp(1)}$. A gluon index ($g$) used in the text is omitted from all functions. The functions $f_{1..}$ and $g_{1..}$ are sum and difference of matrix elements of gluon fields with different circular polarization, while the functions $h_{1..}$ involve gluon fields with different polarizations (becoming diagonal elements if one uses linear polarization). Finally the subscripts L and T indicate the polarization of the nucleon states between which the gluon field operators are evaluated. On the chosen helicity basis matrix elements between transversely polarized nucleons are off-diagonal.

VII. MISCELLANEOUS MATERIAL

A. Inclusion of color

In the previous paragraphs, we already mentioned the *summation over color* for the distribution correlator Φ and the *averaging over color* in the fragmentation correlator Δ. In fact color has to be made explicit and it must be considered as an 'observable'. The latter only means that the distribution of partons entering the hard process and the fragmentation of partons into final state hadrons can be considered separate in time from the (instantaneous) hard process. This will be made more explicit in subsequent sections. Thus including color indices, we would have a color structure like $\langle X^{\bar{r}}(\underline{3}^*)|\psi^r(\xi)|P\rangle$ and so on. The color also must be made explicit in the unit operator that is used in the correlator,

$$\begin{aligned} \mathbb{I} &= \sum_X |X\rangle\langle X| \\ &= \sum_Y |Y(\underline{1})\rangle\langle Y(\underline{1})| + \sum_{X,J} |X(\underline{3}^*), J(\underline{3})\rangle\langle X(\underline{3}^*), J(\underline{3})| + \sum_{X,J} |X(\underline{8}), J(\underline{8})\rangle\langle X(\underline{8}), J(\underline{8})| + \ldots \end{aligned} \tag{216}$$

We used suggestively the notations Y for a colorless final state, X for remnants of hadrons and J for jets emerging from the hard processes. It is clear that this is a sloppy way of writing, because a colorless bound state of course is a subset of some of the 'colored' states. The jets include final state hadrons as discussed for the fragmentation correlators. For more complex situations like hard exclusive processes or diffractive processes terms like

$$\mathbb{I} = \cdots + \sum_{Y_1,Y_2} |Y_1(\underline{1}), Y_2(\underline{1})\rangle\langle Y_1\underline{1}), Y_2(\underline{1})| + \sum_{Y,J_1,J_2} |Y(\underline{1}), J_1(\underline{3}), J_2(\underline{3}^*)\rangle\langle Y(\underline{1}), J_1(\underline{3}), J_2(\underline{3}^*)| + \ldots \tag{217}$$

may be relevant.

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Exercise: What are the relevant color structures in final states in the following processes:

(a) inclusive deep inelastic scattering (DIS) with $q\gamma^* \to q$ as subprocess,

(b) the Drell-Yan process with $q\bar{q} \to \gamma^* \to \ell\bar{\ell}$ as subprocess,

(c) electron-positron annihilation with $e^-e^+ \to q\bar{q}$ as hard subprocess,

(d) Higgs production in hadron-hadron scattering with as hard subprocess $gg \to H \to \gamma\gamma$.

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Let's take as an example the Drell-Yan process involving two correlators. The amplitude of the process involves

$$\mathcal{M} \sim \langle X_1 X_2\, \ell\bar{\ell}|\overline{\psi}^{\,r}(\xi)\Gamma^{rr'}\psi^{r'}(\xi)|P_1P_2\rangle. \tag{218}$$

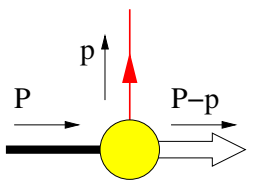
(we for completeness allowed color dependence in the hard vertex Γ, which gives

$$\begin{aligned} d\sigma \;&\sim\; |\mathcal{M}|^2 = \langle P_1P_2|\overline{\psi}^{\,s}(0)\Gamma^{*ss'}\psi^{s'}(0)|X_1^{r''}X_2^{s''}\,\ell\bar{\ell}\rangle\langle X_1^{r''}X_2^{s''}\,\ell\bar{\ell}|\overline{\psi}^{\,r}(\xi)\Gamma^{rr'}\psi^{r'}(\xi)|P_1P_2\rangle \\ &= \langle P_1|\overline{\psi}^{\,s}(0)|X_1^{r''}\rangle\langle X_1^{r''}|\psi^{r'}(\xi)|P_1\rangle\Gamma^{*ss'}(q\bar{q}\to\ell\bar{\ell})\langle P_2|\psi^{s'}(0)|X_2^{s''}\rangle\langle X_2^{s''}|\overline{\psi}^{\,r}(\xi)|P_2\rangle\Gamma^{rr'}(q\bar{q}\to\ell\bar{\ell}) + \ldots \\ &= \Phi^{r's}(P_1)\Gamma^{*ss'}\overline{\Phi}^{s'r'}(P_2)\Gamma^{rr'} = \mathrm{Tr}_c[\Phi(P_1)\Gamma^*\overline{\Phi}(P_2)\Gamma] = \frac{1}{N_c}\,\Phi(P_1)\overline{\Phi}(P_2)\Gamma^*\Gamma(q\bar{q}\to\ell\bar{\ell}). \end{aligned} \tag{219}$$

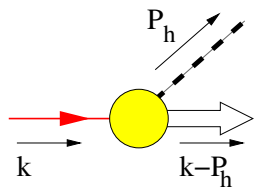
The latter is possible if the hard part is a simple color connector in such a way that the color trace in essence is $\mathrm{Tr}_c[I]$, which is rewritten as $\mathrm{Tr}_c[I]\,\mathrm{Tr}_c[I]/N_c$ and the correlators are taken to be $\Phi(P_1) = \Phi^{rr}(P_1) = \mathrm{Tr}_c[\Phi(P_1)]$.

Regions of importance in parton kinematics

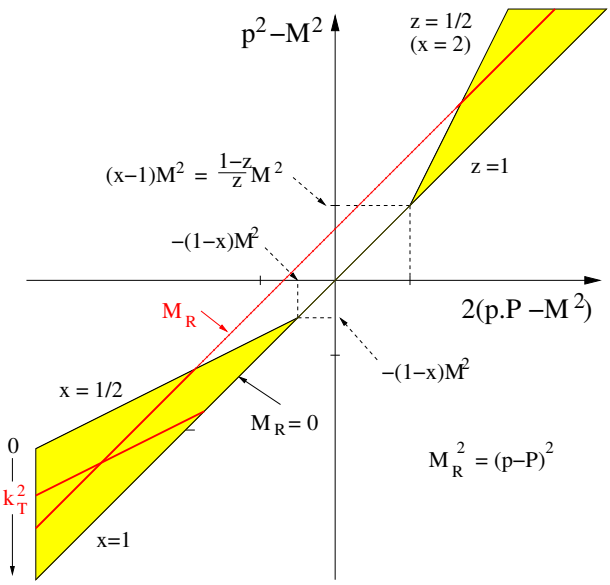
We want to illustrate the kinematics for partons and translate it to physically intuitive quantities, the off-shellness p^2 for partons or the invariant mass squared $M_R^2 = (P-p)^2$ of the residual (spectator) hadronic system and the (spacelike) transverse momentum squared $p_T^2 = -\boldsymbol{p}_T^2$. We can do this for partons in hadrons (distributions), but also for the production of hadrons from partons (fragmentation),



$$\begin{aligned} &p = x\,P + p_T + \sigma_p\,n \\ &P^+ = P{\cdot}n \equiv 1 \\ &x = p{\cdot}n = p^+ \\ &\sigma_p = p{\cdot}P - x\,M^2 \\ &n_+ \equiv P - \tfrac{1}{2}M^2\,n \\ &p^- = p{\cdot}n_+ = p{\cdot}P - \tfrac{1}{2}\,xM^2 \end{aligned}$$



$$\begin{aligned} &k = z^{-1}\,P_h + k_T + \sigma_h\,n_h \\ &P_h{\cdot}n_h = P_h^- \equiv 1 \\ &z^{-1} = k{\cdot}n_h = k^- \\ &\sigma_k = k{\cdot}P_h - z^{-1}\,M_h^2 \\ &n_- \equiv P_h - \tfrac{1}{2}M_h^2\,n_h \\ &k^+ = k{\cdot}n_- = k{\cdot}P_h - \tfrac{1}{2}\,z^{-1}\,M_h^2 \end{aligned}$$



The kinematic freedom for the partons is illustrated in a plot of the off-shellness p^2 versus $2p{\cdot}P$, where it is convenient to slightly shift the zero-points to get a nice symmetry. The relations

$$\begin{aligned} p^2 - M^2 \;&=\; 2x\left(p{\cdot}P - M^2\right) + p_T^2 \\ &\qquad - (1-x)^2\,M^2, \qquad (220) \\ p^2 - M^2 \;&=\; 2\left(p{\cdot}P - M^2\right) + M_R^2, \qquad (221) \end{aligned}$$

determine lines of constant M_R or p_T^2. They are used to map out the physical region determined by $p_T^2 = -\boldsymbol{p}_T^2 \le 0$ and $M_R^2 \ge 0$. We note that for $x = 1$ the region disappears shrinking to the line $M_R = 0$ or $x = 1$. We note the symmetry $x = 1/z$ in jumping from distributions of partons into hadrons to fragmentation of partons into hadrons. We assume that the dominant contributions come from the regions where $p^2 \sim p{\cdot}P \sim M_R^2 \sim M^2$, thus Φ disappearing as a function of these variables.

B. Distribution correlators in high-energy processes

In a hard process, the different importance of the various components allows up to specific orders in $1/Q$, an integration over some components of the parton momenta. The fact that the main contribution in $\Phi(p;P)$ is assumed to come from regions where $p{\cdot}P \leq M^2$, whereas the momenta have characteristic scale Q, allows performing the σ-integration up to M^2/Q^2 contributions (and possible contributions from non-integrable tails). Transverse momentum dependent (TMD) correlators are light-front correlators, integrated over $\sigma \equiv p{\cdot}P - xM^2$,

$$\Phi_{ij}(x,p_T;n) \;=\; \int d(p{\cdot}P)\;\Phi_{ij}(p;P) = \int d\sigma_p\, d\tau_p\, \delta(\tau_p - 2x\,\sigma_p - p_T^2 - x^2M^2)\,\Phi_{ij}(p;P) \tag{222}$$

$$= \int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\; e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle\Big|_{LF}\;, \tag{223}$$

where we have suppressed the dependence on hadron momentum P and made the off-shellness $\tau = p^2$ explicit. The argument of the delta function is useful when one wants to study which regions in off-shellness or in the residual mass spectrum $M_R^2 = (P-p)^2$ contribute for given x and p_T^2. The subscript LF refers to light-front, implying $\xi{\cdot}n = 0$. The light-cone correlators are the correlators containing the parton distribution functions depending only on the light-cone momentum fraction x,

$$\Phi_{ij}(x;n) \;=\; \int d(p{\cdot}P)\,d^2p_T\;\Phi_{ij}(p;P) = \int d\sigma_p\, d\tau_p\,\theta(2x\,\sigma_p - \tau_p + x^2M^2)\,\Phi_{ij}(p;P) \tag{224}$$

$$= \int \frac{d(\xi{\cdot}P)}{(2\pi)}\; e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle\Big|_{LC}\;, \tag{225}$$

where the subscript LC refers to light-cone, implying $\xi{\cdot}n = \xi_T = 0$. This integration is generally allowed (again up to M^2/Q^2 contributions and contributions coming from tails, e.g. logarithmic corrections from $1/p_T^2$ tails) if we are interested in hard processes, in which only hard scales (large invariants $\sim Q^2$ or ratios thereof, angles, rapidities) are measured. If one considers hadronic scale observables (correlations or transverse momenta in jets, slightly off-collinear configurations) one will need the TMD correlators.

Twist expansion

The correlators encompass the information on the soft parts. They depend on the hadron and (contained) quark momenta P and p (and spin vectors). The structure of the correlator is reproduced from these momenta incorporating the required Dirac and Lorentz structure. Clearly, it is advantageous to maximize the number of components along the momentum (collinear). For the *soft* scalar objects this means maximizing the number of contractions with n. This leads for nonlocal operators to the dominance of the twist-2 operators

$$\overline{\psi}(0)\not{n}\psi(\xi) \qquad \text{and} \qquad G^{n\alpha}(0)G^{n\beta}(\xi), \tag{226}$$

(the latter with transverse indices α and β). Twist in this case is just equal to the canonical dimension of the operator combination (remember that dim(n) = -1).

The fact that the matrix elements involve operators on the light-front, allows for specific operators, the socalled *good* operators in front-form quantization, an easy interpretation in terms of *partons*. These partons are the quanta created by the good fields. The good fields are

$$\psi_+ = \frac{1}{2}\,\not{n}_+\not{n}_-\,\psi = \frac{1}{2}\,\not{P}\not{n}\,\psi \quad \text{and} \quad A_T^\mu = g_T^{\mu\nu}A_\nu. \tag{227}$$

In front-form quantization the other components of the fields can be expressed in the good fields using equations of motion, at least after imposing the gauge $A^+ = A{\cdot}n = 0$ (where $G^{n\alpha} = \partial^n A_T^\alpha$). More important, however, for the description of hard processes is that matrix elements involving these good fields turn out to be the leading ones in an expansion in the inverse hard scale $1/Q$.

As argued above (and made more explicit in applications to hard scattering processes) the correlators involve collinear momenta (soft with respect to each other), but for use within the hard process an external direction shows up, represented by a null vector n, which will acquire a meaning in the explicit applications or play an intermediary

role. It can be used in the projection of components or in an expansion of fields or field combinations, e.g.

$$\psi = \psi_+ + \psi_- = \frac{1}{2}\,\not{P}\not{n}\,\psi + \frac{1}{2}\,\not{n}\not{P}\,\psi, \tag{228}$$

$$A^\mu = (A{\cdot}n)P^\mu + A_T^\mu + (A{\cdot}P)n^\mu. \tag{229}$$

$$\overline{\psi}\gamma^\mu\psi = (\overline{\psi}\not{n}\psi)P^\mu + \overline{\psi}\gamma_T^\mu\psi + (\overline{\psi}\not{P}\psi)n^\mu. \tag{230}$$

Examples of orders of magnitude of the fields within the matrix elements appearing in hard processes are

$$\langle\, n{\cdot}A \,\rangle \sim n{\cdot}P = 1, \tag{231}$$

$$\langle\, A_T^\alpha \,\rangle \sim \langle\, G_T^{n\alpha} \,\rangle \sim p_T^\alpha \sim M, \tag{232}$$

$$\langle\, \psi \,\rangle \sim M^{3/2}, \tag{233}$$

$$\langle\, \overline{\psi}\,\not{n}\,\psi \,\rangle \sim \langle\, \overline{\psi}\,[\not{n}, \gamma_T^\mu]\,\psi \,\rangle \sim \langle\, G_T^{n\alpha}\, G_T^{n\beta} \,\rangle \sim M^2. \tag{234}$$

$$\langle\, \overline{\psi}\psi \,\rangle \sim \langle\, \overline{\psi}\, D^n\,\psi \,\rangle \sim \langle\, \overline{\psi}\,\gamma_T^\mu\,\psi \,\rangle \sim M^3\ , \tag{235}$$

$$\langle\, \overline{\psi}\, D_T^\alpha\,\psi \,\rangle \sim \langle\, \overline{\psi}\,\not{P}\,\psi \,\rangle \sim M^4\ , \tag{236}$$

These results are obvious because $\langle\, \overline{\psi}\gamma^\mu\,\psi \,\rangle$ in a matrix element of the form $\Phi(p;P)$ must involve the (relevant) momenta p^μ or P^μ, it must be a vector and it must have dimension 3, leaving $M^2\,p^\mu$ or $M^2\,P^\mu$ as possibilities. Knowing the order of magnitude of the momenta as appearing in a hard scattering process, we obtain the above results. The above integrated or the TMD correlators also can depend on n^μ, e.g. appearing via the transverse tensors $g_T^{\mu\nu}$ or $\epsilon_T^{\mu\nu}$ (both being of order unity).

Color gauge invariance

The field combinations considered sofar in the correlators are not color gauge-invariant since they involve the A-fields and, more important, because they involve nonlocal field combinations. At each specific order in Q one of course expects gauge-invariant combinations. Along the light-cone, the leading combinations involve the 'parton fields'

$$\not{n}\,\psi(\xi) = \not{n}\,\psi_+(\xi) \quad \text{and} \quad G^{n\alpha}(\xi),$$

while $A^+ = A^n = n{\cdot}A$ operators appear in gauge links along the light-cone ($\xi^+ = n{\cdot}\xi = \xi_T = 0$),

$$U_{[0,\xi]}^{[n]} = \mathscr{P}\,\exp\left(-i\int_0^\xi d(\eta{\cdot}P)\,n{\cdot}A(\eta)\right), \tag{237}$$

which are needed to connect colored parton fields. Which n appears in a correlator is fixed by the hard process, although some freedom in n may remain. We note that the exponent in the gauge link is in essence built from 'operators' $(n{\cdot}\partial)^{-1}n{\cdot}A$, which are $\mathscr{O}(1)$. Actually, the gauge-invariant correlators will in some cases appear multiplied with the parton momentum, $p^\mu\,\Phi(p;P)$, etc., which implies a derivative ∂_ξ^μ in the matrix element, which is e.g. standard in the matrix elements involving gluon fields $G^{\mu\nu}$. The color gauge-invariant light-cone correlators for quarks and gluons are

$$\Phi_{ij}(x;n) = \int \frac{d(\xi{\cdot}P)}{(2\pi)}\ e^{i\,p\cdot\xi}\ \langle P|\overline{\psi}_j(0)\,U_{[0,\xi]}^{[n]}\,\psi_i(\xi)|P\rangle\Big|_{LC}, \tag{238}$$

$$\Gamma^{\alpha\beta}(x;n) = \int \frac{d(\xi{\cdot}P)}{(2\pi)}\ e^{i\,p\cdot\xi}\ \langle P|\mathrm{Tr}\left(G^{n\beta}(0)\,U_{[0,\xi]}^{[n]}\,G^{n\alpha}(\xi)\,U_{[\xi,0]}^{[n]}\right)|P\rangle\Big|_{LC}, \tag{239}$$

while for the TMD light-front correlators

$$\Phi_{ij}(x,p_T;n,C) = \int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\ e^{i\,p\cdot\xi}\ \langle P|\overline{\psi}_j(0)\,U_{[0,\xi]}^{[n,C]}\,\psi_i(\xi)|P\rangle\Big|_{LF}, \tag{240}$$

$$\Gamma^{\alpha\beta}(x,p_T;n,C,C') = \int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\ e^{i\,p\cdot\xi}\ \langle P|\mathrm{Tr}\left(G^{n\beta}(0)\,U_{[0,\xi]}^{[n,C]}\,G^{n\alpha}(\xi)\,U_{[\xi,0]}^{[n,C']}\right)|P\rangle\Big|_{LF}, \tag{241}$$

where we in passing mention that path dependence (indicated by the arguments C and C') will arise because of the (necessary) transverse piece(s) in the gauge link.

$$\Phi_{\rm quark}(x,p_T;n,C) = \int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\ e^{i\,p\cdot\xi}\ \langle P|\overline{\psi}_j(0)\,U_{[0,\xi]}^{[n,C]}\,\psi_i(\xi)|P\rangle\Big|_{\xi\cdot n=0},$$

$$\Phi_{\rm gluon}^{\alpha\beta}(x,p_T;n,C,C') = \int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\ e^{i\,p\cdot\xi}\ \langle P|\mathrm{Tr}\left(G^{n\beta}(0)\,U_{[0,\xi]}^{[n,C]}\,G^{n\alpha}(\xi)\,U_{[\xi,0]}^{[n,C']}\right)|P\rangle\Big|_{\xi\cdot n=0},$$

Sum rules

Completing the integration over the correlators, one ends up with a local matrix element

$$\mathrm{Tr}\,(\Gamma\Phi) \;=\; \int dx\, d^2p_T\; \Gamma_{ji}(x,p_T)\,\Phi_{ij}(x,p_T) = \langle P|\overline{\psi}_j(0)\,\Gamma_{ji}(i\partial_\xi)\,U^{[n,C]}_{[0,\xi]}\,\psi_i(\xi)|P\rangle\Big|_{\xi=0}\,. \tag{242}$$

In particular when the operator $\overline{\psi}\Gamma U\psi$ is an operator with simple or known expectation values between plane wave states (including possibly spin dependence) this provides interesting sum rules for the functions appearing in the correlator.

C. Fragmentation correlators in high-energy processes

In high-energy processes, it is useful to employ the Sudakov decomposition of the momenta. We write for the parton momentum

$$k = \frac{1}{z}\,P_h + k_T + \left(k{\cdot}P_h - \frac{M_h^2}{z}\right)\,\frac{n}{P_h{\cdot}n}, \tag{243}$$

where $z = P_h{\cdot}n/k{\cdot}n$. The above equation defines the coordinates of k for fixed P_h. One can consider variations of P_h for fixed k, in which case we write

$$P_h = z\,k + P_{h\perp} - \left(k{\cdot}P_h - \frac{M_{h\perp}^2}{z}\right)\,\frac{n}{k{\cdot}n} \tag{244}$$

with $M_{h\perp}^2 = M_h^2 - P_{h\perp}^2$. For a fixed set of light-like vectors $n_\pm$ identifying the lightlike vector in both above equations as $n = n_+$ (and thus $n_- = P_h/(P_h{\cdot}n)$ or $k/(k{\cdot}n)$, respectively), we write the momenta in either one of the following forms

$$\boxed{\begin{aligned} P_h &= \left[P_h^-,\,\frac{M_h^2}{2P_h^-},\,0_T\right] \\ k &= \left[\frac{P_h^-}{z},\,\frac{k{\cdot}P_h - M_h^2/2z}{P_h^-},\,k_T\right] \end{aligned}} \quad\overset{LT}{\longleftrightarrow}\quad \boxed{\begin{aligned} P_h &= \left[P_h^-,\,\frac{M_{h\perp}^2}{2P_h^-},\,P_{h\perp}\right] \\ k &= \left[k^-,\,\frac{k{\cdot}P_h - M_{h\perp}^2/2z}{z\,k^-},\,0\right] \end{aligned}} \tag{244}$$

These two representations are connected by a Lorentz transformation that leaves the minus component unchanged (analogue of Eq. 28). We note that we have used the invariant $k{\cdot}P_h$ in the above, but it is also possible to use k^2 related through $z^2(k^2 - k_T^2) = 2z\,k{\cdot}P_h - M_h^2$.

For integrations this implies

$$\begin{aligned} \int dz\, d^2k_T\, d(k{\cdot}P_h) \;&=\; \int \frac{P_h{\cdot}n}{(k{\cdot}n)^2}\, d(k{\cdot}n)\, d^2k_T\, d(k{\cdot}P_h) = \int z^2\, d^4k \ldots|_{P_h} = \\ &=\; \int \frac{1}{z^2\,(k{\cdot}n)}\, d(P_h{\cdot}n)\, d^2P_{h\perp}\, d(P_h{\cdot}k) = \int \frac{1}{z^2}\, d^4P_h \ldots|_k. \end{aligned} \tag{245}$$

We consider the TMD correlator integrated only over $k{\cdot}P_h$,

$$\begin{aligned} \Delta_{ij}(z,k_T;n,C) \;&=\; \frac{1}{4z}\int \frac{d(k{\cdot}P_h)}{P_h{\cdot}n}\;\Delta_{ij}(k,P_h;C) = \frac{1}{4z}\int \frac{dk^2}{2(k{\cdot}n)}\;\Delta_{ij}(k,P_h;C) = \int \frac{dk^+}{4\,z}\;\Delta_{ij}(k,P_h;C) \\ &=\; \sum_X \int \frac{d\xi^+\, d^2\xi_T}{4z\,(2\pi)^3}\; e^{ik\cdot\xi}\,\langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\Big|_{LF} \end{aligned} \tag{246}$$

where LF refers to the light-front, $\xi^- = \xi{\cdot}n = 0$. The integration in coordinate space can be written in a coordinate independent way,

$$\int \frac{d\xi^+\, d^2\xi_T}{4z\,(2\pi)^3}\; e^{ik\cdot\xi}\ldots = \int \frac{(k{\cdot}n)}{4\,(P_h{\cdot}n)^2}\,\frac{d(\xi{\cdot}P_h)\,d^2\xi_T}{(2\pi)^3}\;\exp\left(\frac{i}{z}P_h{\cdot}\xi + ik_T{\cdot}\xi_T\right)\ldots\;.$$

Integrating over the transverse momenta, we define

$$\begin{aligned}\Delta_{ij}(z;n) &= \int d^2P_{h\perp}\ \Delta_{ij}(z,k_T;n,C) = \frac{z}{4}\int dk^+\, d^2k_T\ \Delta_{ij}(k,P_h;C)\\ &= \sum_X \int \frac{z}{4}\frac{d\xi^+}{(2\pi)}\ e^{ik\cdot\xi}\ \langle 0|U^{[n]}_{[a,\xi]}\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)U^{[n]}_{[0,a]}|0\rangle\Big|_{LC}\end{aligned} \tag{247}$$

where LC refers to the light-cone, $\xi^- = \xi\cdot n = \xi_T = 0$. Instead of ξ^+ one can use

$$\int \frac{z}{4}\frac{d\xi^+}{(2\pi)}\ e^{ik\cdot\xi}\ldots = \int \frac{1}{4\,(k\cdot n)}\frac{d(\xi\cdot P_h)}{(2\pi)}\ \exp\left(\frac{i}{z}P_h\cdot\xi\right)\ldots\ .$$

We note that issues on dependence on n can be taken over from the distribution functions, including issues on gauge choices and the possibility to get for time-like axial gauges a natural regulator from the transverse momenta.

Sum rules for fragmentation functions

Since for fragmentation correlators, the hadrons are produced, one can construct observables by summing and integrating over them including particular (in principle spin dependent) weights. To construct the observable, we assume the parton momentum k to be fixed, varying P_h and P_x. The sum rule then is

$$\begin{aligned}&\sum_h \int dz\, d^2P_{h\perp}\ O^h_{\alpha\beta}(z,P_{h\perp})\,\Delta_{ij,\beta\alpha}\left(z,\tfrac{P_{h\perp}}{z}\right)\\ &\quad= \sum_{h,X}\int \frac{d^4P_h}{(2\pi)^4}\,(2\pi)\delta(P_h^2-M_h^2)\,\theta(P_h\cdot n)\int\frac{dk^2}{8\pi\,(k\cdot n)}\\ &\qquad\qquad\times\int d^4\xi\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)|P_h,\alpha,X\rangle\, O^h_{\alpha\beta}(P_h)\,\langle P_h,\beta,X|\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\\ &\quad= \int\frac{dk^2}{8\pi\,(k\cdot n)}\int d^4\xi\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)\,\hat{O}\,\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\\ &\quad= \int\frac{dk^+}{4\pi}\int d^4\xi\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)\,\hat{O}\,\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\\ &\quad= \frac{1}{2}\int d\xi^+\, d^2\xi_T\ e^{ik\cdot\xi}\ \langle 0|U^{[n,C]}_{[a,\xi]}\psi_i(\xi)\,\hat{O}\,\overline{\psi}_j(0)U^{[n,C]}_{[0,a]}|0\rangle\Big|_{LF}\end{aligned} \tag{248}$$

where the operator $\hat{O}$ is

$$\hat{O} = \sum_{h,X}\int\frac{dz\, d^2P_{h\perp}}{2z(2\pi)^3}|P_h,\alpha,X\rangle\, O^h_{\alpha\beta}(P_h)\,\langle P_h,\beta,X|. \tag{249}$$

If the operator $\hat{O}$ is also known at the parton level,

$$\hat{O} = \sum_s\int\frac{dp^-\, d^2p_T}{2p^-\,(2\pi)^3}\ b^\dagger(p,s)o(p,s)b(p,s), \tag{250}$$

we obtain (again at fixed k)

$$\begin{aligned}\sum_h\int dz\, d^2P_{h\perp}\ O^h_{\alpha\beta}(z,P_{h\perp})\,\Delta_{ij,\beta\alpha}\Big|_k &= \frac{1}{2(k\cdot n)}\sum_s u_i(k,s)o(k,s)\overline{u}_j(k,s)\\ &= \left[\frac{\not{k}}{2(k\cdot n)}\left(\frac{o(k,s)+o(k,-s)}{2}+\gamma_5\not{s}\,\frac{o(k,s)-o(k,-s)}{2}\right)\right]_{ij}.\end{aligned} \tag{251}$$

The simplest application of this is considering the fragmentation of unpolarized quarks into hadrons with $O^h_{\alpha\beta} = P_h\cdot n\,\delta_{\alpha\beta}$ corresponding to the the operator $\hat{O} = \hat{P}\cdot n$. Using the leading order parametrization

$$\Delta_{ij,\alpha\alpha}(z,k_T;P_h,S_h;n) = \frac{1}{2(P_h\cdot n)}\,D_1(z,k_T^2)\,(\not{P}_h)_{ij}, \tag{252}$$

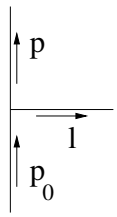
for the correlator (summed over spins), the sumrule becomes

$$\sum_h \int dz\, z D_1^{q\to h}(z) = 1, \tag{253}$$

which is precisely the momentum sumrule for fragmentation functions.

D. Large transverse momenta

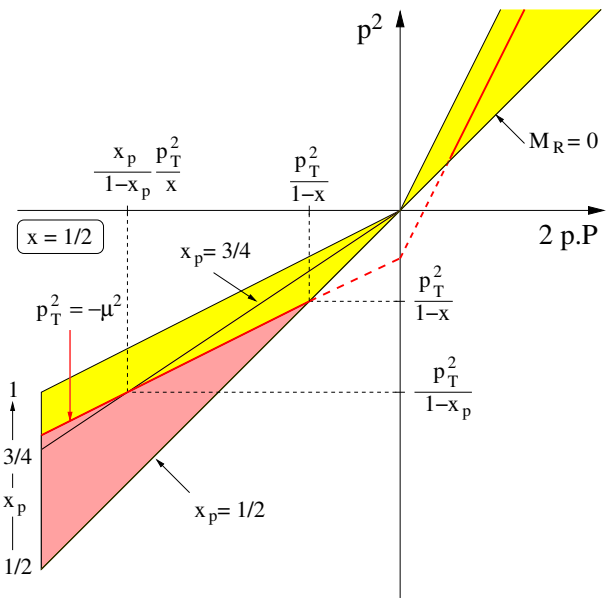
We started out with the assumption that the support of the correlators Φ and Δ is restricted to regions where the scalar products of the momenta involved are of hadronic size, or stated differently the fall-off as a function of these invariants should be sufficiently fast. To check consistency requires consideration of large transverse momenta generated after emission of hard partons. These will produce 'their own' hadrons and can at that stage be treated as on-shell partons. As our first case we look at parton(p_0) $\to$ parton(p) + parton(l) (with $p = p_0 - l$), the emission of an on-shell parton with momentum l by a parton with momentum p_0 (of which we will first neglect p_{0T}). The momentum fraction is reduced from $p_0.n = x/x_p$ ($x \le x_p \le 1$) to the lower value $p.n = x$ and producing at the same time a (moderately large) transverse momentum p_T. We neglect the $\mathcal{O}(M)$ contributions.



$$\begin{aligned}
p_0 &\approx \frac{x}{x_p} P \\
l &\approx (1-x_p)\, p_0 - p_T - \frac{p_T^2}{2\,(1-x_p)\,(p_0{\cdot}n)}\, n \approx \frac{(1-x_p)}{x_p}\, x\, P - p_T - \frac{x_p}{1-x_p}\,\frac{p_T^2}{2x}\, n, \\
p &\approx x_p\, p_0 + p_T + \frac{p_T^2}{2\,(1-x_p)\,(p_0{\cdot}n)}\, n \approx x\, P + p_T + \underbrace{\frac{x_p}{1-x_p}\,\frac{p_T^2}{2x}}_{\sigma_p}\, n.
\end{aligned}$$

For the invariant momenta we have

$$p_0^2 \approx l^2 \approx 0 \qquad \text{and} \qquad p^2 \approx \frac{p_T^2}{1-x_p}.$$



The figure to the left shows the support region as also discussed previously, but now neglecting all $\mathcal{O}(M)$ contributions. Thus $p{\cdot}P \approx \sigma_p \approx p^-$ and

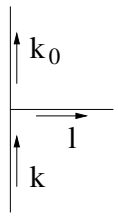
$$p^2 \approx 2x\,\sigma_p + p_T^2.$$

Although we assumed Φ to vanish for large values of the variables $p{\cdot}P$ and p^2, these can take larger values, e.g. after parton braching, as discussed above. In that case the additional variables entering are the fraction $p_0{\cdot}n = x/x_p$ of the original parton and (considered below) its transverse momentum p_{0T} with $x \le x_p \le 1$. By varying x_p from 1 (minimal loss of longitudinal momentum along P) to $x_p = x$ (maximal loss) we scan the full physical region of $\Phi(x, p_T)$. For given x and p_T, the values of the invariants are fixed for a given x_p,

$$\tau_p \approx p^2 \approx \frac{p_T^2}{1-x_p},$$

$$\sigma_p \approx p{\cdot}P \approx \frac{x_p}{1-x_p}\,\frac{p_T^2}{2x}.$$

The situation for fragmentation is analogous.



$$
\begin{aligned}
k_0 &\approx \frac{z_k}{z}\,P_h \\
l &\approx \frac{(1-z_k)}{z_k}\,k_0 - k_T - \frac{z_k\,k_T^2}{2\,(1-z_k)\,(k_0{\cdot}n)}\,n \approx \frac{(1-z_k)}{z}\,P_h - k_T - \frac{z\,k_T^2}{2(1-z_k)}\,n, \\
k &\approx z_k^{-1}\,k_0 + k_T + \frac{z_k\,k_T^2}{2\,(1-z_k)\,(k_0{\cdot}n)}\,n \approx z^{-1}\,P_h + k_T + \underbrace{\frac{z\,k_T^2}{2(1-z_k)}}_{\sigma_k}\,n.
\end{aligned}
$$

For the invariant momenta, we have

$$
k^2 \approx l^2 \approx 0 \qquad \text{and} \qquad k^2 \approx \frac{z_k}{1-z_k}\,k_T^2.
$$

We will now give the explicit integrations including also small transverse momenta p_{0T} for distributions (and k_{0T} for fragmentation). We get

$$
\begin{aligned}
\Phi(x,p_T) &= \int d\sigma_p\,d\tau_p\,\delta(\tau_p - 2x\,\sigma_p - p_T^2)\int d^4l\,\delta(l^2)\;\ldots\Phi_0(p_0) && (254)\\
&= \int d\sigma_p\,d\tau_p\,\delta(\tau_p - 2x\,\sigma_p - p_T^2)\int d(p_0{\cdot}n)\,d^2p_{0T}\,d(p_0{\cdot}P)\,\delta(l^2)\;\ldots\Phi_0(p_0), && (255)
\end{aligned}
$$

The integration over $p_0{\cdot}n = x/x_p$ is easily turned into an x_p-integration, the integration over $p_0{\cdot}P$ can be performed to get $\Phi_0(x/x_p, p_{0T})$ and in the evaluation of l^2 we have (note $l_T = p_{0T} - p_T$). Using

$$
l^2 = (p_0 - p)^2 = (p_T - p_{0T})^2 - \frac{1-x_p}{x_p}\,2x\,\sigma_p, \tag{256}
$$

we find

$$
\begin{aligned}
\Phi(x,p_T) &= \int d\sigma_p\,d\tau_p\,dx_p\,d^2p_{0T}\,\frac{x}{x_p^2}\,\delta\left(\tau_p - 2x\,\sigma_p - p_T^2\right) \\
&\qquad\qquad \times\,\delta\left(p_T^2 - 2p_{0T}{\cdot}p_T + p_{0T}^2 - \frac{1-x_p}{x_p}\,2x\,\sigma_p\right)\;\ldots\Phi_0\left(\frac{x}{x_p}, p_{0T}\right) \\
&= \int d\sigma_p\,d\tau_p\,dx_p\,d^2p_{0T}\,\frac{x}{x_p^2}\,\delta\left(\tau_p - 2x\,\sigma_p - p_T^2\right) \\
&\qquad\qquad \times\,\delta\left(\frac{p_T^2}{x_p} - 2p_{0T}{\cdot}p_T + p_{0T}^2 - \frac{1-x_p}{x_p}\,\tau_p\right)\;\ldots\Phi_0\left(\frac{x}{x_p}, p_{0T}\right) \\
&= \int_x^1 \frac{dx_p\,d^2p_{0T}}{2x_p\,(1-x_p)}\;\ldots\Phi_0\left(\frac{x}{x_p}, p_{0T}\right),
\end{aligned} \tag{257}
$$

where in the integrand invariants like τ_p and σ_p are fixed,

$$
\tau_p = p^2 = \frac{1}{1-x_p}\,p_T^2 - \frac{2x_p}{1-x_p}\,p_{0T}{\cdot}p_T + \frac{x_p}{1-x_p}\,p_{0T}^2, \tag{258}
$$

$$
2x\,\sigma_p = p^2 - p_T^2 = \frac{x_p}{1-x_p}\,p_T^2 - \frac{2x_p}{1-x_p}\,p_{0T}{\cdot}p_T + \frac{x_p}{1-x_p}\,p_{0T}^2. \tag{259}
$$

We note that

$$
\frac{1}{p^2} = \frac{(1-x_p)}{p_T^2}\left(1 + 2\,x_p\,\frac{p_{0T}{\cdot}p_T}{p_T^2} + 4\,x_p^2\,\frac{(p_{0T}{\cdot}p_T)^2}{(p_T^2)^2} - x_p\,\frac{p_{0T}^2}{p_T^2} + \mathcal{O}(p_{0T}^3)\right) \tag{260}
$$

$$
\frac{1}{(p^2)^2} = \frac{(1-x_p)^2}{(p_T^2)^2}\left(1 + 4\,x_p\,\frac{p_{0T}{\cdot}p_T}{p_T^2} + 12\,x_p^2\,\frac{(p_{0T}{\cdot}p_T)^2}{(p_T^2)^2} - 2\,x_p\,\frac{p_{0T}^2}{p_T^2} + \mathcal{O}(p_{0T}^3)\right). \tag{261}
$$

For fragmentation functions, we get

$$
l^2 = (k_0 - k)^2 = (k_T - k_{0T})^2 - (1-z_k)\,2z^{-1}\,\sigma_k, \tag{262}
$$

and we find

$$\begin{aligned}
\Delta(z,k_T) &= \int d\sigma_k\, d\tau_k\, dz_k\, d^2k_{0T}\, \frac{1}{z\, z_k}\, \delta\left(\tau_k - 2z^{-1}\,\sigma_k - k_T^2\right) \\
&\qquad \times \delta\left(k_T^2 - 2k_{0T}\cdot k_T + k_{0T}^2 - (1-z_k)\, 2z^{-1}\,\sigma_k\right)\; \ldots \Delta_0\left(\frac{z}{z_k}, k_{0T}\right) \\
&= \int d\sigma_k\, d\tau_k\, dz_k\, d^2k_{0T}\, \frac{1}{z\, z_k}\, \delta\left(\tau_k - 2z^{-1}\,\sigma_k - k_T^2\right) \\
&\qquad \times \delta\left(z_k\, k_T^2 - 2k_{0T}\cdot k_T + k_{0T}^2 - (1-z_k)\,\tau_k\right)\; \ldots \Delta_0\left(\frac{z}{z_k}, k_{0T}\right) \\
&= \int_x^1 \frac{dz_k\, d^2k_{0T}}{2(1-z_k)}\; \ldots \Delta_0\left(\frac{z}{z_k}, k_{0T}\right),
\end{aligned} \tag{263}$$

where in the integrand invariants like τ_k and σ_k are fixed,

$$\tau_k = k^2 = \frac{z_k}{1-z_k}\, k_T^2 - \frac{2}{1-z_k}\, k_{0T}\cdot k_T + \frac{1}{1-z_k}\, k_{0T}^2, \tag{264}$$

$$2z^{-1}\,\sigma_k = k^2 - k_T^2 = \frac{1}{1-z_k}\, k_T^2 - \frac{2}{1-z_k}\, k_{0T}\cdot k_T + \frac{1}{1-z_k}\, k_{0T}^2. \tag{265}$$

We note that

$$\frac{1}{k^2} = \frac{(1-z_k)}{z_k\, k_T^2}\left(1 + 2\,\frac{p_{0T}\cdot p_T}{z_k\, k_T^2} + 4\,\frac{(k_{0T}\cdot k_T)^2}{z_k^2\,(k_T^2)^2} - \frac{k_{0T}^2}{z_k\, k_T^2} + \mathcal{O}(k_{0T}^3)\right) \tag{266}$$

$$\frac{1}{(k^2)^2} = \frac{(1-z_k)^2}{z_k^2\,(k_T^2)^2}\left(1 + 4\,\frac{k_{0T}\cdot k_T}{z_k\, k_T^2} + 12\,\frac{(k_{0T}\cdot k_T)^2}{z_k^2\,(k_T^2)^2} - 2\,\frac{k_{0T}^2}{z_k\, k_T^2} + \mathcal{O}(k_{0T}^3)\right). \tag{267}$$

Appendix A: The gauge link

As indicated gauge link are essential ingredients to have proper definitions of color gauge-invariant correlators. They arise from insertions of quark-gluon-quark correlators and of these the leading ones are expected to be correlators containing $A{\cdot}n$ fields. The general $A^\mu(\eta)$ field in the correlator in Eq. 22 can be written as

$$A^\mu = \frac{(A{\cdot}n)}{P{\cdot}n}\, P^\mu + A_T^\mu + \frac{(P{\cdot}n)(A{\cdot}P) - (A{\cdot}n)M^2}{(P{\cdot}n)^2}\, n^\mu.$$

This expansion can be written down for $A^\mu(x)$ or $A^\mu(p)$ To see, how quark-gluon-quark correlators are turned into color gauge-invariant objects, it is convenient to look at the momentum space field and rewrite the momentum in terms of the gluon momentum

$$\begin{aligned}
A^\mu(p) &= \int d^4\eta\; e^{i\,p\cdot\eta}\, A^\mu(\eta) \\
&= \int d^4\eta\; e^{ip\cdot\eta}\left[\frac{(A(\eta){\cdot}n)}{p{\cdot}n}\, p^\mu \right. \\
&\qquad + \frac{(p{\cdot}n)\, A_T^\mu(\eta) - p_T^\mu\,(A(\eta){\cdot}n)}{p{\cdot}n} \\
&\qquad \left. + \frac{(p{\cdot}n)(A(\eta){\cdot}P) - (p{\cdot}P)(A(\eta){\cdot}n)}{(p{\cdot}n)(P{\cdot}n)}\, n^\mu\right].
\end{aligned}$$

In the correlator the momentum $p^\mu \longrightarrow i\partial^\mu(\eta)$, so

$$
\begin{aligned}
A^\mu(p) &= \frac{1}{p{\cdot}n}\int d^4\eta\; e^{i\,p\cdot\eta}\Bigg[A^n(\eta)\,p^\mu \\
&\qquad + i\partial^n(\eta)\,A^\mu_{\scriptscriptstyle T}(\eta) - i\partial^\mu_{\scriptscriptstyle T}(\eta)\,A^n(\eta) \\
&\qquad + \frac{i\partial^n(\eta)A^P(\eta) - i\partial^P(\eta)A^n(\eta)}{P{\cdot}n}\,n^\mu\Bigg] \\
&= \frac{1}{p{\cdot}n}\left[A^n(p)\,p^\mu + i\,G^{n\mu}_{\scriptscriptstyle T}(p) + \frac{iG^{nP}(p)}{P{\cdot}n}\,n^\mu\right]
\end{aligned}
$$

The first term will lead to gauge links along the n-direction. The introduction of the gluon momentum p^μ, rather than staying with the hadron momentum is done not only because it renders the subleading parts gauge invariant but it provides the necessary and convenient starting point for using Ward identities for the A^n insertions (gluon with polarization along the parton/hadron momentum). It will turn out that the insertions on the external legs of the hard part give the path dependence.

To see this, consider field combinations denoted as

$$
\underbrace{U^{[n]}_{-}\psi}(p^+) = U^{[n]}_{-}(p^+)\,\psi(p^+) = \int d\xi^-\; e^{i\,p^+\cdot\xi^-}\,U^{[n]}_{[-\infty,\xi^-]}\psi(\xi^-),
$$

where $\xi^- = \xi{\cdot}P$ and $p^+ = p{\cdot}n$, which are the only relevant components that need to be considered here. Explicitly one has (with $A^+ = A{\cdot}n$)

$$
\underbrace{U^{[n]}_{-}\psi}(p^+) = \sum_{N=0}^{\infty}(-i)^N\int_{-\infty}^{\infty}d\xi^-\int_{-\infty}^{\xi}d\eta_N^-\int_{\eta_N^-}^{\xi}d\eta_{N-1}^-\ldots\int_{\eta_2^-}^{\xi}d\eta_1^-\;A^+(\eta_N^-)\ldots A^+(\eta_1^-)\psi(\xi)\,e^{i\,p^+\xi^-},
$$

in which the arguments run between $-\infty < \eta_N^- < \eta_{N-1}^- < \ldots < \eta_1^- < \xi^-$, implemented through $\theta(\eta_{N-1}^- - \eta_N^-)\ldots\theta(\eta_1^- - \eta_2^-)\theta(\xi^- - \eta_1^-)$, which can be rewritten as momentum-space integrations,

$$
\begin{aligned}
\underbrace{U^{[n]}_{-}\psi}(p^+) &= \sum_{N=0}^{\infty}(-i)^N\int_{-\infty}^{\infty}d\xi^-\int_{-\infty}^{\infty}d\eta_N^-\int_{-\infty}^{\infty}d\eta_{N-1}^-\ldots\int_{-\infty}^{\infty}d\eta_1^- \\
&\qquad\times\int\frac{dp_N^+}{-2\pi\,i}\ldots\int\frac{dp_1^+}{-2\pi\,i}\;\frac{e^{-i\,p_N^+(\eta_{N-1}^- - \eta_N^-)}}{p_N^+ + i\epsilon}\ldots\frac{e^{-i\,p_1^+(\xi^- - \eta_1^-)}}{p_1^+ + i\epsilon}\;A^+(\eta_N^-)\ldots A^+(\eta_1^-)\psi(\xi^-)\,e^{i\,p^+\xi^-} \\
&= \sum_{N=0}^{\infty}\int\frac{dp_N^+}{2\pi}\ldots\int\frac{dp_1^+}{2\pi}\,\frac{A^+(p_N^+)}{(p_N^+ + i\epsilon)}\,\frac{A^+(p_{N-1}^+ - p_N^+)}{(p_{N-1}^+ + i\epsilon)}\ldots\frac{A^+(p_1^+ - p_2^+)}{(p_1^+ + i\epsilon)}\,\psi(p^+ - p_1^+). \\
&= \sum_{N=0}^{\infty}\int\frac{dp_N^+}{2\pi}\ldots\int\frac{dp_1^+}{2\pi}\,\frac{A^+(p_N^+)}{(p_N^+ + i\epsilon)}\,\frac{A^+(p_{N-1}^+)}{(p_N^+ + p_{N-1}^+ + i\epsilon)}\ldots\frac{A^+(p_1^+)}{(p_N^+ + \ldots + p_1^+ + i\epsilon)} \\
&\qquad\times\psi(p^+ - p_1^+ - \ldots - p_N^+). \\
&= \sum_{N=0}^{\infty}\int\frac{dp_1^+}{2\pi}\ldots\int\frac{dp_N^+}{2\pi}\,\frac{A^+(p_1^+)}{(p_1^+ + i\epsilon)}\,\frac{A^+(p_2^+)}{(p_1^+ + p_2^+ + i\epsilon)}\ldots\frac{A^+(p_N^+)}{(p_1^+ + \ldots + p_N^+ + i\epsilon)} \\
&\qquad\times\psi(p^+ - p_1^+ - \ldots - p_N^+).
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
\underbrace{U^{[n]}_{-}\psi}(p) &= \int d^4\xi\;\exp\left(i\,p{\cdot}\xi\right)\,\mathcal{P}\exp\left(-ig\int_{-\infty}^{\xi\cdot P}d(\eta{\cdot}P)\;n{\cdot}A(\eta)\right)\psi(x) \\
&= \sum_{N=0}^{\infty}\int\frac{d^4p_N}{(2\pi)^4}\ldots\int\frac{d^4p_1}{(2\pi)^4}\,\frac{A^n(p_N)}{(x_N + i\epsilon)}\,\frac{A^n(p_{N-1} - p_N)}{(x_{N-1} + i\epsilon)}\ldots\frac{A^n(p_1 - p_2)}{(x_1 + i\epsilon)}\,\psi(p - p_1) \\
&= \sum_{N=0}^{\infty}\int\frac{d^4p_1}{(2\pi)^4}\ldots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)}{(x_1 + i\epsilon)}\,\frac{A^n(p_2)}{(x_1 + x_2 + i\epsilon)}\ldots\frac{A^n(p_N)}{(x_1 + \ldots + x_N + i\epsilon)}\,\psi\left(p - \sum_{i=1}^{N}p_i\right),
\end{aligned}
\tag{268}
$$

where $x_i = p_i{\cdot}n$. We thus can expand

$$\underbrace{U_{\pm}^{[n]}}\psi(p) = \sum_{M=0}^{\infty} \underbrace{U_{\pm}^{[n](M)}}\psi(p), \tag{269}$$

where $U_{\pm}^{[n](0)} = 1$. The gauge link and the terms in its expansion, not only have a particular structure in coordinate or momentum space, but they also have a charge structure. In particular for applications in non-abelian gauge theories, it is convenient to expand the gauge link, like the field $A^\mu = A^{a\mu}T_a$, in terms of color matrices,

$$U_{\pm}^{[n]} = U_{\pm}^{a[n]}\, T_a, \tag{270}$$

which is possible for each of the terms in the expansion of the gauge link, but also works for the full gauge link.

It is possible to use a more symmetric expression for the gauge link in momentum space. The momenta $p_1, \ldots, p_N$ are integration variables. We can use the relations

$$\frac{1}{(x_1+x_2+i\epsilon)}\left[\frac{1}{(x_1+i\epsilon)}+\frac{1}{(x_2+i\epsilon)}\right] = \frac{1}{(x_1+i\epsilon)(x_2+i\epsilon)}, \tag{271}$$

$$\begin{aligned}\frac{1}{(x_1+x_2+x_3+i\epsilon)}\underbrace{\left[\frac{1}{(x_1+x_2+i\epsilon)(x_1+i\epsilon)}+\ldots\right]}_{\text{6 permutations}} &= \frac{1}{(x_1+x_2+x_3+i\epsilon)}\underbrace{\left[\frac{1}{(x_1+i\epsilon)(x_2+i\epsilon)}+\ldots\right]}_{\text{3 permutations}} \\ &= \frac{1}{(x_1+i\epsilon)(x_2+i\epsilon)(x_3+i\epsilon)},\end{aligned} \tag{272}$$

and its generalization to more momenta, to symmetrize the result. This simply works in the abelian case, when all permutations of $A^n(p_{\tau(1)})\ldots A^n(p_{\tau(N)})$ are identical, but can also be used in the non-abelian case. In that case one has for two fields (omitting the prescription)

$$\begin{aligned}\frac{A^n(p_1)\,A^n(p_2)}{x_1(x_1+x_2)} &= \frac{1}{2}\frac{A^n(p_1)\,A^n(p_2)}{x_1(x_1+x_2)}+\frac{1}{2}\frac{A^n(p_2)\,A^n(p_1)}{x_2(x_1+x_2)} \\ &= \frac{1}{4}\frac{\{A^n(p_1),A^n(p_2)\}}{x_1\,x_2}+\frac{1}{4}\frac{(x_2-x_1)\,[A^n(p_1),A^n(p_2)]}{x_1\,x_2\,(x_1+x_2)} = \frac{1}{2}\frac{A^n(p_1)\,A^n(p_2)}{x_1\,x_2},\end{aligned}$$

where the commutator term is not important, being proportional to $(x_2-x_1)\,\delta(x_1-x_2)$. Thus we have

$$\begin{aligned}\underbrace{U_{-}^{[n]}}\psi(p) &= \int d^4\xi\ \exp(i\,p{\cdot}\xi)\ \mathcal{P}\exp\left(-ig\int_{-\infty}^{\xi\cdot P} d(\eta{\cdot}P)\ n{\cdot}A(\eta)\right)\psi(x) \\ &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(x_1+i\epsilon)(x_2+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right),\end{aligned} \tag{273}$$

which is now by construction symmetrized in color space. For a link along n coming from $+\infty$ one has

$$\begin{aligned}\underbrace{U_{+}^{[n]}}\psi(p) &= \int d^4\xi\ \exp(i\,p{\cdot}\xi)\ \mathcal{P}\exp\left(-ig\int_{\infty}^{\xi\cdot P} d(\eta{\cdot}P)\ n{\cdot}A(\eta)\right)\psi(x) \\ &= \sum_{N=0}^{\infty}\int\frac{d^4p_N}{(2\pi)^4}\cdots\int\frac{d^4p_1}{(2\pi)^4}\,\frac{A^n(p_N)}{(-x_N+i\epsilon)}\,\frac{A^n(p_{N-1}-p_N)}{(-x_{N-1}+i\epsilon)}\cdots\frac{A^n(p_1-p_2)}{(-x_1+i\epsilon)}\,\psi(p-p_1). \\ &= \sum_{N=0}^{\infty}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)}{(-x_1+i\epsilon)}\cdots\frac{A^n(p_N)}{(-x_1-\ldots-x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right), \\ &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(-x_1+i\epsilon)(-x_2+i\epsilon)\ldots(-x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right),\end{aligned} \tag{274}$$

One also sees that a term $U_{\pm}^{[n](M)}$ is the consecutive action of M simple $U_{\pm}^{[n](1)}$-links. The conjugate link is

$$\begin{aligned}
\overline{\psi}(p)\,U_{-}^{[n]\dagger} &= \int d^4\xi\ \exp\left(-i\,p{\cdot}\xi\right)\,\overline{\psi}(\xi)\,\mathcal{P}\exp\left(+ig\int_{-\infty}^{\xi\cdot P} d(\eta{\cdot}P)\ n{\cdot}A(\eta)\right) \\
&= \sum_{N=0}^{\infty}\frac{(-)^N}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\overline{\psi}\left(p+\sum_{i=1}^{N}p_i\right)\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(p_1^+ + i\epsilon)(p_2^+ + i\epsilon)\ldots(p_N^+ + i\epsilon)} \\
&= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\int\frac{d^4p_N}{(2\pi)^4}\,\overline{\psi}\left(p-\sum_{i=1}^{N}p_i\right)\,\frac{A^n(-p_1)\,A^n(-p_2)\ldots A^n(-p_N)}{(p_1^+ - i\epsilon)(p_2^+ - i\epsilon)\ldots(p_N^+ - i\epsilon)}.
\end{aligned} \tag{275}$$

The links along the n-direction are actually essential to render the insertion of other gluon components gauge-invariant. To see this, one must realize that including the n-links the relevant pieces in the correlator contain

$$\ldots i\partial^\mu(\eta)\ldots U_{[\ldots\eta]}^{[n]}\,A^\alpha(\eta)\,U_{[\eta,\ldots]}^{[n]}\cdots\ .$$

In order to evaluate the pieces we give a number of useful relations for gauge links

$$i\,\partial_\xi^n\,U_{[0,\xi]}^{[n]} = U_{[0,\xi]}^{[n]}\,iD^n(\xi), \tag{276}$$

$$ig\,G^{n\alpha}(\xi) = [iD^n(\xi), iD_T^\alpha(\xi)] = [iD^n(\xi), g\,A_T^\alpha(\xi)] - [i\partial_\xi^\alpha, g\,A^n(\xi)], \tag{277}$$

$$[i\partial_\xi^n, U_{[\eta,\xi]}^{[n]}\,gA_T^\alpha(\xi)\,U_{[\xi,\eta]}^{[n]}] = U_{[\eta,\xi]}^{[n]}\left(ig\,G^{n\alpha}(\xi) + [i\partial_\xi^\alpha, gA^n(\xi)]\right)U_{[\xi,\eta]}^{[n]}, \tag{278}$$

$$iD_T^\alpha(\eta)\,U_{[\eta,\xi]}^{[n]} - U_{[\eta,\xi]}^{[n]}\,iD_T^\alpha(\xi) = \int_{\eta\cdot P}^{\xi\cdot P} d(\zeta{\cdot}P)\ U_{[\eta,\zeta]}^{[n]}\,\underbrace{[iD^n(\zeta), iD_T^\alpha(\zeta)]}_{ig\,G^{n\alpha}(\zeta)}\,U_{[\zeta,\xi]}^{[n]}. \tag{279}$$

In particular one sees that (with α a transverse component),

$$i\partial^n(\eta)U_{[\ldots\eta]}^{[n]}\,A^\alpha(\eta)\,U_{[\eta,\ldots]}^{[n]} \quad\Longrightarrow\quad U_{[\ldots\eta]}^{[n]}\,[iD^n(\eta), A^\alpha(\eta)]U_{[\eta,\ldots]}^{[n]} \tag{280}$$

while

$$-i\partial^\alpha(\eta)U_{[\ldots\eta]}^{[n]}\,A^n(\eta)\,U_{[\eta,\ldots]}^{[n]} \quad\Longrightarrow\quad -U_{[\ldots\eta]}^{[n]}\,[i\partial^\alpha(\eta), A^n(\eta)]U_{[\eta,\ldots]}^{[n]} \tag{281}$$

and for the combination

$$i\partial^n(\eta)U_{[\ldots\eta]}^{[n]}\,A^\alpha(\eta)\,U_{[\eta,\ldots]}^{[n]} - i\partial^\alpha(\eta)U_{[\ldots\eta]}^{[n]}\,A^n(\eta)\,U_{[\eta,\ldots]}^{[n]} \quad\Longrightarrow\quad U_{[\ldots\eta]}^{[n]}\,iG^{n\alpha}(\eta)\,U_{[\eta,\ldots]}^{[n]}. \tag{282}$$

Similarly

$$i\partial^n(\eta)U_{[\ldots\eta]}^{[n]}\,A^P(\eta)\,U_{[\eta,\ldots]}^{[n]} - i\partial^P(\eta)U_{[\ldots\eta]}^{[n]}\,A^n(\eta)\,U_{[\eta,\ldots]}^{[n]} \quad\Longrightarrow\quad U_{[\ldots\eta]}^{[n]}\,iG^{nP}(\eta)\,U_{[\eta,\ldots]}^{[n]}. \tag{283}$$

It is useful to have the above relations also in momentum space. We have (everywhere absorbing g in A)

$$\int d^4\xi\ e^{i\,p\cdot\xi}\,\psi(\xi) = \psi(p), \tag{284}$$

$$\int d^4\xi\ e^{i\,p\cdot\xi}\,i\partial_\xi^\mu\psi(\xi) = p^\mu\,\psi(p), \tag{285}$$

$$\int d^4\xi\ e^{i\,p\cdot\xi}\,A^\mu(\xi)\psi(\xi) = \int\frac{d^4p_0}{(2\pi)^4}\,A^\mu(p_0)\,\psi\left(p-p_0\right) = A^\mu\,\psi(p), \tag{286}$$

$$\int d^4\xi\ e^{i\,p\cdot\xi}\,A^\mu(\xi)\,A^\nu(\xi)\,\psi(\xi) = \int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,A^\mu(p_1)\,A^\nu(p_2)\,\psi\left(p-p_1-p_2\right), \tag{287}$$

$$\begin{aligned}
\int d^4\xi\ e^{i\,p\cdot\xi}\,A^\mu(\xi)\,A^\nu(\xi) &= \int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,A^\mu(p_1)\,A^\nu(p_2)\,(2\pi)^4\,\delta^4\left(p-p_1-p_2\right), \\
&= \int\frac{d^4p_1}{(2\pi)^4}\,A^\mu(p_1)\,A^\nu(p-p_1),
\end{aligned} \tag{288}$$

$$\begin{aligned}
&\int d^4\xi\ e^{i\,p\cdot\xi}\,iG^{n\alpha}(\xi) = iG^{n\alpha}(p) \\
&\qquad = x\,A_T^\alpha(p) - p_T^\alpha\,A^n(p) + \int\frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,[A^n(p_1), A_T^\alpha(p_2)]\,(2\pi)^4\,\delta^4\left(p-p_1-p_2\right),
\end{aligned} \tag{289}$$

For the Fourier transforms of the fields including a gauge link one has besides Eqs 268 or 273,

$$
\begin{aligned}
\underbrace{U_-^{[n]} i\partial^n}\psi(p) &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\,A^n(p_2)\ldots A^n(p_N)}{(x_1+i\epsilon)(x_2+i\epsilon)\ldots(x_N+i\epsilon)}\left(x-\sum_{i=1}^{N}x_i\right)\psi\left(p-\sum_{i=1}^{N}p_i\right)\\
&= x\,\underbrace{U_-^{[n]}\psi}(p)\\
&\quad-\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)}{(x_1+i\epsilon)}\cdots A^n(p_i)\cdots\frac{A^n(p_N)}{(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right)\\
&= x\,\underbrace{U_-^{[n]}\psi}(p)\\
&\quad-\sum_{N=1}^{\infty}\frac{1}{(N-1)!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\ldots A^n(p_{N-1})}{(x_1+i\epsilon)\ldots(x_{N-1}+i\epsilon)}\,A^n(p_N)\,\psi\left(p-\sum_{i=1}^{N}p_i\right).\\
&= x\,\underbrace{U_-^{[n]}\psi}(p)-\underbrace{U_-^{[n]}A^n\psi}(p).
\end{aligned}
$$

Using Eq. 268 one can of course also obtain this result. Thus one finds

$$
\underbrace{U_-^{[n]} iD^n\psi}(p) = i\partial^n\,\underbrace{U_-^{[n]}\psi}(p) \tag{290}
$$

For a transverse derivative one obtains

$$
\begin{aligned}
\underbrace{U_-^{[n]} i\partial_T^\alpha\psi}(p) &= p_T^\alpha\,\underbrace{U_-^{[n]}\psi}(p)-\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\\
&\qquad\frac{A^n(p_1)\ldots p_{iT}^\alpha A^n(p_i)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_i+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right).
\end{aligned}
$$

Using

$$
\begin{aligned}
[A^n(p_1)\ldots A^n(p_N),A_T^\alpha(p_0)] &= A^n(p_1)\ldots A^n(p_N)\,A_T^\alpha(p_0)-A_T^\alpha(p_0)\,A^n(p_1)\ldots A^n(p_N)\\
&= \sum_{i=1}^{N}A^n(p_1)\ldots A^n(p_{i-1})\,[A^n(p_i),A_T^\alpha(p_0)]\,A^n(p_{i+1})\ldots A^n(p_N),
\end{aligned}
$$

we get

$$
\begin{aligned}
&\underbrace{U_-^{[n]}\,A_T^\alpha}\,\psi(p)-\underbrace{A_T^\alpha\,U_-^{[n]}}\,\psi(p)\\
&=\sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{d^4p_0}{(2\pi)^4}\left[\frac{A^n(p_1)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_N+i\epsilon)},A_T^\alpha(p_0)\right]\psi\left(p-p_0-\sum_{i=1}^{N}p_i\right)\\
&=\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{d^4p_0}{(2\pi)^4}\,\frac{A^n(p_1)\ldots[A^n(p_i),A_T^\alpha(p_0)]\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_i+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-p_0-\sum_{i=1}^{N}p_i\right).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
p_T^\alpha\,\underbrace{U_-^{[n]}\psi}(p) &= \sum_{N=0}^{\infty}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,p_T^\alpha\,\frac{A^n(p_1)\ldots A^n(p_N)}{(x_1+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right)\\
&= \sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_{i-1}}{(2\pi)^4}\,\frac{d^4p_{i+1}}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\\
&\qquad\frac{A^n(p_1)\ldots A^n(p_{i-1})}{(x_1+i\epsilon)\ldots(x_{i-1}+i\epsilon)}\,p_T^\alpha\,\frac{A^n(p_{i+1})\ldots A^n(p_N)}{(x_{i+1}+i\epsilon)\ldots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right),
\end{aligned}
$$

and

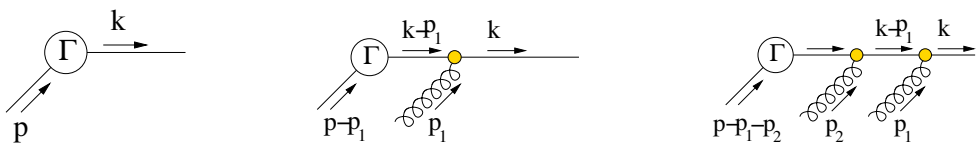
$$\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\dots A^n(p_{i-1})}{(x_1+i\epsilon)\dots(x_{i-1}+i\epsilon)}\,A_T^\alpha(p_i)\,\frac{A^n(p_{i+1})\dots A^n(p_N)}{(x_{i+1}+i\epsilon)\dots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right)$$
$$=\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\dots x_i\,A_T^\alpha(p_i)\dots A^n(p_N)}{(x_1+i\epsilon)\dots(x_i+i\epsilon)\dots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right).$$

Combining the results,

$$\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\dots iG^{n\alpha}(p_i)\dots A^n(p_N)}{(x_1+i\epsilon)\dots(x_i+i\epsilon)\dots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right)$$
$$=\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\dots A^n(p_{i-1})}{(x_1+i\epsilon)\dots(x_{i-1}+i\epsilon)}\,A_T^\alpha(p_i)\,\frac{A^n(p_{i+1})\dots A^n(p_N)}{(x_{i+1}+i\epsilon)\dots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right)$$
$$+\,\underbrace{U_-^{[n]}\,A_T^\alpha}\,\psi(p)-\underbrace{A_T^\alpha\,U_-^{[n]}}\,\psi(p)+\underbrace{U_-^{[n]}i\partial_T^\alpha}\psi(p)-p_T^\alpha\,\underbrace{U_-^{[n]}}\psi(p).$$
$$=\sum_{N=1}^{\infty}\sum_{i=1}^{N}\frac{1}{N!}\int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_N}{(2\pi)^4}\,\frac{A^n(p_1)\dots A^n(p_{i-1})}{(x_1+i\epsilon)\dots(x_{i-1}+i\epsilon)}\,A_T^\alpha(p_i)\,\frac{A^n(p_{i+1})\dots A^n(p_N)}{(x_{i+1}+i\epsilon)\dots(x_N+i\epsilon)}\,\psi\left(p-\sum_{i=1}^{N}p_i\right)$$
$$+\,\underbrace{U_-^{[n]}iD_T{}^\alpha}\psi(p)-i\underbrace{D_T^\alpha\,U_-^{[n]}}\psi(p).$$

Appendix B: Calculation of gauge link

We want to consider a hard proces and see which gluon correlators need to be resummed to get a gauge-invariant correlator including a gauge link. We first look at the insertions onto *one* fermion line. At this stage nonabelian effects do not matter. Working with momentum representation of the fields, we consider of the $A^\mu(p)$ insertion only the collinear term, to be precise $A^n(p)p^\mu/(p.n)$. Furthermore, first consider the situation that the hard process is a simple (constant) vertex (like a γ^α in deep inelastic scattering). Thus, we are going to resum the diagrams



The first term gives as relevant contribution

$$A_0=\overline{\psi}(k)\Gamma\psi(p),$$

where the $\psi(p)$ and $\overline{\psi}(k)$ are fields belonging to the correlators of initial (momentum p) and final state quark (momentum k) respectively. The first gauge link contribution is

$$A_1=\int\frac{d^4p_1}{(2\pi)^4}\,\overline{\psi}(k)\,\frac{-i\,\not p_1\,A^k(p_1)}{p_1{\cdot}k}\,\frac{i(\not k-\not p_1)}{(-2k{\cdot}p_1+i\epsilon)}\,\Gamma\,\psi(p-p_1).$$

The numerator becomes $\not p_1(\not k-\not p_1)=\not p_1\not k=\{\not k,\not p_1\}=2\,k{\cdot}p_1$. The added term is zero since $\overline{\psi}(k)\not k\approx 0$. Thus one has (note that the sign of $k{\cdot}p_1$ is plus),

$$A_1\;=\;\overline{\psi}(k)\int\frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{(-k{\cdot}p_1+i\epsilon)}\,\Gamma\,\psi(p-p_1)=\overline{\psi}(k)\,\underbrace{U_+^{[k](1)}\,\Gamma\psi}(p) \tag{291}$$

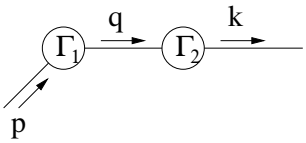
The 2-gluon term becomes

$$
\begin{aligned}
A_2 &= \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\;\overline{\psi}(k)\,\frac{A^k(p_1)}{p_1{\cdot}k}\,\frac{A^k(p_2)}{p_2{\cdot}k}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\not{p}_2\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2}\,\Gamma\,\psi(p-p_1-p_2)\\
&= \int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\;\overline{\psi}(k)\,\frac{A^k(p_1)}{p_1{\cdot}k}\,\frac{A^k(p_2)}{p_2{\cdot}k}\,\frac{\not{p}_1\,(\not{k}-\not{p}_1)}{(-2k{\cdot}p_1+i\epsilon)}\,\frac{\not{p}_2\,(\not{k}-\not{p}_1-\not{p}_2)}{(-2k{\cdot}p_1-2k{\cdot}p_2+i\epsilon)}\,\Gamma\,\psi(p-p_1-p_2)\\
&= \overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,\frac{A^k(p_1)}{(-k{\cdot}p_1+i\epsilon)}\,\frac{A^k(p_2)}{(-k{\cdot}p_1-k{\cdot}p_2+i\epsilon)}\,\Gamma\,\psi(p-p_1-p_2)\\
&= \overline{\psi}(k)\,\underbrace{U_+^{[k](2)}\,\Gamma\psi}(p),
\end{aligned}
\tag{292}
$$

etc. and the total result becoming

$$
\sum_{N=0}^{\infty} A_N = \overline{\psi}(k)\,\underbrace{U_+^{[k]}\,\Gamma\psi}(p) = \overline{\psi}(k)\,U_+^{a[k]}(p)\,\Gamma\psi(p). = \overline{\psi}(k)\,T_a\,\Gamma\,U_+^{a[k]}\psi(p).
\tag{293}
$$

The next situation to be investigated is the momentum dependence if there is momentum dependence in the basic diagram,



$$
A_0 = i\,\overline{\psi}(k)\,\Gamma_2\,\frac{1}{\not{q}}\,\Gamma_1\,\psi(p).
\tag{294}
$$

The two one-gluon insertions are

$$
\begin{aligned}
A_1 &= i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{k{\cdot}p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1+i\epsilon}\,\Gamma_2\,\frac{1}{\not{q}-\not{p}_1}\,\Gamma_1\,\psi(p-p_1)\\
&\quad + i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\Gamma_2\,\frac{A^k(p_1)}{k{\cdot}p_1}\,\frac{1}{\not{q}}\,\not{p}_1\,\frac{1}{\not{q}-\not{p}_1}\,\Gamma_1\,\psi(p-p_1)\\
&= i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{k{\cdot}p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1+i\epsilon}\,\Gamma_2\,\frac{1}{\not{q}}\,\Gamma_1\,\psi(p-p_1)\\
&\quad + i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{k{\cdot}p_1}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1+i\epsilon}\,\Gamma_2\,\left[\frac{1}{\not{q}-\not{p}_1}-\frac{1}{\not{q}}\right]\,\Gamma_1\,\psi(p-p_1)\\
&\quad + i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\Gamma_2\,\frac{A^k(p_1)}{k{\cdot}p_1}\,\left[\frac{1}{\not{q}-\not{p}_1}-\frac{1}{\not{q}}\right]\,\Gamma_1\,\psi(p-p_1).
\end{aligned}
$$

As before we have

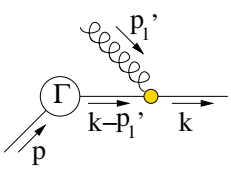
$$
\not{p}_1\,\frac{1}{\not{k}-\not{p}_1+i\epsilon} \approx \frac{2k{\cdot}p_1}{-2k{\cdot}p_1+i\epsilon},
$$

which actually becomes equal to -1 if it is multiplied by a function that vanishes when $p_1 \to 0$, so the second and third line in the above expression cancel and provided[59] $[A^k(p_1), \Gamma_2] = 0$, we are left with

$$
A_1 = i\,\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{A^k(p_1)}{-k{\cdot}p_1+i\epsilon}\,\Gamma_2\,\frac{1}{\not{q}}\,\Gamma_1\,\psi(p-p_1) = i\,\overline{\psi}(k)\,\underbrace{U_+^{[k](1)}\Gamma_2\frac{1}{\not{q}}\,\Gamma_1\psi}(p).
\tag{295}
$$

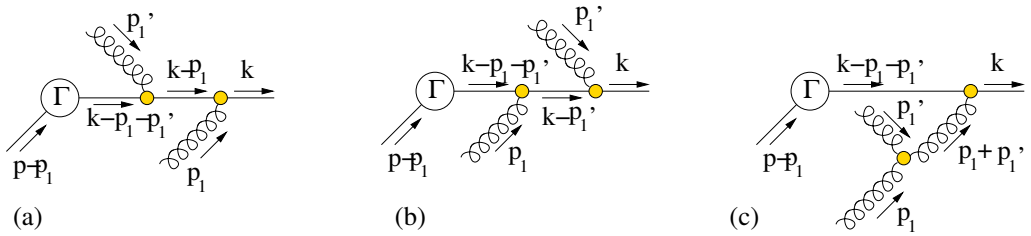
There are three two-gluon insertions, yielding the second order term in the link, etc.

Going back, we consider the insertion on a line coming from a different correlator in the basic diagram with just one vertex Γ. Starting with $A_0 = [\overline{\psi}(k)\,\Gamma\,\psi(p)]\ldots\psi(p')$, we find



$$
\begin{aligned}
A_1 &= \left[\overline{\psi}(k)\int \frac{d^4p_1'}{(2\pi)^4}\,\frac{A^k(p_1')}{k{\cdot}p_1'}\,\not{p}_1'\,\frac{1}{\not{k}-\not{p}_1'+i\epsilon}\,\Gamma\,\psi(p)\right]\ldots\psi(p'-p_1'),\\
&= \left[\overline{\psi}(k)\int \frac{d^4p_1'}{(2\pi)^4}\,\frac{A^k(p_1')}{-k{\cdot}p_1'+i\epsilon}\,\Gamma\,\psi(p)\right]\ldots\psi(p'-p_1'),\\
&= \left[\overline{\psi}(k)\,\underbrace{U_+^{[k](1)}\,\Gamma\,\psi(p)\right]\ldots\psi}(p') = \left[\overline{\psi}(k)\,U_+^{[k](1)}(p)\,\Gamma\,\psi(p)\right]\ldots\psi(p').
\end{aligned}
\tag{296}
$$

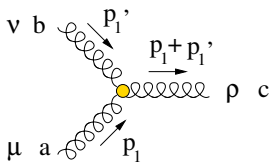
Next we consider the situation of two insertions onto a final state leg with momentum k coming from different correlators, for which there are two (or in general for a nonabelian case, three) contributions,



For the third diagram (relevant in the non-abelian case) we use the propagator

$$D^{\rho\sigma}(k) = \frac{i\, g_s^{\rho\sigma}(k)}{k^2 + i\,\epsilon},$$

where $g_s^{\mu\nu}(k) = -g^{\mu\nu} + k^\mu\, k^\nu/k^2$. Furthermore, we need the QCD vertex



$$V^{\mu\nu\rho}_{abc}(p_1, p_1', p_1 + p_1') = i\, f_{abc}\left[(p_1 - p_1')^\rho\, g^{\mu\nu} + (2p_1 + p_1')^\nu\, g^{\mu\rho} - (p_1 + 2p_1')^\mu\, g^{\nu\rho}\right].$$

The contraction with $p_{1\mu}$ yields

$$p_{1\mu}\, V^{\mu\nu\rho}_{abc}(p_1, p_1', p_1 + p_1') = i\, f_{abc}\left[(p_1 + p_1')^2\, g_s^{\nu\rho}(p_1 + p_1') - p_1'^2\, g_s^{\nu\rho}(p_1')\right],$$

Contraction with $p_{1\mu}$ and $p_{1\nu}'$ gives

$$p_{1\mu}\, p_{1\nu}'\, V^{\mu\nu\rho}_{abc}(p_1, p_1', p_1 + p_1') = \frac{i}{2}\, f_{abc}\,(p_1 + p_1')^2\,(p_1 - p_1')^\rho$$

and including the color contractions with $A^n_a(p_1)$ and $A^n_b(p_1')$ as well as the propagator for leg $p_1 + p_1'$, we get the matrix-valued result

$$p_{1\mu}\, p_{1\nu}'\, T_c\, A^n_a(p_1)\, A^n_b(p_1') V^{\mu\nu\rho'}_{abc}(p_1, p_1', p_1 + p_1')\, D^{\rho'\rho}(p_1 + p_1') = -\frac{i}{2}\,[A^n(p_1), A^n(p_1')]\,(p_1 - p_1')^\rho.$$

For the three diagrams we then obtain

$$\begin{aligned}
A_{11} \;=\; & \left[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \not{p}_1\, \frac{1}{\not{k} - \not{p}_1}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \not{p}_1'\, \frac{1}{\not{k} - \not{p}_1 - \not{p}_1'}\, \Gamma\, \psi(p - p_1)\right] \ldots \psi(p' - p_1') \\
+ \; & \left[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \not{p}_1'\, \frac{1}{\not{k} - \not{p}_1'}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \not{p}_1\, \frac{1}{\not{k} - \not{p}_1 - \not{p}_1'}\, \Gamma\, \psi(p - p_1)\right] \ldots \psi(p' - p_1') \\
- \; & \left[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{[A^k(p_1), A^k(p_1')]}{2\,(k{\cdot}p_1)(k{\cdot}p_1')}\, (\not{p}_1 - \not{p}_1')\, \frac{1}{\not{k} - \not{p}_1 - \not{p}_1'}\, \Gamma\, \psi(p - p_1)\right] \ldots \psi(p' - p_1')
\end{aligned}$$

Let us investigate the singularity structure. Using $x_1 \equiv k{\cdot}p_1$, $x_1' \equiv k{\cdot}p_1'$, and $\alpha_1 \equiv p_1{\cdot}p_1'$ one has a part (coming from diagrams (a) and (c) above) containing

$$\begin{aligned}
& \frac{A^k(p_1)A^k(p_1')}{x_1\, x_1'}\, \frac{\left[-x_1\, x_1' \not{p}_1' - \frac{1}{2}\, x_1\, x_1'(\not{p}_1 - \not{p}_1')\right]}{(-x_1 + i\epsilon)(-x_1' + i\epsilon)}\, \frac{1}{\not{k} - \not{p}_1 - \not{p}_1'} \\
& \quad = \frac{1}{2}\, \frac{A^k(p_1)A^k(p_1')}{(-x_1 + i\epsilon)(-x_1' + i\epsilon)}\, (-\not{p}_1 - \not{p}_1')\, \frac{1}{\not{k} - \not{p}_1 - \not{p}_1'} = \frac{1}{2}\, \frac{A^k(p_1)A^k(p_1')}{(-x_1 + i\epsilon)(-x_1' + i\epsilon)},
\end{aligned}$$

and similarly a part with the fields in opposite order. Hence the result for A_{11} can be rewritten as

$$
\begin{aligned}
A_{11} \;=\;& \frac{1}{2}\big[\overline{\psi}(k)\int \frac{d^4p_1'}{(2\pi)^4}\, \underbrace{U_+^{[k](1)}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \not{p}_1'\, \frac{1}{\not{k}-\not{p}_1'}\, \Gamma\psi(p)}\big]\ldots\psi(p'-p_1') \\
&- \frac{1}{2}\big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \not{p}_1' \left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} - \frac{1}{\not{k}-\not{p}_1'}\right] \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1') \\
&+ \frac{1}{2}\big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \not{p}_1 \left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} - \frac{1}{\not{k}-\not{p}_1}\right] \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1') \\
&+ \frac{1}{2}\big[\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\, \underbrace{U_+^{[k](1)}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \not{p}_1\, \frac{1}{\not{k}-\not{p}_1}\, \Gamma\,\psi(p-p_1)\big]\ldots\psi}(p') \\
&- \frac{1}{2}\big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \not{p}_1 \left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} - \frac{1}{\not{k}-\not{p}_1}\right] \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1') \\
&+ \frac{1}{2}\big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \not{p}_1' \left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} - \frac{1}{\not{k}-\not{p}_1'}\right] \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1') \\
&- \frac{1}{2}\big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{[A^k(p_1), A^k(p_1')]}{(k{\cdot}p_1)(k{\cdot}p_1')}\, (\not{p}_1-\not{p}_1')\, \frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\, \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1')
\end{aligned}
$$

We note that in the lines 2, 3 and 5, 6 of the above equation, the prescription doesn't matter because the part multiplying the product of A-fields vanishes for $p_1 \to 0$ and $p_1' \to 0$. Thus including line 7 all terms cancel and we have

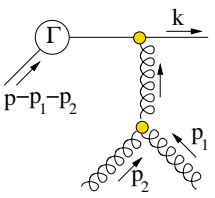
$$
A_{11} = \overline{\psi}(k)\, U_+^{[k](11)}(p,p')\,\Gamma\,\psi(p)\ldots\psi(p') = \frac{1}{2}\big[\overline{\psi}(k)\underbrace{\{U_+^{[k](1)}, \underbrace{U_+^{[k](1)}\}\,\Gamma\,\psi}(p)\big]\ldots\psi}(p'), \tag{297}
$$

i.e. the gaugelinks on (fermion) leg with momentum k is the (color) symmetrized product of gaugelinks.
The abelian case (without third diagram)

$$
\begin{aligned}
A_{11} \;=\;& \big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \not{p}_1\, \frac{1}{\not{k}-\not{p}_1}\, \not{p}_1'\, \frac{1}{\not{k}-\not{p}_1'}\, \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1') \\
&+ \big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \not{p}_1\, \frac{1}{\not{k}-\not{p}_1}\, \not{p}_1' \left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} - \frac{1}{\not{k}-\not{p}_1'}\right] \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1') \\
&+ \big[\overline{\psi}(k)\int \frac{d^4p_1\, d^4p_1'}{(2\pi)^8}\, \frac{A^k(p_1')}{k{\cdot}p_1'}\, \frac{A^k(p_1)}{k{\cdot}p_1}\, \not{p}_1' \left[\frac{1}{\not{k}-\not{p}_1-\not{p}_1'} - \frac{1}{\not{k}-\not{p}_1'}\right] \Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1'),
\end{aligned}
$$

where the first two lines come from the first contribution. Because $[A^k(p_1), A^k(p_1')] = 0$ the result is

$$
A_{11} = [\overline{\psi}(k)\,\Gamma\, U_+^{[k](1)}\psi(p)]\ldots U_+^{[k](1)}\psi(p') \tag{298}
$$



Returning to the A_2-term of a link for $\psi(p)$ we note that the contribution of the nonabelian diagram in which two legs are attached to *one* soft part is actually already included in the soft part with one collinear gluon. Its contribution

$$
-\frac{1}{2}\left[\frac{A^k(p_1)}{k{\cdot}p_1}, \frac{A^k(p_2)}{k{\cdot}p_2}\right](\not{p}_1-\not{p}_2)\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2} = -\frac{1}{2}\,\frac{[A^k(p_1), A^k(p_2)]}{(-x_1+i\epsilon)(-x_2+i\epsilon)}\,\frac{x_2-x_1}{x_1+x_2},
$$

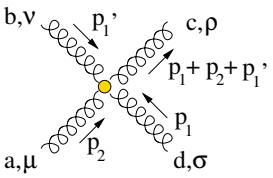
is actually just the vanishing antisymmetric term discussed before (see Eq. 273).

Without further calculation, the result for the contribution from gluons from 3 different soft parts will be

$$
\begin{aligned}
A_{111} \;=\;& \overline{\psi}(k)\, U_+^{[k](111)}(p,p',p'')\;\ldots\;\psi(p)\ldots\psi(p')\;\ldots\;\psi(p'') \\
=\;& \frac{1}{3!}\,\overline{\psi}(k)\,\left\{U_+^{[k](1)}(p), U_+^{[k](1)}(p'), U_+^{[k](1)}(p'')\right\}\;\ldots\;\psi(p)\;\ldots\;\psi(p')\;\ldots\;\psi(p''),
\end{aligned} \tag{299}
$$

where A, B, C indicates the symmetric combination. For this we would actually also need the four-point gluon vertex.

This four-point gluon vertex also is needed in the final case we will explicitly investigate, which is two gluons collinear with momentum p and one collinear with momentum p'. We now need a number of diagrams and the four-point vertex, contracted with three momenta. The latter is given by

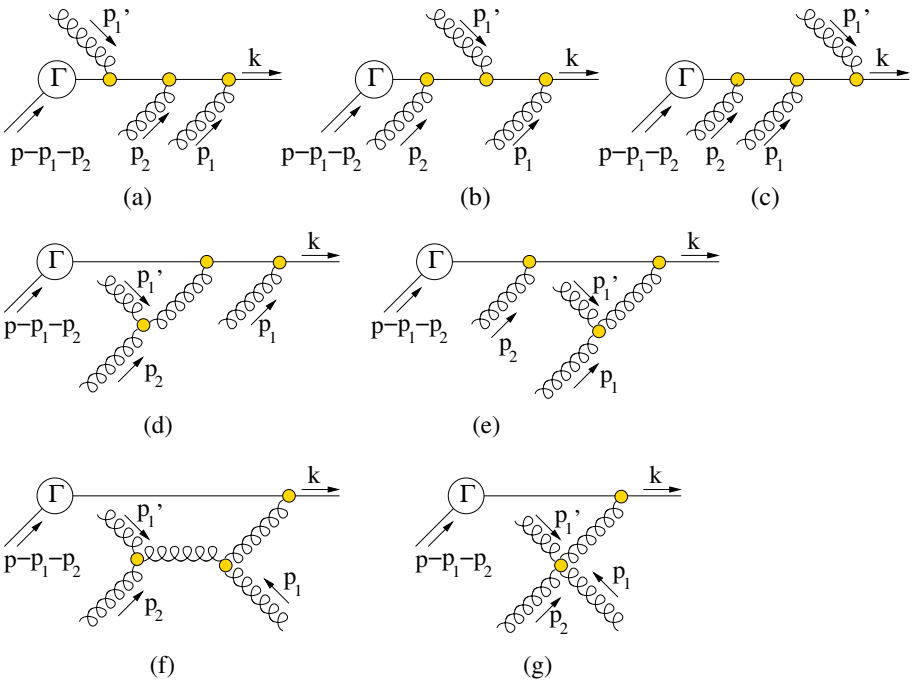


$$
\begin{aligned}
V^{\mu\nu\rho\sigma}_{abcd} &= -i\,g^2\big[f_{abe}\,f_{cde}\,(g^{\mu\rho}\,g^{\nu\sigma}-g^{\mu\sigma}\,g^{\nu\rho}) \\
&\quad + f_{ace}\,f_{bde}\,(g^{\mu\nu}\,g^{\rho\sigma}-g^{\mu\sigma}\,g^{\nu\rho}) \\
&\quad + f_{ade}\,f_{bce}\,(g^{\mu\nu}\,g^{\rho\sigma}-g^{\mu\rho}\,g^{\nu\sigma})\big] \\
&= -i\,g^2\big[f_{abe}\,f_{cde}\,(2\,g^{\mu\rho}\,g^{\nu\sigma}-g^{\mu\sigma}\,g^{\nu\rho}-g^{\mu\nu}\,g^{\rho\sigma}) \\
&\quad + f_{ace}\,f_{bde}\,(2\,g^{\mu\nu}\,g^{\rho\sigma}-g^{\mu\sigma}\,g^{\nu\rho}-g^{\mu\rho}\,g^{\nu\sigma})\big]
\end{aligned}
$$

The contraction with $p_1'^{\,\nu}\,A^n(p_1')$, $p_2^\mu\,A^n(p_2)$ and $p_1^\sigma\,A^n(p_1)$ yields

$$
\begin{aligned}
&p_1^\sigma p_2^\mu p_1'^{\,\nu}\,A^n_d(p_1)\,A^n_a(p_2)\,A^n_b(p_1')\,T_c\,V^{\mu\nu\rho\sigma}_{abcd} = \\
&\quad ig^2\Big[[A^n(p_1),[A^n(p_2),A^n(p_1')]]\,\big(2p_2^\rho\,p_1{\cdot}p_1'-p_1^\rho\,p_2{\cdot}p_1'-p_1'^{\rho}\,p_2{\cdot}p_1\big) \\
&\qquad + [A^n(p_2),[A^n(p_1),A^n(p_1')]]\,\big(2p_1^\rho\,p_2{\cdot}p_1'-p_2^\rho\,p_1{\cdot}p_1'-p_1'^{\rho}\,p_2{\cdot}p_1\big)\Big].
\end{aligned}
$$

The contributions to the gauge-link are the following



For the seven diagrams involving three collinear gluons, two collinear to p, one to p', we obtain the contributions

$$
A_{21} = \big[\overline{\psi}(k)\int \frac{d^4p_1}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,\frac{d^4p_1'}{(2\pi)^4}\,\ldots\,\Gamma\,\psi(p-p_1)\big]\ldots\psi(p'-p_1')
$$

with three basically 'abelian' contributions,

$$
\begin{aligned}
(a) &= \frac{A^k(p_1)}{k{\cdot}p_1}\,\frac{A^k(p_2)}{k{\cdot}p_2}\,\frac{A^k(p_1')}{k{\cdot}p_1'}\,\not p_1\,\frac{1}{\not k-\not p_1}\,\not p_2\,\frac{1}{\not k-\not p_1-\not p_2}\,\not p_1'\,\frac{1}{\not k-\not p_1-\not p_2-\not p_1'} \\
(b) &= \frac{A^k(p_1)}{k{\cdot}p_1}\,\frac{A^k(p_1')}{k{\cdot}p_1'}\,\frac{A^k(p_2)}{k{\cdot}p_2}\,\not p_1\,\frac{1}{\not k-\not p_1}\,\not p_1'\,\frac{1}{\not k-\not p_1-\not p_1'}\,\not p_2\,\frac{1}{\not k-\not p_1-\not p_2-\not p_1'} \\
(c) &= \frac{A^k(p_1')}{k{\cdot}p_1'}\,\frac{A^k(p_1)}{k{\cdot}p_1}\,\frac{A^k(p_2)}{k{\cdot}p_2}\,\not p_1'\,\frac{1}{\not k-\not p_1'}\,\not p_1\,\frac{1}{\not k-\not p_1-\not p_1'}\,\not p_2\,\frac{1}{\not k-\not p_1-\not p_2-\not p_1'}
\end{aligned}
$$

of which the singularity structure is kept in contributions (a_0), (b_0) and (c_0) and we get of which the singularity structure is (using $D = 1/(\not{k} - \not{p}_1 - \not{p}_2 - \not{p}'_1)$)

$$
\begin{aligned}
(a) &= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p'_1)}{x_1\,x_2\,x'_1}\,\frac{x_1\,x_2\,\not{p}'_1\,D}{(-x_1+i\epsilon)(-x_1-x_2+i\epsilon)} \\
&= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p'_1)}{x_1\,x_2\,x'_1}\,\frac{x_2}{(x_1+x_2)}\,\not{p}'_1\,D \\
(b) &= \frac{A^k(p_1)\,A^k(p'_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}\,\frac{x_1}{(-x_1+i\epsilon)}\,\not{p}'_1\,\frac{1}{\not{k}-\not{p}_1-\not{p}'_1}\,\not{p}_2\,D \\
&= \frac{A^k(p_1)\,A^k(p'_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}(-\not{p}'_1)\,\frac{1}{\not{k}-\not{p}_1-\not{p}'_1}\,\not{p}_2\,D \\
&= \frac{A^k(p_1)\,A^k(p'_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}\,\frac{x'_1}{(x_1+x'_1-\alpha_1)}\,\not{p}_2\,D \\
(c) &= \frac{A^k(p'_1)\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}\,\frac{x'_1}{(-x'_1+i\epsilon)}\,\not{p}_1\,\frac{1}{\not{k}-\not{p}_1-\not{p}'_1}\,\not{p}_2\,D \\
&= \frac{A^k(p'_1)\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}(-\not{p}_1)\,\frac{1}{\not{k}-\not{p}_1-\not{p}'_1}\,\not{p}_2\,D \\
&= \frac{A^k(p'_1)\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}\,\frac{(x_1-\alpha_1)}{(x_1+x'_1-\alpha_1)}\,\not{p}_2\,D
\end{aligned}
$$

We have here introduced $\alpha_1 = p'_1{\cdot}p_1$, $\alpha_2 = p'_1{\cdot}p_2$, and $\alpha = \alpha_1 + \alpha_2$. A useful relation is $\not{p}_1\not{p}'_1\not{p}_2 = \alpha_1\,\not{p}_2 = \alpha_2\,\not{p}_1$, also implying $\alpha_1\,x_2 = \alpha_2\,x_1$ or $x_1/x_2 = \alpha_1/\alpha_2$. We note that summing these three diagrams in the abelian case simply give

$$
\begin{aligned}
[(a)+(b)+(c)]_{\text{abelian}} &= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p'_1)}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x'_1+i\epsilon)}\,[-\not{p}'_1-(x_1+x_2)\not{p}_2]\;D \\
&= \frac{A^k(p_1)\,A^k(p_2)\,A^k(p'_1)}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x'_1+i\epsilon)} = \frac{1}{2}\,\frac{A^k(p_1)\,A^k(p_2)\,A^k(p'_1)}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x'_1+i\epsilon)},
\end{aligned}
\tag{300}
$$

giving $A_{21} = [\overline{\psi}(k)\,\Gamma\,U_+^{[k](2)}\psi(p)]\ldots U_+^{[k](1)}\psi(p')$. In the nonabelian case, we also have two effectively 'three-gluon' contributions,

$$
\begin{aligned}
(d) &= -\frac{1}{2}\,\frac{A^k(p_1)}{k{\cdot}p_1}\,\left[\frac{A^k(p_2)}{k{\cdot}p_2},\frac{A^k(p'_1)}{k{\cdot}p'_1}\right]\not{p}_1\,\frac{1}{\not{k}-\not{p}_1}\,(\not{p}_2-\not{p}'_1)\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}'_1} \\
&= +\frac{1}{2}\,\frac{A^k(p_1)\,[A^k(p_2),A^k(p'_1)]}{x_1\,x_2\,x'_1}\,(\not{p}_2-\not{p}'_1)\,D \\
(e) &= -\frac{1}{2}\left[\frac{A^k(p_1)}{k{\cdot}p_1},\frac{A^k(p'_1)}{k{\cdot}p'_1}\right]\frac{A^k(p_2)}{k{\cdot}p_2}\,(\not{p}_1-\not{p}'_1)\,\frac{1}{\not{k}-\not{p}_1-\not{p}'_1}\,\not{p}_2\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}'_1} \\
&= +\frac{1}{2}\,\frac{[A^k(p_1),A^k(p'_1)]\,A^k(p_2)}{x_1\,x_2\,x'_1}\,\frac{(x_1-x'_1-\alpha_1)}{(x_1+x'_1-\alpha_1)}\,\not{p}_2\,D
\end{aligned}
$$

The sum of these two gives

$$
\begin{aligned}
(d)+(e) &= \frac{1}{2}\,\frac{A^k(p_1)\,A^k(p_2)\,A^k(p'_1)}{x_1\,x_2\,x'_1}\,(\not{p}_2-\not{p}'_1)\,D \\
&\quad+\frac{1}{2}\,\frac{A^k(p_1)\,A^k(p'_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}\left(\not{p}'_1-\frac{2\,x'_1}{(x_1+x'_1-\alpha_1)}\not{p}_2\right)D \\
&\quad-\frac{1}{2}\,\frac{A^k(p'_1)\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x'_1}\,\frac{(x_1-x'_1-\alpha_1)}{(x_1+x'_1-\alpha_1)}\,\not{p}_2\,D.
\end{aligned}
$$

An alternative in this calculation is to not add (d) and (e), but rather to add (e) to (b) and (c) to avoid having to deal with poles in $1/(\not{k}-\not{p}_1-\not{p}_1')$. One gets

$$\begin{aligned}
(b)+(c)+(e) &= \frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\left[-\not{p}_1'-\frac{1}{2}\,(\not{p}_1-\not{p}_1')\right]\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D \\
&\quad+\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\left[-\not{p}_1+\frac{1}{2}\,(\not{p}_1-\not{p}_1')\right]\frac{1}{\not{k}-\not{p}_1-\not{p}_1'}\,\not{p}_2\,D \\
&= \frac{1}{2}\,\frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_2\,D+\frac{1}{2}\,\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_2\,D.
\end{aligned}$$

Including the contributions of (a) and (d), we have

$$\begin{aligned}
(a)+\ldots+(e) &= \frac{1}{2}\,\frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{x_1\,x_2\,x_1'}\left[\not{p}_2\,D+\frac{x_2-x_1}{x_1+x_2}\,\not{p}_1'\,D\right] \\
&\quad+\frac{1}{2}\,\frac{A^k(p_1)\,A^k(p_1')\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_1'\,D+\frac{1}{2}\,\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{x_1\,x_2\,x_1'}\,\not{p}_2\,D.
\end{aligned}$$

Finally, there are two types of 'four-gluon' contributions,

$$\begin{aligned}
(f) &= \frac{1}{2}\left[\frac{A^k(p_1)}{k{\cdot}p_1},\left[\frac{A^k(p_2)}{k{\cdot}p_2},\frac{A^k(p_1')}{k{\cdot}p_1'}\right]\right](p_2-p_1')_\lambda \\
&\quad\times\left[(p_1+p_2+p_1')^2\,g_s^{\lambda\tau}(p_1+p_2+p_1')-(p_2+p_1')^2\,g_s^{\lambda\tau}(p_2+p_1')\right] \\
&\quad\times iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'} \\
&= -\left[\frac{A^k(p_1)}{k{\cdot}p_1},\left[\frac{A^k(p_2)}{k{\cdot}p_2},\frac{A^k(p_1')}{k{\cdot}p_1'}\right]\right](\alpha_1-\alpha_2)\,(p_2-p_1')^\tau \\
&\quad\times iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'},
\end{aligned}$$

$$\begin{aligned}
(g) &= -\left[\frac{A^k(p_1)}{k{\cdot}p_1},\left[\frac{A^k(p_2)}{k{\cdot}p_2},\frac{A^k(p_1')}{k{\cdot}p_1'}\right]\right](2\,\alpha_1\,p_2^\tau-\alpha_2\,p_1^\tau) \\
&\quad\times iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'} \\
&\quad-\left[\frac{A^k(p_2)}{k{\cdot}p_2},\left[\frac{A^k(p_1)}{k{\cdot}p_1},\frac{A^k(p_1')}{k{\cdot}p_1'}\right]\right](2\,\alpha_2\,p_1^\tau-\alpha_1\,p_2^\tau) \\
&\quad\times iD_{\tau\rho}(p_1+p_2+p_1')\,\gamma^\rho\,\frac{1}{\not{k}-\not{p}_1-\not{p}_2-\not{p}_1'}.
\end{aligned}$$

Using the contractions

$$\begin{aligned}
-\alpha\,p_2^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \alpha\,\not{p}_2-\tfrac{1}{2}\,\alpha_2\,(\not{p}_1+\not{p}_2+\not{p}_1') \\
&= \alpha_1\,\not{p}_2-\alpha_2\,\not{p}_1+\tfrac{1}{2}\,\alpha_2\,(\not{p}_1+\not{p}_2-\not{p}_1')=\tfrac{1}{2}\,\alpha_2\,(\not{p}_1+\not{p}_2-\not{p}_1') \\
-\alpha\,p_1^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \alpha_2\,\not{p}_1-\alpha_1\,\not{p}_2+\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1')=\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1') \\
-\alpha\,p_1'^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \tfrac{1}{2}(\not{p}_1+\not{p}_2-\not{p}_1'), \\
-\alpha(p_2-p_1')^\tau\,g_{s\,\tau\rho}(p_1+p_2+p_1')\gamma^\rho &= \alpha_1\,\not{p}_2-\alpha_2\,\not{p}_1-\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1'),=-\tfrac{1}{2}\,\alpha_1\,(\not{p}_1+\not{p}_2-\not{p}_1'),
\end{aligned}$$

realizing that with p_1 and p_2 being integration variables,

$$\begin{aligned}
[A^k(p_1),[A^k(p_2),A^k(p_1')]]\,F(p_1,p_2) &= A^k(p_1)A^k(p_2)A^k(p_1')\,F(p_1,p_2)+A^k(p_1')A^k(p_1)A^k(p_2)\,F(p_2,p_1) \\
&\quad-A^k(p_1)A^k(p_1')A^k(p_2)\,\left(F(p_1,p_2)+F(p_2,p_1)\right), \\
[A^k(p_2),[A^k(p_1),A^k(p_1')]]\,F(p_1,p_2) &= A^k(p_1)A^k(p_2)A^k(p_1')\,F(p_2,p_1)+A^k(p_1')A^k(p_1)A^k(p_2)\,F(p_1,p_2) \\
&\quad-A^k(p_1)A^k(p_1')A^k(p_2)\,\left(F(p_1,p_2)+F(p_2,p_1)\right),
\end{aligned}$$

and using

$$C = (\not{p}_1 + \not{p}_2 - \not{p}_1')\,\frac{1}{\not{k} - \not{p}_1 - \not{p}_2 - \not{p}_1'} = \frac{(x_1 + x_2 - x_1' + [\not{p}_1', \not{p}_1] + [\not{p}_1', \not{p}_2]}{(-x_1 - x_2 - x_1' + \alpha)},$$

we can rewrite the last two contributions as

$$\begin{aligned}
(f) &= -\frac{1}{4}\,\frac{[A^k(p_1),[A^k(p_2),A^k(p_1')]]}{x_1\,x_2\,x_1'}\,\frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha^2}\,C \\
&= -\frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha^2}\,C \\
&\quad + \frac{1}{4}\,\frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{(\alpha_1-\alpha_2)^2}{\alpha^2}\,C \\
&\quad - \frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{\alpha_2(\alpha_2-\alpha_1)}{\alpha^2}\,C \\
(g) &= -\frac{1}{4}\,\frac{[A^k(p_1),[A^k(p_2),A^k(p_1')]]}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C - \frac{1}{4}\,\frac{[A^k(p_2),[A^k(p_1),A^k(p_1')]]}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C \\
&= -\frac{1}{2}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C \\
&\quad + \frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C \\
&\quad - \frac{1}{2}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{\alpha_1\,\alpha_2}{\alpha^2}\,C
\end{aligned}$$

Adding the contributions and using $x_1/x_2 \approx \alpha_1/\alpha_2$ we get

$$\begin{aligned}
(f)+(g) &= -\frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\,\frac{x_1}{(x_1+x_2)}\,(\not{p}_1 + \not{p}_2 - \not{p}_1')\,D \\
&\quad + \frac{1}{4}\,\frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'}\,(\not{p}_1 + \not{p}_2 - \not{p}_1')\,D \\
&\quad - \frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{x_2}{(x_1+x_2)}\,(\not{p}_1 + \not{p}_2 - \not{p}_1')\,D.
\end{aligned}$$

The full result becomes

$$\begin{aligned}
(a)+\ldots+(g) &= -\frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{x_1\,x_2\,x_1'}\left[\frac{x_2}{(x_1+x_2)} + \frac{x_2-x_1}{x_1+x_2}\right] \\
&\quad - \frac{1}{4}\,\frac{A^k(p_1)A^k(p_1')A^k(p_2)}{x_1\,x_2\,x_1'} \\
&\quad - \frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{x_1\,x_2\,x_1'}\,\frac{x_2}{(x_1+x_2)} \\
&= \frac{1}{4}\,\frac{A^k(p_1)A^k(p_2)A^k(p_1')}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x_1'+i\epsilon)} \\
&\quad \frac{1}{4}\,\frac{A^k(p_1)\,A^k(p_1'),A^k(p_2)}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)} \\
&\quad \frac{1}{4}\,\frac{A^k(p_1')A^k(p_1)A^k(p_2)}{(-x_1+i\epsilon)\,(-x_1-x_2+i\epsilon)\,(-x_1'+i\epsilon)} \\
&= \frac{1}{8}\,\frac{A^k(p_1)\,A^k(p_2)\,A^k(p_1')}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)} \\
&\quad \frac{1}{4}\,\frac{A^k(p_1)\,A^k(p_1'),A^k(p_2)}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)} \\
&\quad \frac{1}{8}\,\frac{A^k(p_1')\,A^k(p_1)\,A^k(p_2)}{(-x_1+i\epsilon)\,(-x_2+i\epsilon)\,(-x_1'+i\epsilon)}.
\end{aligned}$$

So we get

$$A_{21} = [\overline{\psi}(k)\, U_+^{[k](21)}(p,p')\, \Gamma \ldots \psi(p)] \ldots \psi(p') \tag{301}$$

$$= [\overline{\psi}(k) \left[\frac{1}{4}\, U_+^{[k](2)}(p)\, U_+^{[k](1)}(p') + \frac{1}{4}\, U_+^{[k](1)}(p)\, U_+^{[k](1)}(p')\, U_+^{[k](1)}(p) + \frac{1}{4}\, U_+^{[k](1)}(p')\, U_+^{[k](2)}(p) \right] \Gamma \ldots \psi(p)] \ldots \psi(p') \tag{302}$$

$$= [\overline{\psi}(k) \left[\frac{1}{8}\, U_+^{[k](1)}(p)\, U_+^{[k](1)}(p)\, U_+^{[k](1)}(p') + \frac{1}{4}\, U_+^{[k](1)}(p)\, U_+^{[k](1)}(p')\, U_+^{[k](1)}(p) + \frac{1}{8}\, U_+^{[k](1)}(p')\, U_+^{[k](1)}(p)\, U_+^{[k](1)}(p) \right] \Gamma \ldots \psi(p)] \ldots \psi(p') \tag{303}$$

This reproduces the correct (expected) results in the abelian case, and in the limit that $U^{[k](1)}(p')$ is replaced by a $F^{\alpha k}(p_1')$, or that $U^{[k](1)}(p)$ is replaced by a $F^{\alpha k}(p)$.

Acknowledgments

The material in these lectures is just a selection from the work that is going on in the field of TMDs. Most of it can also be found in recent published lecture notes [48]. We have tried to focus on some basics, which is the structure of the matrix elements. We have not addressed the rich phenomenology of azimuthal asymmetries [6, 7, 13, 49]. Neither have we addressed aspects going beyond tree-level, which is important to get to the full field theoretical treatment of TMDs including evolution [3, 24, 28].

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