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Spin Structure Functions

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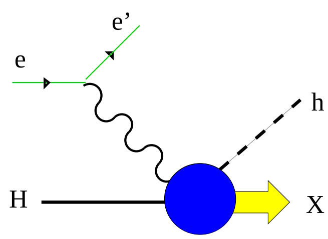
This chapter deals with properties of hadrons in high-energy scattering processes. We specifically study electroweak scattering processes. One specific process, namely lepton-hadron scattering will be dealt with in detail. A signal for an electroweak process is the presence of leptons which do not feel strong interactions. They allow a separation of the the scattering amplitude for the process into a leptonic part and a hadronic part, where the leptonic part, involving elementary particles is known. The structure of the hadronic part is constrained by its (Lorentz) structure and fundamental symmetries and can be parameterized in terms of a number of structure functions. The emerging expression for the scattering amplitude can be used to calculate the cross sections in terms of these structure functions. In turn one can make a theoretical study of the structure functions. Part of this can be done rigorous with as only input (assumption if one wants) the known interactions of the hadronic constituents, quarks and gluons, within the standard model. For this both the electroweak couplings of the quarks needed to describe the interactions with the leptonic part via the exchange of photon, Z^0 or $W^\pm$ bosons as well as the strong interactions of the quarks among themselves via the exchange of gluons described in the QCD part of the standard model are important. For a general reference we refer to the book of Roberts [1].

1.1 Leptoproduction

In this section we discuss the basic kinematics of a particular hard electroweak processes, namely the scattering of a high-energy lepton, e.g. an electron, muon or neutrino from a hadronic target, $\ell(k)H(P) \to \ell'(k')X$. In this process at least one hadron is involved. If one does not care about the final state, counting every event irrespective of what is happening in the scattering process, one talks about an inclusive measurement. If one detects specific hadrons in coincidence with the scattered lepton one talks about semi-inclusive measurements or more specifically 1-particle inclusive, 2-particle inclusive, depending on the number of particles that are detected.



$$q^2 = (k-k')^2 \equiv -Q^2 \leq 0$$
$$2P\cdot q \equiv 2M\,\nu \equiv \frac{Q^2}{x_B}$$
$$2P_h\cdot q \equiv -z_h\,Q^2$$

$$P\cdot k = \frac{P\cdot q}{y} = \frac{Q^2}{2x_B y}$$

The variable x_B is the Bjorken scaling variable. In this scattering process a hadron is probed with a spacelike (virtual) photon, for which one could consider a frame in which the momentum only has a spatial component. This shows that the spatial resolving power of the probing photon is of the order $\lambda \approx 1/Q$. Roughly spoken one probes a nucleus (1 - 10 fm) with $Q \approx 10-100$ MeV, baryon or meson structure (with sizes in the order of 1 fm) with $Q \approx 0.1-1$ GeV and one probes deep into the nucleon (< 0.1 fm) with $Q > 2$ GeV.

Exercise Rewrite the invariant mass squared of the hadronic final state, W^2, in terms of invariants and use that $W^2 \geq M^2$ to show that $0 \leq x_B \leq 1$, with $x_B = 1$ corresponding to elastic scattering.

Leptoproduction is characteristic for a large number of other processes involving particles (leptons) for which the interactions are fully known together with hadrons. The electroweak interactions with the constituents of the hadrons (quarks), however, are also known. This opens the way to study how quarks are embedded in the hadrons (e.g. in leptoproduction or in the Drell-Yan process, $A(P_A)B(P_B) \to \ell(k)\bar{\ell}(k')X$) or to study how quarks fragment into hadrons (in leptoproduction and e^+e^- annihilation into hadrons).

For inclusive unpolarized electron scattering the cross section, assuming one-photon exchange, is given by

$$E' \frac{d\sigma}{d^3k'} = \frac{1}{s-M^2} \frac{\alpha^2}{Q^4} L^{(S)}_{\mu\nu} \, 2MW^{\mu\nu}, \tag{1.1}$$

where $L^{(S)}_{\mu\nu}$ is the symmetric lepton tensor,

$$L^{(S)}_{\mu\nu}(k,k') = \text{Tr}\ [\gamma_\mu(\not{k}'+m)\gamma_\nu(\not{k}+m)] = 2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu}. \tag{1.2}$$

and $W_{\mu\nu}$ is the hadron tensor, which contains the information on the hadronic part of the scattering process,

$$2M\, W_{\mu\nu}(P,q) = \frac{1}{2\pi} \sum_n \int \frac{d^3P_n}{(2\pi)^3\, 2E_n}\ \langle P|J^\dagger_\mu(0)|P_n\rangle \\ \times \langle P_n|J_\nu(0)|P\rangle\ (2\pi)^4\, \delta^4(P+q-P_n), \tag{1.3}$$

where $|P\rangle$ represents a target with momentum P.

Exercise. Show, using $\delta^4(P+q-P_n) = \int d^4x\ \exp(iP\cdot x + iq\cdot x - iP_n\cdot x)$, shifting the argument of the current, $J_\mu(x) = \exp(iP_{op}\cdot x) J_\mu(0) \exp(-iP_{op}\cdot x)$ and using completeness for the intermediate states that the hadron tensor can be written as the expectation value of the product of currents $J_\mu(x)J_\nu(0)$. Then, adding a second term $\propto J_\nu(0)J^\dagger_\mu(0)\delta^4(P-q-P_n)$, which in the physical region ($\nu > 0$) is zero because of the spectral conditions of the intermediate states n ($P^0_n > M$) one can after a similar procedure combine the terms to

$$2M\, W_{\mu\nu} = \frac{1}{2\pi} \int d^4x\ e^{i\,q\cdot x}\, \langle P|[J^\dagger_\mu(x), J_\nu(0)]|P\rangle. \tag{1.4}$$

Including polarization this expression remains valid if summation and averaging over spins is understood.

What is one actually probing in leptoproduction? In the leptoproduction $e + H \to e' + X$ the (unobserved) final state X can be the target (elastic scattering) or an excitation thereof. In a plot of the two independent variables ν and Q^2 (see Fig. 1.1) elastic scattering corresponds to a line given by $\nu = Q^2/2M$, where M is the mass of the target. As we have seen the behavior of the cross section along this line, where the ratio $x_B = 1$, is (besides the $1/Q^4$ of the Mott cross section) proportional to a form factor squared, measuring the expectation value of the electromagnetic current $\langle P'|J^\mu(x)|P\rangle$. Exciting the nucleon gives rise to inelastic contributions in the cross section at $\nu > Q^2/2M$, starting at the threshold $W = M + M_\pi$.

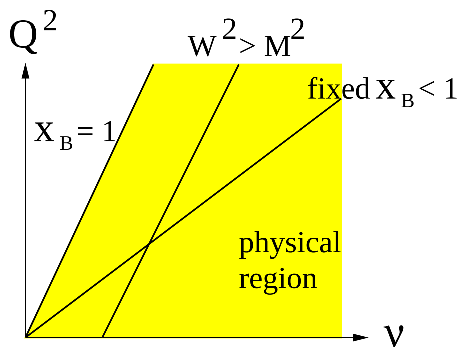


Figure 1.1. *The physical region in deep inelastic scattering.*

Note that as a function of x_B any resonance contribution will move closer to the elastic limit when Q^2 increases. When Q^2 and the energy transfer ν are high enough the cross section will reflect elastic scattering off the pointlike constituents of the nucleon and the cross section will become equal to an incoherent sum of the electron-quark cross section. This is known as the deep inelastic scattering region, in which one finds Bjorken scaling. The cross section, or more precisely the structure functions, become functions of one (kinematic) variable x_B, which is identified with the momentum fraction of the struck quark in the nucleon, enabling measurement of quark distributions. We will make this explicit below. The picture will break down in the limiting cases, such as $x_B \to 1$, where it becomes dual to the summation over resonances (see chapter xx) or $x_B \to 0$, corresponding (for fixed Q^2) to $\nu \to \infty$. In this region one may employ Regge theory (see chapter xx).

It is also possible to consider in more detail the space-time correlations that are probed. As already indicated q is space-like. But from the kinematics of deep inelastic scattering one can see that the process probes the lightcone. Sitting in the nucleon rest-frame we note that both $q^0 = \nu$ and $q^3 = \sqrt{Q^2 + \nu^2}$ go to infinity but working at finite x_B one sees that taking the sum and the difference only one of them goes to infinity*. Choosing $\boldsymbol{q}$ along the negative z-axis one has $q^- = (\nu + |\boldsymbol{q}|)/\sqrt{2} \to \infty$ and $q^+ = (\nu - |\boldsymbol{q}|)/\sqrt{2} \approx -M_N x_B/\sqrt{2}$. This corresponds in the hadronic tensor which involves a Fourier transform of the product of currents to $|x^+| \approx 1/q^- \to 0$ and $|x^-| \approx 1/|q^+| \to 1/Mx_B$ or $|\boldsymbol{x}| \approx |t| \approx 1/Mx_B$. Thus, depending on the value of x_B the distances and times not necessarily are small, but one has $x^2 = x^+x^- - \boldsymbol{x}_\perp^2 \approx -\boldsymbol{x}_\perp^2 \leq 0$, while on the other hand causality requires that $x^2 \geq 0$. Therefore, one sees that deep inelastic scattering probes the lightcone, $x^2 \approx 0$.

* Lightcone coordinates are defined $a^\pm = (a^0 \pm a^3)/\sqrt{2}$. The scalar product of two vectors is given by $a \cdot b = a^+b^- + a^-b^+ - a^1b^1 - a^2b^2$.

1.2 Structure functions and cross section

The simplest thing one can do with the hadron tensor is to express it in standard tensors and functions depending on the invariants, the *structure functions*. Instead of the traditional choice using tensors, $g_{\mu\nu}$, $P_\mu P_\nu$ and $\epsilon_{\mu\nu\rho\sigma}\, q^\rho P^\sigma$ multiplying structure functions W_1, W_2 and W_3 depending on ν and Q^2, we immediately go to a dimensionless representation. First we define a Cartesian basis of vectors [2], starting with the natural space-like momentum (defined by q). Using the target hadron momentum P^μ one can construct an orthogonal four vector $\tilde{P}^\mu = P^\mu - (P\cdot q/q^2)\, q^\mu$, which is timelike with length $\tilde{P}^2 = \kappa\, P\cdot q$ with

$$\kappa = 1 + \frac{M^2\, Q^2}{(P\cdot q)^2} = 1 + \frac{4\, M^2 x_B^2}{Q^2}. \tag{1.5}$$

The quantity κ takes into account mass corrections $\propto M^2/Q^2$ which will vanish for large Q^2 ($\kappa \to 1$). Thus we define

$$Z^\mu \equiv -q^\mu, \tag{1.6}$$

$$T^\mu \equiv -\frac{q^2}{P\cdot q}\, \tilde{P}^\mu = q^\mu + 2x_B\, P^\mu. \tag{1.7}$$

For these vectors we have $Z^2 = -Q^2$ and $T^2 = \kappa\, Q^2$ and we will often use the normalized vectors $\hat{z}^\mu = -\hat{q}^\mu = Z^\mu/Q$ and $\hat{t}^\mu = T^\mu/Q\sqrt{\kappa}$. With respect to these vectors one can also define transverse tensors,

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu \hat{q}^\nu - \hat{t}^\mu \hat{t}^\nu, \tag{1.8}$$

$$\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho \hat{q}_\sigma \tag{1.9}$$

To get the parametrization of hadronic tensors, such as the one in Eq. 1.4, including for generality also an (axial) spin vector S (see section 1.5), we use the general symmetry property,

$$W_{\mu\nu}(q,P,S) = W_{\nu\mu}(-q,P,S) \tag{1.10}$$

as well as properties following from hermiticity, parity and time-reversal invariance,

$$W^*_{\mu\nu}(q,P,S) = W_{\nu\mu}(q,P,S), \tag{1.11}$$

$$W_{\mu\nu}(q,P,S) = \overline{W}_{\nu\mu}(\bar{q},\bar{P},-\bar{S}) \qquad \text{[Parity]}, \tag{1.12}$$

$$W^*_{\mu\nu}(q,P,S) = \overline{W}_{\mu\nu}(\bar{q},\bar{P},\bar{S}) \qquad \text{[Time reversal]}, \tag{1.13}$$

where $\bar{p} = (p^0, -\boldsymbol{p})$. Finally we use current conservation implying $q^\mu W_{\mu\nu} = W_{\mu\nu} q^\nu = 0$. Note that depending on the situation not all constraints can be applied. For inclusive unpolarized leptoproduction one obtains as the most general form for the symmetric tensor,

$$\begin{aligned} M\, W^{\mu\nu\,(S)}(P,q) &= \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) F_1(x_B, Q^2) + \frac{\tilde{P}^\mu \tilde{P}^\nu}{P\cdot q}\, F_2(x_B, Q^2) \\ &= -g_\perp^{\mu\nu}\, \underbrace{F_1(x_B,Q^2)}_{F_T} + \hat{t}^\mu \hat{t}^\nu\, \underbrace{\left(-F_1 + \frac{\kappa}{2x_B}\, F_2\right)}_{F_L}, \end{aligned} \tag{1.14}$$

where the structure functions F_1, F_2 or the transverse and longitudinal structure functions, $F_T = F_1$ and F_L, depend only on the for the hadron part relevant invariants Q^2 and x_B. This is the structure for the electromagnic (photon exchange) part of the electroweak interaction. For the weak (W- or Z-exchange) part both vector and axial vector currents with different parity behavior come in. In that case also the following antisymmetric tensor is allowed,

$$\begin{aligned} M\, W^{\mu\nu\,(A)}(q,P) &= \frac{i\epsilon^{\mu\nu\rho\sigma} P_\rho q_\sigma}{(P\cdot q)}\, F_3(x_B, Q^2) \\ &= i\,\kappa\,\epsilon_\perp^{\mu\nu}\, F_3(x_B, Q^2). \end{aligned} \tag{1.15}$$

It appears in the part of the tensor in which one of the currents in the product is a vector current and the other an axial vector current.

The cross section is obtained from the contraction of lepton and hadron tensors. It is convenient to expand also the lepton momenta k and $k' = k - q$ in $\hat{t}$, $\hat{z}$ and a perpendicular component using the scaling variable $y = P\cdot q/P\cdot k$ (in the target restframe reducing to $y = \nu/E$). The result (including target mass corrections) is

$$\begin{aligned} k^\mu &= \frac{2-y}{y}\,\frac{1}{\kappa}\, T^\mu - \frac{1}{2}\, Z^\mu + k_\perp^\mu \\ &= \frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)}{2y}\,\frac{Q}{\sqrt{\kappa}}\,\hat{t}^\mu + \frac{\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}}{y}\,\frac{Q}{\sqrt{\kappa}}\,\hat{\ell}^\mu \\ &\overset{Q^2\to\infty}{\longrightarrow} \frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)\,Q}{2y}\,\hat{t}^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}^\mu, \end{aligned} \tag{1.16}$$

where $\hat{\ell}^\mu = k_\perp^\mu/|\boldsymbol{k}_\perp|$, is a spacelike unit-vector in the perpendicular direction lying in the (lepton) scattering plane. The kinematics in the frame where virtual photon and target are collinear (including target rest frame)

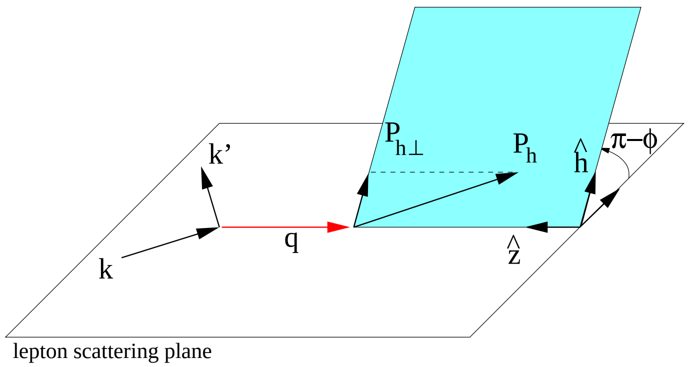


Figure 1.2. *Kinematics for lepton-hadron scattering. Transverse directions indicated with a $\perp$ index are orthogonal to P and q, e.g. the orthogonal component of the momentum of a produced hadron has been indicated as example. Similarly one can consider the orthogonal component of the spin vector of the target.*

is illustrated in Fig. 1.2. With the definition of $\hat{\ell}$, we obtain neglecting mass corrections ($\kappa = 1$) for unpolarized leptons the symmetric leptonic tensor

$$L^{\mu\nu\,(S)} = \frac{Q^2}{y^2}\Bigg[-2\left(1-y+\frac{1}{2}\,y^2\right)g_\perp^{\mu\nu} + 4(1-y)\hat{t}^\mu\hat{t}^\nu + 4(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y}\;\hat{t}^{\{\mu}\hat{\ell}^{\nu\}}\Bigg] \tag{1.17}$$

The explicit contraction of lepton and hadron tensors gives for electromagnetic scattering (only symmetric tensor) the result

$$\begin{aligned}\frac{d\sigma^{ep}}{dx_B dy} &= \frac{4\pi\,\alpha^2\,x_B\,s}{Q^4}\left[\left(1-y+\frac{1}{2}\,y^2\right)F_T(x_B,Q^2) + (1-y)\,F_L(x_B,Q^2)\right] \\ &= \frac{2\pi\,\alpha^2\,s}{Q^4}\left[(1-y)\,F_2(x_B,Q^2) + x_B\,y^2\,F_1(x_B,Q^2)\right]. \end{aligned} \tag{1.18}$$

We have now used the known photon coupling to the lepton and parametrized our ignorance for what happened with the hadron in a hadronic tensor. The fact that we know how the photon interacts with the (quark) constituents of the hadrons will be used later to relate the structure functions to quark properties. In the same way one also knows for the weak interaction processes leading to antisymmetric part containing F_3 in the tensor for unpolarized hadrons, how the Z^0 and W couple to quarks. To describe weak interactions also the antisymmetric part of the lepton tensor is needed, which we will also encounter when we discuss polarization.

1.3 Virtual photon cross sections

The tensor $W_{\mu\nu}$ also appears in the total cross section for $\gamma^* H \to$ everything, where γ^* indicates a virtual photon. For a given virtuality Q^2 of the photon this cross section depends on only one variable, $W^2 = (P+q)^2$, or equivalently on the variable $\nu = P \cdot q/M$,

$$\sigma^{\gamma^* H}(\nu) = \frac{4\pi^2\alpha}{K}\,\epsilon^{\mu *} W_{\mu\nu}\epsilon^{\nu}, \tag{1.19}$$

where $4M\,K$ is the photon flux factor. This flux factor only is physical for real photons ($Q^2 = 0$). One convention is to take $4M\,K = 4\sqrt{(p\cdot q)^2 - p^2 q^2}$, i.e. $K = \sqrt{\nu^2 + Q^2}$. Other possibilities are to take the real photon result $4M\,K = 4\,P\cdot q$ or $K = \nu$ (Hand convention). Another convention that has been used is to equate the final state invariant mass squared $W^2 = (P+q)^2$, i.e. take the result $4M\,K = 2(W^2 - M^2)$ for a massless photon and equate W^2 to the invariant mass in the case of a virtual photon, $W^2 = 2\,P\cdot q + M^2 - Q^2$ or $K = \nu - Q^2/2M$.

Being a (total) cross section for (virtual) photoabsorption, the hadronic tensor is related to the forward (virtual) Compton amplitude through the optical theorem,

$$W_{\mu\nu} = \frac{1}{\pi}\,\text{Im}\;T_{\mu\nu}, \tag{1.20}$$

where

$$2M\,T_{\mu\nu}(P,q) = i\int d^4x\; e^{iq\cdot x}\langle P|\mathcal{T} J_\mu(x) J_\nu(0)|P\rangle, \tag{1.21}$$

Using the photon polarization vectors ϵ^μ_α, where α indicates one of the polarization directions (perpendicular to q^μ),

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}\,(0,\mp 1,-i,0) = \mp\frac{1}{\sqrt{2}}(\epsilon_x \pm i\,\epsilon_y), \tag{1.22}$$

$$\epsilon^\mu_L = \frac{1}{\sqrt{Q^2}}\,(q^3,0,0,q^0), \tag{1.23}$$

one gets two transverse structure functions and one longitudinal structure function, $F_\alpha = \epsilon^{\mu *}_\alpha\, M W_{\mu\nu}\epsilon^\nu_\alpha$. Because of the fact that they are cross sections for (virtual) photons, these structure functions, F_+, F_- and F_L are positive. We have

$$F_T = \frac{1}{2}\,(F_+ + F_-) = F_1, \tag{1.24}$$

$$F_L = \frac{F_2}{2x_B} - F_1, \tag{1.25}$$

$$F_3 = F_+ - F_-. \tag{1.26}$$

1.4 Symmetry properties of the structure functions

For the Compton scattering process, $\gamma^*(q) + H(P) \to \gamma^*(q') + H(P')$, the amplitude $T_{\mu\nu}(P, q, q')$ can be expanded, similar as $W_{\mu\nu}$, in terms of amplitudes T_1 and T_2 that depend on the invariants in the scattering process. These are (except the for a given process fixed photon virtuality) two of the (three) Mandelstam variables for the $\gamma^* N$ process, $s = (P+q)^2$, $t = (P-q)^2$ and $u = (P-q')^2$ (Note that $s + t + u = 2\,M^2 - 2\,Q^2$).

First we review the symmetries and analytic properties of the amplitudes T_i and the structure functions W_i. Crossing symmetry relates the amplitudes

$$T^{ab\to cd}(p_a, p_b, p_c, p_d) = T^{\bar{c}b\to\bar{a}d}(-p_c, p_b, -p_a, p_d). \tag{1.27}$$

For the virtual Compton amplitude $T_1^{\gamma^*\bar{H}\to\gamma^*\bar{H}}(u,s) = T_1^{\gamma^* H\to\gamma^* H}(s,u)$. For the elastic $\gamma^* H$ and $\gamma^* \bar{H}$ amplitudes themselves the crossing properties for the photon imply $T_1(s,u) = T_1(u,s)$. From the optical theorem the relation between the total cross section for $\gamma^* H \to X$ to the elastic $\gamma^* H \to \gamma^* H$ amplitude, specifically the structure function W_1 is found as the discontinuity over the cut in the physical region, which is the imaginary part of the forward ($t = 0$) amplitude T_1. For the forward amplitude ($t = 0$) the variables s and u are related and ν equals $\nu = (s-u)/4M$. Thus T_1 is a symmetric function of ν. One has

$$\sigma_T^{\gamma^* H}(\nu) = \frac{4\pi\alpha}{K}\,\mathrm{Im}\,T_1(s+i\epsilon, u) = \frac{4\pi\alpha}{K}\,\mathrm{Im}\,T_1(\nu + i\epsilon), \tag{1.28}$$

$$\begin{aligned}\sigma_T^{\gamma^* \bar{H}}(\nu) &= \frac{4\pi\alpha}{K}\,\mathrm{Im}\,T_1(u+i\epsilon, s) = \frac{4\pi\alpha}{K}\,\mathrm{Im}\,T_1(-\nu + i\epsilon)\\ &= -\frac{4\pi\alpha}{K}\,\mathrm{Im}\,T_1(\nu + i\epsilon),\end{aligned} \tag{1.29}$$

where the last equality follows from the symmetry of T_1, $T_1(\nu) = T_1(-\nu)$ and the fact that T_1 is a real analytic function, $T_1(\nu) = T_1^*(\nu^*)$. Note that as $W_{\mu\nu}$ is defined as the commutator of the currents the cross section for the 'crossed' part enters with a negative sign. As an analytic function of ν, however, one precisely has

$$W_1(\nu, Q^2) = \frac{1}{\pi}\,\mathrm{Im}\ T_1(\nu + i\epsilon) = -W_1(-\nu, Q^2), \tag{1.30}$$

which can also be derived from the translation invariance properties of the commutator defining $W^{\mu\nu}$. The analytic behavior and symmetry in the Bjorken variable $x = Q^2/2M\nu$ is exactly the same as that in ν, i.e. $W_1(x) = -W_1(-x)$ is antisymmetric in x.

Exercise. A simple example in which all of these properties are illustrated

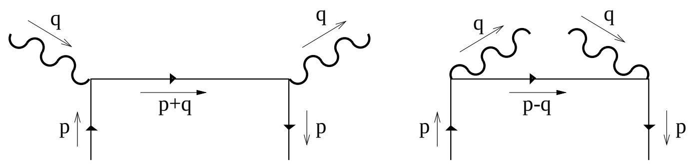


Figure 1.3. *The Compton amplitude for a point fermion.*

is the forward Compton amplitude for scattering off a point fermion with mass m and charge $Q/e = e_f$ (fig. 1.3). The amplitude T_1 is the coefficient of $g_{\mu\nu}$ in the amplitude $T_{\mu\nu}$,

$$2m\,T_{\mu\nu} = e_f^2\,\frac{\bar{u}(p)\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu u(p)}{(p+q)^2 - m^2 + i\epsilon} + [\mu \leftrightarrow \nu,\; q \leftrightarrow -q]. \tag{1.31}$$

Show that it equals

$$\begin{aligned} 2m\,T_1 &= e_f^2\left(\frac{s-u}{2(s-m^2)} + \frac{u-s}{2(u-m^2)}\right) \\ &= e_f^2\left(\frac{1}{1-x} + \frac{1}{1+x}\right), \end{aligned} \tag{1.32}$$

of which the imaginary part precisely is the structure function for a pointlike fermion,

$$2m\,W_1 = e_f^2\left[\delta(1-x) - \delta(1+x)\right]. \tag{1.33}$$

1.5 Polarized leptoproduction

For spin-polarized leptons in the initial state we have

$$\begin{aligned} L_{\mu\nu}^{(s)} &= \mathrm{Tr}\,\left[\gamma_\mu(\not{k}'+m)\gamma_\nu(\not{k}+m)\,\frac{1\pm\gamma_5\,\not{s})}{2}\right] \\ &= 2\,k_\mu k'_\nu + 2\,k'_\mu k_\nu - Q^2\,g_{\mu\nu} \pm 2i\,m\,\epsilon_{\mu\nu\rho\sigma}q^\rho s^\sigma. \end{aligned} \tag{1.34}$$

Note that for light particles or particles at high energies helicity states ($\hat{s}$ $= \hat{\boldsymbol{k}}$) become chirality eigenstates. For $L_{\mu\nu}$ the equivalence is easily seen because for $s^\mu = (|\boldsymbol{k}|/m, E\hat{\boldsymbol{k}}/m)$ ($s^2 = -1$ and $s \cdot p = 0$) one obtains in the limit $E \approx |\boldsymbol{k}|$ the result $m\,s^\mu \approx k^\mu$. Then the leptonic tensor for helicity states ($\lambda_e = \pm$) becomes

$$L_{\mu\nu}^{(\lambda_e=\pm 1/2)} \approx L_{\mu\nu}^{(R/L)} = L_{\mu\nu}^{(S)} + \lambda_e\,L_{\mu\nu}^{(A)}. \tag{1.35}$$

where the antisymmetric lepton tensor is given by

$$L^{(A)}_{\mu\nu}(k,k') = \text{Tr}\ \left[\gamma_\mu\gamma_5 \not{k}'\gamma_\nu\not{k}\right] 2i\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma . \tag{1.36}$$

Expanding in the Cartesian set $\hat{t}$, $\hat{z}$ and the vector $\hat{\ell}$ in the same way as for the symmetric part, we have for the antisymmetric part of the leptonic tensor[†] the result

$$L^{\mu\nu\,(A)} = \frac{Q^2}{y^2}\left[-i\,y(2-y)\,\epsilon_\perp^{\mu\nu} - 2i\,y\sqrt{1-y}\ \hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\right]. \tag{1.37}$$

One can use polarized leptons in deep inelastic $\vec{e}p \to X$ to probe the antisymmetric tensor for unpolarized hadrons, containing the F_3 structure function. This contribution comes in via the interference between the γ and Z interference term.

In the situation where the target is polarized, one has several more structure functions as compared to the case of an unpolarized target. For a spin 1/2 particle the initial state is described by a 2-dimensional spin density matrix $\rho = \sum_\alpha |\alpha\rangle p_\alpha \langle\alpha|$ describing the probabilities p_α for a variety of spin possibilities. This density matrix is hermitean with $\text{Tr}\,\rho = 1$. It can in the target rest frame be expanded in terms of the unit matrix and the Pauli matrices,

$$\rho_{ss'} = \frac{1}{2}\left(1 + \boldsymbol{S}\cdot\boldsymbol{\sigma}_{ss'}\right), \tag{1.38}$$

where $\boldsymbol{S}$ is the spin vector. When $|\boldsymbol{S}| = 1$ one has a pure state (only one state $|\alpha\rangle$ and $\rho^2 = \rho$), when $|\boldsymbol{S}| \le 1$ one has an ensemble of states. For the case $|\boldsymbol{S}| = 0$ one has simply an averaging over spins, corresponding to an unpolarized ensemble. To include spin one could generalize the hadron tensor to a matrix in spin space, $\tilde{W}^{\mu\nu}_{s's}(q,P) \propto \langle P,s'|J^\mu|X\rangle\langle X|J^\nu|P,s>$ depending only on the momenta or one can look at the tensor $\sum_\alpha p_\alpha \tilde{W}^{\mu\nu}_{\alpha\alpha}(q,P)$. The latter is given by

$$W^{\mu\nu}(q,P,S) = \text{Tr}\left(\rho(P,S)\tilde{W}^{\mu\nu}(q,P)\right), \tag{1.39}$$

[†] A useful relation is

$$\epsilon_{\mu\nu\rho\sigma}\,g_{\alpha\beta} = \epsilon_{\alpha\nu\rho\sigma}\,g_{\mu\beta} + \epsilon_{\mu\alpha\rho\sigma}\,g_{\nu\beta} + \epsilon_{\mu\nu\alpha\sigma}\,g_{\rho\beta} + \epsilon_{\mu\nu\rho\alpha}\,g_{\sigma\beta}$$

or for a vector $a_\perp$ orthogonal to $\hat{t}$ and $\hat{q}$,

$$\epsilon^{\mu\nu\rho\sigma}\hat{z}_\rho a_{\perp\sigma} = \hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho},$$
$$\epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho a_{\perp\sigma} = -\hat{z}^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho}.$$

with the spacelike spin vector S appearing *linearly* and in an arbitrary frame satisfying $P \cdot S = 0$. It has invariant length $-1 \le S^2 \le 0$. It is convenient to write

$$S^\mu = \frac{S_L}{M}\left(P^\mu - \frac{M^2}{P\cdot q}\,q^\mu\right) + S_\perp^\mu, \tag{1.40}$$

with

$$S_L \equiv \frac{M\,(S\cdot q)}{(P\cdot q)}. \tag{1.41}$$

For a pure state one has $S_L^2 + \boldsymbol{S}_\perp^2 = 1$. Parity requires that the polarized part of the tensor, i.e. the part containing the spin vector, enters in an antisymmetric tensor of the form

$$\begin{aligned} M\,W^{\mu\nu\,(A)}(q,P,S) &= S_L\,\frac{i\epsilon^{\mu\nu\rho\sigma}q_\rho P_\sigma}{(P\cdot q)}\,g_1 + \frac{M}{P\cdot q}\,i\epsilon^{\mu\nu\rho\sigma}q_\rho S_{\perp\sigma}\,(g_1+g_2) \\ &= -iS_L\,\epsilon_\perp^{\mu\nu}\,g_1 - i\,\frac{2M}{Q}\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}\,x_B(g_1+g_2). \end{aligned} \tag{1.42}$$

It contains two structure functions $g_1(x_B,Q^2)$ and $g_2(x_B,Q^2)$. One also uses $g_T \equiv g_1 + g_2$. The resulting cross section is

$$\frac{d\Delta\sigma_{LL}}{dx_B\,dy} = \frac{4\pi\,\alpha^2}{Q^2}\,\lambda_e\left[S_L\,(2-y)\,g_1 - |\boldsymbol{S}_\perp|\,\cos\phi_S^\ell\,\frac{2M}{Q}\,\sqrt{1-y}\,x_B(g_1+g_2)\right]. \tag{1.43}$$

(Note that in all of the above formulas mass corrections proportional to M^2/Q^2 have been neglected).

A special case of inclusive scattering is the situation in which the final state is identical to the initial state, elastic scattering. In that case the final state four momentum is $P' = P+q$ and is fixed to be $(P+q)^2 = M^2$, i.e. $x_B = 1$. We can still use the formalism for inclusive leptoproduction but the hadron tensor becomes becomes

$$2M\,W_{\mu\nu}(q,P) = \underbrace{\langle P|J_\mu(0)|P'\rangle\langle P'|J_\nu(0)|P\rangle}_{H_{\mu\nu}(P;P')}\,\frac{1}{Q^2}\,\delta(1-x_B). \tag{1.44}$$

These current matrix elements have been discussed in chapter xx.

1.6 The parton model

The intuitive approach

In the intuitive derivation of the parton model one convolutes the γ^*-quark cross section with a momentum distribution of quarks in the nucleon.

Exercise. Show that the $\gamma^* q$ cross section is given by

$$\hat{\sigma}(\gamma^* q) = \frac{4\pi^2\alpha}{2p\cdot q}\,\epsilon^*_\mu w^{\mu\nu}\epsilon_\nu$$
$$w_{\mu\nu}(p,q) = \frac{1}{2M}\left[\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)Q^2 + 4\,\tilde{p}_\mu\tilde{p}_\nu\right]\,\delta(2\,p\cdot q + q^2)$$

(note that there are in principle ambiguities here because of the flux factor for virtual photons) and deduce

$$\hat{\sigma}_T(\gamma^* q) = 4\pi^2\alpha\; e_q^2\;\delta(2p\cdot q - Q^2), \tag{1.45}$$
$$\hat{\sigma}_L(\gamma^* q) = 4\pi^2\alpha\; e_q^2\;\frac{4m^2}{Q^2}\,\delta(2p\cdot q - Q^2) \ll \hat{\sigma}_T. \tag{1.46}$$

In the next step this partonic cross section is folded with the probability function for finding partons in the target. For this purpose it is convenient to give the explicit momenta as lightcone components, $p = [p^-, p^+, \boldsymbol{p}_\perp]$ where $p^\pm = (p^0 \pm p^3)/\sqrt{2}$ or $p = p^- n_- + p^+ n_+ + p_T$ in terms of two lightlike vectors satisfying $n_-^2 = n_+^2 = 1$ and $n_+\cdot n_- = 0$ (thus one has $p^\pm = p\cdot n_\mp$).

Exercise. Derive the lightcone expansion for the external vectors P and q,

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} q = \frac{Q}{\sqrt{2}}\,n_- \;-\; \frac{Q}{\sqrt{2}}\,n_+ \\ \\ P = \frac{x_B M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right. .$$

Note that $n_\pm$ are not unique. Give the vectors $n_\pm$ for the target rest frame ($\boldsymbol{P} = \boldsymbol{0}$). What is the effect of a boost on the vectors $n_\pm$.

In particular the first representation with lightlike vectors shows that when Q^2 becomes large, the nucleon momentum is 'on the scale Q in essence lightlike. While the hard momentum has both components proportional to Q, this is not the case for P and one has $P^- \ll q^-$. In deep-inelastic scattering the plus components are of the same order with ratio precisely being the scaling variable $x_B = -q^+/P^+$.

Exercise. Including finite Q effects the 'better' scaling variable is actually $x_N = -q^+/P^+$ (Nachtmann variable). Show that sticking to the exact definition $x_B = Q^2/2P\cdot q$ one has

$$x_N = \frac{2x_B}{1 + \sqrt{1 + \frac{4\,M^2 x_B^2}{Q^2}}}$$

Next we turn to the parton momentum. Compared with the hard scale Q it is in essence also lightlike. It is useful to expand $p = p^- \, n_- + x\, P^+ n_+ + p_T$, where the *lightcone momentum fraction.* $x \equiv p^+/P^+$ has been introduced.

Exercise. Show that on-shell ($p^2 = m^2$) one has

$$p^- = \frac{m^2 + \boldsymbol{p}_T^2}{2p^+} = \frac{m_\perp^2}{2x\, P^+},$$

while in a hadron one has

$$p^- = \frac{2\, p \cdot P - x\, M^2}{2x\, P^+},$$
$$\boldsymbol{p}_T^2 = (1-x)\, M^2 - M_R^2,$$

where $M_R^2 = (P-p)^2$.

Under the assumption that all invariants $p \cdot P \sim M_R^2 \sim p^2 \sim P^2 = M^2$ one sees that for the expansion in terms of $n_\pm$ one has for a quark in a hadron (as one would have for an on-shell quark) that $p^+ \sim P^+ \sim Q$, while $p^- \sim M^2/Q$ and $p_T^2 \sim M^2$. This is sufficient to derive the parton model results.

Using the cross section for $\gamma^* q$ (elastic scattering) given above,

$$\hat{\sigma}_T = \frac{4\pi^2 \alpha}{Q^2}\, \delta(1 - x_p). \tag{1.47}$$

where $x_p = -q^+/p^+$ and introducing probabilities $f_i(x)$ for finding partons carrying momentum fraction $x = p^+/P^+ = x_B/x_p$ of the target lightcone momentum, leads to[‡]

$$\begin{aligned} \sigma_T &= \sum_i e_i^2 \int dx\, f_i(x)\, \frac{4\pi^2\alpha}{Q^2}\, \delta\left(1 - \frac{x_B}{x}\right) \\ &= \frac{4\pi^2\alpha}{Q^2} \sum_i e_i^2\, x_B\, f_i(x_B). \end{aligned} \tag{1.48}$$

Comparing with

$$\sigma_T = \frac{8\pi^2\alpha}{Q^2}\, x_B\, F_1, \tag{1.49}$$

[‡] We note that the probability involves in fact $f_i(x)\, dx/x$, but we need to fold conting rates which requires that we need the cross section multiplied with flux factors. The ratio of the flux factors for quarks and hadrons is $p \cdot q / P \cdot q \approx p^+/P^+ = x$. Hence we need to weigh the cross section with $f_i(x)\, dx$.

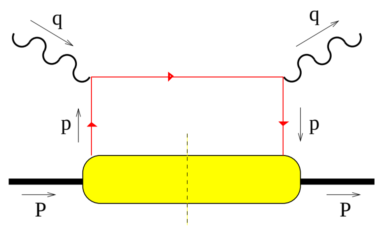


Figure 1.4. *The handbag diagram for inclusive deep inelastic scattering off a hadronic target.*

we get

$$F_1(x_B) = \frac{1}{2}\sum_i e_i^2\, f_i(x_B). \tag{1.50}$$

As $\hat{\sigma}_L \propto 1/Q^2 \to 0$ one obtains $F_L = 0$ or the Callan-Gross relation,

$$F_2(x_B) = 2x_B\, F_1(x_B). \tag{1.51}$$

the diagrammatic approach

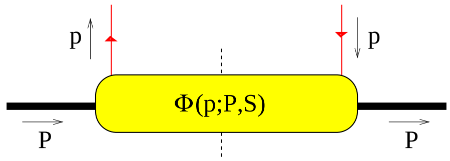
Another way in which the parton model is obtained is by just considering the socalled quark handbag diagram (see Fig. 1.4) and its antiquark equivalent. These diagrams turn out to be the leading ones out of a full set in which the connection to hadrons is left as an unknown quantity [3,4]. The basic expression corresponding to the handbag diagram restricting us to the quark part is

$$\begin{aligned} 2M\, W^{\mu\nu}(P,q) = \\ &= \sum_q e_q^2 \int dp^-\, dp^+\, d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu(\not p + \not q + m)\gamma^\nu\right)\ \delta((p+q)^2 - m^2) \\ &\approx \sum_q e_q^2 \int dp^-\, dp^+\, d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\, \frac{\not q}{2q^-}\,\gamma^\nu\right)\ \delta(p^+ + q^+) \\ &\approx -g_\perp^{\mu\nu}\,\frac{1}{2}\ \int dp^-\, d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\gamma^+\,\Phi(p)\right)\Big|_{p^+ = x_B\,P^+} + \ldots, \end{aligned} \tag{1.52}$$

where $\Phi(p)$ is the forward antiquark-target scattering amplitude,

$$\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\ e^{i\,p\cdot\xi}\langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle, \tag{1.53}$$

diagrammatically represented by



Comparing with the general form of the hadronic tensor, we read off (including now also the antiquark part)

$$2\,F_1(x_B) = 2M\,W_1(x_B,Q^2) = \sum_q e_q^2\,\left[q(x_B) + \bar{q}(x_B)\right], \tag{1.54}$$

with

$$q(x) = \frac{1}{4\pi}\int d\xi^-\;e^{ixP^+\xi^-}\,\langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0}\,, \tag{1.55}$$

$$\overline{q}(x) = \frac{1}{4\pi}\int d\xi^-\;e^{-ixP^+\xi^-}\,\langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0}\,, \tag{1.56}$$

satisfying $\overline{q}(x) = -q(-x)$ (see Exercise below). The result is (as expected) a lightcone correlation function of quark fields.

Exercise. Show that the antiquark distributions are given by $\bar{q}(x) = -q(-x)$. To do this start with the 'proper' definition of antiquark distributions,

$$\Phi^c_{ij}(p) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{ip\cdot\xi}\,\langle PS|\overline{\psi}^c_j(0)\psi^c_i(\xi)|PS\rangle, \tag{1.57}$$

with $\psi^c(\xi) = C\overline{\psi}^T(\xi)$. Show that one finds $\overline{\Phi}(p) = -C(\Phi^c)^T C^\dagger$. One also needs to use the anticommutation relations for fermions, to obtain $\overline{\Phi}_{ij}(p) = -\Phi_{ij}(-p)$, which leads to the crossing relations for quark and antiquark distributions.

The operator in coordinate space

The parton result for the structure functions can also be derived by inserting free currents in the hadronic tensor for the current commutator and using the expression for the free field commutator.

Exercise. Use the anticommutation relations for free quark fields, given by $\{\psi(\xi),\overline{\psi}(0)\} = \frac{1}{2\pi}\,\not{\partial}\,\delta(\xi^2)\,\epsilon(\xi^0)$ to derive for the $g_{\mu\nu}$ contribution in the current-current commutator for quarks

$$[J_\mu(\xi), J_\nu(0)] = [: \overline{\psi}(\xi)\gamma_\mu\psi(\xi) :, : \overline{\psi}(0)\gamma_\nu\psi(0) :]$$
$$= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(\xi^2)\,\epsilon(\xi^0)\right] : \overline{\psi}(\xi)\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\psi(\xi) : \quad (1.58)$$

An important feature, evident in the free-current commutator, is the light-cone dominance. By sandwiching the commutator between physical states and taking the Fourier transform, it is a straightforward calculation to obtain again the hadron tensor and the same result as in the diagrammatic approach above. Details can be found in [5].

Flavor dependence

The explicit flavor and spin dependence of the structure functions in electroweak processes depend on the probe being a γ, Z^0 or $W^\pm$ boson. We are in the situation, however, that we know the currents in terms of the quark fields. Omitting the coupling constants e or $\sqrt{G_F}$, the standard model currents coupling to fermions are

$$J_\mu^{(\gamma)} = : \overline{\psi}(x)\,Q\,\gamma_\mu\,\psi(x) :, \quad (1.59)$$
$$J_\mu^{(Z)} = : \overline{\psi}(x)\,(I_W^3 - Q\sin^2\theta_W)\,\gamma_{\mu L}\,\psi(x) : - : \overline{\psi}(x)\,Q\sin^2\theta_W\,\gamma_{\mu R}\,\psi(x) :$$
$$= : \overline{\psi}(x)\,(I_W^3 - 2\,Q\sin^2\theta_W)\,\gamma_\mu\,\psi(x) : - : \overline{\psi}(x)\,I_W^3\,\gamma_\mu\gamma_5\,\psi(x) :, \quad (1.60)$$
$$J_\mu^{(W)} = : \overline{\psi}(x)\,I_W^\pm\,\gamma_{\mu L}\,\psi(x) :, \quad (1.61)$$

where $\gamma_{\mu R/L} = \gamma_\mu(1 \pm \gamma_5)$.

Using the electromagnetic current, one obtains for the one-photon exchange contribution to ep scattering the following expression in terms of the distribution functions $u_p(x)$, $d_p(x)$, etc.,

$$\frac{F_2^{ep}(x)}{x} = 2\,F_1^{ep}(x) =$$
$$= \frac{4}{9}\,(u_p(x) + \overline{u}_p(x)) + \frac{1}{9}\,(d_p(x) + \overline{d}_p(x)) + \frac{1}{9}\,(s_p(x) + \overline{s}_p(x)) + \ldots$$
$$\equiv \frac{4}{9}\,(u(x) + \overline{u}(x)) + \frac{1}{9}\,(d(x) + \overline{d}(x)) + \frac{1}{9}\,(s(x) + \overline{s}(x)) + \ldots \quad ,(1.62)$$

where the last line phrases the convention to use the proton as reference hadron for distribution functions. Using isospin summetry, $u_p = d_n$, $d_p = u_n$, $s_p = s_n$ one has for en scattering

$$\frac{F_2^{en}(x)}{x} = 2\,F_1^{en}(x) =$$

$$\frac{1}{9}\left(u(x)+\overline{u}(x)\right)+\frac{4}{9}\left(d(x)+\overline{d}(x)\right)+\frac{1}{9}\left(s(x)+\overline{s}(x)\right)+\ldots \quad (1.63)$$

As the difference between quarks and antiquarks contributes to the quantum numbers, it is convenient to divide the quark distribution in a valence part and a sea part

$$q(x) = q_v(x) + q_s(x), \quad (1.64)$$

where

$$q_v(x) \equiv q(x) - \overline{q}(x). \quad (1.65)$$

The quark distributions are positive definite (see also section 1.7), so for instance

$$\frac{1}{4} \le \frac{F_2^{ep}}{F_2^{ep}} = \frac{(u+\overline{u}) + 4(d+\overline{d}) + (s+\overline{s}) + \ldots}{4(u+\overline{u}) + (d+\overline{d}) + (s+\overline{s}) + \ldots} \le 4. \quad (1.66)$$

If one looks at the experimental result one sees near $x \approx 0$ a ratio that is about 1, indicating dominance of sea quarks with $u\overline{u}$ and $d\overline{d}$ pairs in equal amounts. Near $x \approx 1$ the valence quarks dominate. Naively one might expect $u = 2\,d$ and all others zero, i.e. a ratio of 2/3. The experimentally observed limit for $x \to 1$ tends to 1/4, the lower limit, which is reached for $d \ll u$, i.e. dominance of the u-quark in the proton (and therefore the d-quark in the neutron).

It is clear that in order to determine the quark distributions, several processes are needed. At present the various quark and also gluon distributions are well-known (see Fig. 1.5, taken from Ref. [6]). For compilations we refer to Refs [7]

1.7 Properties of quark distributions

Interpretation as densities

To convince oneself that the above expressions for $q(x)$ and $\overline{q}(x)$ actually can be interpreted as quark momentum density one needs to realize that $\overline{\psi}(\xi)\gamma^+\psi(0) = \sqrt{2}\,\psi_+^\dagger(\xi)\psi_+(0)$ where $\psi_\pm = P_\pm\,\psi$ are projections obtained with projection operators onto *good* quark states [8], $P_\pm = \frac{1}{2}\gamma^\mp\gamma^\pm$. One then can insert a complete set of states and obtain

$$\begin{aligned} q(x) &= \int \frac{d\xi^-}{2\pi\sqrt{2}}\, e^{ip\cdot\xi}\, \langle P,S|\psi_+^\dagger(0)\psi_+(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0} \\ &= \frac{1}{\sqrt{2}}\sum_n |\langle P_n|\psi_+|P\rangle|^2\, \delta\left(P_n^+ - (1-x)P^+\right), \end{aligned} \quad (1.67)$$

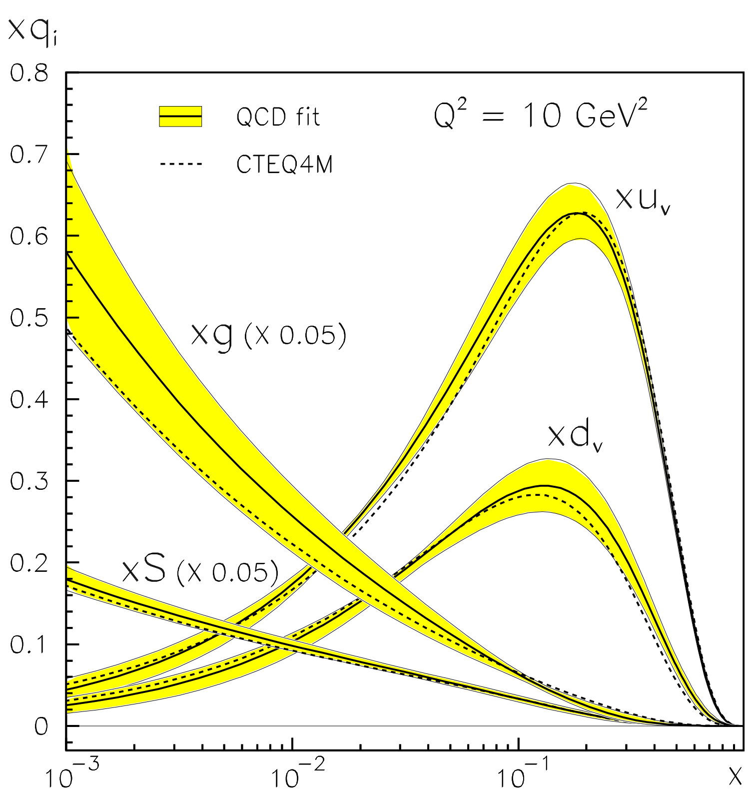


Figure 1.5. The parton momentum densities for gluons, valence and sea quarks. Given are $x\,g(x)$, $x\,S(x) = 2x(\bar{u}(x) + \bar{d}(x) + \bar{s}(x))$ (both divided by a factor 20), $x\,u_v(x)$ and $x\,d_v(x)$ versus x at a scale $\mu = 10$ GeV. The full curves show the results from a fit to world data including QCD corrections (see section 1.9). This analysis is taken from Ref. [6]. The dashed curves are from the CTEQ-4 parton distribution set.

which represents the probability that a quark is annihilated from $|P\rangle$ giving a state $|n\rangle$ with $P_n^+ = (1-x)P^+$. Since $P_n^+ \geq 0$ one sees that $x \leq 1$. From the antiquark distribution $\overline{q}(x)$ and its relation to $-q(-x)$ one obtains $x \geq -1$, thus showing that the support of the functions is $-1 \leq x \leq 1$ [9,10].

Polarized parton densities

Analogously to the unpolarized structure functions one can obtain for the polarized structure functions

$$2\, g_1(x_B) = \sum_q e_q^2 \left[\Delta q(x_B) + \Delta\overline{q}(x_B)\right] \tag{1.68}$$

where

$$S_L \Delta q(x) = \frac{1}{4\pi}\int d\xi^- \; e^{ixP^+\xi^-} \langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0} \tag{1.69}$$

$$S_L \Delta\overline{q}(x) = \frac{1}{4\pi}\int d\xi^- \; e^{-ixP^+\xi^-} \langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0} \tag{1.70}$$

a correlation existing in a hadron with the lightcone component of the spin vector $S_L \neq 0$. This represents the difference of chiral even and odd quarks (in infinite momentum frame quarks parallel or antiparallel to proton spin). The corresponding quark fields are projected out by

$$P_{R/L} = \frac{1}{2}(1 \pm \gamma_5), \tag{1.71}$$

which commute with the projectors $P_\pm$. In this way one obtains distributions $q_R(x)$ and $q_L(x)$ for which $q(x) = q_R(x) + q_L(x)$ and $\Delta q(x) = q_R(x) - q_L(x)$.

Sum rules

As probability distributions the quark distribution functions satisfy a number of obvious sum rules, such as

$$\int_0^1 dx\;\; [u(x) - \overline{u}(x)] = n_u = 2, \tag{1.72}$$

$$\int_0^1 dx\;\; \left[d(x) - \overline{d}(x)\right] = n_d = 1, \tag{1.73}$$

$$\int_0^1 dx\;\; [s(x) - \overline{s}(x)] = n_s = 0, \tag{1.74}$$

corresponding to the (net) number of each of these quark species in the proton. On the basis of this one finds a number of sum rules for the structure functions, such as the *Gottfried sum rule* [11] which is based on an isospin-symmetric sea distribution, $\overline{u}(x) = \overline{d}(x)$,

$$\begin{aligned} S_G &= \int_0^1 \frac{dx}{x}\; [F_2^{ep}(x) - F_2^{en}(x)] \\ &= \frac{1}{3}\int_0^1 dx[u(x) - d(x) + \overline{u}(x) - \overline{d}(x)] = \frac{1}{3}. \end{aligned} \tag{1.75}$$

The experimental result (NMC [12], giving 0.240 ± 0.016 indicates that $\overline{u}(x) \neq \overline{d}(x)$.

Exercise. Show that building a proton and neutron from 'bare' 3-quark nucleons and a pion-bare nucleon component (also coupling to total isospin 1/2) with probability $P_{\pi N}$, leads to

$$S_G = \frac{1}{3}\left(1 - \frac{4}{3}\,P_{\pi N}\right).$$

If one has found the explicit quark distributions one can determine the quantity obtained by weighing the sum over all quarks with the momentum,

$$\int_0^1 dx\, x\, \Sigma(x) \equiv \int_0^1 dx\; x[u(x) + \overline{u}(x) + d(x) + \overline{d}(x) + s(x) + \overline{s}(x) + \ldots] = \epsilon_q. \tag{1.76}$$

It represents the total momentum fraction of the proton carried by quarks. It must obviously be smaller than one.

For the polarized structure functions one can obtain similar estimates using the naive flavor-spin structure of the proton based on $SU(6)$ symmetry,

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\left(2\,u_\uparrow u_\uparrow d_\downarrow - u_\uparrow u_\downarrow d_\uparrow - u_\downarrow u_\uparrow d_\uparrow + \;[d \text{ at places 1 and 2}]\right). \tag{1.77}$$

From this wave function one finds in terms of a normalized one-quark distribution $q(x)$ the 'naive' results

$$\begin{aligned} u_\uparrow(x) &= \frac{5}{3}\,q(x), & u_\downarrow(x) &= \frac{1}{3}\,q(x), \\ d_\uparrow(x) &= \frac{1}{3}\,q(x), & d_\downarrow(x) &= \frac{2}{3}\,q(x), \end{aligned} \tag{1.78}$$

or

$$\begin{aligned} u(x) &= 2\,q(x), & \Delta u(x) &= \frac{4}{3}\,q(x), \\ d(x) &= q(x), & \Delta d(x) &= -\frac{1}{3}\,q(x), \end{aligned} \tag{1.79}$$

and all other distributions (strange quarks or antiquarks) are zero. In this case one obtains naive sum rule results for the polarized distributions,

$$\int_0^1 dx\ [\Delta u(x) + \Delta\overline{u}(x)] = \Delta u = \frac{4}{3}, \tag{1.80}$$

$$\int_0^1 dx\ [\Delta d(x) + \Delta\overline{d}(x)] = \Delta d = -\frac{1}{3}. \tag{1.81}$$

Note that the sum $\Delta\Sigma = \Delta u + \Delta d + \Delta s + \ldots$ represents (as probability distributions) the total number of quarks parallel to the proton spin. If the proton spin comes from the quark spins, as is the case for the above $SU(6)$ wave function multiplying a spherically symmetric spatial wave function, one expects this to be one. It leads to

$$\Gamma_1^p = \int_0^1 dx\ g_1^p(x) = \frac{1}{2}\left[\frac{4}{9}\,\Delta u + \frac{1}{9}\,\Delta d + \frac{1}{9}\,\Delta s\right] = \frac{5}{18} \approx 0.28, \tag{1.82}$$

$$\Gamma_1^n = \int_0^1 dx\ g_1^n(x) = \frac{1}{2}\left[\frac{4}{9}\,\Delta d + \frac{1}{9}\,\Delta u + \frac{1}{9}\,\Delta s\right] = 0, \tag{1.83}$$

which is in disagreement with the experimental result [13], $\Gamma_1^p \approx 0.15$ and the result $\Gamma_1^n \approx -0.04$, obtained from the deuteron sum rule $\Gamma_1^d = (\Gamma_1^p + \Gamma_1^n)\left(1 + \frac{3}{2}\omega_D\right)$ with $\omega_D \approx 0.05$ being the D-wave probability in the deuteron.

The much smaller than expected result stimulated a vigorous experimental program. Including not only inclusive measurements but also semi-inclusive ones (see section 1.11), one has obtained the picture of the polarizations for the different flavors shown in Fig. 1.6 (taken from Ref. [14]), indicating in particular for small x-values a strong deviation from the above naive expectations $\Delta u/u = 2/3$ and $\Delta d/d = -1/3$.

The importance of these sum rules becomes clearer when one starts with the expressions for the distribution functions in terms of matrix elements of bilocal operator combinations. One has

$$\int_{-1}^1 dx\ q(x) = \int_0^1 dx\ [q(x) - \overline{q}(x)] = \frac{\langle P|\overline{\psi}(0)\gamma^+\psi(0)|P\rangle}{2P^+} = n_q, \tag{1.84}$$

where n_q is the coefficient in the expectation value $\langle P|\overline{\psi}(x)\gamma^\mu\psi(0)|P\rangle = 2\,n_q\,P^\mu$. The coefficient n_q is precisely the quark number because the vector currents are used to obtain the quantum numbers for flavor (upness, downness, strangeness, etc.). In general one obtains

$$\int_{-1}^1 dx\ x^{n-1}\ q(x) = \int_0^1 dx\ x^{n-1}\,[q(x) + (-)^n\,\overline{q}(x)]$$

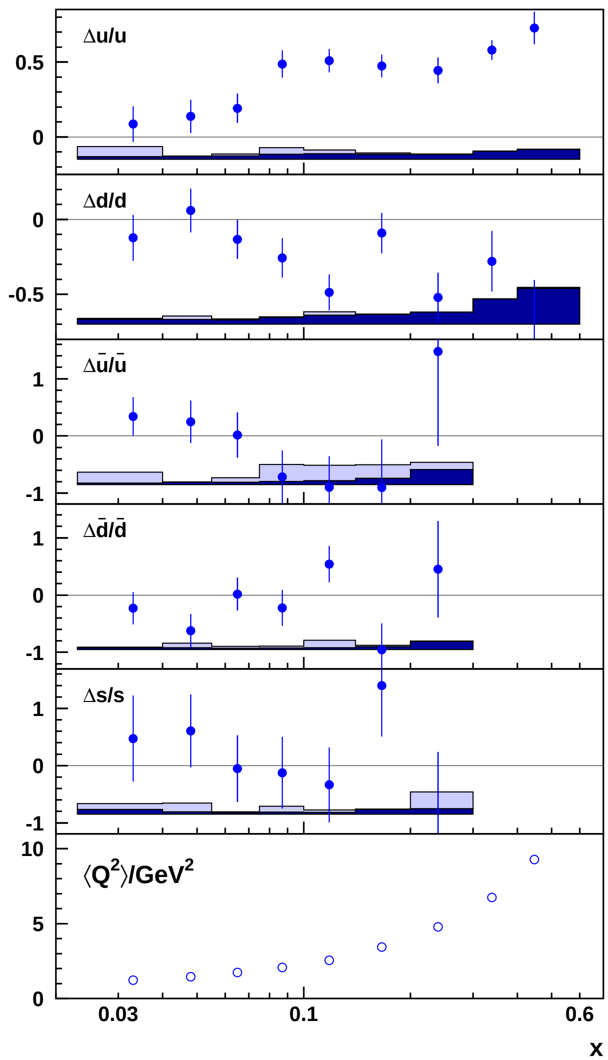


Figure 1.6. The quark polarizations obtained from inclusive and semi-inclusive spin measurements by the HERMES collaboration at DESY [14]. The error bars indicate statistical uncertainties while the bands indicate systematic uncertainties in which the light gray band specifies the part due to errors in the fragmentation process. The lower plot gives the scales at which the asymmetries in the various bins are measured.

$$= \frac{1}{2P^+} \langle P|\overline{\psi}(0)\,\gamma^+ \left(\frac{i\partial^+}{P^+}\right)^{n-1} \psi(0)|P\rangle. \qquad (1.85)$$

The moments of the structure functions are related to expectation values of particular quark operators. In a field theory these matrix elements depend on a renormalization scale μ^2 and thus a similar renormalization scale dependence must be present for the structure functions. In the next sections these QCD corrections will be discussed in more detail. In some cases such as the rule in Eq. 1.84 the result is scale independent This is true if the operator combination corresponds to a conserved current. The situation is different for the second moment that appeared in the momentum sumrule in Eq. 1.76,

$$\int_{-1}^{1} dx\ x\ q(x,\log\mu^2) = \int_0^1 dx\ x\,[q(x,\log\mu^2) + \overline{q}(x,\log\mu^2)]$$
$$= \frac{\langle P|\overline{\psi}(0)\,i\gamma^+\partial^+\psi(0)|P\rangle_{(\mu^2)}}{2\,(P^+)^2} = \epsilon_q(\mu^2), \quad (1.86)$$

where ϵ_q is defined in $\langle P|\overline{\psi}(0)\,i\gamma_\mu\partial_\nu\psi(0)|P\rangle = 2\,\epsilon_q\,P_\mu P_\nu + \ldots$ and is the relative contribution of quarks to the energy momentum tensor of the proton (the dots indicate trace terms $\propto M^2\,g_{\mu\nu}$). Only the first moment of the sum including quark *and* gluon distributions in the proton is scale independent as the local operator turns out to be the energy momentum stress tensor of QCD.

For polarized structure functions the lowest moment is given by

$$\int_{-1}^{1} dx\ \Delta q(x,\log\mu^2) = \int_0^1 dx\ [\Delta q(x,\log\mu^2) + \Delta\overline{q}(x,\log\mu^2)]$$
$$= \frac{\langle P|\overline{\psi}(0)\gamma^+\gamma_5\psi(0)|P\rangle_{(\mu^2)}}{2\,P^+} = \Delta q(\mu^2) \qquad (1.87)$$

The quantity Δq appears in $\langle P|\overline{\psi}(0)\gamma^+\gamma_5\psi(0)|P\rangle = 2M\,\Delta q\,S^\mu$, the matrix element of the axial current, where S^μ is the spin vector for the nucleon. Also this is in general not independent of the renormalization scale. In particular the flavor singlet axial current is not conserved (because of the Adler-Bardeen-Jackiw-anomaly). It implies, however, a breaking independent of the flavor of the quarks. For the non-singlet axial currents that are important in the flavor-changing weak decays of baryons, e.g. for the neutron $\propto \tau_+\,\gamma^\mu\gamma_5$, the current is conserved and the corresponding matrix elements are scale independent. From the neutron decay one deduces the (scale independent) flavor nonsinglet combinations

$$\Delta q^3 = \Delta u(\mu^2) - \Delta d(\mu^2) = G_A = 1.259, \qquad (1.88)$$

while from hyperon decays one finds (using $SU(3)$ symmetry),

$$\Delta q^8 = \Delta u(\mu^2) + \Delta d(\mu^2) - 2\,\Delta s(\mu^2) \approx 0.6, \tag{1.89}$$

In terms of these combinations and the scale-dependent singlet combination,

$$\Delta\Sigma(\mu^2) = \Delta u(\mu^2) + \Delta d(\mu^2) + \Delta s(\mu^2), \tag{1.90}$$

one has

$$\Gamma_1^{p/n}(x) == \frac{1}{9}\,\Delta\Sigma(\mu^2) \pm \frac{1}{12}\,\Delta q^3 + \frac{1}{36}\,\Delta q^8. \tag{1.91}$$

A sum rule involving only flavor non-singlet combinations is for example the *polarized Bjorken sum rule* [15],

$$\int_0^1 dx\;[g_1^{ep}(x) - g_1^{en}(x)] = \frac{1}{6}\Delta q^3 = \frac{G_A}{6} \approx 0.21, \tag{1.92}$$

in reasonable agreement with experiment. Note that this result is a factor 0.75 smaller than the naive expectation for which $G_A = \Delta q^3 = 5/3$ instead of $G_A \approx 1.26$. A long-known explanation for this reduction is the relativistic nature of quarks in hadrons implying a sizable p-wave contribution in the lower components of the quark spinor that reduces the spinor densities $\overline{\psi}\gamma_5\gamma_3\psi = \psi^\dagger\sigma_z\psi$.
Using the result for Γ_1^p or Γ_1^n as the third input one can solve for $\Delta\Sigma$, leading to $\Delta\Sigma \approx 0.2$, very small compared to the naive expectations $\Delta\Sigma$ being of the order of 0.75 (taking the same reduction factor for relativistic quarks as for G_A). This was known as the 'proton spin puzzle'. At present we know that the scale dependence is important and that the deep inelastic measurements imply $\Delta\Sigma(20\text{ GeV}^2) \approx 0.2$.

1.8 The operator product expansion

The connection of structure functions and quark distribution functions to local operators via sum rules is more formally grounded in the operator product expansion (OPE). The basic idea of the OPE is that the product of two operators simplifies in the limit that their arguments coincide, Applying this to the time-ordered product of two currents, suppressing the current indices, one writes writes

$$\mathcal{T} J(x)J(0) \propto \sum_{\beta,n} C_n^{(\beta)}(x^2;g)\,x^{\mu_1}\ldots x^{\mu_n}\,O^{(\beta)}_{\mu_1\ldots\mu_n}(0). \tag{1.93}$$

The functions $C_n^\beta(x^2;g)$ are invariant functions depending on x^2 and the parameters of the theory. In the (well-defined) Euclidean formulation this corresponds to a short-distance expansion. The $O^{\beta}_{\mu_1\ldots\mu_n}$ are a set of local operators. For a unique expansion, one needs a set of (irreducible) symmetric and traceless operators.

An explicit example is the $g_{\mu\nu}$ contribution in the current-current commutator for free quark currents,

$$\begin{aligned}
[J_\mu(x), J_\nu(0)] &= [:\overline{\psi}(x)\gamma_\mu\psi(x):, :\overline{\psi}(0)\gamma_\nu\psi(0):] \\
&= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(x^2)\,\epsilon(x^0)\right] : \overline{\psi}(x)\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\psi(x) : + \ldots \\
&= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(x^2)\,\epsilon(x^0)\right] \sum_{n=0}^{\infty} \frac{x^{\mu_1}\ldots x^{\mu_n}}{n!} \\
&\qquad \times : [\partial_{\mu_1}\ldots\partial_{\mu_n}\overline{\psi}(0)]\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\,\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(0) : \\
&+ \ldots . \qquad (1.94)
\end{aligned}$$

Inserting the expansion in the expectation value of the time ordered product, such as in the Compton amplitude in Eq. 1.21 one finds after Fourier transforming $C(x^2)$,

$$\begin{aligned}
2M\,T(q^2,\nu) &\propto \sum_{\beta,n} \tilde{C}_n^{(\beta)}(q^2;g,\mu^2)\,(-)^n\,\frac{q^{\mu_1}\ldots q^{\mu_n}}{(q^2)^n}\,\langle P|O^{(\beta)}_{\mu_1\ldots\mu_n}(0)|P\rangle_{(\mu^2)} \\
&= \sum_{\beta,n} \tilde{C}_n^{(\beta)}(q^2;g,\mu^2)\,\Theta_n^{(\beta)}(\mu^2)\left[\left(\frac{1}{2x}\right)^n + \mathcal{O}\left(\frac{1}{Q^2}\right)\right], \qquad (1.95)
\end{aligned}$$

The scale dependent quantities $\Theta^{(\beta)}$ appear in the expectation values of the operators,

$$\langle P|O^{(\beta)}_{\mu_1\ldots\mu_n}(0)|P\rangle_{(\mu^2)} = \Theta^{(\beta)}(\mu^2)[P_{\mu_1}\ldots P_{\mu_n} + \ldots], \qquad (1.96)$$

and the dots are dictated by the tracelessness of O.

Let us illustrate this once more for the Compton amplitude for an elementary fermion. In calculating matrix elements one encounters on the right hand side the matrix elements of local operators. In momentum space, the contribution from the T-ordered product of currents corresponds to

$$\langle|\mathcal{T}J(x)J(0)|\rangle \propto = \gamma_\mu\,\frac{1}{\not{p} + \not{q} + i\epsilon}\,\gamma_\nu. \qquad (1.97)$$

One can expand this in the following way (restricting us to the $g_{\mu\nu}$ part),

$$\langle|\mathcal{T}J(x)J(0)|\rangle \propto g_{\mu\nu}\sum_{n=0}^{\infty}\not{q}\left(\frac{\not{p}\,\not{q}}{-q^2}\right)^n + \ldots,$$

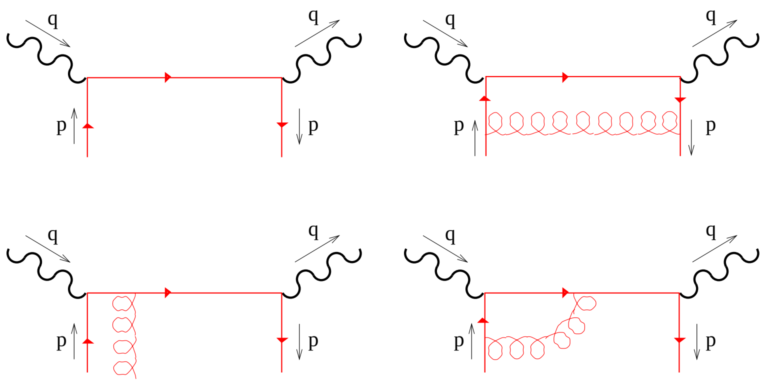


Figure 1.7. The $\mathcal{O}(g^2)$ contributions to the time-ordered product of currents.

which correspond precisely to the matrix elements of local operators of the form $\overline{\psi}\gamma^{\mu}\partial^{\mu_1}\dots\partial^{\mu_n}\psi$, proportional to $p^{\mu_1}\dots p^{\mu_n}$, multiplied with a coefficient in momentum space (see Eq. 1.95), in our case

$$\tilde{C}_n^{\mu_1\cdots\mu_n}(q) = 1. \tag{1.98}$$

The lowest order corresponds to the operators appearing in the expansion in Eq. 1.94. From a similar collinear expansion of the result of the diagrams in Fig. 1.7 one can find the momentum space coefficients $C(q^2, g)$ at $\mathcal{O}(g^2)$ for QCD including interactions.

The important features of the OPE are:

- The OPE yields an expansion in which the target dependence is in the matrix elements. The coefficient functions are independent of the target (factorization!). Their singularity structure plays a crucial role.
- In the operator expectation values (Green's functions) the renormalization scale μ enters. Since the amplitude in Eq. 1.95 cannot depend on a renormalization scale, this dependence must be precisely cancelled by the scale dependence in $\tilde{C}_n^{(\beta)}(q^2; g, \mu)$. The latter scale dependence can be calculated using perturbative QCD.

The systematics of the expansion can be further investigated by looking at a dimensional analysis of the various quantities. For instance the canonical dimension of the coefficients is

$$d\left[C_n^{(\beta)}(x^2)\right] = 6 - d\left[O_n^{(\beta)}\right] + n, \tag{1.99}$$

$$d\left[\tilde{C}_n^{(\beta)}(q^2)\right] = 2 - d\left[O_n^{(\beta)}\right] + n \equiv 2 - t, \tag{1.100}$$

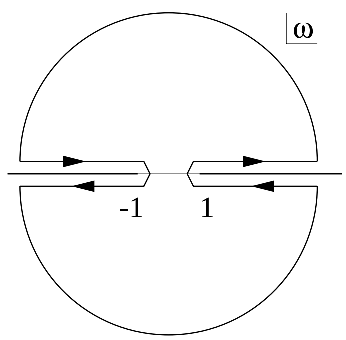
where

$$t = d\left[O_n^{(\beta)}\right] - n \tag{1.101}$$

is referred to as the *twist* of the operator O. The dimension of $\tilde{C}(q^2; g, \mu^2)$ must be provided by q^2, allowing for logarithmic behavior arising from the anomalous dimensions of the operator (as we will see below),

$$\tilde{C}_n^{(\beta)}(q^2; g, \mu^2) \equiv \frac{c_n^{(\beta)}\left(\log(q^2/\mu^2)\right)}{(q^2)^{t-2}}. \tag{1.102}$$

In order to make contact with the deep inelastic structure functions one needs to invert Eq. 1.95 to obtain the coefficients. For this we use the analytic properties of the forward Compton amplitude discussed for the free Compton amplitude in section 1.4. Using the quantity $\omega \equiv 1/x$ one has the properties $T_1(\omega) = T_1(-\omega)$ and $T_1^*(\omega^*) = T_1(\omega)$ (a real-analytic function). Considering the following contour in the complex ω plane with branch cuts in the physical regions (where Im $T_1 \neq 0$)



one obtains

$$\begin{aligned}
T_1(\omega) &= \frac{1}{2\pi i}\left[\int_{-\infty}^{-1} d\omega' \, \frac{T(\omega' + i\epsilon)}{\omega' - \omega} + \int_{-1}^{-\infty} d\omega' \, \frac{T(\omega' - i\epsilon)}{\omega' - \omega}\right. \\
&\quad \left. + \int_{\infty}^{1} d\omega' \, \frac{T(\omega' - i\epsilon)}{\omega' - \omega} + \int_{1}^{\infty} d\omega' \, \frac{T(\omega' + i\epsilon)}{\omega' - \omega}\right] \\
&= \frac{1}{\pi}\int_1^\infty d\omega' \left[\frac{\text{Im } T(\omega' + i\epsilon)}{\omega' + \omega} + \frac{\text{Im } T(\omega' + i\epsilon)}{\omega' - \omega}\right] \\
&= \frac{2}{\pi}\int_1^\infty d\omega' \, \frac{\omega' \, \text{Im } T(\omega')}{\omega'^2 - \omega^2}.
\end{aligned} \tag{1.103}$$

In case of bad convergence one can use a subtracted relation,

$$T_1(\omega) - T_1(0) = \frac{2\omega^2}{\pi} \int_1^\infty d\omega' \, \frac{\mathrm{Im}\, T(\omega')}{\omega'\,(\omega'^2 - \omega^2)}, \qquad (1.104)$$

etc. Expanding around zero one finds

$$T_1(\omega) = \frac{2}{\pi} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \int_1^\infty \frac{d\omega'}{\omega'} \left(\frac{\omega}{\omega'}\right)^n \mathrm{Im}\; T_1(\omega'). \qquad (1.105)$$

Note that this expansion around $\omega = 1/x = 0$ is in fact a short distance expansion since in deep inelastic processes, the relevant contributions in the current-current matrix elements came from $|\boldsymbol{x}| \approx t \approx 1/Mx$. Recall that the structure functions $W_i(q^2,\nu)$ or $F_i(x,q^2)$ are found as the imaginary part of the forward Compton amplitude, to be precise $(1/\pi)\,\mathrm{Im}\; T(q^2,\nu)$. Thus

$$2M\,T(q^2,\nu) \propto 2i \sum_{n \text{ even}} M_n(q^2) \left(\frac{1}{x}\right)^n, \qquad (1.106)$$

where

$$M_n(q^2) = \frac{1}{\pi} \int_0^1 dx\, x^{n-1}\, \mathrm{Im}\; T(q^2,\nu). \qquad (1.107)$$

Thus one sees that the moments of the structure functions $F(x,Q^2) = \mathrm{Im}\, T(x,Q^2)/\pi$ are given by

$$M_n(Q^2) = \int_0^1 dx\, x^{n-1}\, F(x,Q^2) \;\propto\; \frac{c_n^{(\beta)}\,(\log(q^2/\mu^2))\;\; \Theta_n^{(\beta)}(\mu^2)}{(q^2)^{t-2}}. \qquad (1.108)$$

The dominant contributions in the moments and thus also the structure functions come from twist two operators. They come in two classes, quark operators

$$O^F_{\mu_1\cdots\mu_n} = i^{n-1}\mathcal{S}\,\overline{\psi}\,\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_n}\,\psi + \ldots, \qquad (1.109)$$

where $\mathcal{S}$ indicates symmetrization in the indices, D_μ are covariant derivatives; furthermore, there are also gluon operators

$$O^G_{\mu_1\cdots\mu_n} = (2\,i)^{n-2}\mathcal{S}\,F_{\mu_1\lambda} D_{\mu_2} \ldots D_{\mu_{n-1}}\,F_{\mu_n}{}^{\lambda} + \ldots\;. \qquad (1.110)$$

The quark operators can of course be supplemented with flavor operators. Note that the sum rules derived for the parton distributions written as correlation functions and the expansion of the field operators in the correlation functions are precisely the (free) quark operators from the above set.

Twist 4 operators contributing to the unpolarized structure functions are for instance of the form $\mathcal{S}\,\overline{\psi}\gamma_{\mu_1}\psi\,\overline{\psi}\gamma_{\mu_2}\psi$ or $\overline{\psi}[D^\rho, F_{\rho\mu_1}]\gamma_{\mu_2}\psi$. They specifically contribute to the lowest moments of the contributions in deep inelastic scattering arising from diagrams in which more than two partons emerge from the soft scattering part.

1.9 QCD corrections in deep inelastic scattering

The actual scale dependence of the coefficient functions for various operators can be straightforwardly been calculated in QCD. It is for a particular operator governed by the anomalous dimension function $\gamma_i(g)$, independent of the particular process where the operator shows up. One obtains

$$c_i(Q^2; g, \mu) = c_i(\mu^2; g(\tau), \mu)\ \exp\left(-\int_0^\tau d\tau' \gamma_i(g(\tau'))\right) \tag{1.111}$$

$$= c_i(\mu^2; g(\tau), \mu)\ \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{\gamma_{0i}/2b_0}, \tag{1.112}$$

$$= c_i(1; g(\tau))\ \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{d_i}, \tag{1.113}$$

where $\tau = \log(Q^2/\mu^2)$ and γ_{0i} is the coefficient in the leading term in $\gamma_i(g) \approx (\gamma_{0i}/4\pi)\alpha_s$, while b_0 multiplies the leading term in the β function§, $b_0 = (33 - 2f)/3$, and $d_i \equiv \gamma_{0i}/2b_0$.

The next to leading order (NLO) is a matter of calculating the γ function for the appropriate operators and calculating the coefficient functions,

$$\gamma_i(g) = \gamma_{0i}\left(\frac{\alpha_s}{4\pi}\right) + \gamma_{1i}\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots, \tag{1.114}$$

$$c_i(1; g) = 1 + B_i\left(\frac{\alpha_s}{4\pi}\right) + \ldots\ . \tag{1.115}$$

§ The β-function describes the running coupling constant and can be expanded in α_s. The leading order result, $\partial\alpha_s(\tau)/\partial\tau = -(b_0/4\pi)\,\alpha_s^2$ gives

$$\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(\mu^2)} + \frac{b_0}{4\pi}\,\tau,$$

implying that for $Q^2 \to \infty$ one has

$$\alpha_s(Q^2) = \frac{4\pi}{b_0\ \log(Q^2/\Lambda^2)},$$

with the scale invariant definition $-(b_0/4\pi)\ \log\Lambda^2 \equiv \alpha_s^{-1}(\mu^2) - (b_0/4\pi)\ \log\mu^2$.

Many of these coefficients are now know to this or even higher orders, which has been an enormous task.

For sum rules connected themselves to a conserved current the anomalous dimension $\gamma_i(g) = 0$ and the results for the coefficients B_i immediately give the NLO corrections to sum rules. The Gottfried sum rule gets a very small correction. The polarized Bjorken sum rule gets a correction $(1 - \alpha_s/\pi)$, or explicitly (including also the next order [16]),

$$\int_0^1 dx\, \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] = \frac{G_A}{6}\left(1 - \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2}\,(-4.5833 + f/3)\right), \tag{1.116}$$

where f is the number of (active) flavors. This gives an excellent explanation of the experimental result $\Gamma_1^p - \Gamma_1^n \approx 0.19$ being somewhat smaller than $G_A/6 = 0.21$.

Exercise. As we have seen, the g_1^p sum rule by itself involves the singlet combination $\Delta\Sigma$ connected to the singlet axial current, i.e. the combination which is not conserved because of the anomaly. Thus, we need to include the scale dependence for which we need the anomalous dimension that is given by

$$\gamma_1^\Sigma(\alpha_s) = \gamma_1^\Sigma\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots = 16f\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots,$$

Derive the leading order solution

$$\Delta\Sigma(Q^2) = \exp\left(\frac{\gamma_1^\Sigma}{8\pi\, b_0}(\alpha_s(Q^2) - \alpha_s(Q_0^2))\right)\,\Delta\Sigma(Q_0^2). \tag{1.117}$$

Together with the coefficient B^Σ this gives the corrections for the singlet contribution in the spin sum rules

$$\Gamma_1^S(Q^2) = \left(1 - \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2}\,(-4.5833 + 1.16248f)\right)\,\Delta\Sigma(Q^2). \tag{1.118}$$

Including this the deviation of the proton sumrule from the naive result can be pretty well understood.

1.10 Evolution equations

It is possible to extend the intuitive folding picture that we have used to derive the parton model to obtain the QCD correction in a different approach, which provides also a practical way to calculate the coefficient functions $c_i(g)$ and gamma functions $\gamma_i(g)$.

Extending the result for the transverse structure function written as a delta function contribution,

$$\sigma_T = \sum_q \int_0^1 dz\, q(z) \left[e_q^2 \frac{4\pi^2\alpha}{Q^2}\, x\,\delta(x-z) + \delta\hat{\sigma}_q(z,Q^2) + \ldots \right], \tag{1.119}$$

one obtains contributions from the process $\gamma^* q \to Gq$, which is no longer proportional to $\delta(x-z)$ because of the fact that there are two particles in the final state. The amplitude is

$$|\mathcal{M}|^2 = 32\pi^2\, e_q^2\, \alpha\alpha_s\, \frac{4}{3} \left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}} \right) \tag{1.120}$$

and contributes

$$\delta\sigma(\hat{s}) = e_q^2\, \frac{2\pi\alpha\alpha_s}{(\hat{s}+Q^2)^2}\, \frac{4}{3} \left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} - \frac{2\hat{t}Q^2 + 2\hat{s}Q^2 + 2Q^4}{\hat{s}\hat{t}} \right) d\hat{t} \tag{1.121}$$

We now express the momenta (using lightcone components) for $\gamma^*(q)$ + quark (k) $\to$ gluon (p_G) + quark (p_q) as

$$q \equiv \Big[\frac{Q^2}{xA\sqrt{2}}, -\frac{xA}{\sqrt{2}}, \mathbf{0}_\perp \Big] \qquad \text{(photon)}, \tag{1.122}$$

$$k \equiv \Big[\frac{m^2}{zA\sqrt{2}}, \frac{zA}{\sqrt{2}}, \mathbf{0}_\perp \Big] \approx \Big[0, \frac{zA}{\sqrt{2}}, \mathbf{0}_\perp \Big] \qquad \text{(parton)}, \tag{1.123}$$

$$p_G = \Big[\zeta q^-, \frac{\boldsymbol{p}_\perp^2}{2\zeta q^-}, \boldsymbol{p}_\perp \Big] \approx \Big[\frac{\zeta Q^2}{xA\sqrt{2}}, 0, \boldsymbol{p}_\perp \Big], \tag{1.124}$$

$$p_q = \Big[(1-\zeta)q^-, \frac{\boldsymbol{p}_\perp^2}{2(1-\zeta)q^-}, -\boldsymbol{p}_\perp \Big] \approx \Big[\frac{(1-\zeta)Q^2}{xA\sqrt{2}}, 0, -\boldsymbol{p}_\perp \Big]. \tag{1.125}$$

Note that z can be written as x/ξ, where $\xi \equiv -q^+/k^+ = x/z$ is the Bjorken scaling variable for the subprocess. We will neglect all particles masses, in which case $p_q^2 = 0$ gives

$$\boldsymbol{p}_\perp^2 = \frac{\zeta(1-\zeta)(1-\xi)}{\xi}\, Q^2. \tag{1.126}$$

The invariants for the subprocess become

$$\hat{s} = (k+q)^2 = \frac{1-\xi}{\xi}\, Q^2 = \frac{1}{\zeta(1-\zeta)}\, \boldsymbol{p}_\perp^2, \tag{1.127}$$

$$\hat{t} = (p_G - q)^2 = -\frac{1-\zeta}{\xi}\, Q^2 = -\frac{1}{\zeta(1-\xi)}\, \boldsymbol{p}_\perp^2, \tag{1.128}$$

The kinematic range of the process $\hat{s} \geq 0$, $-(\hat{s}+Q^2) \leq t \leq 0$ in principle restricts the ranges of (ξ, ζ) to $0 \leq \xi \leq 1$ and $0 \leq \zeta \leq 1$. The cross section becomes

$$\delta\hat{\sigma}_q(\xi, Q^2) = e_q^2 \, \frac{2\pi\alpha\alpha_s}{Q^2} \, \frac{4}{3} \, \xi \left[\frac{1-\zeta-2\,\xi}{1-\xi} + \frac{1+\xi^2}{(1-\xi)(1-\zeta)} \right] d\zeta. \qquad (1.129)$$

Integrating for inclusive scattering over the final state, i.e. integrating over ζ one sees that there are singular points, specifically for $\zeta = 1$, which corresponds to a gluon radiated with $\boldsymbol{p}_\perp = 0$. These divergences are therefore referred to as *collinear divergences*. There are again several ways of regularizing, either by giving quarks and gluons masses or by dimensional regularization. In this case a $\boldsymbol{p}_\perp$ cutoff also provides a regularization. The restriction $\boldsymbol{p}_\perp^2 \geq \mu^2$ modifies the allowed region in the (ξ, ζ) plane. For a given ξ (not to close to unity) the integration is limited to

$$\frac{\xi}{1-\xi} \, \frac{\mu^2}{Q^2} \leq \zeta \leq 1 - \frac{\xi}{1-\xi} \, \frac{\mu^2}{Q^2}. \qquad (1.130)$$

With this regularization the result becomes

$$\delta\hat{\sigma}_q(\xi, Q^2) = e_q^2 \, \frac{2\pi\alpha\alpha_s}{Q^2} \, \xi \left[B(\xi) + P_{qq}(\xi) \, \log\left(\frac{Q^2}{\mu^2}\right) \right]. \qquad (1.131)$$

where

$$P_{qq}(\xi) = \frac{4}{3} \, \frac{1+\xi^2}{(1-\xi)}, \qquad (1.132)$$

is the *splitting functions* coming from the collinear $1/(1-\zeta)$ singularity, appearing proportional to $\alpha_s \log Q^2$ and $B(\xi)$ is the part appearing proportional to α_s. Combining the results of the (leading) contribution (omitting for now the term $B(\xi)$) one has

$$\sigma_T = \frac{4\pi^2\alpha}{Q^2} \, x \, \sum_q e_q^2 \int_x^1 \frac{dz}{z} \, q(z) \left[\delta\left(1-\frac{x}{z}\right) + \frac{\alpha_s}{2\pi} \, P_{qq}\left(\frac{x}{z}\right) \, \log\left(\frac{Q^2}{\mu^2}\right) \right]. \qquad (1.133)$$

This can be rewritten in parton form

$$\sigma_T = \frac{4\pi^2\alpha}{Q^2} \, x \, \sum_q e_q^2 \, q(x, \log Q^2), \qquad (1.134)$$

where the functions $q(x, \log Q^2)$ satisfy

$$\frac{\partial q(x, \log Q^2)}{\partial \log Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \, q(z) \, P_{qq}\left(\frac{x}{z}\right). \qquad (1.135)$$

As in the case of integrating over ζ above one still encounters divergences, in this case for $\xi = x/z \to 1$. This also gives rise to $\log(Q^2/\mu^2)$ corrections. They are considered separately by considering the $1/(1-\xi)$ appearing in the splitting functions as functionals,

$$\begin{aligned}\int_0^{1-\delta} dx\,\frac{f(x)}{1-x} &= -\int f(x)\,d\log(1-x)\\ &= \int_0^{1-\delta} dx\,\frac{f(x)-f(1)}{1-x} \;-\; f(1)\,\log\delta\\ &= \int_0^{1} dx\,\frac{f(x)-f(1)}{1-x} \;-\; \log\delta\int dx\,f(x)\,\delta(1-x)\\ &\equiv \int_0^{1} dx\,\frac{f(x)}{(1-x)_+} \;-\; \log\delta\int dx\,f(x)\,\delta(1-x)\end{aligned}$$

or

$$\frac{1}{1-x} = \frac{1}{(1-x)_+} - \delta(1-x)\,\log\delta. \tag{1.136}$$

Including these and other singular contributions (vertex corrections to the process $\gamma^* + q \to q$) one then has

$$q(x,Q^2) = \int_x^1 \frac{dz}{z}\,q(z)\Bigg\{\delta\left(1-\frac{x}{z}\right) \tag{1.137}$$

$$+\frac{\alpha_s}{2\pi}\log\left(\frac{Q^2}{\mu^2}\right)\left[a\,\delta\left(1-\frac{x}{z}\right)+P_{qq}\left(\frac{x}{z}\right)\right]\Bigg\}, \tag{1.138}$$

The piece between curly brackets can be seen as the probability density $\mathcal{P}_{qq}$ of finding a quark inside a quark with fraction $\xi = x/z$ of the parent quark to first order in α_s. Instead of calculating all contributions and checking all cancellations, it is easier to see what the final result must be from

$$\int d\xi\,\mathcal{P}_{qq}(\xi) = 1. \tag{1.139}$$

Keeping the form in Eq. 1.133 one can write the full piece between square brackets in Eq. 1.138 as

$$P_{qq}(\xi) = \frac{4}{3}\,\frac{1+\xi^2}{(1-\xi)_+} + 2\,\delta(1-\xi). \tag{1.140}$$

While this splitting function describes how QCD corrections arising from $q \to qG$ splitting are incorporated into the parton distributions (fig. 1.8a), one needs in addition other splitting functions such as P_{qG} describing how quark and gluon distribution functions mix (fig. 1.8b). They are calculated

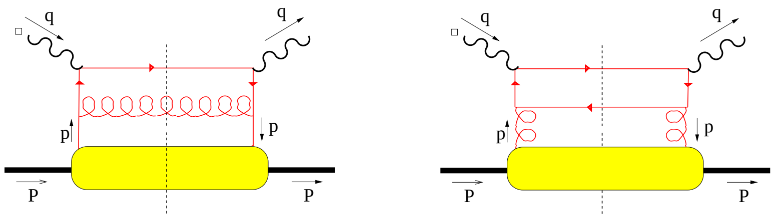


Figure 1.8. Ladder diagrams used to calculate the asymptotic behavior of the correlation functions.

from the proces $\gamma^* G \to q\overline{q}$. Since gluons are flavor-blind, nonsinglet and valence distribution functions are not affected by such corrections.

The splitting function for the polarized distribution functions is given by

$$\Delta P_{qq}(x) = P_{qq}(x) = \frac{4}{3}\,\frac{1+\xi^2}{(1-\xi)_+} + 2\,\delta(1-\xi), \tag{1.141}$$

Solutions of the evolution equations

In solving the evolution equations the moments play an important role, while they also establish the connection with the OPE. Rewriting the evolution equation (Eq. 1.135) as

$$\frac{dq(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_0^1 dz \int_0^1 dy\delta(x-yz)\,P_{qq}(y)\,q(z,\tau), \tag{1.142}$$

and using the moments

$$M_n(\tau) \equiv \int_0^1 dx\, x^{n-1}\, q(x,\tau), \tag{1.143}$$

$$A_n \equiv \int_0^1 dy\, y^{n-1}\, P_{qq}(y,\tau) \;=\; \frac{4}{3}\left[-\frac{1}{2} + \frac{1}{n(n+1)} - 2\sum_{j=2}^{n}\frac{1}{j}\right] \tag{1.144}$$

one has

$$\frac{dM_n(\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\,A_n\,M_n(\tau). \tag{1.145}$$

Using the leading order QCD result for $\alpha_s(\tau)$ (which reads $d\alpha_x/d\tau = -(b_0/4\pi)\,\alpha_s$) this is easily solved, giving the result

$$\frac{M_n(Q^2)}{M_n(\mu^2)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{d_n}, \tag{1.146}$$

where $d_n = -2\,A_n/b_0$. Comparison with Eqs 1.112 and 1.113 show that the moments of the splitting functions are up to a factor equal to the coefficients γ_0, namely $A_n = -\gamma_{0n}/4$. As an example, consider the second moment of the quark distributions, for which $A_2 = -16/9$. The result for d_2 is $d_2 = 32/9\,b_0 = 32/81$ for three flavors. The fraction of momentum carried by valence quarks thus vanishes for $Q^2 \to \infty$ as

$$M_2(Q^2) = \int dx\, x[q(x, \log Q^2) - \overline{q}(x, \log Q^2)] \;\propto\; (\alpha_s(Q^2))^{d_2}. \qquad (1.147)$$

For sea quarks and gluons mixing occurs and one finds that the combination corresponding to the total momentum involving the second moment of the singlet quark distribution and the second moment of the gluon distributions does not vanish, but is Q^2-independent.

In principle the evolution of the structure functions can be done using the evolution equations. Using the moments is actually quite convenient, although one needs all of them. A very useful method in practice is to parametrize the distributions at one Q^2 with a function for which the moments can be easily calculated, evolve the moments and apply an inverse Mellin transform. One has

$$q(x, Q^2) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, M_n(Q^2), \qquad (1.148)$$

where c must be such that M_n has no singularities in the complex n-plane right of the line $\mathcal{R}e\, n = c$. A similar relation exists for the moments of the splitting functions, $A_n = -\gamma_{0n}/4$,

$$P_{qq}(x) = \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, \gamma_{0n}, \qquad (1.149)$$

We have discussed the extensions to higher orders in Eqs 1.114 and 1.115 involving γ_1 and B. The γ_1 coefficients are precisely moments of the next order in α_s in the splitting functions. The coefficients B correspond to the moments of the additional pieces appearing in the expressions for the structure functions, such as the $B(x)$ appearing in the calculation of F_1 in Eq. 1.131.

While the splitting functions are universal (process independent), the contributions $B(x)$ are in general process dependent. An example of a contribution of this type is the longitudinal structure function (for which the dominant parton model result is zero). Calculating both the $\alpha_s\, B(x)$ contributions for F_1 and F_2 in electroproduction gives for instance the first nonvanishing contribution in the longitudinal structure function,

$$F_L(x, Q^2) = \frac{\alpha_s(Q^2)}{\pi}\left[\frac{4}{3}\int_x^1 \frac{dz}{z}\,\left(\frac{x}{z}\right)^2 F_2(z, Q^2)\right.$$

$$+\left(2\sum_q e_q^2\right)\int_x^1 \frac{dz}{z}\,\left(\frac{x}{z}\right)^2\left(1-\frac{x}{z}\right)\,z\,G(z,Q^2)\Bigg].\tag{1.150}$$

1.11 Quark correlation functions in 1PI leptoproduction

We now consider the case in which one particle is detected in coincidence with the scattered lepton, *one-particle-inclusive* or 1PI leptoproduction. The kinematics of this process is already in the picture given before (Fig. 1.2). With a target hadron (momentum P) and a detected hadron h in the final state (momentum P_h) one has a situtation in which two hadrons are involved and the operator product expansion cannot be used. Within the framework of QCD and knowing that the photon or Z^0 current couples to the quarks, it is possible to write down a diagrammatic expansion for leptoproduction, with in the deep inelastic limit ($Q^2 \to \infty$) as relevant diagrams only the ones given in Fig. 1.9 for 1-particle inclusive scattering.

In analogy with the case of inclusive scattering, we also in 1-particle inclusive scattering parametrize the momenta with the help of two lightlike vectors, which are choosen now along the hadron momenta,

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ P_h^2 = M_h^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \\ 2\,P_h\cdot q = -z_h\,Q^2 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} P_h = \frac{z_h\,Q}{\sqrt{2}}\,n_- + \frac{M_h^2}{z_h\,Q\sqrt{2}}\,n_+ \\ \\ q = \frac{Q}{\sqrt{2}}\,n_- - \frac{Q}{\sqrt{2}}\,n_+ + q_T \\ \\ P = \frac{x_B M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right.$$

An additional invariants z_h comes in. Note that the expansion is appropriate for the socalled current fragmentation, in which case the produced hadron is *hard* with respect to the target momentum, i.e. $P\cdot P_h \sim Q^2$. The minus component p^- is irrelevant in the lower soft part, while the plus component k^+ is irrelevant in the upper soft part. Note that after the choice of P and P_h one can no longer omit a transverse component in the other vector, in the consideration above put in the momentum transfer q. One sees that one has (up to mass effects) the relation

$$q_T^\mu = q^\mu + x_B\,P^\mu - \frac{P_h^\mu}{z_h} \equiv -Q_T\,\hat{h}^\mu.\tag{1.151}$$

This relation allows the experimental determination of the 'transverse momentum' effect from the external vectors q, P and P_h which are in general not collinear. The vector $\hat{h}$ defines the orientation of the hadronic plane in Fig. 1.2.

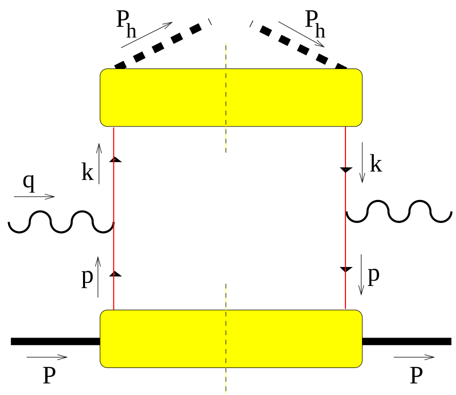


Figure 1.9. *The simplest (parton-level) diagrams for semi-inclusive scattering, of which we consider one-particle inclusive leptoproduction. Note that also the diagram with opposite fermion flow has to be added.*

An important consequence in the theoretical approach (Fig. 1.9) is that one can no longer simply integrate over the transverse components of the quark momenta.

Structure functions and cross sections

For an unpolarized (or spin 0) hadron in the final state the symmetric part of the tensor is given by

$$
\begin{aligned}
M\mathcal{W}_S^{\mu\nu}(q,P,P_h) = & -g_\perp^{\mu\nu}\,\mathcal{H}_T + \hat{t}^\mu \hat{t}^\nu\,\mathcal{H}_L \\
& + \hat{t}^{\{\mu}\hat{h}^{\nu\}}\,\mathcal{H}_{LT} + \left(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\right)\,\mathcal{H}_{TT}. \qquad (1.152)
\end{aligned}
$$

Noteworthy is that also an antisymmetric term in the tensor is allowed,

$$
M\mathcal{W}_A^{\mu\nu}(q,P,P_h) = -i\hat{t}^{[\mu}\hat{h}^{\nu]}\,\mathcal{H}'_{LT}. \qquad (1.153)
$$

Clearly the lepton tensor in Eq. 1.17 or 1.36 is able to distinguish all the structures in the semi-inclusive hadron tensor.

The symmetric part gives the cross section for unpolarized leptons,

$$
\begin{aligned}
\frac{d\sigma_{OO}}{dx_B dy\, dz_h d^2 q_T} = \frac{4\pi\,\alpha^2\, s}{Q^4}\, x_B z_h \Bigg\{ & \left(1 - y + \frac{1}{2}y^2\right)\mathcal{H}_T + (1-y)\,\mathcal{H}_L \\
& - (2-y)\sqrt{1-y}\,\cos\phi_h^\ell\,\mathcal{H}_{LT} \\
& + (1-y)\,\cos 2\phi_h^\ell\,\mathcal{H}_{TT}\Bigg\} \qquad (1.154)
\end{aligned}
$$

while the antisymmetric part gives the cross section for a polarized lepton (note the target is not polarized!)

$$\frac{d\sigma_{LO}}{dx_B dy\, dz_h d^2 q_T} = \lambda_e \frac{4\pi\,\alpha^2}{Q^2}\, z_h \sqrt{1-y}\, \sin\phi_h^\ell\ \mathcal{H}'_{LT}. \tag{1.155}$$

Of course many more structure functions appear for polarized targets or if one considers polarimetry in the final state. In this case the (theoretically) most convenient way to describe the spin vector of the target is via an expansion of the form

$$S^\mu = -S_L \frac{M x_B}{Q\sqrt{2}}\, n_- + S_L \frac{Q}{M x_B \sqrt{2}}\, n_+ + S_T. \tag{1.156}$$

One has up to $\mathcal{O}(1/Q^2)$ corrections $S_L \approx M\,(S\cdot q)/(P\cdot q)$ and $S_T \approx S_\perp$, where the subscript $\perp$ still refers to perpendicular to q and P. For a pure state one has $S_L^2 + \boldsymbol{S}_T^2 = 1$, in general this quantity being less or equal than one.

The parton model approach

The expression for $\mathcal{W}_{\mu\nu}$ can be rewritten as a nonlocal product of currents and it is a straightforward exercise to show by inserting the currents $j_\mu(x) =: \overline{\psi}(x)\gamma_\mu\psi(x):$ that for 1-particle inclusive scattering one obtains in tree approximation

$$\begin{aligned}
&2M\mathcal{W}_{\mu\nu}(q;PS;P_hS_h)\\
&\quad = \frac{1}{(2\pi)^4}\int d^4x\ e^{iq\cdot x}\,\langle PS|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x):\sum_X |X;P_hS_h\rangle\\
&\qquad\qquad\qquad \times\langle X;P_hS_h|:\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|PS\rangle\\
&\quad = \frac{1}{(2\pi)^4}\int d^4x\ e^{iq\cdot x}\,\langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle(\gamma_\mu)_{jk}\\
&\qquad\qquad \langle 0|\psi_k(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li}\\
&\qquad + \frac{1}{(2\pi)^4}\int d^4x\ e^{iq\cdot x}\,\langle PS|\psi_k(x)\overline{\psi}_l(0)|PS\rangle(\gamma_\nu)_{li}\\
&\qquad\qquad \langle 0|\overline{\psi}_j(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\psi_i(0)|0\rangle(\gamma_\mu)_{jk},\\
&\quad = \int d^4p\, d^4k\, \delta^4(p+q-k)\ \mathrm{Tr}\left(\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right) + \left\{\begin{array}{c} q\leftrightarrow -q\\ \mu\leftrightarrow\nu\end{array}\right\},
\end{aligned} \tag{1.157}$$

where

$$\Phi_{ij}(p) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ip\cdot\xi} \, \langle PS|\overline{\psi}_j(0)\psi_i(\xi)|PS\rangle,$$
$$\Delta_{kl}(k) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{ik\cdot\xi} \, \langle 0|\psi_k(\xi) \sum_X |X; P_h S_h\rangle\langle X; P_h S_h|\overline{\psi}_l(0)|0\rangle.$$

Note that in Φ (quark production) a summation over colors is assumed, while in Δ (quark decay) an averaging over colors is assumed. The quantities Φ and Δ correspond to the blobs in Fig. 1.9 and parametrize the soft physics, leading to the definitions of distribution and fragmentation functions [17,18]. Soft refers to all invariants of momenta being small as compared to the hard scale, i.e. for $\Phi(p)$ one has $p^2 \sim p \cdot P \sim P^2 = M^2 \ll Q^2$.

In general many more diagrams have to be considered in evaluating the hadron tensors, but in the deep inelastic limit they can be neglected or considered as corrections to the soft blobs. We return to this later. One situation we want to mention here are target fragmentation parts involving matrix elements of the form $\langle PS|\psi_j(x) \sum_X |X; P_h S_h\rangle \; \langle X; P_h S_h|\psi_i(0)|PS\rangle$, known as *fracture functions*. They are relevant in the situation where $P \cdot P_h \sim M^2$ (target fragmentation region), which can be distinguished experimentally from the region we are interested in, $P \cdot P_h \sim Q^2$ (current fragmentation region).

1.12 Collinear parton distributions

The form of Φ is constrained by hermiticity, parity and time-reversal invariance. The quantity depends besides the quark momentum p on the target momentum P and the spin vector S and one must have

$$[\text{Hermiticity}] \quad \Rightarrow \quad \Phi^\dagger(p,P,S) = \gamma_0 \, \Phi(p,P,S) \, \gamma_0, \tag{1.158}$$
$$[\text{Parity}] \quad \Rightarrow \quad \Phi(p,P,S) = \gamma_0 \, \Phi(\bar{p},\bar{P},-\bar{S}) \, \gamma_0, \tag{1.159}$$
$$[\text{Time reversal}] \;\Rightarrow\; \Phi^*(p,P,S) = (-i\gamma_5 C) \, \Phi(\bar{p},\bar{P},\bar{S}) \, (-i\gamma_5 C), \tag{1.160}$$

where $C = i\gamma^2\gamma_0$, $-i\gamma_5 C = i\gamma^1\gamma^3$ and $\bar{p} = (p^0, -\boldsymbol{p})$.

To obtain the leading contribution in inclusive deep inelastic scattering one can integrate over the component p^- and the transverse momenta (see discussion in the section where the parton model has been derived). This integration restricts the nonlocality in $\Phi(p)$. The relevant soft part then is a particular Dirac trace of the quantity

$$\Phi_{ij}(x) = \int dp^- \, d^2p_T \;\, \Phi_{ij}(p,P,S)$$

$$= \int \frac{d\xi^-}{2\pi}\, e^{ip\cdot\xi}\, \langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \qquad (1.161)$$

depending on the lightcone fraction $x = p^+/P^+$. When one wants to calculate the leading order in $1/Q$ for a hard process, one looks for leading parts in M/P^+ because $P^+ \propto Q$. The leading contribution [19] turns out to be proportional to $(M/P^+)^0$,

$$\Phi(x) = \frac{1}{2}\left\{ f_1(x)\, \not{n}_+ + S_L\, g_1(x)\, \gamma_5\, \not{n}_+ + h_1(x)\, \frac{\gamma_5\, [\, \not{S}_\perp,\, \not{n}_+]}{2} \right\}. \qquad (1.162)$$

The precise expression of the functions $f_1(x)$, etc. as integrals over the amplitudes can be easily written down after tracing with the appropriate Dirac matrix,

$$f_1(x) = \int \frac{d\xi^-}{4\pi}\, e^{ip\cdot\xi}\, \langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \qquad (1.163)$$

$$S_L\, g_1(x) = \int \frac{d\xi^-}{4\pi}\, e^{ip\cdot\xi}\, \langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \qquad (1.164)$$

$$S_T^i\, h_1(x) = \int \frac{d\xi^-}{4\pi}\, e^{ip\cdot\xi}\, \langle P,S|\overline{\psi}(0)\, i\sigma^{i+}\gamma_5\, \psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \qquad (1.165)$$

Including flavor indices, the functions $f_1^q(x) = q(x)$ and $g_1^q(x) = \Delta q(x)$ are precisely the functions that we encountered before.

The third function in the above parametrization is known as *transversity* or *transverse spin distribution* [20]. Including flavor indices one also denotes $h_1^q(x) = \delta q(x)$. In the same way as we have seen for $f_1(x)$ and $g_1(x)$, the function h_1 can be interpreted as a density, but one needs instead of the projectors on quark chirality states, $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$, those on quark transverse spin states, $P_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^i\gamma_5)$. One has

$$f_1(x) = f_{1R}(x) + f_{1L}(x) = f_{1\uparrow}(x) + f_{1\downarrow}(x), \qquad (1.166)$$

$$g_1(x) = f_{1R}(x) - f_{1L}(x), \qquad (1.167)$$

$$h_1(x) = f_{1\uparrow}(x) - f_{1\downarrow}(x). \qquad (1.168)$$

This results in some trivial bounds such as $f_1(x) \geq 0$ and $|g_1(x)| \leq f_1(x)$. We already did discuss the support and charge conjugation properties of $f_1(x)$. The analysis for all these functions shows that the support is in all cases $-1 \leq x \leq 1$, while the charge conjugation properties of the functions are $\overline{f}(x) = -f(-x)$ (C-even) for f_1 and h_1 and $\overline{f}(x) = +f(-x)$ (C-odd) for g_1.

Exercise. Show that the Dirac structure for h_1 in terms of chirality states is $\overline{\psi}_R\psi_L$ and $\overline{\psi}_L\psi_R$. Such functions are called *chiral-odd*. Explain why chiral-odd functions cannot be measured in inclusive deep inelastic scattering.

While the evolution equations for $q(x)$ and $\Delta q(x)$ require quark-quark and quark-gluon splitting functions, the evolution for $\delta q(x)$ does not involve mixing with gluon distributions because the chiral-odd nature of $\delta q(x)$. The splitting function is given by

$$\delta P_{qq}(\xi) = \frac{4}{3}\,\frac{2\,\xi}{(1-\xi)_+} + 2\,\delta(1-\xi). \tag{1.169}$$

1.13 Bounds on the distribution functions

The trivial bounds on the distribution functions ($|h_1(x)| \le f_1(x)$ and $|g_1(x)| \le f_1(x)$) can be sharpened. For instance one can look explicitly at the structure in Dirac space of the correlation function Φ_{ij}. Actually, we will look at the correlation functions $(\Phi\,\gamma_0)_{ij}$, which involves at leading order matrix elements $\psi^\dagger_{+j}(0)\psi_{+i}(\xi)$. One has in Weyl representation ($\gamma^0 = \rho^1$, $\gamma^i = -i\rho^2\sigma^i$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \rho^3$) the matrices

$$P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_+\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad P_+\gamma^1\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The good projector only leaves two (independent) Dirac spinors, one righthanded (R), one lefthanded (L). On this basis of good R and L spinors the for hard scattering processes relevant matrix $(\Phi\not{n}_-)$ is given by

$$(\Phi\,\not{n}_-)_{ij}(x) = \begin{pmatrix} f_1 + S_L\,g_1 & (S_T^1 + i\,S_T^2)\,h_1 \\ & \\ (S_T^1 - i\,S_T^2)\,h_1 & f_1 - S_L\,g_1 \end{pmatrix} \tag{1.170}$$

One can also turn the S-dependent correlation function Φ into a matrix in the nucleon spin space via the standard spin 1/2 density matrix $\rho(P,S)$. The relation is $\Phi(x;P,S) = \mathrm{Tr}\,[\Phi(x;P)\,\rho(P,S)]$. Writing

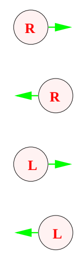
$$\Phi(x;P,S) = \Phi_O + S_L\,\Phi_L + S_T^1\,\Phi_T^1 + S_T^2\,\Phi_T^2, \tag{1.171}$$

one has on the basis of spin 1/2 target states with $S_L = +1$ and $S_L = -1$ respectively

$$\Phi_{ss'}(x) = \begin{pmatrix} \Phi_O + \Phi_L & \Phi_T^1 - i\,\Phi_T^2 \\ \Phi_T^1 + i\,\Phi_T^2 & \Phi_O - \Phi_L \end{pmatrix} \tag{1.172}$$

Exercise. Show by generalizing $\Phi(p)$ to a matrix elements between states $\langle P, s|$ and $|P, s'\rangle$ that for the matrix $M = (\Phi \not n_-)^T$ (transposed in Dirac space) one has $v^\dagger M v \geq 0$ for any direction v in Dirac space.
mbox
On the basis $+R$, $-R$, $+L$ and $-L$ the matrix in quark $\otimes$ nucleon spin-space becomes

$$(\Phi(x)\,\not n_-)^T = \begin{pmatrix} f_1 + g_1 & 0 & 0 & 2\,h_1 \\ 0 & f_1 - g_1 & 0 & 0 \\ 0 & 0 & f_1 - g_1 & 0 \\ 2\,h_1 & 0 & 0 & f_1 + g_1 \end{pmatrix} \tag{1.173}$$



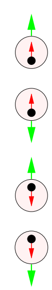
Of this matrix any diagonal matrix element must always be positive, hence the eigenvalues must be positive, which gives a bound on the distribution functions stronger than the trivial bounds, namely

$$|h_1(x)| \leq \frac{1}{2}\,(f_1(x) + g_1(x)) \tag{1.174}$$

known as the Soffer bound [21].

Exercise. Show that a change to the transverse quark spin basis gives the quark production matrix

$$(\Phi(x)\,\not n_-)^T = \begin{pmatrix} f_1 + h_1 & 0 & 0 & g_1 + h_1 \\ 0 & f_1 - h_1 & g_1 - h_1 & 0 \\ 0 & g_1 - h_1 & f_1 - h_1 & 0 \\ g_1 + h_1 & 0 & 0 & f_1 + h_1 \end{pmatrix} \tag{1.175}$$



1.14 Transverse momentum dependent correlation functions

Without integration over p_T, the soft part is

$$\Phi(x, \boldsymbol{p}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ip\cdot\xi}\, \langle P,S|\overline{\psi}(0)\psi(\xi)|P,S\rangle \Big|_{\xi^+=0}. \qquad (1.176)$$

For the leading order results one can write down parametrizations which for the parts involving unpolarized targets (O), longitudinally polarized targets (L) and transversely polarized targets (T) up to parts proportional to M/P^+ take the form [22,23]

$$\Phi_O(x, \boldsymbol{p}_T) = \frac{1}{2}\left\{ f_1(x, \boldsymbol{p}_T)\, \not{n}_+ + h_1^\perp(x, \boldsymbol{p}_T)\, \frac{i\,[\not{p}_T,\, \not{n}_+]}{2M} \right\} \qquad (1.177)$$

$$\Phi_L(x, \boldsymbol{p}_T) = \frac{1}{2}\left\{ S_L\, g_{1L}(x, \boldsymbol{p}_T)\, \gamma_5\, \not{n}_+ + S_L\, h_{1L}^\perp(x, \boldsymbol{p}_T) \frac{\gamma_5\,[\not{p}_T,\, \not{n}_+]}{2M} \right\} \qquad (1.178)$$

$$\begin{aligned}\Phi_T(x, \boldsymbol{p}_T) = \frac{1}{2}\Bigg\{ & f_{1T}^\perp(x, \boldsymbol{p}_T)\, \frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_+^\nu p_T^\rho S_T^\sigma}{M} \\ & + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\, g_{1T}(x, \boldsymbol{p}_T)\, \gamma_5\, \not{n}_+ + h_{1T}(x, \boldsymbol{p}_T)\, \frac{\gamma_5\,[\not{S}_T,\, \not{n}_+]}{2} \\ & + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\, h_{1T}^\perp(x, \boldsymbol{p}_T)\, \frac{\gamma_5\,[\not{p}_T,\, \not{n}_+]}{2M} \Bigg\}. \end{aligned} \qquad (1.179)$$

All functions appearing here have a natural interpretation as densities. This is seen as discussed before for the $\boldsymbol{p}_T$-integrated functions. Now it includes densities such as the density of longitudinally polarized quarks in a transversely polarized nucleon (g_{1T}) and the density of transversely polarized quarks in a longitudinally polarized nucleon ($h_{1L}^\perp$).

Upon integration over p_T not all functions survive. We are then left with Eq. 1.162 with $f_1(x) = \int d^2p_T\ f_1(x, p_T)$, $g_1(x) = \int d^2p_T\ g_{1L}(x, p_T)$ and $h_1(x) = \int d^2p_T \left[h_{1T}(x) + \frac{\boldsymbol{p}_T^2}{2M^2}\, h_{1T}^\perp(x, p_T) \right]$. The explicit treatment of transverse momenta also provides also a way to include the evolution of quark distribution and fragmentation functions. The assumption that soft parts vanish sufficiently fast as a function of the invariants $p\cdot P$ and p^2, which at constant x implies a sufficiently fast vanishing as a function of $\boldsymbol{p}_T^2$, simply turns out not to be true. Assuming that the result for $\boldsymbol{p}_T^2 \ge \mu^2$ is given by the diagram shown in Fig. 1.8 one finds that the extra distribution written in terms of p_T becomes

$$f_1(x, \boldsymbol{p}_T^2) \overset{\boldsymbol{p}_T^2 \ge \mu^2}{\Longrightarrow} \frac{1}{\pi\, \boldsymbol{p}_T^2}\, \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dy}{y}\, P_{qq}\left(\frac{x}{y}\right)\, f_1(y; \mu^2), \qquad (1.180)$$

which gives $f_1(x;\mu^2) \equiv \pi \int_0^{\mu^2} d\boldsymbol{p}_T^2\ f_1(x,\boldsymbol{p}_T^2)$ a logarithmic scale dependence. Actually we find that different functions survive when one integrates over $\boldsymbol{p}_T$ weighting with p_T^α, e.g.

$$\begin{aligned}\Phi_\partial^\alpha(x) &\equiv \int d^2p_T\, \frac{p_T^\alpha}{M}\, \Phi(x,\boldsymbol{p}_T)\\ &= \frac{1}{2}\left\{ -g_{1T}^{(1)}(x)\, S_T^\alpha\, \not{n}_+\gamma_5 - S_L\, h_{1L}^{\perp(1)}(x)\, \frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2}\right.\\ &\qquad \left. - f_{1T}^{\perp(1)}\, \epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_-^\nu S_T^\rho - h_1^{\perp(1)}\, \frac{i[\gamma^\alpha,\,\not{n}_+]}{2}\right\},\end{aligned} \tag{1.181}$$

involving *transverse moments* defined as

$$g_{1T}^{(1)}(x) = \int d^2p_T\, \frac{\boldsymbol{p}_T^2}{2M^2}\, g_{1T}(x,\boldsymbol{p}_T), \tag{1.182}$$

and similarly for the other functions. The functions $h_1^\perp$ and $f_{1T}^\perp$ are *T-odd.* As we will explain in the section on color gauge invariance they do not to vanish because time reversal invariance cannot be used for the transverse moments. Also for fragmentation functions they will not vanish. The T-odd functions correspond to unpolarized quarks in a transversely polarized nucleon ($f_{1T}^\perp$) or transversely polarized quarks in an unpolarized hadron ($h_1^\perp$). The easiest way to interpret the functions is by considering their place in the quark production matrix $(\Phi(x,p_T)\,\not{n}_-)^T$, which becomes [24]

$$\begin{pmatrix} f_1 + g_{1L} & \frac{|p_T|}{M}\, e^{i\phi}\, g_{1T} & \frac{|p_T|}{M}\, e^{-i\phi}\, h_{1L}^\perp & 2\, h_1 \\ \frac{|p_T|}{M}\, e^{-i\phi}\, g_{1T}^* & f_1 - g_{1L} & \frac{|p_T|^2}{M^2}\, e^{-2i\phi}\, h_{1T}^\perp & -\frac{|p_T|}{M}\, e^{-i\phi}\, h_{1L}^{\perp *} \\ \frac{|p_T|}{M}\, e^{i\phi}\, h_{1L}^{\perp *} & \frac{|p_T|^2}{M^2}\, e^{2i\phi}\, h_{1T}^\perp & f_1 - g_{1L} & -\frac{|p_T|}{M}\, e^{i\phi}\, g_{1T}^* \\ 2\, h_1 & -\frac{|p_T|}{M}\, e^{i\phi}\, h_{1L}^\perp & -\frac{|p_T|}{M}\, e^{-i\phi}\, g_{1T} & f_1 + g_{1L} \end{pmatrix}.$$

In this representation T-odd functions appear as imaginary parts, $f_{1T}^\perp = -\mathrm{Im}\, g_{1T}$ and $h_1^\perp = \mathrm{Im}\, h_{1L}^\perp$.

1.15 Fragmentation functions

Just as for the distribution functions one can perform an analysis of the soft part describing the quark fragmentation [18]. One needs

$$\Delta_{ij}(z,\boldsymbol{k}_T) = \sum_X \int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ik\cdot\xi}\, Tr\langle 0|\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)|0\rangle\Big|_{\xi^-=0}. \tag{1.183}$$

For the production of unpolarized (or spin 0) hadrons h in hard processes one needs to leading order in $1/Q$ the $(M_h/P_h^-)^0$ part of the correlation function,

$$\Delta_O(z,\boldsymbol{k}_T) = z\,D_1(z,\boldsymbol{k}'_T)\,\not{n}_- + z\,H_1^\perp(z,\boldsymbol{k}'_T)\,\frac{i\,[\not{k}_T,\not{n}_-]}{2M_h}. \tag{1.184}$$

The arguments of the fragmentation functions D_1 and $H_1^\perp$ are $z = P_h^-/k^-$ and $\boldsymbol{k}'_T = -z\boldsymbol{k}_T$. The first is the (lightcone) momentum fraction of the produced hadron, the second is the transverse momentum of the produced hadron with respect to the quark. The fragmentation function D_1 is the equivalent of the distribution function f_1. It can be interpreted as a quark decay function, giving the probability of finding a hadron h in a quark. The quantity $n_h = \int dz\; D_1(z)$ is the number of hadrons.

Exercise. Show that the normalization of the fragmentation functions is given by $\sum_h \int dz\; z\,D_1^{q\to h}(z)$ by relating this quantity to a local operator. One needs to eliminate the hadrons in the intermediate state via the momentum operator

$$P^\mu = \sum_{h,X} |P_h,X\rangle\, P_h^\mu\,\langle P_h,X|.$$

The function $H_1^\perp$, interpretable as the difference between the numbers of unpolarized hadrons produced from a transversely polarized quark depending on the hadron's transverse momentum, is allowed because of the non-applicability of time reversal invariance [25]. This is natural for the fragmentation functions [26,27] because of the appearance of out-states $|P_h,X\rangle$ in the definition of Δ, in contrast to the plane wave states appearing in Φ. The function $H_1^\perp$ is of interest because it is chiral-odd. This means that it can be used to probe the chiral-odd quark distribution function h_1, which can be achieved e.g. by measuring a particular azimuthal asymmetry of produced pions in the current fragmentation region.

The spin structure of fragmentation functions is also conveniently summarized by explicitly giving it on a R and L chiral quark basis, for which we find for decay into spin zero hadrons,

$$
(\Delta(z, k_T)\not{n}_+)^T = \left(\begin{array}{cc} D_1 & i\frac{|k_T|\, e^{-i\phi}}{M_h}\, H_1^{\perp} \\ -i\frac{|k_T|\, e^{+i\phi}}{M_h}\, H_1^{\perp} & D_1 \end{array}\right) \begin{array}{c} \mathrm{R} \\ \mathrm{L} \end{array} \tag{1.185}
$$

$$
\begin{array}{cc} \mathrm{R} & \mathrm{L} \end{array}
$$

Examples of azimuthal asymmetries

Transverse momentum dependence shows up in the azimuthal dependence in the SIDIS cross section (via $\hat{h}$ or transverse spin vectors), in most cases requiring polarization of beam and/or target or requiring polarimetry [28, 29]. Examples of leading azimuthal asymmetries, appearing for polarized leptoproduction are

$$
\left\langle \frac{Q_T}{M} \sin(\phi_h^\ell - \phi_S^\ell) \right\rangle_{OT} = \frac{2\pi\alpha^2\, s}{Q^4}\, |\boldsymbol{S}_T| \left(1 - y + \frac{1}{2}\, y^2\right) \sum_{a,\bar{a}} e_a^2\, x_B\, f_{1T}^{\perp(1)a}(x_B) D_1^a(z_h). \tag{1.186}
$$

$$
\left\langle \frac{Q_T}{M_h} \sin(\phi_h^\ell + \phi_S^\ell) \right\rangle_{OT} = \frac{4\pi\alpha^2\, s}{Q^4}\, |\boldsymbol{S}_T|\, (1 - y) \sum_{a,\bar{a}} e_a^2\, x_B\, h_1^a(x_B) H_1^{\perp(1)a}(z_h). \tag{1.187}
$$

The notation $\langle W \rangle$ is the q_T-integrated cross section with weight W. The factor Q_T is included, because it together with the direction $\hat{h}$ combines to q_T, allowing a defolding of the cross section in distribution and fragmentation parts (one of them weighted with transverse momentum). Note that both of these asymmetries involve T-odd functions, which can only appear in single spin asymmetries. The latter can easily be checked from the conditions on the hadronic tensor, which are the same as those in Eq. 1.11 to 1.13. They require an odd number of spins vectors entering in the symmetric part and an even number of spins entering in the antisymmetric part of the hadron tensor. The results of single spin asymmetries in SIDIS measurements on a transversely polarized target from HERMES [30] are shown in Fig. 1.10. An extended review of transverse momentum dependent functions and transversity can be found in Ref. [31]

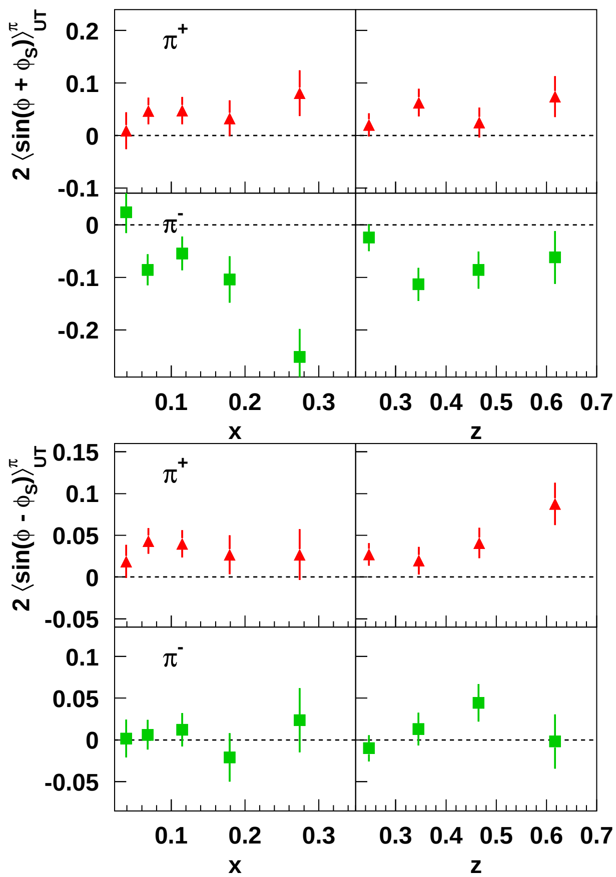


Figure 1.10. Weighted asymmetries for the Collins and Sivers angles (see Eqs 1.186 and 1.187) obtained in semi-inclusive single spin asymmetries measured on a transversely polarized Hydogen target by the HERMES collaboration at DESY [30]. The error bars represent the statistical uncertainties.

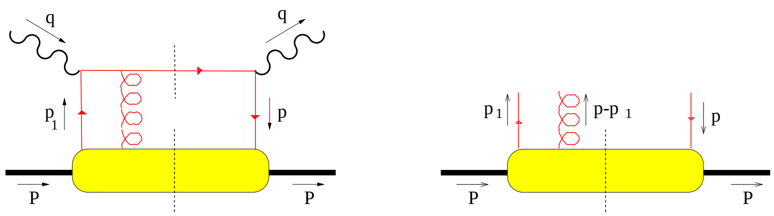


Figure 1.11. Examples of gluonic diagrams that must be included at subleading order in lepton hadron inclusive scattering (left) and the soft part entering this process (right).

1.16 Inclusion of subleading contributions

If one proceeds up to order $1/Q$ one also needs terms in the parametrization of the soft part proportional to M/P^+. Limiting ourselves to the $\boldsymbol{p}_T$-integrated correlations one needs [19]

$$\Phi(x) = \frac{1}{2}\left\{ f_1(x)\; \not{n}_+ + S_L\, g_1(x)\,\gamma_5\; \not{n}_+ + h_1(x)\,\frac{\gamma_5\,[\,\not{S}_T,\; \not{n}_+]}{2}\right\}$$
$$+ \frac{M}{2P^+}\left\{ e(x) + g_T(x)\,\gamma_5\;\not{S}_T + S_L\, h_L(x)\,\frac{\gamma_5\,[\,\not{n}_+,\; \not{n}_-]}{2}\right\} \quad (1.188)$$

We will use inclusive scattering off a transversely polarized nucleon ($|\boldsymbol{S}_\perp| = 1$) as an example to show how higher twist effects can be incorporated in the cross section. The hadronic tensor for a transversely polarized nucleon is zero in leading order in $1/Q$. At order $1/Q$ one obtains a contribution from the handbag diagram, which turns out to involve the transverse moments in Φ_∂^α in Eq. 1.181. There is a second contribution at order $1/Q$, however, coming from diagrams as the one shown in Fig. 1.11. For these gluon diagrams one needs bilocal matrix elements containing $1/Q$ one only needs the matrix element of the bilocal combinations $\overline{\psi}(0)\, gA_T^\alpha(\xi)\,\psi(\xi)$ and $\overline{\psi}(0)\, gA_T^\alpha(0)\,\psi(\xi)$. The $\Phi_A^\alpha(x)$ and $\Phi_\partial^\alpha(x)$ contributions sum to $\Phi_D^\alpha(x)$ involving matrix elements of bilocal combinations $\overline{\psi}(0)\, iD_T^\alpha\,\psi(\xi)$ for which one can Use the QCD equations of motion to relate them to the functions appearing in Φ,

$$\Phi_D^\alpha(x) = \frac{M}{2}\left\{ -\left(x\, g_T - \frac{m}{M}\, h_1\right)\; S_T^\alpha\, \not{n}_+\gamma_5 \right.$$
$$\left. - S_L\left(x\, h_L - \frac{m}{M}\, g_1\right)\; \frac{[\gamma^\alpha, \not{n}_+]\gamma_5}{2}\right\}. \quad (1.189)$$

The distribution function g_T e.g. shows up in the corresponding structure function of polarized inclusive deep inelastic scattering

$$2M\, W_A^{\mu\nu}(q,P,S_T) = i\,\frac{2Mx_B}{Q}\,\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}\,g_T(x_B), \tag{1.190}$$

leading for the structure function $g_T(x_B,Q^2)$ defined in Eq. 1.42 to the result

$$g_T(x_B,Q^2) = \frac{1}{2}\sum_q e_q^2\left(g_T^q(x_B) + g_T^{\bar{q}}(x_B)\right). \tag{1.191}$$

In the process of integrating the correlation functions over p^-, p_T and finally over p^+, consecutively restraining the nonlocality to lightfront-separated fields, lightcone-separated fields and local fields, interesting relations can be derived. For instance, the correlators $\int dx\,\Phi^{[\gamma^\mu\gamma_5]}(x)$ must yield $g_A\,S_T^\mu$ (for any μ), which means that the functions in the nonlocal correlators $\Phi^{[\gamma^+\gamma_5]}(x)$ and $\Phi^{[\gamma^\alpha\gamma_5]}(x)$ (α transverse) yield the same result after integration over x, $\int dx\,g_1(x) = \int dx\,g_T(x)$ or $\int dx\,g_2(x) = 0$, known as the Burkhardt-Cottingham sumrule [32]. For quark-quark correlators, similar considerations yield relations between the subleading functions and the transverse momentum dependent leading functions, referred to as *Lorentz invariance relations*, such as [33,28]

$$g_T = g_1 + \frac{d}{dx}\,g_{1T}^{(1)}, \tag{1.192}$$

although these relations may be too naive if one includes gauge links (see section 1.17). An interesting result is obtained by combining this relation with an often used approximation, in which the interaction-dependent part Φ_A^α is set to zero. In that case the difference $\Phi_D^\alpha - \Phi_\partial^\alpha$ vanishes. Using Eqs 1.189 and 1.181 this gives

$$x\,\tilde{g}_T = x\,g_T - g_{1T}^{(1)} - \frac{m}{M}\,h_1 = 0, \tag{1.193}$$

Exercise. As an application of the relations between twist-three functions and transverse momentum dependent functions in combination with the Lorentz invariance relations, one can eliminate $g_{1T}^{(1)}$ using Eq. 1.192 and obtain a relation between g_T, g_1 and $\tilde{g}_T$ (assuming sufficient neat behavior of the functions). Show that this relation for for $g_2 = g_T - g_1$ takes the form

$$\begin{aligned} g_2(x) = &-\left[g_1(x) - \int_x^1 dy\,\frac{g_1(y)}{y}\right] + \frac{m}{M}\left[\frac{h_1(x)}{x} - \int_x^1 dy\,\frac{h_1(y)}{y^2}\right] \\ &+ \left[\tilde{g}_T(x) - \int_x^1 dy\,\frac{\tilde{g}_T(y)}{y}\right]. \end{aligned} \tag{1.194}$$

Neglecting the interaction-dependent part, $\tilde{g}_T(x) = 0$ one obtains the Wandzura-Wilczek approximation [34] for g_2, which in particular when one neglects the quark mass term provides a simple and often used estimate for g_2. It has become the standard with which experimentalists compare the results for g_2.

1.17 Color gauge invariance

We have sofar disregarded two issues. The first issue is that the correlation function Φ discussed in previous sections involve two quark fields at different space-time points and hence are not color gauge invariant. The second issue are the gluonic diagrams similar as the ones we have discussed in the previous section (see Fig. 1.11), among which also correlation functions appear involving matrix elements with longitudinal (A^+) gluon fields,

$$\overline{\psi}_j(0)\, gA^+(\eta)\, \psi_i(\xi).$$

These do not lead to any suppression. The reason is that because of the $+$-index in the gluon field the matrix element is proportional to P^+, p^+ or $M\,S^+$ rather than the proportionality to $M\,S_T^\alpha$ or p_T^α that one gets for a gluonic matrix element with transverse gluons.

A straightforward calculation, however, shows that the gluonic diagrams with one or more longitudinal gluons involve matrix elements (soft parts) of operators $\overline{\psi}\psi$, $\overline{\psi}\,A^+\,\psi$, $\overline{\psi}\,A^+A^+\,\psi$, etc. that can be resummed into a correlation function

$$\Phi_{ij}(x) = \int \frac{d\xi^-}{2\pi}\, e^{ip\cdot\xi}\, \langle P,S|\overline{\psi}_j(0)\,\mathcal{U}(0,\xi)\,\psi_i(\xi)|P,S\rangle\bigg|_{\xi^+=\xi_T=0}, \tag{1.195}$$

where $\mathcal{U}$ is a gauge link operator

$$\mathcal{U}(0,\xi) = \mathcal{P}\exp\left(-i\int_0^{\xi^-} d\zeta^-\, A^+(\zeta)\right) \tag{1.196}$$

(path-ordered exponential with path along $-$-direction). Et voila, the unsuppressed gluonic diagrams combine into a color gauge invariant correlation function [35]. We note that at the level of operators, one expands

$$\overline{\psi}(0)\psi(\xi) = \sum_n \frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(0), \tag{1.197}$$

in a set of local operators (cf Eq. 1.94), but only the expansion of the nonlocal combination with a gauge link

$$\overline{\psi}(0)\,\mathcal{U}(0,\xi)\,\psi(\xi) = \sum_n \frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)D_{\mu_1}\ldots D_{\mu_n}\psi(0), \tag{1.198}$$

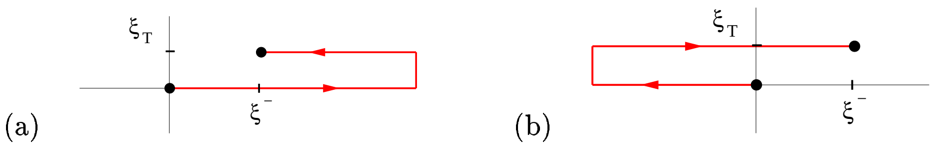


Figure 1.12. The gauge link structure in the quark-quark correlator Φ in SIDIS (a) and DY (b) respectively

is an expansion in terms of local gauge invariant operators. The latter operators are precisely the local (quark) operators that appear in the operator product expansion applied to inclusive deep inelastic scattering (cf Eq. 1.109).

For the p_T-dependent functions, one finds that inclusion of A^+ gluonic diagrams leads to a color gauge invariant matrix element with links running via $\xi^{=} = \pm\infty$ [36,37]. For instance in lepton-hadron scattering one finds

$$\Phi(x, \boldsymbol{p}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ip\cdot\xi}\, \langle P, S|\overline{\psi}(0)\,\mathcal{U}^{[+]}(0,\xi)\,\psi(\xi)|P, S\rangle\bigg|_{\xi^+=0}, \quad (1.199)$$

where the link $\mathcal{U}^{[+]}$ is shown in Fig. 1.12a. We note that the gauge link involves transverse gluons [38,39], showing that one in processes involving more hadrons the effects of transverse gluons are not necessarily suppressed, as has also been shown in explicit model calculations [40].

Moreover, depending on the process the gauge link can also run via minus infinity, involving the link in Fig. 1.12b. This is for instance the case in Drell-Yan processes. The transverse momentum dependent distribution functions also are no longer constrained by time-reversal, as the time reversal operation interchanges the $U^{[+]}$ and $U^{[-]}$ links, leading to the appearance of T-odd functions in Eq. 1.181. The process dependence of the gauge link, however, points to particular sign changes when single spin azimuthal asymmetries in sem-inclusive leptoproduction are compared to those in for instance Drell-Yan scattering. For such effects the measurement of transverse momentum dependence is a must, since the specific link structure does not matter in p_T-integrated functions, in which both links in Fig. 1.12 reduce to the same straight-line link connecting 0 and ξ.

References

[1] R.G. Roberts, *The structure of the proton*, Cambridge University Press (1990)

[2] R. Meng, F.I. Olness and D.E. Soper, Nucl. Phys. B 371 (1992) 79.

[3] R.K. Ellis, W. Furmanski and R. Petronzio, Nucl. Phys. B 212 (1983) 29; Nucl. Phys. B 207 (1982) 1.

[4] A.V. Efremov and O.V. Teryaev, Sov. J. Nucl. Phys. 39 (1984) 962.

[5] R.L. Jaffe in *Relativistic Dynamics and Quark Nuclear Physics*, eds M.B. Johnson and A. Picklesheimer Wiley, New York (1986).

[6] M. Botje, Eur. Phys. J. C 14 (2000) 285

[7] Durham-RAL databases, http://durpdg.dur.ac.uk/HEPDATA/
Particle Data Group, http://pdg.web.cern.ch/pdg/

[8] J.B. Kogut and D.E. Soper, Phys. Rev. D1 (1970) 2901.

[9] R.L. Jaffe, Nucl. Phys. B 229 (1983) 205.

[10] M. Diehl and T. Gousset, Phys. Lett. B 428 (1998) 359.

[11] K. Gottfried, Phys. Rev. Lett. 18 (1967) 1174.

[12] NMC Collaboration, D. Allasia et al., Phys. Lett. B249 (1990) 366

[13] The first surprising results were reported by the EMC Collaboration, J. Ashman et al., Phys. Lett. B206 (1988) 364; Nucl. Phys. B328 (1989) 1

[14] A. Airapetian et al. (HERMES Collaboration), Phys. Rev. D 71 (2005) 012003

[15] J.D. Bjorken, Phys. Rev. 179 (1969) 1547

[16] J. Kodaira et al., Phys. Rev. D20 (1979) 627; S.A. Larin, F.V. Tkachev and J.A.M. Vermaseren, Phys. Lett. B259 (1991) 345; S.A. Larin and J.A.M. Vermaseren, Phys. Rev. Lett. 66 (1991) 862

[17] D.E. Soper, Phys. Rev. D 15 (1977) 1141; Phys. Rev. Lett. 43 (1979) 1847.

[18] J.C. Collins and D.E. Soper, Nucl. Phys. B 194 (1982) 445.

[19] R.L. Jaffe and X. Ji, Nucl. Phys. B 375 (1992) 527.

[20] X. Artru and M. Mekhfi, Z. Phys. 45 (1990) 669, J.L. Cortes, B. Pire and J.P. Ralston, Z. phys. C55 (1992) 409.

[21] J. Soffer, Phys. Rev. Lett. 74, 1292 (1995).

[22] J.P. Ralston and D.E. Soper, Nucl. Phys. B152 (1979) 109.

[23] R.D. Tangerman and P.J. Mulders, Phys. Rev. D51 (1995) 3357.

[24] A. Bacchetta, M. Boglione, A. Henneman and P.J. Mulders, Phys. Rev. Lett. 85 (2000) 712

[25] J. Collins, Nucl. Phys. B396 (1993) 161.

[26] K. Hagiwara, K. Hikasa and N. Kai, Phys. Rev. D27 (1983) 84.

[27] R.L. Jaffe and X. Ji, Phys. Rev. Lett. 71 (1993) 2547; Phys. Rev. D57 (1998) 3057.

[28] P.J. Mulders and R.D. Tangerman, Nucl. Phys. B 461 (1996) 197; Nucl. Phys. B 484 (1997) 538 (E).

[29] D. Boer and P.J. Mulders, Phys. Rev. D 57 (1998) 5780.

[30] A. Airapetian et al. (HERMES Collaboration), Phys. Rev. Lett. 94 (2005) 012002

[31] V. Barone, A. Drago and P.G. Ratcliffe, Phys. Rept. 359 (2002) 1.
Transverse polarisation of quarks in hadrons

[32] H. Burkhardt and W.N. Cottingham, Ann. Phys. (N.Y.) 56 (1976) 453.

[33] A.P. Bukhvostov, E.A. Kuraev and L.N. Lipatov, Sov. Phys. JETP 60 (1984) 22.

[34] S. Wandzura and F. Wilczek, Phys. Rev. D16 (1977) 707.

[35] A.V. Efremov and A.V. Radyushkin, Theor. Math. Phys. 44 (1981) 774.

[36] D. Boer and P.J. Mulders, Nucl. Phys. B 569 (2000) 505.

[37] J.C. Collins, Phys. Lett. B 536 (2002) 43.

[38] A.V. Belitsky, X. Ji and F. Yuan, Nucl. Phys. B 656 (2003) 165.

[39] D. Boer, P.J. Mulders and F. Pijlman, Nucl. Phys. B 667 (2003) 201.

[40] S.J. Brodsky, D.S. Hwang and I. Schmidt, Phys. Lett. B 530 (2002) 99;
Nucl. Phys. B 642 (2002) 344.