Structure of Hadrons in Hard Scattering Processes

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Contents

1 Quantum chromodynamics

The basic framework for the strong interactions is the nonabelian gauge theory based on the color symmetry group $SU(N_c)$ with $N_c = 3$, Quantum Chromodynamics (QCD). The lagrangian of this field theory is in essence very simple. It is built from fermion fields ψ belonging to the *basic representation* $\underline{3}$ of the $SU(3)$ color group and gauge fields belonging to the *conjugate representation*, which are introduced to make the theory locally gauge invariant.

In these lectures we will address a number of issues that are in between nonperturbative approaches, such as lattice calculations or modelling and the perturbative approaches. The aim is to gain as much information on hadrons as possible. Such information can range from the spectrum of hadrons to global properties like the charges or charge distributions to more detailed information like the momentum distributions of the quarks and gluons. Important, however, is that we want to carefully analyse, which information is reliable. This requires the analysis of the various operators, in particular quark and gluon operators, that play a role in QCD, look at their expectation values, the scale dependence of these matrix elements and the way they are measured.

Background and more details can be found in many field theory and particle physics books. In particular we suggest:

1. S. Weinberg, *The quantum theory of fields, I and II*, Cambridge University Press.
2. R.G. Roberts, *The structure of the proton*, Cambridge University Press.
3. M.E. Peshkin and D.V. Schroeder, *An introduction to Quantum Field Theory*, Addison-Wesly, 1995.
4. G. Sterman, *An introduction to Quantum Field Theory*, Cambridge university Press
5. T. Muta, *Foundations of Quantum Chromodynamics*, World Scientific 1987.

2 The color group $SU(N_c)$

The transformations in $SU(N_c)$ are expanded in the $N_G = N_c^2 - 1$ generators T_a,

$$U(\theta) = e^{i\theta_a\,T_a}. \tag{1}$$

The eight 3×3 defining hermitean traceless matrices needed for $SU(3)$ form the *basic representation* to which the quarks belong. To be precise, the color charges F_a for the quarks and antiquarks (*conjugate representation*) are the following representation of the generators

$$T_a(q): \quad F_a(q) = \frac{\lambda_a}{2}, \qquad T_a(\bar{q}): \quad F_a(\bar{q}) = -\frac{\lambda_a^*}{2}, \tag{2}$$

the standard set being the eight Gell-Mann matrices λ_a $(a = 1, \ldots, 8)$. One makes the choice Tr $(T_a T_b)$ $= T_F\,\delta_{ab}$ which in the case of $SU(3)$ is $T_F = 1/2$ since Tr $\lambda_a \lambda_b = 2\,\delta_{ab}$. In general the representation matrices or charges satisfy the commutation relations

$$[T_a, T_b] = i\,f_{abc}\,T_c, \tag{3}$$

where f_{abc} are the (totally antisymmetric) structure constant of the *non-abelian* group.

We note that the above is a special case of semi-simple Lie algebras, i.e. a Lie algebra $\underline{G}$ that does not have nonzero abelian ideals (invariant subgroups $\underline{I}$ for which $[\underline{I}, \underline{G}] \in \underline{I}$). We have the following properties (see e.g. Weinberg)

- Cartan's criterion: $\underline{G}$ is semi-simple if and only if the Killing form constructed from the generators as Tr $(T_a^\dagger T_b)$ is a matrix for which the determinant is nonzero.

- Choosing the Killing form diagonal, the structure constants are are totally antisymmetric.
- A semi-simple Lie algebra is a direct sum of *simple* Lie algebras (with no invariant subgroups at all) or $U(1)$ groups.

Exercise. Use the Jacobi-identity for the generators, $[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] = 0$ to derive a relation between the structure constants and then show that the matrices $(T_a)_{bc} = -i\, f_{abc}$ form a representation. This N_G-dimensional representation is called the *adjoint* or *regular* representation. Show that it is self-conjugate.

3 The lagrangian of QCD

The lagrangian (omitting gauge terms) is given by

$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu}(x) F^{\mu\nu\, a}(x) + \overline{\psi}(x)(i\, \not{D} - m)\psi(x), \tag{4}$$

where

$$D_\mu \psi(x) = \partial_\mu \psi(x) - ig\, A^a_\mu(x) F_a\, \psi(x), \tag{5}$$

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g\, f_{abc} A^b_\mu(x) A^c_\nu(x). \tag{6}$$

Introducing matrices $A_\mu \equiv A^a_\mu T_a$ (thus $A^a_\mu = \frac{1}{T_F}\,\mathrm{Tr}\,(A_\mu\, T_a)$) and in the same way matrices D_μ and $F_{\mu\nu}$ one obtains,

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}(x) F^{\mu\nu}(x) + \overline{\psi}(x)(i\, \not{D} - m)\psi(x), \tag{7}$$

where[1]

$$i\, D_\mu = i\, \partial_\mu\, 1 + g\, A_\mu, \tag{8}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\, [A_\mu, A_\nu] = \frac{i}{g}\, [D_\mu, D_\nu]. \tag{9}$$

and we use $F_{\mu\nu}F^{\mu\nu}$ for $\frac{1}{T_F}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) = F^a_{\mu\nu}F^{\mu\nu\, a}$.

Using the matrix notation, it is straightforward to check how the color gauge invariance is implemented. The fermion field and (by definition) its covariant derivative transforms like

$$\psi(x) \longrightarrow U(\theta(x))\, \psi(x), \tag{10}$$

$$D_\mu \psi(x) \longrightarrow U(\theta(x))\, D_\mu \psi(x), \tag{11}$$

or $D_\mu \longrightarrow U\, D_\mu\, U^{-1}$. This implies for the $A_\mu(x)$ field that

$$A_\mu(x) \longrightarrow U(\theta(x))\, A_\mu\, U^{-1}(\theta(x)) - \frac{i}{g}\, (\partial_\mu U(\theta(x))\, U^{-1}(\theta(x)). \tag{12}$$

For the group $U(1)$ this reduces to the familiar $A_\mu \to A_\mu + \frac{1}{g}\, \partial_\mu \theta(x)$. The matrix form for $F_{\mu\nu}$ immediately shows its behavior under gauge transformations

$$F_{\mu\nu}(x) \longrightarrow U(\theta(x))\, F_{\mu\nu}(x)\, U^{-1}(\theta(x)), \tag{13}$$

and the invariance of the quadratic term for the gauge fields in the lagrangian.

[1] Note that it can sometimes be useful to absorb the coupling constant in A_μ, i.e. $A_\mu \equiv g\, A^a_\mu F_a$.

The equations of motion

Having the lagrangian one easily obtains the equations of motion via the action principle

$$(i\,\not{D} - m)\psi(x) = 0 \tag{14}$$
$$D_\mu F^{\mu\nu} = j_M^\nu, \tag{15}$$

where $j_M^{\nu a}(x) = g\,\overline{\psi}(x)\,\gamma^\nu F_a\,\psi(x)$ is the matter color current. The explicit expressions for $D_\mu F^{\mu\nu}$ are

$$D_\mu F^{\mu\nu} = [D_\mu, F^{\mu\nu}] = \partial_\mu F^{\mu\nu} - ig\,[A_\mu, F^{\mu\nu}] \tag{16}$$

or

$$D_\mu F^{\mu\nu\,a} = \partial_\mu F^{\mu\nu\,a} + g\,f_{abc} A_\mu^b\,F^{\mu\nu\,c}, \tag{17}$$

and similarly for other quantities belonging to the color conjugate representation.

Exercise. Show that $D_\nu j_M^\nu = [D_\nu, j_M^\nu] = 0$ for the matter part of the color current (appearing in the above equations), while $\partial_\nu j_C^\nu = 0$ for the full color current $j_C^\nu = j_M^\nu + ig[A_\mu, F^{\mu\nu}]$.

Exercise. Use the Jacobi-identity to show the Bianchi identity $D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0$.

4 A geometric picture of gauge theories

A geometric picture of gauge theories is useful for comparison with general relativity and for topological considerations. Consider the space $\prod_x {}^xG$ (a *fibre bundle*). At each space-time point x there is an internal space G (say spin or isospin). In each of these spaces a reference frame is defined. ${}^x\psi(x)$ denotes a field vector $\psi(x)$ which belongs to a representation of G, i.e. forms a vector in the internal space (see fig. 1). The superscript x denotes that it is expressed with respect to the frame at point x, i.e. the basis of ${}^xG$. Let fields $A_\mu^a(x)$ determine the 'parallel displacements' in the internal spaces, i.e. connect the basis for ${}^xG$ and ${}^{x+dx}G$,

$${}^{x+dx}\psi(x) \quad = \quad (1 + ig\,dx^\mu\,A_\mu(x))\;{}^x\psi(x), \tag{18}$$

which connects two identical vectors, but expresses them with respect to different bases.

The 'true' difference in the vector $\psi(x)$ and $\psi(x+dx)$ is denoted with the covariant derivative connecting the vectors expressed with respect to the same basis, i.e.

$${}^{x+dx}\psi(x+dx) = (1 + dx^\mu\,D_\mu)^{x+dx}\,\psi(x), \tag{19}$$

which because of the presence of the 'connection' A_μ differs from the total change between ${}^x\psi(x)$ and ${}^{x+dx}\psi(x+dx)$,

$${}^{x+dx}\psi(x+dx) = (1 + dx^\mu\,\partial_\mu)\;{}^x\psi(x). \tag{20}$$

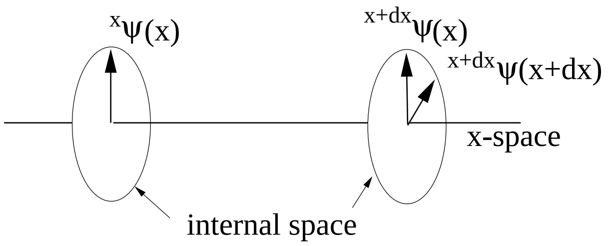


Figure 1: The vectors belonging to internal space located at each point in (one-dimensional) space

The three equations given so far immediately give the familiar $D_\mu = \partial_\mu - ig\, A_\mu(x)$. We note that local gauge invariance requires that we can modify all local systems via a (local) unitary transformation $U(x)$. The relation in Eq. 19, should be independent of such transformations, requiring that the 'connection' $A_\mu(x)$ is such that $D_\mu \longrightarrow U(x) D_\mu U^{-1}(x)$.

A 'constant' vector that only rotates because of the arbitrary definitions of local frames satisfies $D_\mu\, \psi(x) = 0$. Considering a path $x^\mu(s)$ from a given point (0) to point x one has

$$\frac{dx^\mu}{ds}\left[\partial_\mu - ig\, A_\mu(x(s))\right]\psi(x(s)) = 0, \tag{21}$$
$$\frac{d\psi(x(s))}{ds} = ig\, A_\mu(x(s))\, \psi(x(s))\, \frac{dx^\mu}{ds}, \tag{22}$$

which is solved by the path-ordered exponential

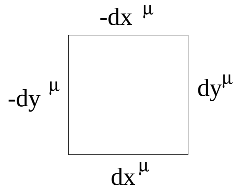
$$\psi(x) = \underbrace{\mathcal{P}\, e^{ig\int_0^x ds^\mu\, A_\mu(s)}}_{\mathcal{U}(x,0)}\psi(0), \tag{23}$$

explicitly given by

$$\begin{aligned}\mathcal{U}(x,0) &= \sum_{n=0}^{\infty}(-i)^n \int_0^s ds_1\, \frac{dx^{\mu_1}}{ds_1}\int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n\, \frac{dx^{\mu_n}}{ds_n}\, A_{\mu_1}(x(s_1))\ldots A_{\mu_n}(x(s_n)) \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_0^s ds_1\, \frac{dx^{\mu_1}}{ds_1}\int_0^{s} ds_2 \ldots \int_0^{s} ds_n\, \frac{dx^{\mu_n}}{ds_n}\, \mathcal{P}\, A_{\mu_1}(x(s_1))\ldots A_{\mu_n}(x(s_n)).\end{aligned}$$

This gives rise to a (path dependent) phase in each point.

In principle such a phase in a given point is not observable. However, if two different paths to the same point give different phases the effects can be observed. What is this physical effect by which the 'connection' A_μ can be observed? For this consider the phase around a closed loop,



For the constant vector, it is given by

$$\begin{aligned}\psi(x) &= (1 - i\, dy^\rho D_\rho)(1 - i\, dx^\sigma D_\sigma)(1 + i\, dy^\nu D_\nu)(1 + i\, dx^\mu D_\mu)\psi(x) \\ &= (1 + \, dx^\mu dy^\nu\, [D_\mu, D_\nu])\, \psi(x) \\ &= (1 - ig\, d\sigma^{\mu\nu}\, F_{\mu\nu})\, \psi(x),\end{aligned} \tag{24}$$

where $F_{\mu\nu} = (i/g)[D_\mu, D_\nu]$. Similarly as the definition of the covariant derivative the effect is thus independent of the local choice of basis and we have the transformation law $G_{\mu\nu} \longrightarrow U(x) G_{\mu\nu} U^{-1}(x)$. In geometric language the effect on parallel transport of a vector depends on the 'curvature' $G_{\mu\nu}$. Only if this quantity is nonzero a physically observable effect of A_μ exists. If it is zero one has

$$\mathcal{P}\, e^{ig\oint ds^\mu\, A_\mu(s)} = 1 \tag{25}$$

which for an abelian field gives the familiar $\oint dx^\mu A_\mu(x) = 0$. Equivalently A_μ can be considered as a pure gauge effect, which means that by an appropriate transformation $U(x)$ it can be gauged away.

5 Symmetries and conserved currents

Symmetry transformations together with the action principle lead to Noether currents. If the symmetry transformation leaves the action invariant the currents are conserved. For variations

$$\Delta\phi_i(x) \equiv \phi'_i(x') - \phi_i(x), \tag{26}$$
$$\Delta x^\mu \equiv x'^\mu - x^\mu, \tag{27}$$

one obtains besides the equations of motion a surface term contribution

$$\delta A = Q(\sigma_1) - Q(\sigma_2) \tag{28}$$

where

$$Q(\sigma) = \int_\sigma d\sigma_\mu \underbrace{\left[-\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\Delta\phi + \overbrace{\left[-\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\phi}\,\partial^\nu\phi + \mathcal{L}\,g^{\mu\nu}\right]}^{T^{\mu\nu}} \Delta x_\nu \right]}_{-j^\mu(x)}. \tag{29}$$

Here $T^{\mu\nu}$ is the energy-momentum tensor, while the current $j^\mu(x)$ is the Noether current. Two special cases for the surface are $\sigma = (t, R^3)$, i.e. $x^0 = t =$ constant (3-space) or $x^+ = (x^0 + x^3)/\sqrt{2} = \tau =$ constant (lightfront) in which case one obtains the charges

$$Q(t) = \int d^3x\; j^0(x), \tag{30}$$
$$Q(\tau) = \int dx^-\, d^2x_T\; j^+(x). \tag{31}$$

In classical field theories $Q(\sigma)$ is a conserved quantity, in quantum field theories it becomes the generator of the symmetry transformations in the Hilbert space. Since one can use Stokes and the definition of the action to write

$$\delta A = \int_{\sigma_1}^{\sigma_2} d^4x\; \partial_\mu j^\mu(x) = \int_{\sigma_1}^{\sigma_2} d^4x\; \delta\mathcal{L}(x), \tag{32}$$

one sees that the current is conserved if the lagrangian is invariant. In fact the symmetry breaking terms determine

$$\partial_\mu j^\mu(x) = \delta\mathcal{L}(x), \tag{33}$$

often useful if the symmetry-breaking terms go with specific parameters, e.g. masses or explicit x-dependent terms in the lagrangian.

Internal symmetries

For transformations, other than space-time transformations, e.g. gauge transformations $\delta\phi = i\,\theta_a\,T_a\phi(x)$, one has the current

$$j^\mu(x) = -i\,\theta_a \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}}_{\Pi^\mu(x)} T_a\phi(x), \tag{34}$$

and the generators of the symmetry become

$$Q_a(t) = -i\int d^3x\; \Pi^0(x)\,T_a\phi(x). \tag{35}$$

Exercise. Show that the canonical equal-time commutation relations $[\phi(\boldsymbol{x},t),\Pi^0(\boldsymbol{y},t)] = i\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$ imply that the the charges expressed in terms of the quantum fields satisfy

$$[Q_a(t),Q_b(t)] = i\,f_{abc}\,Q_c(t), \tag{36}$$
$$[Q_a(t),\phi(\boldsymbol{x},t)] = T_a\phi(\boldsymbol{x},t). \tag{37}$$

In fact color and flavor symmetries in QCD are examples. We will consider in more detail also chiral symmetry in QCD.

6 Topological currents

Besides the Noether currents related to symmetries, there may exist conserved quantities related to classes of solutions, socalled topological currents. To illustrate this we will give a simple example in 1+1 dimensions, the Sine-Gordon theory with

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\theta\partial^\mu\theta) - U(\theta) \tag{38}$$
$$\stackrel{\text{static}}{=} -\frac{1}{2}\left(\frac{\partial\theta}{\partial x}\right)^2 - U(\theta) = -\mathcal{H}, \tag{39}$$

with $U(\theta) = m^2(1-\cos\theta)$. Note that the integral over this static expression is just minus the energy density $(-E[\theta])$ in $1+1$ dimensions or minus the Euclidean lagrangian $(-S_E[\theta])$ in *one* dimension. We look for static solutions which in order to have finite energy must satisfy that $\theta(-\infty) = 2\pi\,n_{-\infty}$ and $\theta(\infty) = 2\pi\,n_\infty$.

Exercise. Show that a solution exists with $\theta(-\infty) = 0$ and $\theta(\infty) = 2\pi$ of the form

$$\theta(x) = 4\,\tan^{-1}\left(e^{mx}\right).$$

Calculate the energy density for this solution and show it equals

$$\mathcal{E}(x) = \frac{16\,m^2\,e^{2mx}}{(1+e^{2mx})^2},$$

which gives as energy $E[\theta] = 8\,m$.

The solutions of the Sine-Gordon theory can be divided in classes as explained in Fig. 2. A solution from a particular class, can be characterized by the topological quantum number

$$Q = \frac{1}{2\pi}\left(\theta(+\infty) - \theta(-\infty)\right). \tag{40}$$

Solutions with different values for Q cannot be continuously deformed into each other, e.g. the $Q=1$ solution cannot continuously be deformed into a solution with $Q=0$. In other words Q is conserved and labels a particular class of solutions. Although not derivable from a symmetry principle, there also exist for this charge a conserved current given by

$$j^\mu = \frac{1}{2\pi}\,\epsilon^{\mu\nu}\partial_\nu\theta, \tag{41}$$

The corresponding charge is

$$Q(t) = \int dx\; j^0(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\;\frac{\partial\theta}{\partial x}, \tag{42}$$

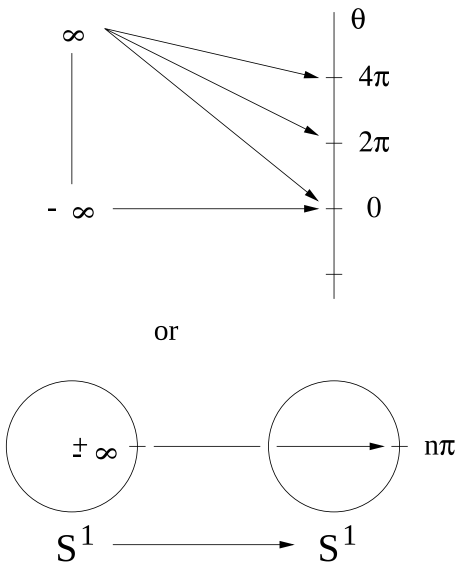


Figure 2: Topological structure of the Sine-Gordon theory. We can consider static localized solutions (approaching the vacuum at infinity). For them the parameter space for θ is the real axis with minima of the potential at values $n\pi$, or equivalently a circle ($0 \leq \theta < 2\pi$). One can always make the choice $\theta(-\infty) = 0$. Thus the solutions are just maps of the real axis with the points at infinity identified (this is the circle S^1) into the parameter-space (S^1). The topology of the maps is $\Pi_1(S^1) = Z$.

indeed only depending on the endpoint values of $\theta(x)$ and hence the same within a class of functions. Typically, topological currents are constructed from the Levi-Civita tensor and derivatives (or in more complex situations the Maurer-Cartan forms).

We note the following useful property of the nontrivial solution connecting two adjacent minima of the potential. By rewriting the Euclidean action or the energy as

$$E[\theta] = \frac{1}{2}\int dx \, \left(\frac{d\theta}{dx} \mp \sqrt{2\,U(\theta)}\right)^2 \pm \int_{\theta(-\infty)}^{\theta(+\infty)} d\theta \, \sqrt{2\,U(\theta)}, \tag{43}$$

one derives the Bogomol'nyi bound

$$E[\theta] \geq \left| \int_{\theta(-\infty)}^{\theta(+\infty)} d\theta \, \sqrt{2\,U(\theta)} \right|. \tag{44}$$

The right hand side is determined by the boundary of the field, i.e. is (a function of) the topological charge. The bound is reached when the left hand side is zero. Indeed $\theta'(x) = \sqrt{2\,U(\theta)}$ is precisely the solution given above, while with $\theta(-\infty) = 0$ one has for the right hand side of the bound the value $8m\,\sin^2(\pi\,Q)$. We note finally that when $U(\theta)$ is considered as a barrier, $e^{-S[\theta]}$ is the probability to tunnel this barrier.

7 Space-time symmetries

Let us first give the appropriate transformations for coordinates and fields under Lorentz transformations. We have

$$\begin{aligned} \Delta\phi_i(x) &= \frac{1}{2}\,\epsilon_{\rho\sigma}(i\,S^{\rho\sigma})_{ij}\phi_j(x), & (45)\\ \Delta x^\mu &= \epsilon^{\mu\nu}x_\nu = \frac{1}{2}\epsilon_{\rho\sigma}(a^{\rho\sigma})^\mu_{\ \nu}\,x^\nu, & (46) \end{aligned}$$

where $(a_{\rho\sigma})_{\mu\nu} = g_{\rho\mu}g_{\sigma\nu} - g_{\rho\mu}g_{\sigma\nu}$. We have the commutation relations

$$\begin{aligned} [a_{\rho\sigma}, a_{\kappa\lambda}] &= g_{\sigma\kappa}\,a_{\rho\lambda} - g_{\sigma\lambda}\,a_{\rho\kappa} - g_{\rho\kappa}\,a_{\sigma\lambda} - g_{\rho\lambda}\,a_{\sigma\kappa}, & (47)\\ [i\,S_{\rho\sigma}, i\,S_{\kappa\lambda}] &= g_{\sigma\kappa}\,iS_{\rho\lambda} - g_{\sigma\lambda}\,iS_{\rho\kappa} - g_{\rho\kappa}\,iS_{\sigma\lambda} - g_{\rho\lambda}\,iS_{\sigma\kappa} & (48) \end{aligned}$$

and we have for scalar fields $iS_{\rho\sigma} = 0$, for fermion fields[2] $(iS_{\rho\sigma})_{ij} = -\frac{1}{4}\left([\gamma_\rho, \gamma_\sigma]\right)_{ij} = \frac{i}{2}(\sigma_{\rho\sigma})_{ij}$, while for vector fields $(iS_{\rho\sigma})_{\mu\nu} = (a_{\rho\sigma})_{\mu\nu}$.

The canonical currents for translations and Lorentz transformations are

$$
\begin{aligned}
T^{\mu\nu} &= -\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\phi}\,\partial^\nu\phi + \mathcal{L}\,g^{\mu\nu}, && (49)\\
M^{\mu\rho\sigma} &= -\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\phi_i}\,(iS^{\rho\sigma})_{ij}\phi_j + x^\rho\,T^{\mu\sigma} - x^\sigma\,T^{\mu\rho}. && (50)
\end{aligned}
$$

The integrated quantities

$$
\begin{aligned}
P^\nu &= \int d^3x\;T^{0\nu} = \int dx^-\,d^2x_T\;T^{+\nu}, && (51)\\
J^{\rho\sigma} &= \int d^3x\;M^{0\rho\sigma} = \int dx^-\,d^2x_T\;M^{0\rho\sigma} && (52)
\end{aligned}
$$

are the ten generators of the Poincaré transformations in the Hilbert space. In particular $H = \int d^3x\;\mathcal{H} = \int d^3x\;T^{00}$ is the Hamiltonian, the generator of time translations.

With the following procedure we can construct a symmetric (equivalent) energy-momentum tensor (Belinfante tensor) and a simplified Lorentz tensor. Define the first term in the Lorentz tensor as

$$
H^{\mu\rho\sigma} \equiv -\frac{\delta\mathcal{L}}{\delta\,\partial_\mu\phi_i}\,(iS^{\rho\sigma})_{ij}\phi_j, \qquad (53)
$$

which is antisymmetric in ρ and σ. It is easy to see that conservation of $M^{\mu\rho\sigma}$ implies that $T^{\mu\nu} - T^{\nu\mu} = -\partial_\rho H^{\rho\mu\nu}$. In order to construct quantity $T_B^{\mu\nu}$ that satisfies $\partial_\mu T_B^{\mu\nu} = \partial_\mu T^{\mu\nu}$ one could try something of the form

$$
T_B^{\mu\nu} = T^{\mu\nu} + \partial_\rho G^{\rho\mu\nu}, \qquad (54)
$$

antisymmetric in ρ and μ. In order that the second term precisely is the antisymmetric part one must have $\partial_\mu G^{\mu[\rho\sigma]} = \partial_\mu H^{\mu\rho\sigma}$. The tensor

$$
G^{\mu\rho\sigma} = \frac{1}{2}\left(H^{\mu\rho\sigma} + H^{\rho\sigma\mu} + H^{\sigma\rho\mu}\right) = -G^{\rho\mu\sigma} \qquad (55)
$$

has the desired properties. After a partial integration and subtracting surface terms one has

$$
M^{\mu\rho\sigma} = x^\rho\,T_B^{\mu\sigma} - x^\sigma\,T_B^{\mu\rho} = G^{\mu[\rho\sigma]} + x^\rho\,T^{\mu\sigma} - x^\sigma\,T^{\mu\rho}. \qquad (56)
$$

We end this section by giving the results for noninteracting theories.

Complex scalar fields

$$
\begin{aligned}
&\mathcal{L} = \partial^\mu\phi^*\,\partial_\mu\phi - M^2\,\phi^*\phi, && (57)\\
&\left(\partial^\mu\partial_\mu + M^2\right)\phi(x) = 0, && (58)\\
&T^{\mu\nu} = T_B^{\mu\nu} = \partial^{\{\mu}\phi^*\partial^{\nu\}}\phi + \mathcal{L}\,g^{\mu\nu} \;\stackrel{e.o.m.}{=}\; \partial^{\{\mu}\phi^*\partial^{\nu\}}\phi && (59)
\end{aligned}
$$

Fermion fields

$$
\begin{aligned}
&\mathcal{L} = \overline{\psi}(x)\left(\frac{i}{2}\overleftrightarrow{\partial\!\!\!/} - M\right)\psi(x), && (60)\\
&(i\,\partial\!\!\!/ - M)\,\psi(x) = 0, && (61)
\end{aligned}
$$

[2]Note $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$

$$
\begin{aligned}
T^{\mu\nu} &= \frac{i}{2}\,\overline{\psi}\,\gamma^{\mu}\,\overset{\leftrightarrow}{\partial}{}^{\nu}\psi + \left(\frac{i}{2}\,\overline{\psi}\,\overset{\leftrightarrow}{\not\partial}\,\psi - m\,\overline{\psi}\psi\right) g^{\mu\nu} \\
&\overset{e.o.m.}{=} \frac{i}{2}\,\overline{\psi}\,\gamma^{\mu}\,\overset{\leftrightarrow}{\partial}{}^{\nu}\psi = \frac{i}{8}\,\overline{\psi}\,(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} + \gamma^{\mu}\gamma^{\rho}\gamma^{\nu} + \gamma^{\nu}\gamma^{\rho}\gamma^{\mu} + \gamma^{\rho}\gamma^{\nu}\gamma^{\mu})\,\overset{\leftrightarrow}{\partial}_{\rho}\,\psi, \qquad (62) \\
H^{\mu\rho\sigma} &= \frac{1}{4}\,\overline{\psi}\,\{\gamma^{\mu},\sigma^{\rho\sigma}\}\,\psi = \frac{i}{8}\,\overline{\psi}\,\{\gamma^{\mu},[\gamma^{\rho},\gamma^{\sigma}]\}\,\psi, \qquad (63) \\
G^{\rho\mu\nu} &= \frac{i}{8}\,\overline{\psi}\,(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} - \gamma^{\rho}\gamma^{\nu}\gamma^{\mu})\,\psi = -\frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\overline{\psi}\gamma_{\sigma}\gamma_{5}\psi, \qquad (64) \\
\partial_{\rho}G^{\rho\mu\nu} &= \frac{i}{8}\,\overline{\psi}\left(\gamma^{\mu}\gamma^{\nu}\,\overset{\leftarrow}{\not\partial} + \gamma^{\mu}\gamma^{\nu}\,\overset{\rightarrow}{\not\partial} - \overset{\leftarrow}{\not\partial}\,\gamma^{\nu}\gamma^{\mu} - \overset{\rightarrow}{\not\partial}\,\gamma^{\nu}\gamma^{\mu}\right)\psi \\
&\overset{e.o.m.}{=} \frac{i}{8}\,\overline{\psi}\left(\gamma^{\mu}\gamma^{\nu}\,\overset{\leftarrow}{\not\partial} - \overset{\leftarrow}{\not\partial}\,\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}\,\overset{\rightarrow}{\not\partial} - \overset{\rightarrow}{\not\partial}\,\gamma^{\nu}\gamma^{\mu}\right)\psi \\
&= \frac{i}{16}\,\overline{\psi}\left([\gamma^{\mu},\gamma^{\nu}]\,\overset{\leftarrow}{\not\partial} - \overset{\leftarrow}{\not\partial}\,[\gamma^{\mu},\gamma^{\nu}] + [\gamma^{\nu},\gamma^{\mu}]\,\overset{\rightarrow}{\not\partial} - \overset{\rightarrow}{\not\partial}\,[\gamma^{\nu},\gamma^{\mu}]\right)\psi \\
&= \frac{i}{16}\,\overline{\psi}\,(\gamma^{\rho}[\gamma^{\mu},\gamma^{\nu}] - [\gamma^{\mu},\gamma^{\nu}]\gamma^{\rho})\,\overset{\leftrightarrow}{\partial}_{\rho}\,\psi, \qquad (65) \\
T_{B}^{\mu\nu} &= \frac{i}{4}\,\overline{\psi}\gamma^{\{\mu}\,\overset{\leftrightarrow}{\partial}{}^{\nu\}}\psi. \qquad (66)
\end{aligned}
$$

Relations that directly follows from the equations of motion (e.o.m.) are $\overline{\psi}\,\Gamma\,\overset{\rightarrow}{\not\partial}\,\psi = -\overline{\psi}\,\overset{\leftarrow}{\not\partial}\,\Gamma\,\psi$ and $\partial_{\rho}\left(\overline{\psi}\{\gamma^{\rho},\Gamma\}\psi\right) = \overline{\psi}\,[\overset{\leftrightarrow}{\not\partial},\Gamma]\,\psi$.

Massless vector fields

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\,F_{\mu\nu}(x)F^{\mu\nu}(x), \qquad (67) \\
D_{\mu}\,F^{\mu\nu} &= [D_{\mu},F^{\mu\nu}] = \partial_{\mu}F^{\mu\nu} - ig\,[A_{\mu},F^{\mu\nu}] = 0, \qquad (68) \\
T^{\mu\nu} &= F^{\mu\rho}\partial^{\nu}A_{\rho} + \left(-\frac{1}{4}\,F_{\rho\sigma}F^{\rho\sigma}\right) g^{\mu\nu}, \qquad (69) \\
H^{\mu\rho\sigma} &= F^{\mu\rho}A^{\sigma} - F^{\mu\sigma}A^{\rho}, \qquad (70) \\
G^{\rho\mu\nu} &= F^{\rho\mu}A^{\nu}, \qquad (71) \\
\partial_{\rho}G^{\rho\mu\nu} &= (\partial_{\rho}F^{\rho\mu})A^{\nu} + F^{\rho\mu}\,\partial_{\rho}A^{\nu} \overset{e.o.m.}{=} F^{\rho\mu}\,\partial_{\rho}A^{\nu} + ig\,[A_{\rho},F^{\rho\mu}]\,A^{\nu} \\
&= F^{\rho\mu}\,(\partial_{\rho}A^{\nu} + ig\,[A^{\nu},A_{\rho}]) = F^{\rho\mu}\,D_{\rho}A^{\nu}, \qquad (72) \\
T_{B}^{\mu\nu} &= F^{\mu\rho}F^{\nu}{}_{\rho} + \left(-\frac{1}{4}\,F_{\rho\sigma}F^{\rho\sigma}\right) g^{\mu\nu}, \qquad (73)
\end{aligned}
$$

8 Currents in QCD

The space-time currents derived from the full lagrangian of QCD are

$$
\begin{aligned}
T^{\mu\nu} \quad &= \quad \frac{i}{2}\,\overline{\psi}\,\gamma^{\mu}\,\overset{\leftrightarrow}{\partial}{}^{\nu}\psi + \left(\frac{i}{2}\,\overline{\psi}\,\overset{\leftrightarrow}{\not D}\,\psi - m\,\overline{\psi}\psi\right) g^{\mu\nu} \\
&\quad + F^{\mu\rho}\partial^{\nu}A_{\rho} + \left(-\frac{1}{4}\,F_{\rho\sigma}F^{\rho\sigma}\right) g^{\mu\nu} \qquad (74) \\
&\overset{e.o.m.}{=} \quad \frac{i}{2}\,\overline{\psi}\,\gamma^{\mu}\,\overset{\leftrightarrow}{\partial}{}^{\nu}\psi + j^{\mu}A^{\nu} \\
&\quad + F^{\mu\rho}\partial^{\nu}A_{\rho} + \left(-\frac{1}{4}\,F_{\rho\sigma}F^{\rho\sigma}\right) g^{\mu\nu} - (\partial_{\rho}F^{\rho\mu} - ig\,[A_{\rho},F^{\rho\mu}])\,A^{\nu}
\end{aligned}
$$

$$= \quad \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{D}^\nu\psi + F^{\mu\rho}F^\nu_{\ \rho} + \left(-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right)g^{\mu\nu}. \tag{75}$$

Note that the currents are the sum of the currents for fermions and massless vector fields (the quark-gluon interaction term does not contain derivatives), but they are not separately gauge invariant. The (also conserved) symmetric Belinfante tensor is

$$T_B^{\mu\nu} \quad = \quad \frac{i}{4}\overline{\psi}\gamma^{\{\mu} \overleftrightarrow{D}^{\nu\}}\psi + F^{\mu\rho}F^\nu_{\ \rho} + \left(-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right)g^{\mu\nu}, \tag{76}$$

and the angular momentum tensor can be expressed in terms of the latter tensor as

$$M^{\mu\rho\sigma} = x^\rho\, T_B^{\mu\sigma} - x^\sigma\, T_B^{\mu\rho}. \tag{77}$$

9 Isospin/flavor symmetry

Neglecting at this point the color interactions, the lagrangian for the quarks consists of the free Dirac lagrangian for each of the types of quarks, called flavors. Including a sum over the different flavors (up, down, strange, etc.) one can write

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - M)\psi, \tag{78}$$

where ψ is extended to a vector in *flavor space* and M is a diagonal matrix,

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \\ \vdots \end{pmatrix}, \qquad M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & \ddots \end{pmatrix} \tag{79}$$

(Note that each of the entries in this vector for ψ is a 4-component Dirac spinor). This lagrangian density then is invariant under unitary (vector) transformations in the flavor space,

$$\psi \longrightarrow e^{i\,\underline{\alpha}\cdot\underline{T}}\psi, \tag{80}$$

which for instance including only two flavors form an $SU(2)_V$ symmetry (isospin symmetry) generated by the Pauli matrices, $\underline{T} = \underline{\tau}/2$. The conserved currents corresponding to this symmetry transformation are found directly using Noether's theorem,

$$\underline{V}^\mu = \overline{\psi}\gamma^\mu\underline{T}\psi. \tag{81}$$

Using the Dirac equation, it is easy to see that one gets

$$\partial_\mu\underline{V}^\mu = i\,\overline{\psi}\,[M,\underline{T}]\,\psi. \tag{82}$$

Furthermore $\partial_\mu\underline{V}^\mu = 0 \Longleftrightarrow [M,\underline{T}] = 0$. From group theory (Schur's theorem) one knows that the latter can only be true, if in flavor space M is proportional to the unit matrix, $M = m\cdot 1$. I.e. $SU(2)_V$ (isospin) symmetry is good if the up and down quark masses are identical. This situation, both are very small, is what happens in the real world.

This symmetry is realized in the Weyl mode. In the spectrum of QCD we see an almost perfect isospin symmetry, e.g. a doublet (isospin 1/2) of nucleons, proton and neutron, with almost degenerate masses (M_p = 938.3 MeV/c^2 and M_n =939.6 MeV/c^2), but also a triplet (isospin 1) of pions, etc.

10 Realization of symmetries

Before continuing we want to discuss recall the two possible ways that a symmetry can be implemented. They are known as the *Weyl mode* or the *Goldstone mode*

Weyl mode. In this mode the lagrangian *and* the vacuum are both invariant under a set of symmetry transformations generated by Q^a, i.e. for the vacuum $Q_a|0\rangle = 0$. In this case the spectrum is described by degenerate representations of the symmetry group. Well-known examples are rotational symmetry and the fact that in the spectrum multiplets appear labeled by angular momentum ℓ (with members labeled by m). The generators Q_a (in that case the rotation operators L_z, L_x and L_y or instead of the latter two L_+ and L_-) are used to label the multiplet members or transform them into one another. A bit more formal, if the generators Q_a generate a symmetry, i.e. $[Q_a, H] = 0$, and $|a\rangle$ and $|a'\rangle$ belong to the same multiplet (there is a Q_a such that $|a'\rangle = Q_a|a\rangle$) then $H|a\rangle = E_a|a\rangle$ implies that $H|a'\rangle = E_a|a'\rangle$, i.e. a and a' are degenerate states. The vacuum is actually a nondegenerate singlet representation of the symmetry group.

Goldstone mode. Also in this mode the lagrangian is invariant but now $Q_a|0\rangle \neq 0$ for a number of generators. This means that they are operators that create states from the vacuum, denoted $|\pi^a(k)\rangle$. The generators thus do not transform states within one multiplet! As the generators for a symmetry are precisely the zero-components of a conserved current $J_a^\mu(x)$ integrated over space, there must be a nonzero expectation value $\langle 0|J_a^\mu(x)|\pi_a;k\rangle$. Using translation invariance and the fact that k_μ is the only four vector on which this matrix element can depend one may write

$$\langle 0|J_a^\mu(x)|\pi_a;k\rangle = f_\pi\, k_\mu\, e^{i\,k\cdot x} \quad (f_\pi \neq 0) \tag{83}$$

for all the states labeled by indices 'a' corresponding to 'broken' generators. Taking the derivative,

$$\langle 0|\partial_\mu J_a^\mu(x)|\pi_a;k\rangle = f_\pi\, k^2\, e^{i\,k\cdot x} = f_\pi\, m_{\pi_a}^2\, e^{i\,k\cdot x}. \tag{84}$$

If the transformations in the lagrangian give rise to a symmetry the Noether currents are conserved, $\partial_\mu J_a^\mu = 0$, irrespective of the fact if they annihilate the vacuum, and one must have $m_{\pi_a} = 0$, i.e. a massless Goldstone boson for each 'broken' generator. Note that for the fields $\pi_a(x)$ one would have the relation $\langle 0|\pi_a(x)|\pi_a;k\rangle = e^{i\,k\cdot x}$, suggesting the stronger relation $\partial_\mu J_a^\mu(x) = f_\pi\, m_{\pi_a}^2\, \pi_a(x)$.

11 Chiral symmetry

Turning back to QCD, one can for fermions consider another set of symmetry transformations, socalled axial vector transformations,

$$\psi \longrightarrow e^{i\,\underline{\alpha}\cdot\underline{T}\gamma_5}\psi, \tag{85}$$

which for instance including only two flavors form $SU(2)_A$ transformations generated by the Pauli matrices, $\underline{T}\gamma_5 = \underline{\tau}\gamma_5/2$. Note that these transformations also work on the spinor indices. The currents corresponding to this symmetry transformation are again found using Noether's theorem,

$$\underline{A}^\mu = \overline{\psi}\gamma^\mu\underline{T}\gamma_5\psi. \tag{86}$$

Using the Dirac equation, it is easy to see that one gets

$$\partial_\mu\underline{A}^\mu = i\,\overline{\psi}\,\{\underline{M},\underline{T}\}\,\gamma_5\,\psi. \tag{87}$$

In this case $\partial_\mu\underline{A}^\mu = 0$ will be true if the quarks have zero mass, which is approximately true for the up and down quarks. Therefore the world of up and down quarks describing pions, nucleons and atomic nuclei has not only an isospin or vector symmetry $SU(2)_V$ but also an (approximate) axial vector symmetry $SU(2)_A$. This combined symmetry is what one calls *chiral symmetry.*

That the massless theory has this symmetry can also be seen by writing it down for the socalled lefthanded and righthanded fermions, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$, in terms of which the Dirac lagrangian becomes

$$\mathcal{L} = i\,\overline{\psi}_L\partial\!\!\!/\psi_L + i\,\overline{\psi}_R\partial\!\!\!/\psi_R - \overline{\psi}_R\underline{M}\psi_L - \overline{\psi}_L\underline{M}\psi_R. \tag{88}$$

If the mass is zero the lagrangian is split into two disjunct parts for L and R showing that there is a direct product $SU(2)_L \otimes SU(2)_R$ symmetry, generated by $\underline{T}_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\underline{T}$, which is equivalent to the V-A symmetry. This symmetry, however, is by nature not realized in the Weyl mode. How can we see this. The chiral fields ψ_R and ψ_L are transformed into each other under parity. Therefore realization in the Weyl mode would require that all particles exist with both positive and negative parity, or, stated equivalently, parity would not play a role in the world. We know that mesons and baryons (such as the nucleons) have a well-defined parity that is conserved. Positive and negative parity hadrons are *not* degenerate.

The conclusion is that the original symmetry of the lagrangian is spontaneously broken (the vacuum is not invariant). Since the vector part of the symmetry is the well-known (in essence unbroken) isospin symmetry, nature has choosen the path

$$SU(2)_L \otimes SU(2)_R \quad \Longrightarrow \quad SU(2)_V,$$

i.e. the lagrangian density is invariant under left (L) and right (R) rotations independently, while the groundstate is only invariant under isospin rotations ($R = L$). From the number of broken generators it is clear that one expects three massless Goldstone bosons, for which the field (according to the discussion in the previous section) has the same behavior under parity, etc. as the quantity $\partial_\mu A^\mu(x)$, i.e. (leaving out the flavor structure) the same as $\overline{\psi}\gamma_5\psi$. The field corresponds to a pseudoscalar particle (spin zero, parity minus). In the real world, where the quark masses are not completely zero, chiral symmetry is not perfect. Still the basic fact that the generators acting on the vacuum give a nonzero result and $f_\pi \neq 0$ remains. The fact that the symmetry is not perfect and the right hand side of Eq. 87 is nonzero, gives rise to a nonzero (but small) mass for the Goldstone bosons according to Eq. 84. The Goldstone bosons of QCD are the pions for which $f_\pi = 93$ MeV and which have a mass of $m_\pi \approx 138$ MeV/c^2, much smaller than any of the other mesons or baryons.

Axial anomaly

Not only for the space-time currents one must be careful in a gauge theory, e.g. the presence of a color current coming from the gluons, the sum of which is conserved. A special case is also the singlet axial current, i.e. having in flavor space $T = 1$. Besides the presence of a mass term, it is not conserved because of quantum corrections,

$$\partial_\mu(\overline{\psi}\,\gamma^\mu\gamma_5\,\psi) = 2i\,m\,\overline{\psi}\,\gamma_5\,\psi + \frac{\alpha_s}{2\pi}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}, \tag{89}$$

where $\tilde{F}_{\mu\nu} = -(1/2)\,\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. Note that the right-hand side can be written as a total derivative,

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu k^\mu \tag{90}$$

with

$$\begin{aligned} k_\mu &= -\epsilon_{\mu\nu\rho\sigma}\,A^\nu\,\left(F^{\rho\sigma} + \frac{1}{3}ig\,[A^\rho, A^\sigma]\right) \\ &= -\epsilon_{\mu\nu\rho\sigma}\,A^\nu\,\left(\partial^\rho A^\sigma - \partial^\sigma A^\rho - \frac{2}{3}ig\,[A^\rho, A^\sigma]\right) \\ &= -2\,\epsilon_{\mu\nu\rho\sigma}\,\left(A^\nu\partial^\rho A^\sigma - \frac{2}{3}ig\,A^\nu A^\rho A^\sigma\right). \end{aligned} \tag{91}$$

Exercise. Show that $\partial_\mu k^\mu = F_{\mu\nu}\tilde{F}^{\mu\nu}$.

A term like this, which arises only at the quantum level, is known as an anomaly term, in this case it is known as the Adler-Bell-Jackiw anomaly or $U(1)_A$ anomaly. The (gauge variant) current k^μ cannot be derived from a symmetry, it is an example of a topological current. Another argument for this term considers the axial current as

$$\overline{\psi}(x)\gamma_\mu\gamma_5\psi(x) = \lim_{\epsilon\to 0}\ \overline{\psi}(x-i\epsilon)\,\gamma_\mu\gamma_5\,\mathcal{U}(x-i\epsilon,x+i\epsilon)\,\psi(x+i\epsilon). \tag{92}$$

A careful handling of this limit leads to the axial anomaly.

12 Hadrons

In low energy QCD, e.g. the spectrum, the relevant degrees of freedom are the hadrons, the bound states of quarks, antiquarks and (possibly) gluons. Some aspects can be studied in lattice approaches in which one, e.g. by discretizing space-time, can try to study Green's functions using computers. Other aspects use specific models. Quark potential models are the earliest models treating quarks explicitly and subjecting them to an interquark potential that confines them. Specifically for the heavy quarks, potential models are useful because they can be treated nonrelativistically using a Schrödinger equation. The latter is one of the main objections agains the use of potential models for the light quarks. Also bag models treat quarks explicitly. They build in confinement in an ad hoc way, by imposing appropriate boundary conditions that ensure confinement of the quark currents to a region of space called the bag. The advantage is that within this bag quarks can be relativistic. In fact they are usually treated as free particles in accordance with the asymptotic freedom of QCD. Chiral symmetry requires the addition of mesons in order to ensure continuity of axial currents in and outside the bag.

Effective field theories are also employed for the study of hadrons. For example, Skyrme models based on an $1/N_c$ expansion of QCD, where N_c is the number of colors. In the limit $N_c \to \infty$, QCD only contains stable mesons with finite masses. Simple counting arguments show that the three-meson (MMM) coupling is proportional to $1/\sqrt{N_c}$ and the MMMM coupling is proportional to $1/N_c$. Baryons in this limit have infinite mass ($\propto N_c$) and finite size ($\propto 1$). The latter behavior is also found for solitons, topologically stable classical solutions arising in field theories with a nontrivial topological structure. An example of such a model is the nonlinear sigma model, with pions as the only degrees of freedom.
[On large N_c expansion and hadrons, see E. Witten, Nucl. Phys. B 160 (1979) 57 and Nucl. Phys. B 223 (1983) 552].

Flavor and spin structure of hadrons

The flavor and spin structure of the experimentally observed hadrons can be accounted for by quarks. Baryons are in essence built from three quarks and mesons are built from a quark and an antiquark. The quarks can be considered as belonging to the basic representation of an $SU(N_f)$ flavor symmetry group. Only for $SU(2)$ this symmetry (isospin) is a very good symmetry as the bare quark masses m_u and m_d are much smaller than the QCD scale parameter. Also for three flavors (u, d and s) the symmetry is useful, specifically because it is often only the quark masses which break the symmetry. For the heavy quarks, another $SU(3)$ symmetry involving the c, b and t quarks can be useful when one considers quantities that are independent of the quark masses. The quarks have spin $j = 1/2$, belonging to the basic representation of $SU(2)$. For symmetry considerations it is often useful to consider the three quark flavors with the two spin degrees of freedom as the basic representation of an $SU_{FS}(6)$ flavor-spin symmetry. Antiquarks belong to the conjugate representations of the various symmetry groups.

The spin and flavor structure of baryons and mesons can be found by combining the basic representations. Starting with the q^3 baryons one has for the spin of three quarks

$$SU_J(2)\colon\quad \frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}, \tag{93}$$

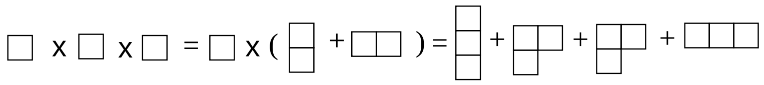


Figure 3: determination of the product of basic representations in $SU(N)$ using Young diagrams

where the spin (j) values are used to denote the (2j+1)-dimensional representations. In this case the decomposition can be read off from the Young diagram rule in Fig. 3, noting that a representation in which three basic entities are coupled antisymmetrically is impossible for $SU(2)$, i.e. all columns with more than two boxes are omitted. Furthermore columns of two boxes, corresponding to an antisymmetrized combinations of spin up and spin down, represent the trivial one-dimensional representation. In combination with other boxes they also can be omitted.

For flavor one has

$$SU_F(3):\quad \underline{3}\otimes\underline{3}\otimes\underline{3} = \underline{1}\oplus\underline{8}\oplus\underline{8}\oplus\underline{10}, \tag{94}$$

where the representations are labeled by their dimensions. For flavor $\otimes$ spin one has

$$SU_{FS}(6):\quad [6]\otimes[6]\otimes[6] = [20]\oplus[70]\oplus[70]\oplus[56], \tag{95}$$

where the representations are also labeled by their dimensions. Besides the above quantum numbers color has to be taken into account. Requiring the hadrons to be color singlets, i.e. belonging to the completely antisymmetric $SU_C(3)$ representation, one deduces that the flavor $\otimes$ spin wave function must be completely symmetric, i.e. belong to the [56] representation. The $SU(3)\otimes SU(2)$ content, $(\underline{f}, s)$, of this representation is

$$[56] \supset (\underline{8}, 1/2) + (\underline{10}, 3/2). \tag{96}$$

The explicit flavor-spin wave functions for the proton can be constructed by symmetrizing the direct product of flavor and spin wave functions. Only the combinations in the [56] representation of $SU(6)$ will give a nonvanishing result. For the proton one gets

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\left(2\,u_\uparrow u_\uparrow d_\downarrow - u_\uparrow u_\downarrow d_\uparrow - u_\downarrow u_\uparrow d_\uparrow + \ [d \text{ at places 1 and 2}]\right).\Big), \tag{97}$$

and for the lambda

$$|\Lambda\uparrow\rangle = \frac{1}{\sqrt{12}}\left(u_\uparrow d_\downarrow s_\uparrow - u_\downarrow d_\uparrow s_\uparrow - d_\uparrow u_\downarrow s_\uparrow + d_\downarrow u_\uparrow s_\uparrow + \ [s_\uparrow \text{ at places 1 and 2}]\right). \tag{98}$$

Other wave functions can be obtained directly by replacements $u \to d$ (e.g. neutron from proton) or $u \to s$ (Σ^- from n). Combining spatial wave functions (accounting for total antisymmetry of the wave function) one gets in a simple quark shell-model the allowed baryon spectrum with ([fs],L^P) states,

$$([56], 0^+)\oplus([70], 1^-)\oplus([56], 0^+)\oplus\ldots, \tag{99}$$

combining to states with $J = L+s, \ldots, L-s$.

For the spin of $q\bar{q}$ mesons all combinations

$$SU_J(2):\quad \frac{1}{2}\otimes\frac{1}{2} = 0\oplus 1, \tag{100}$$

$$SU_F(3):\quad \underline{3}\otimes\underline{3}^* = \underline{1}\oplus\underline{8}, \tag{101}$$

are possible. Including orbital excitations the allowed J^{PC} values are, for $L = 0, 1, 2, \ldots$, states with $J = L-S, \ldots, L+S$, $P = (-)^{L+1}$, and for neutral mesons $C = L+S$. C-parity is useful as it determines the number of photons to which the meson couples, $C = (-)^{n_\gamma}$. For mesons $G-parity$, $G = (-)^{L+S+I}$ is useful, which determines the number of pions to which the meson couples, $C = (-)^{n_\pi}$.

13 Quark potential models

Quark potential models are the oldest models in the study of hadron structure. In their simplest form quarks move in a background potential, leading to a shell model structure. The groundstate baryons, the $1/2^+$ octet and $3/2^+$ decouplet, and the groundstate mesons, the 0^- and 1^- nonets, only contain the lowest s-wave solution. An often used potential is the harmonic oscillator, which also allows treatment of excited baryons and mesons, because it is easy to separate center of mass and relative coordinates. Clearly quark potential models may be expected to work best for heavy quark systems. For such systems more realistic potentials than the abovementioned harmonic oscillator have been used, behaving as $\alpha_s(r)/r$ for short distances in accordance with asymptotic freedom and as $T_0\,r$ corresponding to a linear confining potential for large distances. The typical value of the linear energy density (or string density) is typically $T_0 \approx 0.18$ GeV^2 ≈ 0.9 GeV/fm. This relates to the slope of meson Regge trajectories, $J = \alpha_0 + \alpha'\,M^2$ via the relation $1/\alpha' = 2\pi\,T_0 \approx 1.1$ GeV^2.

The relation between the slope of Regge trajectories and the string tension can be obtained with the following semi-classical argument (noting that this is the limit of the quantum mechanical treatment for large angular momenta). Consider a rotating string, rotating around an axis perpendicular on the string, the ends (with massless quarks) moving with $v = \pm 1$. In the rotating string the energy-density along the string is given by $T_0\,\gamma = T_0/\sqrt{1-v^2}$. This leads (using $dr = dv/\omega$ with ω the angular velocity) to the following expressions for mass and angular momentum,

$$
\begin{aligned}
M &= \frac{T_0}{\omega}\int_{-1}^{1} dv\,\frac{1}{\sqrt{1-v^2}} = \frac{\pi T_0}{\omega},\\
J &= \frac{T_0}{\omega^2}\int_{-1}^{1} dv\,\frac{v^2}{\sqrt{1-v^2}} = \frac{\pi T_0}{2\omega^2}.
\end{aligned}
$$

Thus one finds

$$
M^2 = \frac{\pi^2\,T_0^2}{\omega^2} = 2\pi T_0\,J,
$$

the linear relation between mass squared of orbitally excited states and the angular momentum.

Except for the confinement mechanism another important feature in models with explicit quark degrees of freedom is the hyperfine structure due to the color-magnetic part of the one-gluon exchange interaction. It is of the form

$$
H_{\rm M} = \alpha_s\,M\,\Delta \tag{102}
$$

with

$$
\Delta = -\sum_{i>j}\langle(\boldsymbol{\sigma} F^a)_i\cdot(\boldsymbol{\sigma} F^a)_j\rangle, \tag{103}
$$

where F^a indicate the color charges, for the quarks represented by $F_q^a = \lambda^a/2$, for the antiquarks by $F_{\bar q}^a = -\lambda^{a*}/2$. The quantity M depend on the model. In quark potential models it could be a hyperfine interaction $\propto 1/r^3$

The operator Δ can be calculated straightforwardly. For instance, for a meson we must form a color singlet. Since $\langle F^2\rangle = \langle(F_q + F_{\bar q})^2\rangle = 0$ and $\langle F_q^2\rangle = \langle F_{\bar q}^2\rangle = 4/3$, one obtains $\langle F_q\cdot F_{\bar q}\rangle = -4/3$. Similarly manipulations are well-known for the spin, yielding $\langle\boldsymbol{\sigma}_q\cdot\boldsymbol{\sigma}_{\bar q}\rangle = -3$ for a spin 0 meson and $\langle\boldsymbol{\sigma}_q\cdot\boldsymbol{\sigma}_{\bar q}\rangle = +1$ for a spin 1 meson. Similarly one can find for the baryons $\langle F_1\cdot F_2\rangle = -2/3$ for any pair. For the spin the explicit wave function can be used to find the result for Δ. For the case of two quarks (coupling to a color triplet c = $\underline{3}$), a quark and an antiquark and three quarks (baryons) the expectation value of Δ for some specific configurations, namely a diquark (system of two quarks), mesons and baryons are as follows.

quark configuration	spin	flavor (isospin)	Δ
qq	0	$\underline{3}^*$ (0)	-2
	1	$\underline{6}$ (1)	$+\,2/3$
$q\bar{q}$	0	$\underline{1} \oplus \underline{8}$ $(0 \oplus 1)$	-4
	1	$\underline{1} \oplus \underline{8}$ $(0 \oplus 1)$	$+4/3$
qqq	1/2	$\underline{8}$ (1/2)	-2
	3/2	$\underline{10}$ (3/2)	$+\,2$

Note that the attraction between two quarks in the spin singlet state (color $\underline{3}^*$) has led to the suggestion that in the early-universe quark-gluon phase a color-superconducting phase may have played a role.

14 The bag model

As an example of a model which incorporates asymptotic freedom and confinement is the (MIT) bag model [T. DeGrand, R.L. Jaffe, K. Johnson and J. Kiskis, Phys. Rev. D 12 (1975) 2060]. In essence confinement is obtained by ensuring that quarks cannot flow out of a certain volume, called the *bag*. For this purpose the quark current, $j_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$, is replaced by $j_{B\mu} \equiv j_\mu(x).\theta_B(x)$, where θ_B is a theta-function with the value one within the bag volume. The surface normal is indicated as n_μ, satisfying $\partial_\mu\theta_B(x) = n_\mu\delta_B(x)$. Current conservation in the bag model therefore requires that

$$\partial_\mu j^\mu_B(x) = (\partial_\mu j^\mu(x))\,\theta_B(x) + n_\mu j^\mu(x)\,\delta_B(x). \tag{104}$$

The boundary condition thus is $n_\mu j^\mu = 0$ at the surface of the bag.

Exercise. Show that for conservation of the quark current a sufficient condition is $\not{n}\psi = e^{i\,\alpha\gamma_5}\psi$. Actually only for $\alpha = 0$ and $\alpha = \pi$ a symmetric spectrum of positive and negative energy states emerges. The case $\alpha = 0$ is discussed below. For the case $\alpha = \pi$ there is a fermionic zeromode.

Applied to a free s-wave Dirac spinor,

$$\psi = \frac{N}{\sqrt{4\pi}}\begin{bmatrix} i\,F(r) \\ -\,\boldsymbol{\sigma}\cdot\hat{r}\,G(r)\end{bmatrix} e^{-iEt}, \tag{105}$$

with

$$F(r) = \sqrt{\frac{E+m}{E}}\,j_0(kr), \tag{106}$$

$$G(r) = \sqrt{\frac{E-m}{E}}\,j_1(kr). \tag{107}$$

The confinement condition on this spinor leads for a spherical bag with radius R to the condition

$$\tan x = \frac{x}{1-\lambda-\alpha}, \tag{108}$$

where $x = kR$, $\alpha = ER$ and $\lambda = mR$. The lowest energy solution for a massless quark yields

$$m = 0; \qquad x = \alpha = 2.043, \tag{109}$$

and for what turns out to be an appropriate value for the strange quark ($m \approx 280$ MeV and $R \approx 5\ \mathrm{GeV}^{-1} \approx 1$ fm)

$$\lambda = 1.4; \qquad x = 2.489; \qquad \alpha = 2.856. \tag{110}$$

For a specific quark mode in the bag model the normalization becomes

$$\frac{N^2}{4\pi} = \frac{\alpha(\alpha-\lambda)}{2\alpha(\alpha-1)+\lambda}\,\frac{1}{4\pi\,R^3\,j_0^2(x)}. \tag{111}$$

The equations of motion for the bag model including the boundary conditions can also be obtained in a lagrangian formulation. One needs

$$\mathcal{L} = (i\,\overline{\psi}\partial\!\!\!/\psi - m\,\overline{\psi}\psi - B)\theta_B(x) - \frac{1}{2}\,\overline{\psi}\psi\,\delta_B(x). \tag{112}$$

The bag pressure B is essential to obtain stable solutions. It gives rise to a volume energy $\propto BR^3$ preventing the bag to blow up and balancing the quark kinetic energies $\propto N\,\alpha/R$. Its value is something like $B^{1/4} = 0.15$ GeV.

The hyperfine interaction in the bag model is of the form discussed in the previous paragraph. The quantity M in this case simply is a function of the mode energy which depends on $\lambda_i = m_i R$ for the two quarks involved, $M(m_i, m_j, R) = M(\lambda_i, \lambda_j)$ with $M(0,0) = 0.177$, $M(0,1.4) = 0.143$ and $M(1.4,1.4) = 0.116$.

The bag modes can be used to calculate, e.g. the charge and current densities and their Fourier transforms, the electromagnetic form factors. The structure of the nucleon in terms of the quarks combined with the electromagnetic current in terms of the quarks determines the form factors. For example in the bag model the effects of the quarks is additive with the expectation value of the zero and three-vector components given by

$$\begin{aligned}
\langle \mathrm{Bag}|J^0(x)|\mathrm{Bag}\rangle &= \sum_i e_i\;\rho_i(x) && (113)\\
\rho(x) &= \overline{\psi}(x)\gamma^0\psi(x) = \psi^\dagger(x)\psi(x) = \frac{N^2}{4\pi}(F^2+G^2), && (114)\\
\langle \mathrm{Bag}|\mathbf{J}(x)|\mathrm{Bag}\rangle &= \sum_i e_i\;\mathbf{j}_i(x) && (115)\\
\mathbf{j}(x) &= \overline{\psi}(x)\boldsymbol{\gamma}\psi(x) = \psi^\dagger(x)\rho_1\boldsymbol{\sigma}\psi(x)\\
&= \frac{N^2}{4\pi}\,F.G\,(\boldsymbol{\sigma}\times\hat{r}) \equiv \frac{f(r)}{r^2}\,(\boldsymbol{\sigma}\times\mathbf{r}). && (116)
\end{aligned}$$

Taking the Fourier transform, e.g. of $\mathbf{j}(\mathbf{r})$ we obtain

$$\mathbf{j}(\boldsymbol{q}) = \sum_i e_i\,(i\boldsymbol{\sigma}_i\times\boldsymbol{q})G_M^{\mathrm{bag}}(\boldsymbol{q}), \tag{117}$$

with for $\mu_i^{\mathrm{bag}} = G_M^{\mathrm{bag}}(0)$

$$\begin{aligned}
\mu_i^{\mathrm{bag}} = \int d^3r\;f(r)/3 \quad &= \frac{R}{6}\,\frac{4\alpha+2\lambda-3}{2\alpha(\alpha-1)+\lambda}\\
&= \frac{1}{2m}\,\frac{\lambda}{3}\,\frac{4\alpha+2\lambda-3}{2\alpha(\alpha-1)+\lambda}\\
&\stackrel{m=0}{=} \frac{4x-3}{12x(x-1)}\,R = 0.202\,R. && (118)
\end{aligned}$$

The latter value, $\mu_n^{\mathrm{bag}} = 0.202\,R$ is appropriate for the (almost) massless up and down quarks, for a strange quark with $\lambda = mR = 1.4$ the result is $\mu_s^{\mathrm{bag}} = 0.156\,R = 0.436/2m_s$. The flavor dependence of the magnetic moments are as in the nonrelativistic quark model further determined by the expectation values of $\sum e_i\,\mu_i^{\mathrm{bag}}\,\sigma_{zi}$, i.e. for the proton

$$\begin{aligned}
\frac{\mu_p}{2M} = \frac{G_M^p(0)}{2M} &= \langle p|\sum e_i\,\mu_i^{\mathrm{bag}}\,\sigma_{zi}|p\rangle\\
&= \frac{4}{3}\,e_u\,\mu_u^{\mathrm{bag}} - \frac{1}{3}\,e_d\,\mu_d^{\mathrm{bag}} = \mu_n^{\mathrm{bag}}, && (119)
\end{aligned}$$

and in a similar way obtain for the neutron

$$\begin{aligned}
\frac{\mu_n}{2M} &= -\frac{1}{3}\,e_u\,\mu_u^{\rm bag} + \frac{4}{3}\,e_d\,\mu_d^{\rm bag} = -\frac{2}{3}\,\mu_n^{\rm bag}, \qquad (120)\\
\frac{\mu_\Lambda}{2M} &= e_s\,\mu_s^{\rm bag} = -\frac{1}{3}\mu_s^{\rm bag}. \qquad (121)
\end{aligned}$$

With a characteristic bag radius $R \approx 1$ fm the result for the proton is $\mu_p = 2M\,\mu_n^{\rm bag} \approx 1.9$. The experimental result for the proton is $\mu_p = 2.79$.

One problem of the bag model - or in any other static extended hadron model - is the treatment of the center of mass motion. The calculation has been performed using a static bag, which is not an momentum eigenstate, such as appears in the nucleon current expectation value. At best a bag can be considered as a superposition of plane waves,

$$|Bag\rangle = \int \frac{d^3P}{(2\pi)^3}\;\Phi(\boldsymbol{P})\,|\boldsymbol{P}\rangle. \qquad (122)$$

For such a superposition one has $\langle \boldsymbol{P}\rangle = 0$ but $\langle \boldsymbol{P}^2\rangle \neq 0$. One can estimate the corrections for the magnetic moments yielding a correction of about +10 %.

Note the successful calculation of the ratios of the magnetic moments, $\mu_n/\mu_p = -2/3$ (exp: -0.685) and $\mu_\Lambda/\mu_p = -0.26$ (exp: -0.22), which are a success for the naive additive quark model based on $SU(6)$ symmetry.

The axial current density can also be calculated in the bag model. For a quark one has $A^0 = \psi^\dagger\gamma_5\psi = 0$ and

$$\mathbf{A} = \psi^\dagger\boldsymbol{\sigma}\psi = \frac{N^2}{2\pi}\left[\left(F^2+G^2\right)\boldsymbol{\sigma} + 2G^2\,(\boldsymbol{\sigma}\cdot\hat{r})\hat{r}\right]. \qquad (123)$$

For $G_A(0)$ we obtain for a transition of quark mode i to quark mode j

$$\begin{aligned}
G_{A,i-j}^{\rm bag} &= \frac{N^2}{4\pi}\int d^3r\;\left(F^2 - \frac{1}{3}G^2\right)\\
&= \frac{2x_ix_j}{3(x_i^2-x_j^2)}\,\frac{2(\alpha_i-\alpha_j)+(\lambda_i-\lambda_j)}{\sqrt{2\alpha_i(\alpha_i-1)+\lambda_i}\,\sqrt{2\alpha_j(\alpha_j-1)+\lambda_j}}\\
&\overset{\lambda_i=\lambda_j}{=} \frac{2\alpha^2+4\lambda\alpha-3\lambda}{3\left[2\alpha(\alpha-1)+\lambda\right]}\\
&\overset{\lambda=0}{=} \frac{x}{3(x-1)}. \qquad (124)
\end{aligned}$$

Explicitly one has for massless nonstrange quarks $G_{A,n-n}^{\rm bag} = 0.653$. Also here one needs to include some spin-factors,

$$G_A^{n\rightarrow p} = G_{A,d\rightarrow u}^{\rm bag}\langle p|\sum_i F_i^{d\rightarrow u}\sigma_{zi}|n\rangle = \frac{5}{3}G_{A,d\rightarrow u}^{\rm bag}, \qquad (125)$$

where the flavor operator $F^{d\rightarrow u}$ denotes $(\lambda_1+i\lambda_2)/2$ or for two flavors $(\tau_1+i\tau_2)/2$. It is common to compare the group-theoretical result with the (vector current) matrix element $\langle p|\sum_i F_i^{d\rightarrow u}|n\rangle$, which equals one, i.e. the quark model result is $(G_A/G_V)_{QM} = 5/3$. The bag result, $G_A = \frac{5}{3}G_{A,d\rightarrow u}^{\rm bag} = 1.09$ is in reasonable agreement with the experimental result, specifically if one includes center of mass corrections which in this case can be estimated to be about +6 %.

15 Effective Chiral lagrangians and solitons

We already have discussed chiral symmetry in strong interactions and the fact that it is spontaneously broken,

$$SU_L(2)\otimes SU_R(2)\otimes U_V(1) \;\longrightarrow\; SU_V(2)\otimes U_V(1).$$

This means that there is a (parity changing) triplet of generators for which $Q_i|0\rangle \neq 0$, creating an isotriplet of 0^- massless Goldstone bosons. Considering explicitly QCD with two flavors, omitting all gluons which do not care about the flavor symmetry, we focus on the isospin structure of $\mathcal{L} = i\,\overline{\psi}\partial\!\!\!/\psi - \overline{\psi}\underline{M}\psi$ in which ψ and M now have also an isospin structure,

$$\psi = \left(\begin{array}{c} \psi^u \\ \psi^d \end{array}\right), \qquad \underline{M} = \left(\begin{array}{cc} m_u & 0 \\ 0 & m_d \end{array}\right). \tag{126}$$

Projecting on righthanded and lefthanded fermions $\psi_{R/L} = P_{R/L}\,\psi$ with $P_{R/L} = (1 \pm \gamma_5)/2$ one obtains

$$\mathcal{L} = i\,\overline{\psi}_R\partial\!\!\!/\psi_R + i\,\overline{\psi}_L\partial\!\!\!/\psi_L - \overline{\psi}_R\underline{M}\psi_L - \overline{\psi}_L\underline{M}\psi_R, \tag{127}$$

which shows immediately that if $M = 0$ the lagrangian is invariant under $SU_L(2) \otimes SU_R(2)$ (*chiral symmetry*), explicitly

$$SU_R(2):\ \psi_R \rightarrow V_R\,\psi_R \qquad \text{with} \qquad V_R = \exp\left(i\underline{\alpha}_R \cdot \underline{T}\right), \tag{128}$$

$$SU_L(2):\ \psi_L \rightarrow V_L\,\psi_L \qquad \text{with} \qquad V_L = \exp\left(i\underline{\alpha}_L \cdot \underline{T}\right), \tag{129}$$

with for two flavors $\underline{T} = \boldsymbol{\tau}/2$. In nature only the symmetry $V_R = V_L$ is realized, i.e. $\psi \rightarrow V\,\psi$. The mechanism underlying the spontaneous symmetry breaking may be *instantons* in QCD. The fact that $M \neq 0$ exactly, implies that $m_\pi \neq 0$.

The implications of chiral symmetry in the hadronic world can be studied in socalled effective theories. For instance one expects that in the mesonic sector (which in the large N_c limit is the most important) the Goldstone bosons will be the lightest ones which dominate the physics at low energies. Representing meson fields by a unitary $N_f \times N_f$ matrix,

$$U = \exp\left(i\,\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{f_\pi}\right) = \frac{1}{f_\pi}\left(\sigma + i\,\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right), \tag{130}$$

with for the latter representation $\sigma^2 + i\boldsymbol{\phi}^2 = f_\pi^2$ the simplest term in the lagrangian is

$$\mathcal{L} = \frac{f_\pi^2}{4}\,\mathrm{Tr}\left(\partial_\mu U\,\partial^\mu U^\dagger\right) \approx \frac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} + \ldots, \tag{131}$$

which is invariant under

$$U \longrightarrow V_L\,U\,V_R^\dagger. \tag{132}$$

The symmetry is spontaneously broken, that is the groundstate, which is a constant unitary matrix, say $U_0 \equiv 1$, is only invariant under a combined left-right multiplication, $V_L = V_R$. The symmetry, therefore, is spontaneously broken to $SU_V(2)$, the isospin (or in general flavor) symmetry group. Connected with such a spontaneous symmetry breaking there are 3 (massless) Goldstone modes. With the choice of groundstate ($U_0 = 1$) they are precisely represented as the 2×2 unitary matrices U, so the lagrangian above also represents the result after breaking the symmetry.

Already early in the sixties, Skyrme noted that the possibility of topological excitations exists in this theory. In order to find out if there are classical (time-independent) solutions with finite energy that approach the vacuum, $U(\boldsymbol{r}) \rightarrow 1$ for $|\boldsymbol{r}| \rightarrow \infty$ one must consider the mappings from R^3 with all points at infinity identified ($\sim S^3$) to the parameter-space of the fields, i.e. $SU(2) \sim S^3$. The topological structure of these mappings is $\Pi_3(SU(2)) = \Pi_3(S^3) = Z$. A representative solution in each class is the hedgehog solution,

$$U_c(\boldsymbol{r}) = \exp\left(i\boldsymbol{\tau}\cdot\boldsymbol{e}_\pi\,F(r)\right) = \cos F(r) + i\boldsymbol{\tau}\cdot\boldsymbol{e}_\pi(\boldsymbol{r})\,\sin F(r), \tag{133}$$

with e.g. for the simplest nontrivial solution $\boldsymbol{e}_\pi(\boldsymbol{r}) = \boldsymbol{r}/r$. The angle F is referred to as the chiral angle and satisfies $F(r) \rightarrow 0$ for $r \rightarrow \infty$. To find univalued solutions at the origin one most have $F(0) = n\pi$ with n integer, exhibiting the topological structure Z. The mappings can be characterized by a topological quantum number that has no dynamical origin. Nevertheless it can be connected to a conserved (topological) current, in this case

$$j^{\mu} = \frac{1}{24\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(L_\nu L_\rho L_\sigma), \tag{134}$$

where $L_\mu \equiv U^\dagger \partial_\mu U$ (the Maurer-Cartan forms). The normalization of the current is chosen such that n corresponds to the topological charge,

$$\int d^3x\; j^0(x) = -\int_0^\infty dr\;\frac{2\,F'(r)\,\sin^2 F(r)}{\pi} = \frac{F(0)-F(\infty)}{\pi} = n, \tag{135}$$

where $F' = dF/dr$. The solitons in this model are in fact the groundstates in different sectors of the Hilbert space, already by Skyrme identified with sectors with nonzero baryon number. The topological current is identified as the baryon current. We will not discuss this here in further detail, nor the study of the excitation spectrum in the baryon sector. We only mention the success of the model for the description of groundstate baryon excitations and the nucleon-pion phase shifts, all of this achieved by studying and (semi-classically) quantizing the fluctuations of the classical skyrmion solutions. We note that the simple first order term indicated in the effective lagrangian suffices to discuss the possibility of nontrivial classical solutions, but to obtain stable solutions from the equations of motion requires higher order terms, which by the way are quite natural in an effective theory.

A nice example to illustrate how symmetries can be used to constrain the terms in the effective lagrangian is the mass term. We note first that the mass term in QCD breaks chiral symmetry, seen by writing it in terms of left- and righthanded fermion fields, $-\overline{\psi}\underline{M}\psi = -\overline{\psi}_R\underline{M}\psi_L - \overline{\psi}_L\underline{M}\psi_R$ (note $\underline{M} = \underline{M}^\dagger$ is the diagonal quark mass matrix). Secondly we note that when one would allow $\underline{M}$ to transform under chiral transformations as $\underline{M} \rightarrow V_L\underline{M}V_R^\dagger$ (and thus $\underline{M}^\dagger \rightarrow V_R\underline{M}V_L^\dagger$) chiral invariance is restored. The simplest realization of the above behavior in the effective lagrangian is a term linear in U of the form

$$\mathcal{L}_m^{\mathrm{eff}} = \frac{Bf_\pi^2}{2}\,\mathrm{Tr}\left(\underline{M}\,U + \underline{M}\,U^\dagger\right) = (m_u+m_d)\,B\,\left(f_\pi^2 - \frac{1}{2}\boldsymbol{\pi}^2 + \ldots\right). \tag{136}$$

Equating the vacuum expectation value of the QCD and effective mass term leads to

$$B\,f_\pi^2(m_u+m_d) = -m_u\langle 0|\overline{\psi^u}\psi^u|0\rangle - m_d\langle 0|\overline{\psi^d}\psi^d|0\rangle, \tag{137}$$

or since $B\,(m_u+m_d) = M_\pi^2$ one gets the *Gell-Mann - Oakes - Renner relation*

$$f_\pi^2\,M_\pi^2 = -(m_u+m_d)\langle 0|\overline{\psi^u}\psi^u|0\rangle, \tag{138}$$

(assuming isospin symmetry). From it one sees that chiral symmetry breaking implies $M_\pi^2 \neq 0$ for the hadronic world, corresponding to $\langle 0|\overline{\psi}^u\psi^u|0\rangle \neq 0$ for the (nonstrange) quark fields.

Bag models and chiral lagrangians can be coupled in a consistent way with quarks inside and pions outside of the bag by taking for the boundary condition term in Eq. 112 the form $\overline{\psi}\,U_5\,\psi\,\delta_B(x)$ with $U_5 = \exp\left(\frac{i}{f_\pi}\,\boldsymbol{\tau}\cdot\boldsymbol{\pi}\,\gamma_5\right)$.

16 Instantons and chiral symmetry breaking

The groundstate in QCD is $F_{\mu\nu} = 0$. The field configurations corresponding to this are the A_μ fields,

$$A_\mu^{\mathrm{vac}}(x) = -\frac{i}{g}\,(\partial_\mu U(x))\,U^{-1}(x). \tag{139}$$

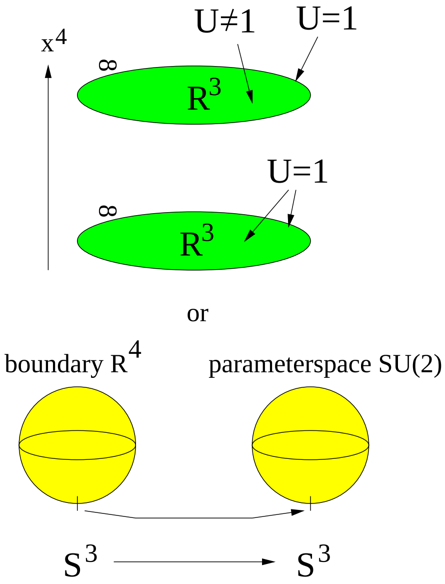


Figure 4: Topological structure of instanton solutions in QCD. This topology is related to field-free solutions obtained from nontrivial A_μ configurations still having a finite action. Without loss of generality we can assume that all points at infinity approach $U(\boldsymbol{x}, x^4 = i\,x^0) \to 1$ (local gauge invariance allows for this choice), hence we can identify spacelike infinity and consider it as S^3. An interpolation to a different vacuum configuration is topologically the same as looking for a nontrivial mapping of R^3 with infinity identified ($\sim S^3$) into $SU(N_c)$ or in fact into an appropriate $SU(2)$ subgroup ($\sim S^3$).

One might look for nontrivial solutions for the vacuum. The matrices $U(\theta)$ belong to $SU(N_c)$. This is a simple compact group with $\Pi_1(G) = \Pi_2(G) = 0$. One has $\Pi_3(SU(N_c)) = Z$. This topology can be realized by considering classical solutions in the Euclidean theory. The solutions satisfy the free equations, $D_\mu F^{\mu\nu} = 0$ and approach $F_{\mu\nu} \to 0$. We look for solutions that have a finite action $S[A]$, and interpolate between two vacuum configurations. Such solutions can be realized in 4-dimensional Euclidean space and are called *instanton solutions* (see Fig. 4). A solution with winding number 1 is

$$
\begin{aligned}
ig\,A^i(x) &= \left(\frac{r^2}{r^2+R^2}\right)\left(\left(\partial_\mu U(x)\right)U^{-1}(x)\right.\\
U(x) &= \left(\frac{x^4 + i\,\boldsymbol{\tau}\cdot\boldsymbol{x}}{r}\right),
\end{aligned}
\tag{140}
$$

where $r = \sqrt{\boldsymbol{x}^2 + (x^4)^2}$ and R is an arbitrary scale factor. The topological charge is given by the Cherns-Simon index (see also previous section)

$$
n = \frac{-i\,g^3}{48\pi^2}\int d^3x\;\epsilon^{0ijk}\left(A^i\,A^j\,A^k\right)
\tag{141}
$$

(note that $(AAA) \equiv \frac{1}{T_F}\,\mathrm{Tr}(AAA) = 2\,\mathrm{Tr}(AAA)$). We note that since $F_{\mu\nu} \to 0$ at spatial infinity one has

$$
\begin{aligned}
n &= \frac{-g^2}{32\pi^2}\int d^3x\;\epsilon^{0ijk}\left[F^{ij}A^k + \frac{2}{3}ig\left(A^i\,A^j\,A^k\right)\right]\\
&= \frac{g^2}{32\pi^2}\int d^3x\;k^0 = \frac{g^2}{32\pi^2}\int d^4x\;\partial_\mu k^\mu = \frac{g^2}{32\pi^2}\int d^4x\;F_{\mu\nu}\tilde{F}^{\mu\nu},
\end{aligned}
\tag{142}
$$

hence n is the difference of charges ΔQ_E at different Euclidean times. The Euclidean action can be written as

$$
S[A] = \frac{1}{4}\int d^4x\;F_{\mu\nu}F^{\mu\nu} = \frac{1}{8}\int d^4x\;\left[F_{\mu\nu}F^{\mu\nu} + \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right].
\tag{143}
$$

The action for the configuration connecting adjacent minima is found using the Bogomol'nyi bound,

$$
S[A] \ge \left|\frac{1}{4}\int d^4x\;F_{\mu\nu}\tilde{F}^{\mu\nu}\right| = \frac{8\pi^2}{g^2}\,|n|.
\tag{144}
$$

It satisfies $F = \pm\tilde{F}$, which indeed (trivially) solves the equations of motion. We note that $e^{-S[A]} = e^{-8\pi^2/g^2}$ is the vacuum to vacuum amplitude between different vacua. The relation with chiral symmetry breaking comes via the axial anomaly. In QCD the current k^μ is not conserved by itself but only in combination with the axial vector current, which is the difference of vector current for right- and left-handed quarks, one has

$$\Delta Q_E = N_L - N_R, \tag{145}$$

where $N_{R/L} = N_{R/L} - \overline{N}_{R/L}$ are the net numbers of right- and left-handed quarks.

17 Forms of quantization

As discussed by Dirac in his seminal paper [P.A.M. Dirac, *Forms of Relativistic Dynamics*, Rev. Mod. Phys. 21 (1949) 392] there are more *forms* of quantization. One can for instance start on a hypersurface x^0 = constant, known as *instant form* or on a hypersurface $x^+ = (x^0 + x^3)/\sqrt{2}$ = constant, known as *front form*. Depending on the choice, there are different sets of *dynamical generators* out of the ten Poincaré generators that leave the hypersurface invariant. These kinematical generators can be choosen to be free of interactions.

In **instant form** there are six kinematical generators, three translations P^i and three rotations $J^i = \frac{1}{2}\,\epsilon_{ijk}\,M^{jk}$. Quantization in instant form starts with (interaction independent) canonical commutation relations $[x^i, p^j] = i\,\delta^{ij}$ and $[s^i, s^j] = i\,\epsilon^{ijk}\,s^k$, with on the $x^0 = t$ hypersurface the kinematical generators given by

$$\boldsymbol{P} = \boldsymbol{p}, \tag{146}$$

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} + \boldsymbol{s}. \tag{147}$$

In **front form** there are seven kinematical generators, three translations P^+ and P_T^i, and furthermore $J^3 = M^{12}$, M^{-+}, M^{1+}, M^{2+}. Quantization in front form starts with (interaction independent) canonical commutation relations $[x^+, p^-] = i$, $[x^i, p^j] = i\,\delta_T^{ij}$ (i = 1,2), with on the $x^+ = \tau$ hypersurface the (simple) kinematical generators given by

$$P^+ = p^+, \tag{148}$$

$$\boldsymbol{P}_T = \boldsymbol{p}_T, \tag{149}$$

$$J^3 = M^{12} = x^1 p^2 - x^2 p^1 + \lambda \tag{150}$$

$$M^{-+} = x^- p^+ - \tau p^-, \tag{151}$$

$$M^{i+} = x^i p^+ - \tau p^i \qquad (i = 1, 2). \tag{152}$$

Spin $\boldsymbol{s}$ and lightcone helicity λ are allowed extensions with vanishing commutators with the x and p operators.

Exercise. Check that the kinematical operators satisfy the commutation relations of the Poincaré group,

$$[P^\mu, P^\nu] = 0, \tag{153}$$

$$[M^{\mu\nu}, P^\rho] = -i\left(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu\right), \tag{154}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - i\left(g^{\mu\sigma}M^{\rho\nu} - g^{\nu\sigma}M^{\rho\mu}\right). \tag{155}$$

In order to discuss the quantization of the QCD field-theoretical lagrangian we need the 'momenta'

$$\overline{\Pi}^\mu_\psi = \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\psi} = \frac{i}{2}\,\overline{\psi}\gamma^\mu, \tag{156}$$

$$\Pi^\mu_\psi = \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\overline{\psi}} = -\frac{i}{2}\,\gamma^\mu\psi, \tag{157}$$

$$\Pi^{\mu}_{A_\nu} = \frac{\delta \mathcal{L}}{\delta\, \partial_\mu A_\nu} = -F^{\mu\nu}. \tag{158}$$

Instant Form quantization

We want to quantize fields by imposing equal time commutation relations with the conjugate momenta. These momenta can be obtained from the above general momenta. We find for the fermion field $\overline{\Pi}_\psi = \delta\mathcal{L}/\delta\,\partial_0\psi = \frac{i}{2}\,\psi^\dagger$. The canonical momentum of the A^μ fields and the remaining components of the field tensor are given by

$$F^{0i} = -E^i = \dot{A}^i + \partial_i A^0 - i\,g\,[A^0, A^i] = \dot{A}^i + D_i A^0, \tag{159}$$

$$F^{ij} = -\partial_i\, A^j + \partial_j\, A^i - i\,g\,[A^i, A^j] = -\epsilon^{ijk}\, B^k \tag{160}$$

where $\dot{A}^\mu = \partial_0 A^\mu$. The absence of a canonical momentum for A^0 implies a constraint

$$\frac{\delta\mathcal{L}}{\delta\, A^0} \;=\; -D_i E^i + j^0 = 0, \tag{161}$$

where $D_i E^i = [D_i, E^i] = \partial_i E^i + ig\,[A^i, E^i]$ and $j^0 = \psi^\dagger\psi$. This equation is the non-abelian Gauss' law. The other equations of motion are

$$\left(i\,\gamma^0 D_0 + i\,\gamma^i D_i - m\right)\psi = 0, \tag{162}$$

$$D_0 F^{0i} + D_j F^{ji} = j^i. \tag{163}$$

Rewriting slightly, we have as equations of motion

$$i\,D_0\psi = -\gamma_0\left(i\,\gamma^j D_j - m\right)\psi = \left(i\,\gamma^j D_j + m\right)\gamma_0\,\psi, \tag{164}$$

$$D_0 E^i = \epsilon^{ikl} D_k B^l - j^i, \tag{165}$$

$$D_j E^j = j^0. \tag{166}$$

Note that the last of these equations is just the constraint implied by the absence of a canonical momentum for A^0. The constraint involves interactions, in this case through $\dot{A}^i$ in E^i. To proceed, one still can make a gauge choice. We just want to mention that the (obvious) choice $A^0 = 0$ in combination with $\partial_\mu\, A^\mu = 0$ has problems (Gribov ambiguity).
[The classical reference on constraints in quantum mechanics is P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer graduate school of Science, 1964]

Front Form quantization

We want to quantize fields by imposing equal time commutation relations with the conjugate momenta using the lightcone time x^+, i.e. $\dot{\psi} = \partial_+\psi = \partial^-\psi$, etc. In terms of the lightcone coordinates one has for the fermion field $\overline{\Pi}_\psi = \delta\mathcal{L}/\delta\,\partial_+\psi = \frac{i}{\sqrt{2}}\,\psi_+^\dagger$, where the *good/bad* fields $\psi_\pm = P_\pm\psi$ are introduced with the projectors $P_\pm \equiv \frac{1}{2}\gamma^\mp\gamma^\pm$. We note that there is no canonical momentum for the bad field ψ_-. For the gluon fields one has (using $i, j, \ldots$ for transverse indices!)

$$F^{-+} = \dot{A}^+ - \partial_- A^- + ig\,[A^+, A^-] = \dot{A}^+ - D_- A^-, \tag{167}$$

$$F^{-i} = \dot{A}^i + \partial_i A^- + ig\,[A^i_T, A^-] = \dot{A}^i_T + D_{Ti} A^- \equiv -E^i_T, \tag{168}$$

$$F^{+i} = \partial_- A^i + \partial_i A^+ - ig\,[A^i_T, A^+] = \partial_- A^i_T + D_i A^+ \equiv B^i_T, \tag{169}$$

$$F^{ij} = \partial_j A^i_T - \partial_i A^j_T - ig\,[A^i_T, A^j_T] \equiv -\epsilon^{ij}_T\, B, \tag{170}$$

the first two being canonical momenta. Note that there is no canonical momentum for A^-. We can simplify by going to the lightcone gauge $A^+ = 0$. In that case the first and third expressions simplify

$$F^{-+} = -\partial_- A^-, \tag{171}$$
$$F^{+i} = B_T^i = \partial_- A_T^i. \tag{172}$$

These equations actually also indicate that choosing $A^+ = 0$ not removes all gauge freedom. Inverting the equations shows that one also needs to specify a boundary condition at infinity. To be precise one needs a prescription for $(\partial_-)^{-1}$, for which we use

$$(\partial_-)^{-1} f(x^-) = \frac{1}{2} \int dy^- \,\epsilon(x^- - y^-) f(y^-), \tag{173}$$

i.e. one imposes an antisymmetric boundary condition $f(-\infty) = -f(\infty)$. The equations of motion made explicit in lightcone coordinates and using the gauge $A^+ = 0$ are

$$\begin{aligned}
&\left(i\gamma^+ D_+ + i\gamma^- D_- + i\gamma^i D_i - m\right)\psi = 0,\\
&D_- F^{-+} + D_i F^{i+} = \partial_- F^{-+} + \partial_i F^{i+} + ig\,[A_T^i, F^{i+}] = j^+,\\
&D_+ F^{+-} + D_i F^{i-} = \dot{F}^{+-} - ig\,[A^-, F^{+-}] + \partial_i F^{i-} + ig\,[A_T^i, F^{i-}] = j^-,\\
&D_+ F^{+i} + D_- F^{-i} + D_j F^{ji} = \dot{F}^{+i} - ig\,[A^-, F^{+i}] + \partial_- F^{-i} + \partial_j F^{ji} + ig\,[A_T^j, F^{ji}] = j^i.
\end{aligned}$$

Using the fields $\psi_\pm$, the equations of motion become

$$2i\, D_+ \psi_+ = (i\,\not{D}_T + m\,)\;(\gamma^-\,\psi_-), \tag{174}$$
$$i\,\partial_-(\gamma^-\psi_-) = (i\,\not{D}_T - m)\;\psi_+, \tag{175}$$
$$-(\partial_-)^2 A^- - D_T \cdot B_T = j^+ = g\sqrt{2}\,\psi_+^\dagger \psi^+, \tag{176}$$
$$D_+(\partial_- A^-) + D_T \cdot E_T = j^- = \frac{g}{\sqrt{2}}\,(\gamma^-\psi_-)^\dagger(\gamma^-\psi_-), \tag{177}$$
$$D_+ B_T^i - \partial_- E_T^i + \epsilon_T^{ij} D_j\, B = j^i = \frac{g}{\sqrt{2}}\left[(\gamma^-\psi_-)^\dagger \gamma^i \psi_+ + \psi_+^\dagger \gamma^i (\gamma^-\psi_-)\right]. \tag{178}$$

The second equation is a constraint not involving interactions, which can be solved as $\psi_-[A_T, \psi_+]$, since there are no time-derivatives involved. The same is true for the third equation, which can be solved as $A^-[A_T, \psi_+]$. To be precise

$$(\gamma^-\psi_-) = (i\partial_-)^{-1}(i\not{D}_T - m)\,\psi_+ \tag{179}$$
$$A^- = (i\partial_-)^{-2}\left[g\,\psi_+^\dagger \psi_+ + D_T \cdot B_T\right]. \tag{180}$$

One is left with two (independent) equations governing the time dependence of a minimal set of fields, ψ_+ and A_T^i. For these (dynamical) fields canonical commutation relations can be imposed independent of the interactions [J.B. Kogut and D.E. Soper, *Quantum electrodynamics in the infinite-momentum frame*, Phys. Rev. D 1 (1970) 2901]. The lagrangian left after gauge fixing can also be written as

$$\begin{aligned}
\mathcal{L}_G &= -\frac{1}{2}\,F^{+-}F^{-+} + \frac{1}{2}\,F^{+i}F^{-i} - \frac{1}{2}\,B^2\\
&= \frac{1}{2}\left(\partial_- A^-\right)^2 - \frac{1}{2}\,E_T \cdot B_T - \frac{1}{2}\,B^2.
\end{aligned} \tag{181}$$

The fermionic part (including the quark-gluon interaction terms) is

$$\begin{aligned}
\mathcal{L}_F &= \frac{i}{\sqrt{2}}\left(\psi_+^\dagger \dot{\psi}_+ - \dot{\psi}_+^\dagger \psi_+\right) + g\sqrt{2}\,\psi_+^\dagger A^- \psi_+ + \frac{1}{2\sqrt{2}}\,(\gamma^-\psi_-)^\dagger \overleftrightarrow{\partial}_-\,(\gamma^-\psi_-)\\
&\quad + \frac{1}{\sqrt{2}}\left[(\gamma^-\psi_-)^\dagger\,(i\not{D}_T - m)\,\psi_+ + \psi_+^\dagger\,(i\not{D}_T + m)\,(\gamma^-\psi_-)\right].
\end{aligned} \tag{182}$$

The nice thing of the front form quantization is the fact that the fields ψ_+ and A_T are dynamical fields, for which canonical equal (lightcone) time commutation relations can be imposed, of which the right hand side is interaction-independent. That means that a free field expansion for ψ_+ is a sensible thing, even if the quanta are confined. This is the basis behind the successful parton model in deep inelastic scattering.

18 Quantization of nonabelian gauge theories

In principle generating functionals are very appropriate to get an intuitive approach to the quantization of gauge field theories. We will summarize the procedure in the language of path integrals (see text books for details). A problem with gauge theories is the fact that the integration over the gauge fields A_μ includes specifically an integration over subsets $\{{}^gA_\mu\}$, which are obtained through a gauge transformation g on $\underline{A}_\mu$,

$$A_\mu \stackrel{g(x)}{\longrightarrow} {}^gA_\mu = U(g)A_\mu U^{-1}(g) - \frac{i}{g}[\partial_\mu U(g)]U^{-1}(g). \tag{183}$$

The action is constant on these gauge orbits and as a consequence the functional integral diverges. One must restrict A_μ using constraints

$$\mathcal{F}^a(A_\mu(x)) = B^a(x), \tag{184}$$

for example $\partial_\mu A^{\mu a} = 0$ (Lorentz gauge) or $\nabla \boldsymbol{A}^a = 0$ (Coulomb gauge). The constraints must provide a unique solution g of the equation $\mathcal{F}^a({}^gA_\mu) = B^a$. Next introduce the quantity $\Delta_\mathcal{F}[A_\mu]$,

$$\Delta_\mathcal{F}^{-1}[A_\mu] \equiv \int \mathcal{D}g\, \prod_{x,a} \delta\left(\mathcal{F}^a({}^gA_\mu(x)) - B^a(x)\right), \tag{185}$$

where $\mathcal{D}g$ is a functional integration over the group. Near the identity where $g(x) \approx 1 + d\theta^a(x)L_a$ one has $\mathcal{D}g \equiv \prod_{x,a} d\theta^a(x)$. The quantity $\Delta_\mathcal{F}$ is invariant under group transformations, $\Delta_\mathcal{F}[A_\mu] = \Delta_\mathcal{F}[{}^gA_\mu]$ because of the invariance of the group measure, $dg = dgg'$. The problematic integration over the gauge orbits now can be removed by writing

$$\begin{aligned}
Z[J] &= \int \mathcal{D}A_\mu \ldots\ \exp\left(iS[A_\mu,\ldots] + i\textstyle\int dx\, J_a^\mu A_\mu^a\right)\\
&= \int \mathcal{D}g\, \mathcal{D}A_\mu \ldots \Delta_\mathcal{F}[A_\mu] \prod_{x,a} \delta(\mathcal{F}^a({}^gA_\mu) - B^a)\ \exp\left(iS[A_\mu,\ldots] + i\textstyle\int dx\, J_a^\mu A_\mu^a\right)\\
&= \int \mathcal{D}g\, \mathcal{D}A_\mu \ldots \Delta_\mathcal{F}[A_\mu] \prod_{x,a} \delta(\mathcal{F}^a(A_\mu) - B^a)\ \exp\left(iS[A_\mu,\ldots] + i\textstyle\int dx\, J_a^\mu A_\mu^a\right)\\
&\propto \int \mathcal{D}A_\mu \prod_{x,a} \delta(\mathcal{F}^a(A_\mu) - B^a) \ldots\ \Delta_\mathcal{F}[A_\mu]\ \exp\left(iS[A_\mu,\ldots] + i\textstyle\int dx\, J_a^\mu A_\mu^a\right).
\end{aligned} \tag{186}$$

This functional is actually independent under an infinitesimal change of B^a as there exists a gauge transformation for which $\mathcal{F}^a({}^gA_\mu) - B^a - \delta B^a = \mathcal{F}^a(A_\mu) - B^a$. Therefore it is possible to write

$$\begin{aligned}
Z[J] &= \int \mathcal{D}B^a\, \exp\left(-\frac{1}{2}i \textstyle\int dx\, (B^a(x))^2\right)\\
&\qquad \times \int \mathcal{D}A_\mu \prod_{x,a} \delta(\mathcal{F}(A_\mu) - B^a) \ldots\ \Delta_\mathcal{F}[A_\mu]\ \exp\left(iS[A_\mu,\ldots] + i\textstyle\int dx\, J_a^\mu A_\mu^a\right)\\
&= \int \mathcal{D}A_\mu\ \ldots\ \Delta_\mathcal{F}[A_\mu]\ \exp\left(i\{S[A_\mu,\ldots] - \frac{1}{2}\textstyle\int dx\, (\mathcal{F}^a(A_\mu))^2\} + i\textstyle\int dx\, J_a^\mu A_\mu^a\right).
\end{aligned} \tag{187}$$

The quantity $\Delta_{\mathcal{F}}[A_\mu]$ can be calculated by considering the action of an infinitesimal gauge transformation on the constraint,

$$\mathcal{F}^a({}^g A_\mu(x)) = \mathcal{F}^a(A_\mu(x)) + \int dy\ [M_{\mathcal{F}}(x,y)]_{ab}\theta^b(y), \tag{188}$$

leading to

$$\Delta_{\mathcal{F}}^{-1}[A_\mu] = \int \prod_{x,a} d\theta^a(x)\ \delta\left(\textstyle\int dy\,[M_{\mathcal{F}}(x,y)]_{ab}\theta^b(y)\right) \quad = \quad \frac{1}{\det M_{\mathcal{F}}} = e^{-\mathrm{Tr}\ \ln M_{\mathcal{F}}}. \tag{189}$$

As an example, consider the Lorentz gauge, $\mathcal{F}^a(A_\mu) = \partial_\mu A^a_\mu = 0$. From the transformation of the gauge fields,

$${}^g A^a_\mu(x) = A^a_\mu(x) - f_{abc}\theta^b(x)A^c_\mu(x) + \frac{1}{g}\,\partial_\mu\theta^a(x) = A^a_\mu(x) + \frac{1}{g}\,D_\mu\theta^a(x) \tag{190}$$

one obtains

$$\partial^\mu\,{}^g A^a_\mu(x) = \partial^\mu A^a_\mu(x) - f_{abc}[\partial^\mu\theta^b(x)]A^c_\mu(x) + \frac{1}{g}\,\partial^\mu\partial_\mu\theta^a(x), \tag{191}$$

or

$$[M_L(x,y)]_{ab} = [\frac{1}{g}\,\partial^\mu\partial_\mu\,\delta_{ab} - f_{abc}A^c_\mu\partial^\mu]\,\delta(x-y). \tag{192}$$

We use a functional integration property to rewrite $\Delta_{\mathcal{F}}[A_\mu] = \det M_{\mathcal{F}}$ as a functional integral involving anticommuting complex scalar fields (ghost fields),

$$\det M_{\mathcal{F}} = \int \mathcal{D}c\,\mathcal{D}c^*\ \exp(-\textstyle\int dx\, c^{a*}[M_{\mathcal{F}}]_{ab}c^b). \tag{193}$$

This equality holds if these ghost fields do not appear in initial or final state. One then ends up with a functional integral derived from the following action,

$$S = S[A_\mu,\ldots] + S_{GF}[A_\mu] + S[c,c^*] \tag{194}$$

where the first term is the (classical) lagrangian density, S_{GF} contains the gauge-fixing constraint and the last term contains the ghost fields.

As a second example, consider the axial gauge, $\mathcal{F}^a(A_\mu) = n^\mu A^a_\mu = 0$ with $n^2 = 1$ (temporal gauge $A^0 = 0$), $n^2 = -1$ (spacelike axial gauge $A^3 = 0$) or $n^2 = 0$ (lightcone gauge $A^+ = 0$) [3]. In this case we obtain

$$n^\mu\,{}^g A^a_\mu(x) = n^\mu A^a_\mu(x) + \frac{1}{g}\,n^\mu\partial_\mu\theta^a(x), \tag{195}$$

i.e. M_L is independent of A_μ. So the ghost fields decouple from the gauge fields and may be integrated out.

19 Feynman rules for QCD

Lorentz gauge

Using the Lorentz gauge and appropriately rescaling the ghost fields, one obtains the QCD lagrangian density for quarks and gluons

$$\mathcal{L} = -\frac{1}{4}\,F^a_{\mu\nu}(x)F^{\mu\nu a}(x) + \overline{\psi}(x)(i\not{D} - m)\psi(x) - \frac{\lambda}{2}(\partial^\mu A^a_\mu)^2 + c^{a*}(x)(\partial^2\,\delta_{ab} - g\,f_{abc}A^c_\mu\partial^\mu)c^b(x), \tag{196}$$

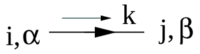
[3] The $\pm$ components of a four vector are $a^\pm = (a^0 \pm a^3)/\sqrt{2}$.

where

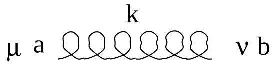
$$D_\mu\psi(x) = \partial_\mu\psi(x) - ig\,A_\mu^a(x)F_a\,\psi(x), \tag{197}$$

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g\,f_{abc}A_\mu^b(x)A_\nu^c(x). \tag{198}$$

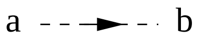
The propagators are derived from the quadratic terms



$$\left(\frac{i\,\delta_{ij}}{\not k - m + i\epsilon}\right)_{\beta\alpha} = \frac{i\,\delta_{ij}\,(\not k + m)_{\beta\alpha}}{k^2 - m^2 + i\epsilon}$$

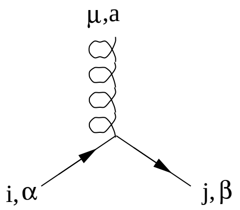


$$-i\,\delta_{ab}\left[\frac{g_{\mu\nu}}{k^2 + i\epsilon} - \left(1 - \frac{1}{\lambda}\right)\frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2}\right].$$

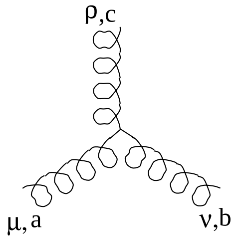


$$\frac{i\,\delta_{ab}}{k^2 + i\epsilon}$$

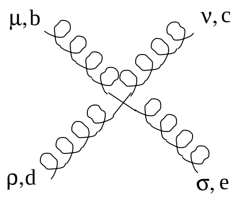
Two special cases for the gauge field propagator are the Feynman propagator with $\lambda = 1$ and the Landau propagator with $\lambda = \infty$. The vertices are derived from the interaction terms in the lagrangian,



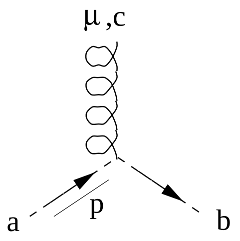
$$i\,g\,(\gamma_\mu)_{\beta\alpha}(F^a)_{ji}$$



$$g\,f_{abc}[(p-q)^\rho\,g^{\mu\nu} + (q-r)^\mu\,g^{\nu\rho} + (r-p)^\rho\,g^{\rho\mu}]$$



$$\begin{aligned}&i\,g^2\,f_{abc}f_{ade}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})\\ &+i\,g^2\,f_{abd}f_{ace}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})\\ &+i\,g^2\,f_{abe}f_{acd}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma})\end{aligned}$$



$$-g\,f_{abc}\,p^\mu$$

Axial gauges

Except for the absence of ghost propagator and ghost-gauge field coupling, the only difference in the Feynman rules is in the gauge field propagator. It is derived from the quadratic term in

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}^a(x)F^{\mu\nu a}(x) + \overline{\psi}(x)(\not D - m)\psi(x) - \frac{\lambda}{2}(n^\mu A_\mu^a)^2, \tag{199}$$

yielding the propagator

$$iG_{ab}^{\mu\nu} = \frac{-i\,\delta_{ab}}{k^2+i\epsilon}\left[g^{\mu\nu} + \left(n^2 + \frac{1}{\lambda}\,k^2\right)\frac{k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right]. \tag{200}$$

In the limit $\lambda \to \infty$, the propagator satisfies $n_\mu\, G_{ab}^{\mu\nu} = 0$, while moreover the contraction of the propagator with k^μ cancels the physical pole,

$$\lim_{k^2\to 0} k^2\, k_\mu G_{ab}^{\mu\nu} = 0. \tag{201}$$

This is referred to as a physical gauge, where there are only two (transverse) degrees of freedom.

Another form of the QCD Lagrangian that can be used is referred to as the planar gauge case with

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}^a(x)F^{\mu\nu a}(x) + \overline{\psi}(x)(i\!\!\not{D} - m)\psi(x) - \frac{\lambda}{2\,n^2}(n^\mu A_\mu^a)\partial^2(n^\nu A_\nu^a), \tag{202}$$

yielding the propagator

$$iG_{ab}^{\mu\nu} = \frac{-i\,\delta_{ab}}{k^2+i\epsilon}\left[g^{\mu\nu} + \left(1 - \frac{1}{\lambda}\right)\frac{n^2\, k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right]. \tag{203}$$

In the limit $\lambda \to \infty$, the propagator is identical to the case above.

In the axial gauge spurious poles $(k\cdot n)^{-1}$ and $(k\cdot n)^{-2}$ appear. If one is to avoid the introduction of extra terms, one must choose a prescription that does not encounter poles while one performs the Wick rotation, e.g. the Mandelstam-Leibbrandt prescription for the lightcone gauge ($n^2 = 0$),

$$\frac{1}{k^+} = \lim_{\epsilon\to 0}\frac{1}{k^+ + i\epsilon k^-} = \lim_{\epsilon\to 0}\frac{k^-}{k^2+i\epsilon} = \lim_{\epsilon\to 0}\frac{1}{k^+ + i\epsilon\,\mathrm{sgn}(k^-)}, \tag{204}$$

which, however, does involve an additional lightcone vector.

20 Playing with color

The gauge group for QCD is the semi-simple Lie group $SU(N_c)$ with $N_G = N_c^2 - 1$ generators T_a that form the basis of the Lie algebra and satisfy

$$[T_a, T_b] = i\,f_{abc}T_c. \tag{205}$$

The quarks belong to the N_c-dimensional defining representation. For the representation matrices F_a Eq. 205 is realized as matrix commutation relations. Furthermore, for a semi-simple Lie group the representation matrices can be choosen such that the socalled Killing form is diagonal,

$$\mathrm{Tr}\,(T_aT_b) = T_F\,\delta_{ab}, \tag{206}$$

which for $SU(N_c)$ is usually choosen to be $T_F = 1/2$. The representation matrices of the defining representation are for $SU(2)$ given in terms of the Pauli matrices, $F_a = \tau_a/2$ (a = 1,2,3) while for $SU(3)$ in the defining representation $\underline{3}$ they are usually expressed in terms of the Gell-Mann matrices, $(F_a)_{ij} = (\lambda_a)_{ij}/2$ (a = 1,...,8). Note that in SU(2) the defining representation is self-conjugate, which is not the case for $SU(3)$. The representation matrices for antiquarks are given by $F_a = -\lambda_a^*/2$ forming the conjugate representation $\underline{3}^*$. A useful relation for semi-simple Lie groups is further

$$i f_{abc} = \frac{1}{T_F}\,\mathrm{Tr}\,(T_aT_bT_c - T_cT_bT_a). \tag{207}$$

The gluons belong to the $N_c^2 - 1$ dimensional adjoint representation, for which the representation matrices are given by

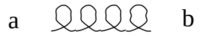
$$(T_a)_{bc} = -i\,f_{abc}. \tag{208}$$

It is straightforward to prove that this is a representation using the Jacobi-identity for the commutator,

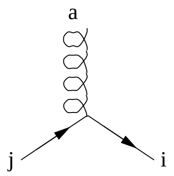
$$f_{abd}f_{dce} + f_{bcd}f_{dae} + f_{cad}f_{dbe} = 0. \tag{209}$$

In calculations it is practical to separate the group theoretical factors from the dynamical parts. For the calculation the following factors are necessary

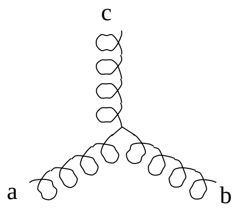
δ_{ij} internal quark line



δ_{ab} internal gluon or ghost line



$(F_a)_{ij}$ quark-quark-gluon vertex

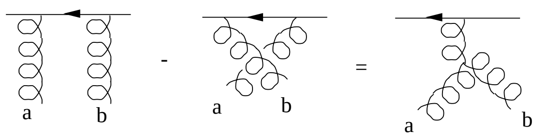


$i\,f_{abc}$ three-gluon vertex

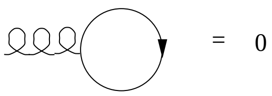
The group theoretic weight of the four-gluon vertex is the product of two three gluon vertex-factors.

The properties of the Lie algebra are easily translated into diagrammatic language. For example

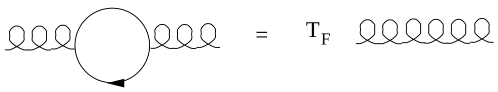
(1) $[F_a, F_b] = i\,f_{abc}F_c$



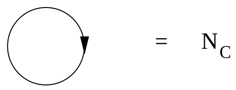
(2) $\mathrm{Tr}\,F_a = 0$



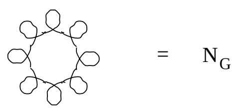
(3) $\mathrm{Tr}\,(F_aF_b) = T_F\,\delta_{ab}$



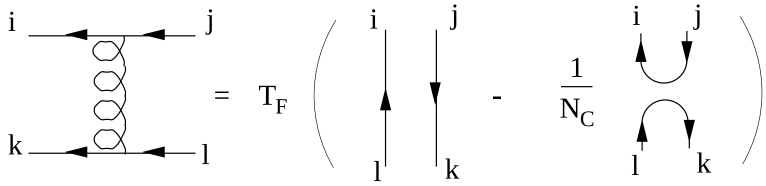
(4) $\delta_{ii} = N_c$



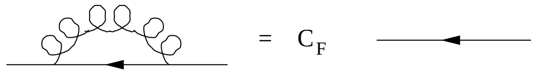
(5) $\delta_{aa} = N_G = N_c^2 - 1$



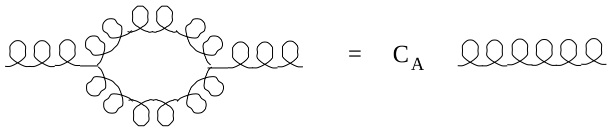
The color flow and the corresponding multiplicative factors appearing in diagrams with quarks and gluons are most easily proven starting with the following relation (proof!)
(6) $\sum_a (F_a)_{ij}\,(F_a)_{kl} = T_F\,\delta_{il}\,\delta_{jk} - \frac{T_F}{N_c}\,\delta_{ij}\,\delta_{kl}$,
diagrammatically



As next relations it is then straightforward to prove
(7) $\sum_a (F_a)_{ij}\,(F_a)_{jk} = C_F\,\delta_{ik}$ with $C_F = T_F\,N_G/N_c = (N_c^2-1)/2N_c$,



(8) $f_{adb}\,f_{adc} = C_A\,\delta_{bc}$ with $C_A = 2T_F(N_G+1)/N_c = N_c$,



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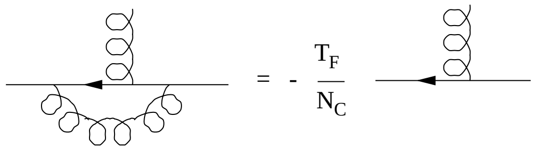

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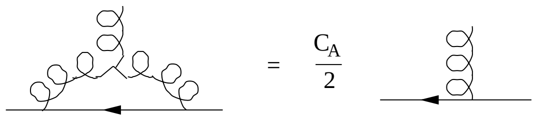
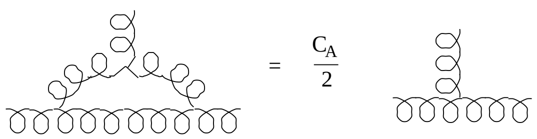


(11)



21 Divergencies in field theories

In calculating the Green's functions in a specific theory one usually encounters infinities. A first analysis of these infinities is obtained from dimensional analysis. Consider the Green's functions with e_b boson fields and e_f fermion fields. The canonical dimensions of the fields and the lagrangian density is

$$d(\mathcal{L}) = 4,$$
$$d(\phi) = d(A_\mu) = 1, \qquad\qquad d(\psi) = 3/2.$$

From this one sees immediately that for the Green's function in coordinate space (G), those in momentum space ($\tilde{G}$), the boson (Δ_F and $\tilde{\Delta}_F$) and fermion (S_F and $\tilde{S}_F$) propagators and the truncated Green's functions (Γ) one has

$$
\begin{aligned}
&d(G) = e_b + \frac{3}{2}\,e_f,\\
&d(\tilde{G}) = 4 - 3\,e_b - \frac{5}{2}\,e_f,\\
&d(\Delta_F) = 2, \qquad d(\tilde{\Delta}_F) = -2,\\
&d(S_F) = 3, \qquad d(\tilde{S}_F) = -1,\\
&d(\Gamma) = 4 - e_b - \frac{3}{2}\,e_f.
\end{aligned}
$$

In the calculation of the truncated Green's functions (in momentum space) one employs vertices, for which we define

$$\delta_i = b_i + \frac{3}{2}\,f_i + d_i - 4, \tag{210}$$

where b_i and f_i are the number of boson and fermion lines at the vertex and d_i is the number of derivatives appearing. Note that $-\delta_i$ indicates precisely the dimension of the coupling constant which has to multiply this vertex, in order for $\mathcal{L}$ to have the right dimension. Examples in 4 dimensions are $\delta(\phi^4) = 0$, $\delta(\bar{\psi}\not{A}\psi) = 0$ and $\delta(\phi^3) =$ -1. In the calculation of a specific diagram a degree of divergence (D) is introduced to 'measure' the divergence, e.g.

$\int dk\, k$	$D = 2$	(quadratically divergent)
$\int dk\, k^{-1}$	$D = 0$	(logarithmically divergent)
$\int dk\, k^{-2}$	$D = -1$	(convergent)

These divergences arising from momentum integrations are called *ultraviolet divergences*. For a given diagram it is easy to find the degree of divergence adding the specific contributions from the integration in the loops (ℓ loops, each adding 4), the internal boson lines (i_b lines, each adding -2), the internal fermion lines (i_f lines, each adding -1) and the derivatives at the vertices (n_i vertices, each adding d_i). The result is

$$D = 4\,\ell - i_f - 2\,i_b + \sum_i n_i\,d_i, \tag{211}$$

which using the relation $e_b + 2\,i_b = \sum_i n_i\,b_i$, $e_f + 2\,i_f = \sum_i n_i\,f_i$ and $\ell = i - v + 1$ (here $v = \sum_i n_i$ is the number of vertices) is rewritten as

$$D = \sum_i n_i\,\delta_i + 4 - e_b - \frac{3}{2}\,e_f, \tag{212}$$

which indeed only differs from $d(\Gamma)$ by the dimensions in the coupling constants at the vertices.

Theories with only vertices with negative δ_i are called *super-renormalizable*. Each truncated Green's function only contains a finite number of divergent graphs. Theories which do contain vertices with $\delta_i \le 0$ are called *renormalizable*. The divergences are present in all diagrams, but only in a finite number of Green's functions (modulo some complications with divergent subdiagrams). The infinities will be absorbed in a redefinition of the parameters multiplying the various terms in the lagrangian. If there are vertices with $\delta_i > 0$ the theory is called *non-renormalizable*. The problem is that all Green's functions contain infinite terms, which requires in general the introduction of an infinite number of new terms in the lagrangian to absorb the infinities.

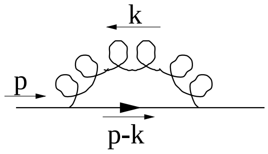
Exercise. Show that in QCD all Green's functions that have $D \ge 0$ are those that also have a tree-level contribution, i.e. that have a corresponding vertex in the lagrangian.

22 Regularization

We will demonstrate the first step in the renormalization procedure for the example of the self-energy contribution to $\Gamma^{(2)}(p)$. Writing the full Green's function

$$\Gamma^{(2)}_{ij}(p) = -i\,(S_F)^{-1}_{ij}(p) = -i[(\not p - m)\delta_{ij} + \Sigma_{ij}(p)], \tag{213}$$

the contribution of the one-loop diagram in the Feynman gauge ($\lambda = 1$ in the gluon propagator),



to the self-energy is given by

$$-i\,\Sigma(p) = -g^2\,C_F \int \frac{d^4k}{(2\pi)^4}\,\frac{\gamma_\mu(\not p - \not k + m)\gamma^\mu}{k^2\,((p-k)^2 - m^2)}. \tag{214}$$

This diagram has therefore $D = 1$, i.e. it is linearly divergent.

Pauli-Villars regulator method

One way of dealing with the divergence is the replacement

$$\frac{1}{k^2 - m^2} \longrightarrow \frac{1}{k^2 - m^2} - \frac{1}{k^2 - M^2} = \frac{m^2 - M^2}{(k^2 - m^2)(k^2 - M^2)}. \tag{215}$$

This wil render a finite expression for the above integral.

Dimensional regularization

A second way to regularize the infinity is dimensional regularization. In this method one extends the dimension of space-time to $n = 4 - 2\epsilon$. The integration is taken

$$\frac{d^4k}{(2\pi)^4} \longrightarrow \frac{d^nk}{(2\pi)^n}, \tag{216}$$

We note that the dimensions of the fields and parameters change accordingly,

$$\begin{aligned}
& d(\mathcal{L}) = n = 4 - 2\epsilon, \\
\text{from } (\partial_\mu\phi)^2 : \quad & d(\phi) = \frac{n-2}{2} = 1 - \epsilon, \\
\text{from } g\phi^4 : \quad & d(g) = 2\epsilon, \\
\text{from } \bar\psi\not\partial\psi : \quad & d(\psi) = \frac{n-1}{2} = \frac{3}{2} - \epsilon, \\
\text{from } g\,\bar\psi\not A\psi : \quad & d(g) = \epsilon.
\end{aligned}$$

Also with γ-matrices we have to be careful, e.g.

$$\begin{aligned}
&\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) = 4\,g_{\mu\nu}, \\
&\gamma_\mu\gamma^\mu = n, \\
&\gamma_\mu\gamma_\nu\gamma^\mu = (2-n)\gamma_\nu,
\end{aligned}$$

Using this for massless quarks and introducing a mass scale M through $g \to g_0\, M^\epsilon$ in order to make the new g_0 dimensionless, the result for $\Sigma(p)$ becomes

$$\Sigma(p) = g_0^2\, C_F\, (2-n)\, M^{2\epsilon} \int \frac{d^n k}{(2\pi)^n\, i}\, \frac{\not p - \not k}{k^2\,(p-k)^2}. \tag{217}$$

The result using standard techniques in dimensional integration, including a rotation into Euclidean space, becomes after expanding around $n = 4$

$$\Sigma(p) = \left(\frac{g_0}{4\pi}\right)^2 C_F\, \not p \left(\frac{1}{\epsilon} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\, M^2}\right)\right) + \mathcal{O}(\epsilon^2), \tag{218}$$

where γ is Euler's constant.

23 Renormalization schemes

The complete result of the one-loop calculation for the inverse quark propagator thus is

$$S_F^{-1}(p) = i\Gamma^{(2)}(p) = \not p \left[1 + \left(\frac{g_0}{4\pi}\right)^2 C_F \left(\frac{1}{\epsilon} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\, M^2}\right)\right)\right]. \tag{219}$$

The presence of the finite (p^2-dependent) terms introduce ambiguities in the way one defines renormalized quantities, such as a renormalized mass or renormalized fields. In this case ($m = 0$) we only deal with the wave function renormalization. Writing

$$\psi = Z_2^{1/2}\, \psi_R, \tag{220}$$

one has for instance

$$G_R^{(2)}(x_1, x_2) = \langle 0 | \mathcal{T} \bar\psi_R(x_1) \psi_R(x_2) | 0 \rangle = Z_2^{-1}\, G^{(2)}(x_1, x_2), \tag{221}$$

$$\Gamma_R^{(2)}(p) = Z_2\, \Gamma^{(2)}(p). \tag{222}$$

The infinity $\propto 1/\epsilon$ is absorbed in the renormalization constant such that the result for $\Gamma_R^{(2)}$ is finite. Possible schemes are

On-shell subtraction scheme

In this cases one requires

$$i\Gamma_R^{(2)}(p) \overset{p^2=m^2}{=} \not p - m \overset{m=0}{=} \not p. \tag{223}$$

For quarks the mass-shell is not very useful and in particular for massless quarks this gives trouble because of the mass singularity in $\ln(-p^2)$.

Off-shell subtraction scheme or momentum-space subtraction scheme (MOM)

In this case one requires at a value $p^2 = -\mu^2 < 0$ that $\Gamma_R^{(2)}$ has the form of the free (massless) propagator,

$$i\Gamma_R^{(2)}(p) \overset{p^2=-\mu^2}{=} \not p. \tag{224}$$

Then

$$Z_2^{-1} = 1 + \left(\frac{g_0}{4\pi}\right)^2 C_F \left(\frac{1}{\epsilon} - \gamma + 1 - \ln\left(\frac{\mu^2}{4\pi\, M^2}\right)\right), \tag{225}$$

and

$$i\Gamma_R^{(2)}(p) \;=\; \not p \left[1 - \left(\frac{g_0}{4\pi}\right)^2 C_F \ln\left(\frac{-p^2}{\mu^2}\right)\right]. \tag{226}$$

Minimal subtraction scheme (MS)

In this case only the pole term $\propto 1/\epsilon$ is removed,

$$Z_2^{-1} = 1 + \left(\frac{g_0}{4\pi}\right)^2 C_F\,\frac{1}{\epsilon}, \tag{227}$$

and

$$i\Gamma_R^{(2)}(p) \;=\; \not p\left[1 - \left(\frac{g_0}{4\pi}\right)^2 C_F\left(\gamma - 1 + \ln\left(\frac{-p^2}{4\pi\,M^2}\right)\right)\right]. \tag{228}$$

This is converted in the result in the MOM-scheme after setting $\mu^2 = 4\pi\,e^{1-\gamma}\,M^2$.

Modified minimal subtraction scheme ($\overline{\text{MS}}$)

In this case the pole term in combination with the constants γ and $\ln 4\pi$ are removed,

$$Z_2^{-1} = 1 + \left(\frac{g_0}{4\pi}\right)^2 C_F\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right), \tag{229}$$

and

$$i\Gamma_R^{(2)}(p) \;=\; \not p\left[1 - \left(\frac{g_0}{4\pi}\right)^2 C_F\left(-1 + \ln\left(\frac{-p^2}{M^2}\right)\right)\right]. \tag{230}$$

Again this can be converted to the results in the other schemes after redefining the momentum scales.

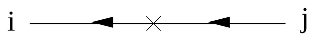
After the renormalization procedure one can obtain finite results for all Green's functions involving the (finite) renormalized fields by re-expressing the lagrangian in terms of renormalized fields and renormalized parameters. For instance, for the kinetic term in the Dirac Lagrangian,

$$\mathcal{L} = \bar\psi(i\not\partial)\psi = Z_2\,\bar\psi_R(i\not\partial)\psi_R = \bar\psi_R(i\not\partial)\psi_R + \mathcal{L}_C, \tag{231}$$

where

$$\mathcal{L}_C = (Z_2 - 1)\,\bar\psi_R(i\not\partial)\psi_R, \tag{232}$$

which gives rise to a new vertex



$(Z_2 - 1)\,(\not p)_{ij}$.

For the full QCD lagrangian density one needs to redefine fields,

$$A^a_\mu = Z_3^{1/2}\,A^a_{R\mu}, \qquad c^a = \tilde Z_3^{1/2}\,c^a_R, \qquad \psi = Z_2^{1/2}\psi_R, \tag{233}$$

and redefine parameters,

$$g = Z_g\,g_R, \qquad \lambda = Z_3^{-1}\,\lambda_R, \qquad m = Z_m\,m_R. \tag{234}$$

The renormalization constant for λ is related to that of the gluon field in order to keep the gauge-fixing term in the same form. Note that there are in principle 8 terms in the lagrangian density of QCD. The fact that five renormalization constants are sufficient is a consequence of gauge invariance. This provides relations between different Green's functions, the socalled Ward-Takahashi identities.

24 The renormalization group

For an n-point Green's function in a theory with fields ϕ and parameters g and m we have seen

$$\Gamma_R^{(n)}(p; g_R, m_R, \mu) = Z_\phi^{n/2}(R)\,\Gamma^{(n)}(p; g, m), \tag{235}$$

where μ is the renormalization scale and the renormalization constants are defined

$$\phi = Z_\phi^{1/2}\,\phi_R, \qquad g = Z_g\,g_R, \qquad m = Z_m\,m_R \tag{236}$$

(Note that the precise definitions may vary depending on the theory, e.g. in a bosonic theory one conventionally has $m^2 = Z_m\,m_R^2$). The renormalization constants and the renormalized quantities depend on the bare parameters and the renormalization scale as we have also seen in the example in the previous section, for instance $g_R = g_R(g,m,\mu)$, $Z_g = Z_g(g,m,\mu)$. For another scheme (and/or renormalization scale) one has

$$\Gamma^{(n)}_{R'}(p;g_{R'},m_{R'},\mu) = Z_\phi^{n/2}(R')\,\Gamma^{(n)}(p;g,m). \tag{237}$$

Between the two schemes we have a finite renormalization constant

$$\begin{aligned}
\phi_{R'} &= Z_\phi^{1/2}(R',R)\,\phi_R = \frac{Z_\phi^{1/2}(R)}{Z_\phi^{1/2}(R')}\,\phi_R,\\
g_{R'} &= Z_g(R',R)\,g_R = \frac{Z_g(R)}{Z_g(R')}\,g_R,\\
m_{R'} &= Z_m(R',R)\,m_R = \frac{Z_m(R)}{Z_m(R')}\,m_R.
\end{aligned} \tag{238}$$

One can define a group of renormalization constants $\{Z_g(R',R)\}$ with a multiplication $Z_g(R'',R) = Z_g(R'',R')\,Z_g(R',R)$, an inversion $[Z_g(R',R)]^{-1} = Z_g(R,R')$ and an identity $Z_g(R,R) = 1$, which all belong to or remain in the group. This abelian group together with similar groups for the other renormalization constants is known as the renormalization group.

25 The renormalization group equations (RGE)

The renormalization schemes R are characterized by a scale μ. The bare parameters are independent of this scale, $dg/d\mu = dm/d\mu = 0$, and also

$$\frac{d}{d\mu}\,\Gamma^{(n)}(p;g,m) = 0. \tag{239}$$

Thus

$$Z_\phi^{-n/2}\left(\frac{\partial}{\partial\mu} + \frac{\partial g_R}{\partial\mu}\,\frac{\partial}{\partial g_R} + \frac{\partial m_R}{\partial\mu}\,\frac{\partial}{\partial m_R}\right)\Gamma^{(n)}_R + \left(\frac{\partial Z_\phi^{-n/2}}{\partial\mu}\right)\Gamma^{(n)}_R = 0, \tag{240}$$

or

$$\left(\mu\,\frac{\partial}{\partial\mu} + \beta\,\frac{\partial}{\partial g_R} - \gamma_m\,m_R\,\frac{\partial}{\partial m_R} - n\,\gamma\right)\Gamma^{(n)}_R = 0, \tag{241}$$

where

$$\beta = \mu\,\frac{\partial g_R}{\partial\mu} = \beta(g_R, m_R/\mu), \tag{242}$$

$$\gamma_m = -\frac{\mu}{m_R}\,\frac{\partial m_R}{\partial\mu} = \gamma_m(g_R, m_R/\mu), \tag{243}$$

$$\gamma = \frac{\mu}{2}\,\frac{\partial \ln Z_\phi}{\partial\mu} = \mu^2\,\frac{\partial \ln Z_\phi}{\partial\mu^2} = \gamma(g_R, m_R/\mu). \tag{244}$$

The (finite) dimensionless functions β, γ_m and γ depend on the (renormalized) parameters and the renormalization scale in dimensionless combinations and the regulators. The dependence on the latter is either direct or indirectly through g_R. In the MS or $\overline{\mathrm{MS}}$ scheme where the dependence on μ enters through $g_0\,\mu^\epsilon$ the functions actually only depend on g_R. Finally note that through the relations in Eq. 234 the functions β and γ_m can also be obtained in terms of derivatives of $\ln Z_g$ and $\ln Z_m$ respectively. Similarly the function γ can be obtained from the scale-dependence of $\Gamma_R^{(2)}$.

We conclude this section by giving for QCD the renormalization group equations for the $1PI$ Green's functions with n_F fermions and n_G gluons,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} - m_R\,\gamma_m(g_R)\frac{\partial}{\partial m_R} - 2\lambda_R\,\delta(g_R)\,\frac{\partial}{\partial\lambda_R}\right.$$
$$\left. -n_G\,\gamma_G(g_R) - n_F\,\gamma_F(g_R)\right]\Gamma_R^{(n_F,n_G)}(p_i;g_R,m_R,\lambda_R,\mu) = 0 \tag{245}$$

with

$$\beta(g_R) = \mu\,\frac{\partial g_R}{\partial\mu}, \tag{246}$$
$$\gamma_m(g_R) = -\frac{\mu}{m_R}\,\frac{\partial m_R}{\partial\mu}, \tag{247}$$
$$\delta(g_R) = -\frac{\mu}{2\lambda_R}\,\frac{\partial\lambda_R}{\partial\mu} = \gamma_G(g_R), \tag{248}$$
$$\gamma_G(g_R) = -\frac{\mu}{2}\,\frac{\partial\ln Z_3}{\partial\mu}, \tag{249}$$
$$\gamma_F(g_R) = -\frac{\mu}{2}\,\frac{\partial\ln Z_2}{\partial\mu}, \tag{250}$$

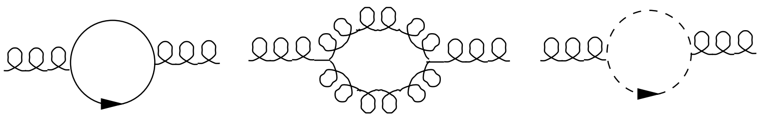
(Remember that the partial derivative here means, keeping parameters $g = g_0\,\mu^\epsilon$, m, and λ fixed).

The γ-function for fermion fields up to order g^2 can be immediately obtained from the calculation of Z_2 or $\Gamma_R^{(2)}$,

$$\gamma_\psi(g) = \left(\frac{g}{4\pi}\right)^2 C_F + \ldots. \tag{251}$$

(Because $\epsilon\,\ln\mu$ is fixed, the ϵ-dependence contains the μ-dependence.)

The β-function of QCD can for instance be obtained from the one-loop vacuum polarization corrections to the gluon propagator,



$\sim$ f T_F $\sim C_A$ $\sim C_A$

which gives for the regularized coupling squared for $|q^2| \gg \mu^2$ in the $\overline{\mathrm{MS}}$ scheme

$$g^2(q^2) \approx g_0^2\left\{1 - \left(\frac{g_0}{4\pi}\right)^2\left[\frac{11\,C_A - 4f\,T_F}{3}\right]\,\ln\left(\frac{|q^2|}{\mu^2}\right)\right\}, \tag{252}$$

where f is the number of flavors, i.e. the number of quarks contributing in the loops. From this one finds the first coefficient (b_0) in the β-function. Including also the next order one has

$$\beta(g) = -b_0\frac{g^3}{32\pi^2} - b_1\,\frac{g^5}{512\pi^4} + \ldots\ , \tag{253}$$

with

$$b_0 = \frac{11\,C_A - 4f\,T_F}{3} \overset{N_c=3}{=} \frac{33-2f}{3}. \tag{254}$$
$$b_1 = \frac{34\,C_A^3 - 4f\,T_F\,(5\,C_A + 3\,C_F)}{3} \overset{N_c=3}{=} \frac{306-38f}{3}. \tag{255}$$

The result for QED is also easily obtained with $g \to e$, $C_A \to 0$, $f \to 1$ and $T_F \to 1$,

$$\beta(e) = \frac{e^3}{24\pi^2} + \dots\ . \tag{256}$$

(Note the sign change!)

The result for γ_m is

$$\gamma_m(g) = \gamma_{m0} \left(\frac{g}{4\pi}\right)^2 + \gamma_{m1} \left(\frac{g}{4\pi}\right)^4 + \dots \tag{257}$$

with

$$\gamma_{m0} = 6\, C_F \overset{N_c=3}{=} 8, \tag{258}$$

$$\gamma_{m1} = C_F \frac{9\, C_F + 97\, C_A - 20 f\, T_F}{3} \overset{N_c=3}{=} \frac{4}{9}(303 - 10f). \tag{259}$$

26 Asymptotic solutions of the RGE.

From the RGE one can find the asymptotic behavior of the renormalized Green's functions, without knowing their precise form. Naive dimensional analysis of the Green's functions shows that for a Green's function with canonical dimension d_Γ (e.g. $d_\Gamma = 4 - n$ for an n-point scalar field Green's function)

$$\Gamma^{(n)}(p_i; g, m, \mu) = \mu^{d_\Gamma} \overline{\Gamma}\left(\frac{p_i}{\mu}; g, \frac{m}{\mu}\right) \tag{260}$$

(Note: parameters and Green's functions are renormalized ones! The same equation also holds in coordinate space!). Consider the rescaled momenta σp_i in the Green's function for $\sigma \to \infty$. Then the mass dependence does not play a role anymore and

$$\Gamma^{(n)}(\sigma p_i; g, \mu) = \mu^{d_\Gamma} \overline{\Gamma}\left(\frac{\sigma p_i}{\mu}; g\right). \tag{261}$$

From

$$\left(\mu \frac{\partial}{\partial \mu} + \sigma \frac{\partial}{\partial \sigma}\right) \overline{\Gamma}\left(\frac{\sigma p_i}{\mu}; g\right) = 0, \tag{262}$$

it follows that

$$\left(\mu \frac{\partial}{\partial \mu} + \sigma \frac{\partial}{\partial \sigma} - d_\Gamma\right) \Gamma^{(n)}(\sigma p_i; g, \mu) = 0, \tag{263}$$

or

$$\left[\sigma \frac{\partial}{\partial \sigma} - \beta(g) \frac{\partial}{\partial g} + n\, \gamma(g) - d_\Gamma\right] \Gamma^{(n)}(\sigma p_i; g, \mu) = 0. \tag{264}$$

Write the Green's function as

$$\Gamma^{(n)}(\sigma p_i; g, \mu) \equiv \sigma^{d_\Gamma} \exp\left(n \int_0^g dg' \frac{\gamma(g')}{\beta(g')}\right) F^{(n)}(\sigma p_i; g, \mu) \tag{265}$$

and $\tau = \ln \sigma$. Then

$$\left(\frac{\partial}{\partial \tau} - \beta(g) \frac{\partial}{\partial g}\right) F^{(n)}(e^\tau p_i; g, \mu) = 0. \tag{266}$$

Consider the *running coupling* $\bar{g}(\tau, g)$ which satisfies

$$\bar{g}(0, g) \equiv g \quad \text{and} \quad \frac{\partial \bar{g}(\tau, g)}{\partial \tau} = \beta(\bar{g}). \tag{267}$$

This function satisfies

$$\left(\frac{\partial}{\partial \tau} - \beta(g) \frac{\partial}{\partial g}\right) \bar{g}(\tau, g) = 0. \tag{268}$$

(Proven by starting with $d\tau = d\bar{g}/\beta(\bar{g})$ for given g, integrating left- and right-hand side (from 0 to τ and g to $\bar{g}$ respectively), and finally differentiating with respect to g). The dependence on g and τ in $F^{(n)}$ is therefore entirely contained in $\bar{g}(\tau, g)$, i.e. the function $F^{(n)}(t, g)$ satisfying Eq. 266 is $F^{(n)}(0, \bar{g}(t, g))$. Applied to the renormalized Green's functions one has then

$$\Gamma^{(n)}(\sigma p_i; g, \mu) = \sigma^{d_\Gamma} \exp\left(n \int_0^g dg' \frac{\gamma(g')}{\beta(g')}\right) F^{(n)}(p_i; \bar{g}(\tau, g), \mu) \tag{269}$$

Finally rewrite

$$\begin{aligned} n \int_0^g dg' \frac{\gamma(g')}{\beta(g')} &= n \int_0^{\bar{g}} dg' \frac{\gamma(g')}{\beta(g')} - n \int_g^{\bar{g}} dg' \frac{\gamma(g')}{\beta(g')} \\ &= n \int_0^{\bar{g}} dg' \frac{\gamma(g')}{\beta(g')} - n \int_0^{\tau} d\tau' \, \gamma(\bar{g}(\tau', g)), \end{aligned}$$

leading to the final result

$$\Gamma^{(n)}(e^\tau p_i; g, \mu) = \exp\left(d_\Gamma \, \tau - n \int_0^\tau d\tau' \, \gamma(\bar{g}(\tau', g))\right) \Gamma^{(n)}(p_i; \bar{g}(\tau, g), \mu). \tag{270}$$

The result tells us that rescaling the momenta ($\tau = 0 \leftrightarrow \sigma = 1$ and $\tau = \infty \leftrightarrow \sigma = \infty$) is equivalent to:

- rescaling with canonical dimension, σ^{4-n} factor.
- replacing the coupling constant by an effective coupling constant $g \to \bar{g}(\tau)$ with $\bar{g}(0) = g$, its behavior governed by the β-function.
- multiplication with an exponential factor, showing the *anomalous dimension* of the fields, which is the result of summing logarithms and is controlled by the γ-function.

We note that the mass parameter can be included in a similar way. The mass m must be replaced by a running mass $\overline{m}$, which for $\tau = 0$ coincides with the renormalized mass m and the running of which is given by

$$\frac{1}{\overline{m}} \frac{\partial \overline{m}}{\partial \tau} = -1 - \gamma_m(\bar{g}), \tag{271}$$

which knowing $\bar{g}(\tau)$ is easily solved

$$\overline{m}(\tau) = m \exp\left(-\tau - \int_0^\tau d\tau' \, \gamma_m(\bar{g}(\tau))\right). \tag{272}$$

27 The running coupling constant

The behavior of the running coupling constant is governed by the β-function,

$$\frac{d\bar{g}(\tau)}{d\tau} = \beta(\bar{g}) \tag{273}$$

Interesting is the behavior if $\beta(\bar{g})$ has a zero at $\bar{g} = g_0$. Depending on the behavior near this zero, the coupling constant will tend towards g_0 for $\tau \to \infty$ (UV stable point) or for $\tau \to -\infty$ (IR stable). Specifically if $\bar{g} = 0$ is a zero of the β function it depends on the sign of the first coefficient what is the behavior as a function of τ. In QED and QCD we have seen that $\beta(\bar{g}) \propto -b_0 \, \bar{g}^3$. If $b_0 > 0$ the point $\bar{g} = 0$ is UV stable, i.e. for $\tau \to \infty$ $\bar{g}(\tau) \to 0$. This situation, which is the case for QCD (at least provided f is not too large) is known as *asymptotic freedom*. Note that QED is not asymptotically free, the coupling constant rises with Q^2.

For QCD it is straightforward to obtain the explicit result for the behavior of the coupling constant, $\alpha_s(\mu^2)$ from the explicit result for the β-function. Writing $\tau = \ln(\mu/\mu_0)$ one looks for solutions of

$$\frac{\partial \alpha_s(\mu^2)}{\partial \ln(\mu^2)} = -\frac{b_0}{4\pi}\,\alpha_s^2 - \frac{b_1}{(4\pi)^2}\,\alpha_s^3 + \dots\ . \tag{274}$$

To first order it is given by

$$\frac{1}{\alpha_s(\mu^2)} = \frac{1}{\alpha_s(\mu_0^2)} + \frac{b_0}{4\pi}\,\ln\left(\frac{\mu^2}{\mu_0^2}\right), \tag{275}$$

or introducing the scale-invariant quantity

$$\frac{1}{\alpha_s(\mu_0^2)} - \frac{b_0}{4\pi}\,\ln\mu_0^2 \equiv -\frac{b_0}{4\pi}\,\ln\Lambda^2, \tag{276}$$

one can write the asymptotic solution

$$\alpha_s(\mu^2) = \frac{4\pi}{b_0\,\ln(\mu^2/\Lambda^2)}. \tag{277}$$

The parameter Λ is known as the *QCD scale parameter*. Note that its definition does not depend on the scale μ but it does depend on the number of flavors and the order of the calculation! Better is therefore, e.g. to fix the scale dependence of QCD by giving, say, $\alpha_s(M_Z^2)$. Experimentally this is about $\alpha_s(M_Z^2) = 0.115 \pm 0.005$.

One should take care in dealing with the running coupling constant, because the scale parameter Λ depends on the number of flavors and the order, i.e. one should properly use $\Lambda^{(f)}_{(order)}$ in conjunction with the equation for the running coupling constant. Putting the transition between $\Lambda^{(f-1)}$ and $\Lambda^{(f)}$ at the threshold at which the new quark flavor (f) must be included (μ_f^2), one has assuming continuity for $\alpha_s(Q^2)$ that

$$\frac{35-2f}{12\pi}\,\ln\left(\frac{\Lambda^{(f-1)2}}{\mu_f^2}\right) \equiv \frac{33-2f}{12\pi}\,\ln\left(\frac{\Lambda^{(f)2}}{\mu_f^2}\right). \tag{278}$$

The result for the scale parameter as a function of the number of flavors changes because of the passing of the various thresholds. Assuming these to be given roughly by the production thresholds for $K\bar{K}$, $D\bar{D}$ or $B\bar{B}$ mesons, which is roughly equal to $4m_q^2$ one finds the $(\overline{MS})$ results below.

f	μ_f^2 [GeV^2]	b_0	$\Lambda^{(f)}_{(1)}$ [MeV]	$\alpha_s(\mu_f^2)$
2		29/3	159	
3	$4M_K^2 \approx 1$	9	140	0.354
4	$4M_D^2 \approx 10$	25/3	110	0.224
5	$4M_B^2 \approx 100$	23/3	75	0.167
6	$4M_t^2 \approx 150{,}000$	7	34	0.096
	M_Z^2			0.115

Also for the second order result a scale invariant definition of Λ can be found,

$$\frac{1}{\alpha_s(\mu^2)} - \frac{b_1}{4\pi\,b_0}\,\ln\left(1 + \frac{4\pi\,b_0}{b_1}\,\frac{1}{\alpha_s(\mu^2}\right) - \frac{b_0}{4\pi}\,\ln\mu_0^2 \equiv -\frac{b_0}{4\pi}\,\ln\Lambda^2. \tag{279}$$

The results corresponding to $\alpha_s(M_Z^2) = 0.115 \pm 0.005$ are $\Lambda^{(5)}_{(1)} = 73^{+26}_{-20}$ MeV and $\Lambda^{(5)}_{(2)} = 204^{+68}_{-55}$ MeV.

Knowing the behavior of $\bar{g}(\tau)$ it is possible to calculate explicitly the behavior of other quantities such as the running mass $\overline{m}(\tau)$. The anomalous behavior is (using first order results) governed by

$$\int_0^\tau d\tau'\gamma_m(\bar{g}(\tau')) = -\int_{\alpha_s(0)}^{\alpha_s(\tau)} \frac{d\alpha_s}{\alpha_s}\,\frac{\gamma_{m0}}{2b_0} = -\frac{\gamma_{m0}}{2b_0}\,\ln\left(\frac{\alpha_s(\tau)}{\alpha_s(0)}\right). \tag{280}$$

The scale dependence of $\overline{m}$ in going from scale μ_0 to μ is therefore

$$\begin{aligned}\overline{m}(\mu) &= \overline{m}(\mu_0)\,\exp\left(-\ln\frac{\mu}{\mu_0} + \frac{\gamma_{m0}}{2b_0}\,\ln\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)\right)\\ &= \frac{\overline{m}(\mu_0)\mu_0}{\mu}\,\left(\frac{\ln(\mu^2/\Lambda^2)}{\ln(\mu_0^2/\Lambda^2)}\right)^{12/(33-2f)}. \end{aligned}\tag{281}$$

First order asymptotic RGE solutions

Also the asymptotic behavior of the Green's functions to first order can be written down explicitly when one knows the running coupling constant. One obtains by evaluating the integral over $\gamma(\bar{g}) \approx (\gamma_0/4\pi)\,\alpha_s$ the result

$$\Gamma^{(n)}\left(\frac{\mu}{\mu_0}\,p;\overline{g}(\mu_0),\frac{\overline{m}(\mu_0)}{\mu_0},\mu_0\right) = \left(\frac{\mu}{\mu_0}\right)^{d_\Gamma}\,\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{n\gamma_0/2b_0}\,\Gamma^{(n)}\left(p;\overline{g}(\mu),\frac{\overline{m}(\mu)}{\mu_0},\mu_0\right)\tag{282}$$

or using the full rescaling properties of the Green's function (Eq. 260)

$$\Gamma^{(n)}\left(p;\overline{g}(\mu_0),\frac{\overline{m}(\mu_0)}{\mu},\mu\right) = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{n\gamma_0/2b_0}\,\Gamma^{(n)}\left(p;\overline{g}(\mu),\frac{\overline{m}(\mu)}{\mu_0},\mu_0\right).\tag{283}$$

28 Composite operators

We want to conclude this general chapter with the treatment of composite operators such as $\phi^2(x)$, $\overline{\psi}(x)\psi(x)$, $\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$ (symmetrized in μ and ν) and $F_{\mu\alpha}F^{\alpha}_{\ \nu}$. These composite operators contain more fields with the same space-time argument. Adding for a composite operator Ω a source term to the lagrangian density,

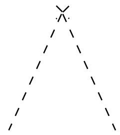
$$\mathcal{L} = \mathcal{L}_0 + \chi(x)\Omega(x),\tag{284}$$

e.g. $\chi\,\overline{\psi}\psi$ or $\frac{1}{2}\,\chi\,\phi^2$ one can obtain Green's functions

$$G_\Omega^{(n)}(x,x_1,\ldots,x_n) = \langle 0|\mathcal{T}\Omega(x)\phi(x_1)\ldots\phi(x_N)|0\rangle.\tag{285}$$

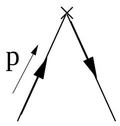
from additional vertices for the composite operators introduced along the same line as interaction terms in the lagrangian. Examples are

ϕ^2



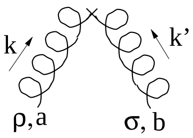
$1,$

$i\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$



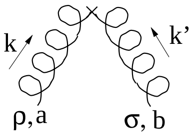
$\frac{1}{2}(\gamma_\mu p_\nu + \gamma_\nu p_\mu),$

$F_{\mu\alpha}F^{\alpha}_{\ \nu}$



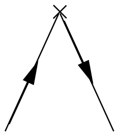
$$\delta_{ab}[q_\rho k_\mu g_{\nu\sigma} + q_\sigma k'_\nu g_{\mu\rho} + k_\mu k'_\nu g_{\rho\sigma} + (k\cdot k')g_{\mu\rho}g_{\nu\sigma}],$$

$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$



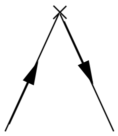
$-2\,\delta_{ab}\,\epsilon_{\mu\nu\rho\sigma}k^\mu k'^\nu,$

$\overline{\psi}\gamma_\mu\psi$



$\gamma_\mu,$

$\overline{\psi}\gamma_\mu\gamma_5\psi$

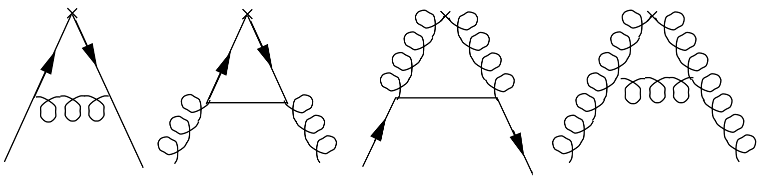


$\gamma_\mu\gamma_5,$

In general the truncated Green's functions will contain divergences, which need to be regulated and renormalized. One has

$$\Gamma^{(n)}_{\Omega,R}(p,p_i) = Z_\Omega\, Z_\phi^{n/2}\, \Gamma^{(n)}_\Omega(p,p_i), \tag{286}$$

where Z_Ω can be calculated in a renormalizable theory. In many cases operators with the same dimension d and the same 'quantum numbers' will mix, e.g. $\Omega_F = i\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$ and $\Omega_G = F^a_{\mu\lambda}F^{a\,\lambda}_{\ \ \nu}$ mix as should be clear from diagrams like



In this case one has

$$\Gamma^{(n)}_{\Omega_i,R}(p,p_i) = (Z_\Omega)_{ij}\, Z_\phi^{n/2}\, \Gamma^{(n)}_{\Omega_j}(p,p_i). \tag{287}$$

It is straightforward to write down the renormalization group equations for the renormalized Green's functions involving composite operators,

$$\left[\left(\mu\,\frac{\partial}{\partial\mu} + \beta(g)\,\frac{\partial}{\partial g} - \gamma_m(g)\,m\,\frac{\partial}{\partial m} - n\,\gamma(g)\right)\delta_{ij} + (\gamma_\Omega)_{ij}\right]\Gamma^{(n)}_{\Omega_i,R} = 0, \tag{288}$$

with the anomalous dimension (matrix) function

$$\gamma_\Omega = \mu\, Z_\Omega\,\frac{\partial Z_\Omega^{-1}}{\partial\mu} = -\mu\,\frac{\partial Z_\Omega}{\partial\mu}\,Z_\Omega^{-1}. \tag{289}$$

As an explicit example we give the result for the anomalous dimension (matrix) function $\gamma(g) = \gamma_0\,(\alpha_s/4\pi)$ for the quark and gluon operators, $\Omega_F = i\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$ and $\Omega_G = F^a_{\mu\lambda}F^{a\,\lambda}_{\ \ \nu} - \frac{1}{4}\,g_{\mu\nu}\,F^a_{\rho\sigma}F^a_{\rho\sigma}$.

$$\gamma(g) = \begin{bmatrix} \gamma_0^{FF} & \gamma_0^{FG} \\ \gamma_0^{GF} & \gamma_0^{GG} \end{bmatrix}\left(\frac{\alpha_s}{4\pi}\right) + \ldots = \begin{bmatrix} \frac{16}{3}\,C_F & -\frac{8}{3}\,f\,T_F \\ -\frac{16}{3}\,C_F & \frac{8}{3}\,f\,T_F \end{bmatrix}\left(\frac{\alpha_s}{4\pi}\right) + \ldots\ . \tag{290}$$

The anomalous dimension determines the asymptotic behavior. Naively one has in this case (assuming vertices with $\delta = 0$) for the momentum space truncated Green's function the naive dimension

$$d(\Gamma_\Omega) = d(\Omega) - e_b - \frac{3}{2}\,e_f, \tag{291}$$

where $d(\Omega)$ is the canonical dimension of Ω, e.g. $d(\phi^2) = 2$. Completely analogous to the behavior of the ordinary Green's functions one obtains in this case

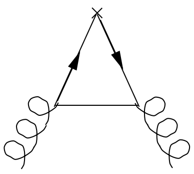
$$\Gamma^{(n)}_\Omega(e^\tau p, e^\tau p_i; g, \mu) = \exp\left(d(\Gamma_\Omega)\tau - n\int_0^\tau d\tau'\,\gamma(\bar{g}(\tau',g)) + \int_0^\tau d\tau'\,\gamma_\Omega(\bar{g}(\tau',g))\right)\,\Gamma^{(n)}_\Omega(p,p_i;\bar{g}(\tau,g),\mu). \tag{292}$$

The first order solution of this is

$$\Gamma_{\Omega}^{(n)}(p,p_i;g(\mu_0),\mu) = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{(n\gamma_0-\gamma_{\Omega 0})/2b_0} \Gamma_{\Omega}^{(n)}(p,p_i;g(\mu),\mu_0). \tag{293}$$

A special case are the Green's functions involving conserved currents. In this case the Ward-Takahashi identies between Green's functions ensure among others that Green's functions containing conserved currents have anomalous dimension zero. Examples of conserved currents are the vector currents $\underline{V}^\mu(x)$ or the conserved currents connected to the Lorentz group, such as the energy momentum tensor. The energy momentum tensor actually is proportional to the sum of the operators Ω_F and Ω_G discussed above. Indeed, as can be seen from the γ-function, the anomalous dimension vanishes for this combination. The asymptotic behavior of Green's functions is not destroyed by the presence of mass terms that break the symmetry. A continuous change in the mass parameter does not modify the anomalous dimensions. Therefore also the nonsinglet axial vector currents are conserved.

A special case mentioned already is the singlet axial current with a unit-matrix structure in flavor space. In this case the conservation equation not only is broken by a mass term, which would not produce an anomalous dimension. At the quantum level, the one loop contribution,



does not vanish, but it is proportional to $\epsilon_{\mu\nu\rho\sigma}k^\mu k'^\nu$, which stands for the insertion of $F_{\mu\nu}\tilde{F}^{\mu\nu}$. To be precise an additional term proportional to this operator is required on the right hand side of the singlet axial operator, as we have already discussed, $\partial_\mu(\overline{\psi}\,\gamma^\mu\gamma_5\,\psi) = \frac{\alpha_s}{2\pi}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$. The anomalous dimension for the operators $\Omega_\Sigma = \sum_f \overline{\psi}\,\gamma_\mu\gamma_5\,\psi$ and $\Omega_\Gamma = (\alpha_s/2\pi)\,k_\mu$, where $\partial_\mu k^\mu = F_{\mu\nu}\tilde{F}^{\mu\nu}$ is found to be

$$\gamma(g) = \begin{bmatrix} \gamma_1^{\Sigma\Sigma} & \gamma_1^{\Sigma\Gamma} \\ \gamma_1^{\Gamma\Sigma} & \gamma_1^{\Gamma\Gamma} \end{bmatrix} \left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots = \begin{bmatrix} 16\,f & 0 \\ 16 & 0 \end{bmatrix} \left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots\,. \tag{294}$$

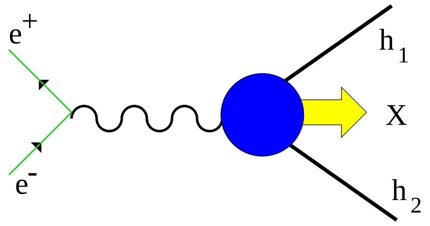
We note that the operators $\Omega_1 = \Omega_\Sigma$ and $\Omega_2 = \Omega_\Sigma - f\,\Omega_\Gamma$ diagonalize the renormalization group equations, the second operator (conserved!) having anomalous dimension zero.

29 Electroweak scattering processes

Electroweak scattering processes are in a sense ideal to study properties of hadrons. Only one specific process, namely lepton-hadron scattering will be dealt with in detail. A signal for an electroweak process is the presence of leptons which do not feel strong interactions. Consider for instance the following three types of processes, The *annihilation* process, $\ell\bar{\ell} \to X$, the *lepton-hadron* scattering process, $\ell H \to \ell' X$, and the *lepton-pair production* process, $AB \to \ell\bar{\ell}X$ (Drell-Yan process). All of these processes involve electroweak currents, coupling to the leptons in a known way. The advantage of electromagnetic processes further lies in the fact that the process is accurately described in terms of the exchange of one photon, as the coupling, $\alpha = e^2/4\pi \approx 1/137$ is weak. The same is true for the weak vector bosons. On the hadronic side, the coupling to the quarks is known, but the structure of hadrons in terms of quarks and gluons is the unknown part. The fact that the coupling to the quarks is known, however, enables the study of quarks and (indirectly) gluons.

First let us consider the variables that are used to describe these scattering processes.

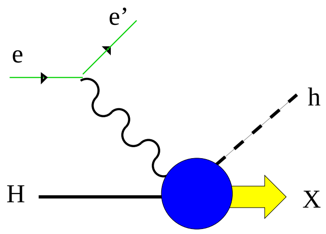
Electron-positron annihilation: $\ell(k)\bar{\ell}(k') \to X$



$$
\begin{aligned}
q^2 = (k+k')^2 &\equiv Q^2 \geq 0 && (295)\\
2\,P_1 \cdot q &\equiv z_1\,Q^2 && (296)\\
\ldots & && (297)\\
2\,P_1 \cdot k &\equiv y\,P_1 \cdot q && (298)
\end{aligned}
$$

In the case of production of hadrons with a timelike (virtual) photon one can consider the rest-frame of the virtual photon, in which case it is clear that Q is a measure of the excitation energy.

Lepton-hadron scattering: $\ell(k)H(P) \to \ell'(k')X$



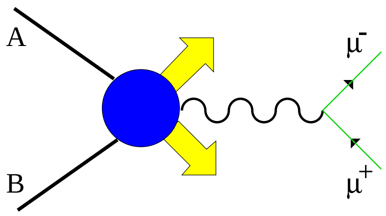
$$
\begin{aligned}
q^2 = (k-k')^2 &\equiv -Q^2 \leq 0 && (299)\\
2P \cdot q \equiv 2M\,\nu &\equiv \frac{Q^2}{x_B} && (300)\\
2P_h \cdot q &\equiv -z_h\,Q^2 && (301)
\end{aligned}
$$

$$P \cdot k = \frac{P \cdot q}{y} = \frac{Q^2}{2x_B y} \qquad (302)$$

The variable x_B is the Bjorken scaling variable. In this scattering process a hadron is probed with a spacelike (virtual) photon, for which one can consider a frame in which the momentum only has a spatial component, from which it is clear that the resolving power of the probing photon is of the order $\lambda \approx 1/Q$. Roughly spoken one probes a nucleus (1 - 10 fm) with $Q \approx 10 - 100$ MeV, baryon or meson structure (with sizes in the order of 1 fm) with $Q \approx 0.1 - 1$ MeV and one probes deep into the nucleon (< 0.1 fm) with $Q > 2$ GeV.

Exercise Rewrite the invariant mass squared of the hadronic final state, W^2, in terms of invariants and use that $W^2 \geq M^2$ to show that $0 \leq x_B \leq 1$, with $x_B = 1$ corresponding to elastic scattering.

Lepton-pair production or Drell-Yan scattering: $A(P_A)B(P_B) \to \ell(k)\bar{\ell}(k')X$



$$
\begin{aligned}
q^2 = (k+k')^2 &\equiv Q^2 \geq 0 && (303)\\
2P_A \cdot q &\equiv \frac{Q^2}{x_A} && (304)\\
2P_B \cdot q &\equiv \frac{Q^2}{x_B} && (305)
\end{aligned}
$$

$$2k \cdot q \equiv \frac{P_A \cdot q}{y} \qquad (306)$$

In Drell-Yan scattering one already has two hadrons to start with. Here Q^2 is the invariant mass of the lepton pair. The above processes are characteristic for a large number of other processes involving particles for which the interactions are fully known on the one side and hadrons on the other side.

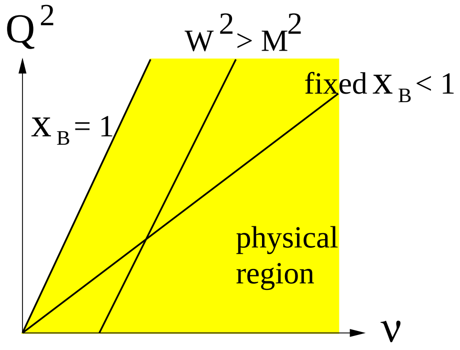


Figure 5: *The physical region in deep inelastic scattering.*

30 Leptoproduction

For inclusive electron scattering the cross section is given by

$$E' \frac{d\sigma}{d^3k'} = \frac{1}{s-M^2}\,\frac{\alpha^2}{Q^4}\,L_{\mu\nu}\,2MW^{\mu\nu}, \tag{307}$$

where $L_{\mu\nu}$ is the lepton tensor,

$$L_{\mu\nu}(k,k';\lambda_e) = 2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda_e\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma. \tag{308}$$

and $W_{\mu\nu}$ is the hadron tensor, which contains the information on the hadronic part of the scattering process,

$$2M\,W_{\mu\nu}(P,q) = \frac{1}{2\pi}\sum_n \int \frac{d^3P_n}{(2\pi)^3\,2E_n}\,\langle P|J_\mu^\dagger(0)|P_n\rangle\,\langle P_n|J_\nu(0)|P\rangle\;(2\pi)^4\,\delta^4(P+q-P_n), \tag{309}$$

where $|P\rangle$ represents a target with momentum P.

Exercise. Show, using $\delta^4(P+q-P_n) = \int d^4x\;\exp(iP\cdot x + iq\cdot x - iP_n\cdot x)$, shifting the argument of the current, $J_\mu(x) = \exp(iP_{op}\cdot x)J_\mu(0)\exp(-iP_{op}\cdot x)$ and using completeness for the intermediate states that the hadron tensor can be written as the expectation value of the product of currents $J_\mu(x)J_\nu(0)$. Then, adding a second term $\propto J_\nu(0)J_\mu^\dagger(0)\delta^4(P-q-P_n)$, which in the physical region ($\nu > 0$) is zero because of the spectral conditions of the intermediate states n ($P_n^0 > M$) one can after a similar procedure combine the terms to

$$2M\,W_{\mu\nu} = \frac{1}{2\pi}\int d^4x\;e^{i\,q\cdot x}\,\langle P|[J_\mu^\dagger(x),J_\nu(0)]|P\rangle, \tag{310}$$

where summation and averaging over spins is understood.

What is one probing in leptoproduction

In the leptoproduction $e + H \rightarrow e' + X$ the (unobserved) final state X can be the target (elastic scattering) or an excitation thereof. In a plot of the two independent variables ν and Q^2 (see Fig. 5) one fixes W^2. In the case of an elementary fermionic target (e.g. an electron or a hypothetical quark target) the system cannot be broken up. In that case, the energy transfer is for given Q^2 fixed at $\nu = Q^2/2M$, where M is the mass of the target. For nucleons the elastic scattering cross section (on top of the $1/Q^4$ of the Mott cross section) is proportional to a form factor squared, measuring the expectation value of the electromagnetic current $\langle P'|J^\mu(x)|P\rangle$. It is also possible to excite the system. This gives rise to inelastic contributions in the cross section at $\nu > Q^2/2M$, starting at the threshold $W = M + M_\pi$. When Q^2 and the energy transfer ν are high enough the cross section will reflect elastic scattering off the pointlike constituents of the nucleon and the cross section will become equal to an incoherent sum of the electron-quark cross section. This is known as the deep inelastic scattering region, in which one finds Bjorken scaling. The cross section, more precisely the structure functions, become functions of one (kinematic) variable $x_{_B}$, which is identified with the momentum fraction of the struck quark in the nucleon, enabling measurement of quark distributions.

It is also possible to consider in more detail the space-time correlations that are probed. As already indicated q is space-like. But from the kinematics of deep inelastic scattering one can see that the process probes the lightcone. Sitting in the nucleon rest-frame we note that both $q^0 = \nu$ and $q^3 = \sqrt{Q^2 + \nu^2}$ go to infinity but working at finite $x_{_B}$ one sees that taking the sum and the difference only one of them goes to infinity. Choosing $\boldsymbol{q}$ along the negative z-axis one has $q^- = (\nu + |\boldsymbol{q}|)/\sqrt{2} \to \infty$ and $q^+ = (\nu - |\boldsymbol{q}|)/\sqrt{2} \approx -M_N x_B/\sqrt{2}$. This corresponds in the hadronic tensor which involves a Fourier transform of the product of currents to $|x^+| \approx 1/q^- \to 0$ and $|x^-| \approx 1/|q^+| \to 1/Mx_B$ or $|\boldsymbol{x}| \approx |t| \approx 1/Mx_B$. Thus, depending on the value of x_B the distances and times not necessarily are small, but one has $x^2 = x^+x^- - \boldsymbol{x}_\perp^2 \approx -\boldsymbol{x}_\perp^2 \le 0$, while on the other hand causality requires that $x^2 \ge 0$. Therefore, one sees that deep inelastic scattering probes the lightcone, $x^2 \approx 0$.

31 Structure functions

The simplest thing one can do with the hadron tensor is to express it in standard tensors and functions depending on the invariants, the *structure functions*. Instead of the traditional choice using tensors, $g_{\mu\nu}$, $P_\mu P_\nu$ and $\epsilon_{\mu\nu\rho\sigma}\, q^\rho P^\sigma$ multiplying structure functions W_1, W_2 and W_3 depending on ν and Q^2, we immediately go to a dimensionless representation. First we define a Cartesian basis of vectors, starting with the natural space-like momentum (defined by q). Using the target hadron momentum P^μ one can construct an orthogonal four vector $\tilde P^\mu = P^\mu - (P\cdot q/q^2)\, q^\mu$, which is timelike. Thus we define

$$Z^\mu \equiv -q^\mu, \tag{311}$$

$$T^\mu \equiv -\frac{q^2}{P\cdot q}\,\tilde P^\mu = q^\mu + 2x_{_B}\, P^\mu. \tag{312}$$

For these vectors we have $Z^2 = -Q^2$ and $T^2 \approx Q^2$ (we neglect target mass effects) and we will often use the normalized vectors $\hat z^\mu = Z^\mu/Q$ and $\hat t^\mu = T^\mu/Q$. With respect to these vectors one can also define transverse tensors,

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat q^\mu \hat q^\nu - \hat t^\mu \hat t^\nu, \tag{313}$$

$$\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}\,\hat t_\rho \hat q_\sigma \tag{314}$$

Using hermiticity and parity invariance for the currents and current conservation (implying $q^\mu W_{\mu\nu} = W_{\mu\nu} q^\nu = 0$) one obtains as the most general form the symmetric tensor

$$\begin{aligned} M\, W^{\mu\nu\,(S)}(P,q) &= \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) F_1(x_{_B},Q^2) + \frac{\tilde P^\mu \tilde P^\nu}{P\cdot q}\, F_2(x_{_B},Q^2) & (315)\\ &= -g_\perp^{\mu\nu}\, \underbrace{F_1(x_{_B},Q^2)}_{F_T} + \hat t^\mu \hat t^\nu\, \underbrace{\left(-F_1 + \frac{\kappa}{2x_{_B}}\, F_2\right)}_{F_L}, & (316)\end{aligned}$$

where the structure functions F_1, F_2 or the transverse and longitudinal structure functions, $F_T = F_1$ and F_L, depend only on the for the hadron part relevant invariants Q^2 and $x_{_B}$. This is the structure for the electromagnic (photon exchange) part of the electroweak interaction. For the weak (W- or Z-exchange) parity invariance can not be used and also the following antisymmetric tensor is allowed,

$$M\,W^{\mu\nu\,(A)}(q,P) = i\,\epsilon_\perp^{\mu\nu}\,F_3(x_B,Q^2). \tag{317}$$

In all equations given here we have omitted target mass effects of order M^2/Q^2.

The explicit contraction with the lepton tensor gives for electromagnetic scattering (only symmetric tensor) the result

$$\begin{aligned}
\frac{d\sigma^{e^\mp p}}{dx_B dy} &= \frac{4\pi\,\alpha^2\,x_B\,s}{Q^4}\left[\left(1-y+\frac{1}{2}y^2\right)F_T(x_B,Q^2) + (1-y)\,F_L(x_B,Q^2)\right] \\
&= \frac{2\pi\,\alpha^2\,s}{Q^4}\left[(1-y)\,F_2(x_B,Q^2) + x_B\,y^2\,F_1(x_B,Q^2)\right].
\end{aligned} \tag{318}$$

We have now used the photon coupling to the lepton. For weak interaction processes we also need the coupling to Z^0 and W, known in the standard model. Important is the fact that we in principle not only know the currents coupling to the leptons, but also the coupling to the constituents of the hadrons, the quarks. It is that knowledge that will be used to further analyse the hadron tensor.

The standard model currents coupling to fermions are

$$J_\mu^{(\gamma)} = \,:\overline{\psi}(x)\,Q\,\gamma_\mu\,\psi(x):, \tag{319}$$

$$\begin{aligned}
J_\mu^{(Z)} &= \,:\overline{\psi}(x)\,(I_W^3 - Q\sin^2\theta_W)\,\gamma_{\mu L}\,\psi(x): - :\overline{\psi}(x)\,Q\sin^2\theta_W\,\gamma_{\mu R}\,\psi(x): \\
&= \,:\overline{\psi}(x)\,(I_W^3 - 2\,Q\sin^2\theta_W)\,\gamma_\mu\,\psi(x): - :\overline{\psi}(x)\,I_W^3\,\gamma_\mu\gamma_5\,\psi(x):,
\end{aligned} \tag{320}$$

$$J_\mu^{(W)} = \,:\overline{\psi}(x)\,I_W^\pm\,\gamma_{\mu L}\,\psi(x):, \tag{321}$$

where $\gamma_{\mu R/L} = \gamma_\mu(1\pm\gamma_5)$. Note that the factors e, $e/2\sin\theta_W\,\cos\theta_W$ and $e/2\sqrt{2}\,\sin\,\theta_W$ that would be expected for the γ, Z^0 and W-couplings to the quarks are conventionally included in the fine structure constant α and the Fermi-coupling $G_F/\sqrt{2} = e^2/8M_W^2\sin^2\theta_W$.

32 Virtual photon cross sections

The tensor $W_{\mu\nu}$ also appears in the total cross section for $\gamma^* H \to$ everything, where γ^* indicates a virtual photon. For a given virtuality Q^2 of the photon this cross section depends on only one variable, $W^2 = (P+q)^2$, or equivalently on the variable $\nu = P\cdot q/M$,

$$\sigma^{\gamma^* H}(\nu) = \frac{4\pi^2\alpha}{K}\,\epsilon^{\mu *}W_{\mu\nu}\epsilon^\nu, \tag{322}$$

where $4M\,K$ is the photon flux factor. This flux factor only is physical for real photons ($Q^2 = 0$). One convention is to take $4M\,K = 4\sqrt{(p\cdot q)^2 - p^2q^2}$, i.e. $K = \sqrt{\nu^2+Q^2}$. Other possibilities are to take the real photon result $4M\,K = 4\,P\cdot q$ or $K = \nu$ (Hand convention). Another convention that has been used is to equate the final state invariant mass squared $W^2 = (P+q)^2$, i.e. take the result $4M\,K = 2(W^2 - M^2)$ for a massless photon and equate W^2 to the invariant mass in the case of a virtual photon, $W^2 = 2\,P\cdot q + M^2 - Q^2$ or $K = \nu - Q^2/2M$.

Being a (total) cross section for (virtual) photoabsorption, the hadronic tensor is related to the forward (virtual) Compton amplitude through the optical theorem,

$$W_{\mu\nu} = \frac{1}{\pi}\,\mathcal{I}m\;T_{\mu\nu}, \tag{323}$$

where

$$2M\,T_{\mu\nu}(P,q) = i\int d^4x\;e^{iq\cdot x}\langle P|\mathcal{T}J_\mu(x)J_\nu(0)|P\rangle, \tag{324}$$

Using the photon polarization vectors ϵ_α^μ, where α indicates one of the polarization directions (perpendicular to q^μ),

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0,\mp 1,-i,0) = \mp\frac{1}{\sqrt{2}}(\epsilon_x \pm i\,\epsilon_y), \tag{325}$$

$$\epsilon^\mu_L = \frac{1}{\sqrt{Q^2}}(q^3,0,0,q^0), \tag{326}$$

one gets the longitudinal and transverse structure functions $F_\alpha = \epsilon_\alpha^{\mu *}\, M W_{\mu\nu}\epsilon_\alpha^\nu$, which, because of the fact that they are cross sections for (virtual) photons, are positive,

$$F_T = \frac{1}{2}(F_+ + F_-) = F_1, \tag{327}$$

$$F_3 = F_+ - F_-, \tag{328}$$

$$F_L = \frac{F_2}{2x_{_B}} - F_1. \tag{329}$$

33 Symmetry properties of the structure functions

For the Compton scattering process, $\gamma^*(q) + H(P) \rightarrow \gamma^*(q') + H(P')$, the amplitude $T_{\mu\nu}(P,q,q')$ can be expanded, similar as $W_{\mu\nu}$, in terms of amplitudes T_1 and T_2 that depend on the invariants in the scattering process. These are (except the for a given process fixed photon virtuality) two of the (three) Mandelstam variables for the $\gamma^* N$ process, $s = (P+q)^2$, $t = (P-q)^2$ and $u = (P-q')^2$ (Note that $s+t+u = 2\,M^2 - 2\,Q^2$).

First we review the symmetries and analytic properties of the amplitudes T_i and the structure functions W_i. Crossing symmetry relates the amplitudes

$$T^{ab\rightarrow cd}(p_a,p_b,p_c,p_d) = T^{\bar c b\rightarrow \bar a d}(-p_c,p_b,-p_a,p_d). \tag{330}$$

For instance, for the virtual Compton amplitude $T_1^{\gamma^*\bar H\rightarrow\gamma^*\bar H}(u,s) = T_1^{\gamma^* H\rightarrow\gamma^* H}(s,u)$. For the elastic $\gamma^* H$ and $\gamma^*\bar H$ amplitudes themselves the crossing properties for the photon imply $T_1(s,u) = T_1(u,s)$. From the optical theorem the relation between the total cross section for $\gamma^* H\rightarrow X$ to the elastic $\gamma^* H\rightarrow\gamma^* H$ amplitude, specifically the structure function W_1 is found as the discontinuity over the cut in the physical region, which is the imaginary part of the forward ($t=0$) amplitude T_1. For the forward amplitude s and u are related and since for $t=0$ the variable ν equals $\nu = (s-u)/4M$, T_1 is a symmetric function of ν. One has

$$\sigma_T^{\gamma^* H}(\nu) = \frac{4\pi\alpha}{K}\,\mathcal{I}m\,T_1(s+i\epsilon,u) = \frac{4\pi\alpha}{K}\,\mathcal{I}m\,T_1(\nu+i\epsilon), \tag{331}$$

$$\begin{aligned}\sigma_T^{\gamma^*\bar H}(\nu) = \frac{4\pi\alpha}{K}\,\mathcal{I}m\,T_1(u+i\epsilon,s) &= \frac{4\pi\alpha}{K}\,\mathcal{I}m\,T_1(-\nu+i\epsilon)\\ &= -\frac{4\pi\alpha}{K}\,\mathcal{I}m\,T_1(\nu+i\epsilon),\end{aligned} \tag{332}$$

where the last equality follows from the symmetry of T_1, $T_1(\nu) = T_1(-\nu)$ and the fact that T_1 is a real analytic function, $T_1(\nu) = T_1^*(\nu^*)$. Note that as $W_{\mu\nu}$ is defined as the commutator of the currents the cross section for the 'crossed' part enters with a negative sign. As an analytic function of ν, however, one precisely has

$$W_1(\nu,Q^2) = \frac{1}{\pi}\,\mathcal{I}m\;T_1(\nu+i\epsilon) = -W_1(-\nu,Q^2), \tag{333}$$

which can also be derived from the translation invariance properties of the commutator defining $W^{\mu\nu}$. The analytic behavior and symmetry in the Bjorken variable $x = Q^2/2M\nu$ is exactly the same as that in ν, i.e. $W_1(x) = -W_1(-x)$ is antisymmetric in x.

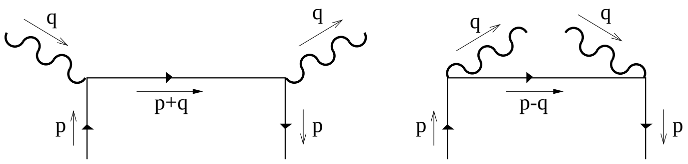


Figure 6: *The Compton amplitude for a point fermion.*

Exercise. A simple example in which all of these properties are illustrated is the forward Compton amplitude for scattering off a point fermion with mass m and charge $Q/e = e_f$ (fig. 6). The amplitude T_1 is the coefficient of $g_{\mu\nu}$ in the amplitude $T_{\mu\nu}$,

$$2m\,T_{\mu\nu} = e_f^2\,\frac{\bar{u}(p)\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu u(p)}{(p+q)^2 - m^2 + i\epsilon} + [\mu \leftrightarrow \nu,\ q \leftrightarrow -q]. \tag{334}$$

Show that it equals

$$\begin{aligned} 2m\,T_1 &= e_f^2\left(\frac{s-u}{2(s-m^2)} + \frac{u-s}{2(u-m^2)}\right) \\ &= e_f^2\left(\frac{1}{1-x} + \frac{1}{1+x}\right), \end{aligned} \tag{335}$$

of which the imaginary part precisely is the structure function for a pointlike fermion,

$$2m\,W_1 = e_f^2\left[\delta(1-x) - \delta(1+x)\right]. \tag{336}$$

34 Polarized leptoproduction

Note that for spin-polarized leptons in the initial state we have

$$\begin{aligned} L_{\mu\nu}^{(s)} &= \mathrm{Tr}\left[\gamma_\mu(\not{k}' + m)\gamma_\nu(\not{k}+m)\,\frac{1 \pm \gamma_5\,\not{s})}{2}\right] \\ &= 2\,k_\mu k'_\nu + 2\,k'_\mu k_\nu - Q^2\,g_{\mu\nu} \pm 2i\,m\,\epsilon_{\mu\nu\rho\sigma}q^\rho s^\sigma. \end{aligned} \tag{337}$$

Note that for light particles or particles at high energies helicity states ($\hat{s} = \hat{\boldsymbol{k}}$) become chirality eigenstates. For $L_{\mu\nu}$ the equivalence is easily seen because for $s^\mu = (|\boldsymbol{k}|/m, E\hat{\boldsymbol{k}}/m)$ ($s^2 = -1$ and $s\cdot p = 0$) one obtains in the limit $E \approx |\boldsymbol{k}|$ the result $m\,s^\mu \approx k^\mu$. Therefore the leptonic tensor for helicity states is

$$L_{\mu\nu}^{(\lambda=\pm 1/2)} \approx L_{\mu\nu}^{(R/L)} = L_{\mu\nu}^{(S)} \pm L_{\mu\nu}^{(A)}. \tag{338}$$

One can use polarized leptons in deep inelastic $\vec{e}p \to X$ to probe the antisymmetric tensor for unpolarized hadrons, containing the F_3 structure function. This contribution comes in via the interference between the γ and Z interference term.

Exercise. Why would one want to use polarized leptons on an unpolarized target to measure F_3. The $\gamma - Z^0$-interference for unpolarized leptons already contains vector and axial vector parts, being of the form $(1-4\sin^2\theta_W)L_{\mu\nu}^{(S)} - L_{\mu\nu}^{(A)}$ and thus allows the measurement of F_3.

In the situation where the target is polarized, one has even with the constraint of parity invariance (thus for γ-exchange) several more structure functions as compared to the case of an unpolarized target. For a spin 1/2 target one can also employ the spin vector S^μ ($S^2 = -1$ and $P \cdot S = 0$). Separating $S_\perp$ satisfying $S_\perp \cdot P = S_\perp \cdot q = 0$ is achieved by writing

$$\begin{aligned} S^\mu &= \frac{S\cdot q}{P\cdot q}\left(P^\mu - \frac{M^2}{P\cdot q}\,q^\mu\right) + S_\perp^\mu \\ &= \frac{\lambda}{M}\left(P^\mu - \frac{M^2}{P\cdot q}\,q^\mu\right) + S_\perp^\mu, \end{aligned} \tag{339}$$

with

$$\lambda \equiv \frac{M\,(S\cdot q)}{(P\cdot q)}, \tag{340}$$

such that $S^2 = -(\lambda^2 + \boldsymbol{S}_\perp^2)$ = -1 and one parametrizes

$$\begin{aligned} M\,W^{\mu\nu\,(A)}(q,P,S) &= \lambda\,\frac{i\epsilon^{\mu\nu\rho\sigma}q_\rho P_\sigma}{(P\cdot q)}\,g_1 + \frac{M}{P\cdot q}\,i\epsilon^{\mu\nu\rho\sigma}q_\rho S_{\perp\sigma}\,(g_1+g_2) \\ &= -i\lambda\,\epsilon_\perp^{\mu\nu}\,g_1 - 2i\,\frac{M}{Q}\,\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}\,x_B(g_1+g_2) \end{aligned} \tag{341}$$

in terms of two structure functions $g_1(x_B, Q^2)$ and $g_2(x_B, Q^2)$. The resulting cross section is

$$\frac{d\Delta\sigma}{dx_B\,dy} = \frac{4\pi\,\alpha^2}{Q^2}\,\lambda_e\left[\lambda(2-y)\,g_1 - 2|\boldsymbol{S}_\perp|\,\cos\phi_S\,\frac{M}{Q}\,\sqrt{1-y}\,x_B(g_1+g_2)\right] \tag{342}$$

(Note that in all of the above formulas mass corrections proportional to M^2/Q^2 have been neglected).

35 Form factors

A special case of inclusive scattering is the situation in which the final state is identical to the initial state, elastic scattering. In that case the final state four momentum is $P' = P + q$ and is fixed to be $(P+q)^2 = M^2$, i.e. $x_B = 1$. We can still use the formalism for inclusive leptoproduction but the hadron tensor becomes becomes

$$2M\,W_{\mu\nu}(q,P) = \underbrace{\langle P|J_\mu(0)|P'\rangle\langle P'|J_\nu(0)|P\rangle}_{H_{\mu\nu}(P;P')}\,\frac{1}{Q^2}\,\delta(1-x_B). \tag{343}$$

One needs the current matrix elements, instead of those of the commutator or the time-ordered product. The elastic scattering contribution at $x_B = 1$ or $\nu = Q^2/2M$ is measured at low energies. The current matrix elements are expressed in form factors, which are Fourier transforms of charge and current distributions. We will consider in more detail the vector and axial vector currents, V_μ and A_μ.

For an elementary fermion, e.g. an electron, the currents are known from the and it is trivial to write down the expectation value of the vector and axial vector currents,

$$\langle p',s'|V_\mu(x)|p,s\rangle = \bar{u}(p',s')\gamma_\mu u(p,s)\,e^{i(p'-p)\cdot x}, \tag{344}$$

$$\langle p',s'|A_\mu(x)|p,s\rangle = \bar{u}(p',s')\gamma_\mu u(p,s)\,e^{i(p'-p)\cdot x}. \tag{345}$$

For composite systems the expectation values of the current operator is in general unknown. One can however write down a most general form and use translation invariance. The latter states

$$\langle P',S'|J_\mu(x)|P,S\rangle = e^{i(P'-P)\cdot x}\langle P',S'|J_\mu(0)|P,S\rangle. \tag{346}$$

The most general form consistent with the requirements of Poincaré invariance, gauge invariance and invariance under parity and time reversal yields for a spin 0 system (e.g. a pion or a ^{12}C nucleus) the form

$$\langle P',S'|V_\mu(0)|P,S\rangle = (P_\mu + P'_\mu)\,F(Q^2), \tag{347}$$

involving one invariant function (*form factor*) depending on the (only) invariant, $q^2 \equiv -Q^2$, where $q = P' - P$. For spin 1/2 systems (such as a nucleon, ^3H or ^3He) the most general expression for the expectation value of the current can be written using free Dirac spinors $U(P,S)$,

$$\langle P', S'|V_\mu(0)|P,S\rangle = \overline{U}(P',S')\left[\gamma_\mu\, F_1(Q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}\, F_2(Q^2)\right] U(P,S), \tag{348}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ and $q = P' - P$. The functions F_1 and F_2 are the Dirac and Pauli form factors.

The elastic cross section

In the case of electromagnetic scattering one obtains for the tensor $H_{\mu\nu}$ the result

$$\begin{aligned}
H_{\mu\nu}(P,P') &= \frac{1}{2}\sum_{S,S'}\langle P,S|J_\mu(0)|P',S'\rangle\langle P',S'|J_\nu(0)|P,S\rangle \\
&= \frac{1}{2} Tr\ [\Gamma_\mu(\not{P}' + M)\Gamma_\nu(\not{P} + M)] \\
&= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) Q^2(F_1+F_2)^2 + 4\,\tilde{P}_\mu\tilde{P}_\nu\,\left(F_1^2 + \frac{Q^2}{4M^2}\,F_2^2\right) \\
&= -g_{\perp\,\mu\nu}\, Q^2 G_M^2 + \hat{t}_\mu\hat{t}_\nu\, 4M^2\, G_E^2
\end{aligned} \tag{349}$$

One thus sees the following *elastic* contribution in the structure functions

$$F_T(x_B, Q^2) = G_M^2(Q^2)\,\delta(1 - x_B), \tag{350}$$

$$F_L(x_B, Q^2) = \frac{4M^2}{Q^2}\, G_E^2(Q^2)\,\delta(1 - x_B). \tag{351}$$

What information is in the form factors

For the forward matrix element one obtains using $\overline{U}(P,S')\gamma_\mu U(P,S) = 2P_\mu\,\delta_{SS'}$,

$$\langle P,S'|V_\mu(0)|P,S\rangle = 2P_\mu\,\delta_{SS'}\,F_1(0). \tag{352}$$

We note that for the (diagonal) vector currents for the quarks, $V_\mu^q = \overline{\psi}^q\gamma_\mu\psi^q$, the quantities $F_1^q(0)$ are (by definition) the quark numbers $F_1^q(0) = n_q$. The matrix element of the electromagnetic current J_μ^{em} (given above) satisfying $F_1^{em}(0) = Q$ (the charge of the hadron state).

Form factors for spacelike q probe the spatial distribution of the charges. Note that in that case there always exist a frame in which q is purely spacelike ($q^0 = 0$). For the interpretation of the significance of the form factors this frame (brick-wall or Breit frame) is therefore particularly useful. Working out the current expression for the nucleon in the Breit frame ($P'^0 = P^0$, $\boldsymbol{P} = -\boldsymbol{q}/2$, $\boldsymbol{P}' = +\boldsymbol{q}/2$), gives

$$\langle P',S'|J_0^{\rm em}(0)|P,S\rangle = 2M\left[F_1 - \frac{Q^2}{4M^2}\,F_2\right] \equiv 2M\,G_E(Q^2), \tag{353}$$

$$\langle P',S'|\boldsymbol{J}^{\rm em}(0)|P,S\rangle = [F_1 + F_2]\,(i\boldsymbol{\sigma}_N \times \boldsymbol{q}) \equiv G_M(Q^2)\,(i\boldsymbol{\sigma}_N \times \boldsymbol{q}), \tag{354}$$

where $\chi_{S'}^\dagger\,\boldsymbol{\sigma}\,\chi_S \equiv \boldsymbol{\sigma}_N$. These expressions show the relevance of the Sachs form factors G_E and G_M. The quantity $e\,G_E(0)$ is the charge Q of the nucleon, $e\,G_M(0)/M$ is the magnetic moment of the nucleon. The quantity $\kappa = G_M(0) - Q = G_M(0) - G_E(0)$ is the anomalous magnetic moment. Note the following forward matrix elements

$$\langle P,S'|J_0^{\rm em}(0)|P,S\rangle = 2M\,F_1(0) = 2M\,G_E(0), \tag{355}$$

$$\langle P,S'|(\boldsymbol{r}\times\boldsymbol{J}^{\rm em})(0)|P,S\rangle = [F_1(0) + F_2(0)]\,\boldsymbol{\sigma}_N = G_M(0)\,\boldsymbol{\sigma}_N, \tag{356}$$

For an elementary fermion one has $F_1(Q^2) = 1$ and $F_2(Q^2) = 0$. For the nucleon one has for the proton $F_1^p(0) = Q_p = 1$ and $F_2^p(0) = \kappa_p = 1.793$ and for the neutron $F_1^n(0) = Q_n = 1$ and $F_2^n(0) = \kappa_n =$ -1.913. Roughly one has for nucleons the dipole-like behavior

$$G_E^p(Q^2) \approx \frac{G_M^p}{\mu_p} \approx \frac{G_M^n}{\mu_n} \approx \frac{1}{(1+Q^2/\Lambda^2)^2}, \tag{357}$$

with $\Lambda^2 \approx 0.71$ GeV^2 and $\mu_N = G_M^N(0) = Q_N + \kappa_N$ is the nucleon magnetic moment.

For the axial current the expectation value for a spin 1/2 state (e.g. nucleon) can be written as

$$\langle P', S'|A_\mu(0)|P, S\rangle \;=\; \overline{U}(P', S')\left[\gamma_\mu\; G_A(Q^2) + q_\mu\; H_A(Q^2)\right]\gamma_5\, U(P, S). \tag{358}$$

The value at $Q^2 = 0$ which using the spin vector $\overline{U}(P,S)\,\gamma_\mu\gamma_5\,U(P,S) = 2M\,S_\mu(P)$ (satisfying $S^2 =$ -1, $P\cdot S = 0$) appears in the forward matrix element,

$$\langle P, S|A_\mu(0)|P, S\rangle \;=\; 2M\, S_\mu(P)\, G_A(0). \tag{359}$$

For the diagonal axial vector currents $A_\mu^q = \overline{\psi^q}\gamma_\mu\gamma_5\psi^q$ the quantity $G_A^q(0) \equiv \Delta q$. Again using the brick wall frame and taking $\boldsymbol{q} = 0$ one sees

$$\langle P, S|\boldsymbol{A}(0)|P, S\rangle \;= 2M\,\boldsymbol{\sigma}_N\, G_A(0). \tag{360}$$

The quantity $G_A(0)$ is the nucleon axial charge. Its value,$G_A(0) = 1.26$, is obtained from neutron β-decay where one measures a flavor-changing axial current, $\langle p|\mathbf{A}_+|n\rangle = 2M\,G_A(0)\langle p|\boldsymbol{\sigma}_N\,\tau_{N+}|n\rangle$. Using isospin invariance one sees that $G_A(0)$ can be expressed as $G_A(0) = \Delta u$ - Δd.

36 The parton model I: incoherent scattering

An other derivation of the parton model is by convoluting the γ^*-quark cross section with a momentum distribution of quarks in the nucleon.

Exercise. Show that the $\gamma^* q$ cross section is given by

$$\hat{\sigma}(\gamma^* q) = \frac{4\pi^2\alpha}{2p\cdot q}\,\epsilon_\mu^*\hat{W}^{\mu\nu}\epsilon_\nu$$

$$\hat{W}_{\mu\nu}(p,q) = \frac{1}{2M}\left[\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)Q^2 + 4\,\tilde{p}_\mu\tilde{p}_\nu\right]\delta(2\,p\cdot q + q^2)$$

(note that there are in principle ambiguities here because of the flux factor for virtual photons) and deduce

$$\hat{\sigma}_T(\gamma^* q) \;=\; 4\pi^2\alpha\; e_q^2\;\delta(2p\cdot q - Q^2), \tag{361}$$

$$\hat{\sigma}_L(\gamma^* q) \;=\; 4\pi^2\alpha\; e_q^2\;\frac{4m^2}{Q^2}\,\delta(2p\cdot q - Q^2) \ll \hat{\sigma}_T. \tag{362}$$

In the next step this partonic cross section is folded with a probability function for finding partons in the target. For this purpose it is convenient to give the explicit momenta as lightcone components, $p = [p^-, p^+, \boldsymbol{p}_\perp]$ where $p^\pm = (p^0 \pm p^3)/\sqrt{2}$ or $p = p^- n_- + p^+ n_+ + p_T$ in terms of two lightlike vectors satisfying $n_-^2 = n_+^2 = 1$ and $n_+\cdot n_- = 0$ (thus one has $p^\pm = p\cdot n_\mp$).

Exercise. Derive the lightcone expansion for the (external) vectors P and q,

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} q = \;\frac{Q}{\sqrt{2}}\,n_- \;-\; \frac{Q}{\sqrt{2}}\,n_+ \\ \\ P = \;\frac{x_B M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right. ,$$

or in lightcone components

$$q = \left[\frac{Q^2}{A\sqrt{2}}, -\frac{A}{\sqrt{2}}, \mathbf{0}_\perp\right],$$
$$P = \left[\frac{x_B M^2}{A\sqrt{2}}, \frac{A}{x_B\sqrt{2}}, \mathbf{0}_\perp\right] \approx \left[0, \frac{A}{x_B\sqrt{2}}, \mathbf{0}_\perp\right].$$

What is the effect of a boost? What is A in the nucleon rest-frame. $A \to \infty$ is referred to as the *infinite momentum frame*. In particular the first representation with lightlike vectors shows that when Q^2 becomes large, the nucleon momentum is 'on the scale Q in essence lightlike. While the hard momentum has both components proportional to Q, thus is not the case for P and one has $P^- \ll q^-$. In deep-inelastic scattering the plus components are of the same order with ratio precisely being the scaling variable $x_B = -q^+/P^+$.

Exercise. Including finite Q effects the 'better' scaling variable is actually $x_N = -q^+/P^+$ (Nachtmann variable). Show that sticking to the exact definition $x_B = Q^2/2P\cdot q$ one has

$$x_N = \frac{2x_B}{1+\sqrt{1+\frac{4\,M^2 x_B^2}{Q^2}}}$$

Next we define the parton momentum. On the hard scale Q it is in essence also lightlike and since it belongs to the hadron $p\cdot P$ should be hadronic, i.e. $\ll Q^2$. This leads us to

$$p = \frac{p^-\, n_-}{+} x\, P^+ n_+ + p_T, \tag{363}$$

where the ratio $x \equiv p^+/P^+$ is called the *lightcone momentum fraction*.

Exercise. Show that on-shell one has

$$p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+} = \frac{m_\perp^2}{2p^+}.$$

To derive the parton model results it is sufficient that $p^- \sim P^- \ll q^-$. Show that this is the case if $p\cdot P \sim p^2 \sim P^2 = M^2$.

Using the cross section for $\gamma^* q$ (elastic scattering) given above,

$$\hat\sigma_T = \frac{4\pi^2\alpha}{Q^2}\,\delta(1-x_p). \tag{364}$$

where $x_p = -q^+/p^+$ and introducing probability functions $f_i(x)$ for finding partons carrying momentum fraction $x = p^+/P^+ = x_B/x_p$ of the target lightcone momentum, leads to

$$\begin{aligned}\sigma_T &= \sum_i e_i^2 \int dx\, f_i(x)\,\frac{4\pi^2\alpha}{Q^2}\,\delta\left(1-\frac{x_B}{x}\right)\\ &= \frac{4\pi^2\alpha}{Q^2}\sum_i e_i^2\, x_B\, f_i(x_B).\end{aligned} \tag{365}$$

Comparing with

$$\sigma_T = \frac{8\pi^2\alpha}{Q^2}\, x_B\, F_1, \tag{366}$$

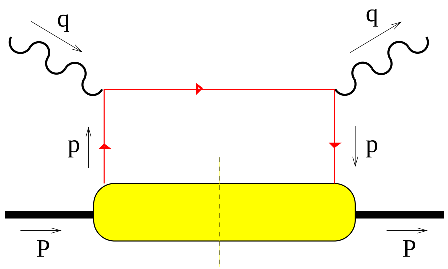


Figure 7: *The handbag diagram for inclusive deep inelastic scattering off a hadronic target.*

we get

$$F_1(x_B) = \frac{1}{2}\sum_i e_i^2\, f_i(x_B). \tag{367}$$

As $\hat\sigma_L \propto 1/Q^2 \to 0$ one obtains $F_L = 0$ or the Callan-Gross relation,

$$F_2(x_B) = 2x_B\, F_1(x_B). \tag{368}$$

37 The parton model II: diagrammatic approach

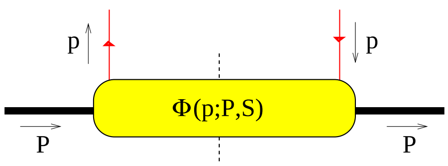
Another way in which the parton model is obtained is by just considering one diagram (which later will turn out to be the leading one), the socalled quark handbag diagram (see Fig. 7) and its antiquark equivalent. The basic expression corresponding to the handbag diagram restricting us to the quark part is

$$\begin{aligned} 2M\,W^{\mu\nu}(P,q) &= \sum_q e_q^2 \int dp^-\, dp^+\, d^2\boldsymbol{p}_\perp\; Tr\left(\Phi(p)\,\gamma^\mu(\not p + \not q + m)\gamma^\nu\right)\,\delta((p+q)^2 - m^2) \\ &\approx \sum_q e_q^2 \int dp^-\, dp^+\, d^2\boldsymbol{p}_\perp\; Tr\left(\Phi(p)\,\gamma^\mu\,\frac{\not q}{2q^-}\,\gamma^\nu\right)\,\delta(p^+ + q^+) \\ &\approx -g_\perp^{\mu\nu}\,\frac{1}{2}\int dp^-\, d^2\boldsymbol{p}_\perp\; Tr\left(\gamma^+\,\Phi(p)\right)\Big|_{p^+ = x_B\,P^+} + \ldots, \end{aligned} \tag{369}$$

where $\Phi(p)$ is the forward antiquark-target scattering amplitude,

$$\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,p\cdot\xi}\langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle, \tag{370}$$

diagrammatically represented by



Comparing with the general form of the hadronic tensor, we read off (including now also the antiquark part)

$$2\,F_1(x_B) = 2M\,W_1(x_B,Q^2) = \sum_q e_q^2\,\left[q(x_B) + \bar q(x_B)\right], \tag{371}$$

with

$$q(x) = \frac{1}{4\pi}\int d\xi^- \; e^{ixP^+\xi^-} \langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0}, \tag{372}$$

$$\overline{q}(x) = \frac{1}{4\pi}\int d\xi^- \; e^{-ixP^+\xi^-} \langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0}, \tag{373}$$

satisfying $\overline{q}(x) = -q(-x)$ (see Exercise below). The result is (as expected) a lightcone correlation function of quark fields.

The operator in coordinate space

The parton result for the structure functions can also be derived by inserting free currents in the hadronic tensor for the current commutator and using the expression for the free field commutator.

Exercise. Use the anticommutation relations for free quark fields, $\{\psi(\xi),\overline{\psi}(0)\} = \frac{1}{2\pi}\,\not\partial\,\delta(\xi^2)\,\epsilon(\xi^0)$ to derive for the $g_{\mu\nu}$ contribution in the current-current commutator for quarks

$$\begin{aligned}[J_\mu(\xi),J_\nu(0)] &= [:\overline{\psi}(\xi)\gamma_\mu\psi(\xi):,:\overline{\psi}(0)\gamma_\nu\psi(0):] \\ &= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(\xi^2)\,\epsilon(\xi^0)\right]:\overline{\psi}(\xi)\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\psi(\xi):\,.\end{aligned} \tag{374}$$

An important feature, evident in the free-current commutator, is the lightcone dominance. By sandwiching the commutator between physical states and taking the Fourier transform, it is a straightforward calculation to obtain again the hadron tensor and the same result as in the diagrammatic approach above.
[Details can be found in R.L. Jaffe in *Relativistic Dynamics and Quark Nuclear Physics*, eds M.B. Johnson and A. Picklesheimer (Wiley, New York, 1986).]

Interpretation as densities

To convince oneself that the above expressions for $q(x)$ and $\overline{q}(x)$ actually can be interpreted as quark momentum density one needs to realize that $\overline{\psi}(\xi)\gamma^+\psi(0) = \sqrt{2}\,\psi_+^\dagger(\xi)\psi_+(0)$ where $\psi_\pm = P_\pm\,\psi$ are projections obtained with projection operators $P_\pm = \frac{1}{2}\gamma^\mp\gamma^\pm$. One then can insert a complete set of states and obtain

$$\begin{aligned} q(x) &= \int \frac{d\xi^-}{2\pi\sqrt{2}}\; e^{ip\cdot\xi}\,\langle P,S|\psi_+^\dagger(0)\psi_+(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0} \\ &= \frac{1}{\sqrt{2}}\sum_n |\langle P_n|\psi_+|P\rangle|^2\;\delta\left(P_n^+ - (1-x)P^+\right), \end{aligned} \tag{375}$$

which represents the probability that a quark is annihilated from $|P\rangle$ giving a state $|n\rangle$ with $P_n^+ = (1-x)P^+$.

Polarized parton densities

Analogously to the unpolarized structure functions one can obtain for the polarized structure functions

$$2\,g_1(x_B) = \sum_q e_q^2\,[\Delta q(x_B) + \Delta\overline{q}(x_B)] \tag{376}$$

where

$$\lambda\Delta q(x) = \frac{1}{4\pi}\int d\xi^-\ e^{ixP^+\xi^-}\langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0}, \tag{377}$$

$$\lambda\Delta\overline{q}(x) = \frac{1}{4\pi}\int d\xi^-\ e^{-ixP^+\xi^-}\langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0}, \tag{378}$$

a correlation existing in a hadron with the lightcone component of the spin vector $\lambda \neq 0$. This represents the difference of chiral even and odd quarks (in infinite momentum frame quarks parallel or antiparallel to proton spin). The corresponding quark fields are projected out by

$$P_{R/L} = \frac{1}{2}(1 \pm \gamma_5), \tag{379}$$

which commute with the projectors $P_\pm$. In this way one obtains distributions $q_R(x)$ and $q_L(x)$ for which $q(x) = q_R(x) + q_L(x)$ and $\Delta q(x) = q_R(x) - q_L(x)$.

Exercise. Show that the antiquark distributions are given by $\bar{q}(x) = -q(-x)$ and $\Delta\bar{q}(x) = \Delta q(-x)$. To do this start with the 'proper' definition of antiquark distributions,

$$\Phi^c_{ij}(p) = \frac{1}{(2\pi)^4}\int d^4\xi\ e^{ip\cdot\xi}\,\langle PS|\overline{\psi}^c_j(0)\psi^c_i(\xi)|PS\rangle, \tag{380}$$

with $\psi^c(\xi) = C\overline{\psi}^T(\tilde{\xi})$. Show that one finds $\overline{\Phi}(p) = -C(\Phi^c)^T C^\dagger$. One also needs to use the anticommutation relations for fermions, to obtain $\overline{\Phi}_{ij}(p) = -\Phi_{ij}(-p)$, which leads to the crossing relations for quark and antiquark distributions.

38 Parton distributions

The explicit flavor and spin dependence of the structure functions in electroweak processes depend on the probe being a γ, Z^0 or $W^\pm$ boson, for which we have discussed the relevant currents. In general we write for the quark currents that we need in the hadronic tensor (i.e. corrected for factors $2\sqrt{2}\sin\theta_W$ and $2\sin\theta_W\cos\theta_W$),

$$\begin{aligned} J^{(V)}_\mu &= \sum_q \overline{\psi}^q(x)\left[g^{(V)}_{Vq}\,\gamma_\mu + g^{(V)}_{Aq}\,\gamma_\mu\gamma_5\right]\psi^q(x) \\ &= \sum_q \overline{\psi}^q(x)\left[\frac{1}{2}\,g^{(V)}_{Rq}\,\gamma_{\mu R} + \frac{1}{2}\,g^{(V)}_{Lq}\,\gamma_{\mu L}\right]\psi^q(x), \end{aligned} \tag{381}$$

From the currents one finds the coefficients expressed in terms of the charges (Q) or weak isospin ($\boldsymbol{I}_W$) of the quarks,

$$g^{(\gamma)}_{Lq} = g^{(\gamma)}_{Rq} = Q, \tag{382}$$

$$g^{(W)}_{Lq} = 2\,I^\pm_W, \quad g^{(W)}_{Rq} = 0, \tag{383}$$

$$g^{(Z)}_{Lq} = 2\,I^3_W - 2\,Q\,\sin^2\theta_W, \quad g^{(Z)}_{Rq} = -2\,Q\,\sin^2\theta_W. \tag{384}$$

Including the flavors explicitly, e.g. the current to be used in $W^{(\gamma\gamma)}_{\mu\nu}$ is

$$J^{(\gamma)}_\mu = \frac{2}{3}\,\overline{\psi}^u\gamma_\mu\psi^u - \frac{1}{3}\,\overline{\psi}^d\gamma_\mu\psi^d - \frac{1}{3}\,\overline{\psi}^s\gamma_\mu\psi^s + \ldots. \tag{385}$$

One obtains for ep scattering the following expression in terms of the distribution functions $u_p(x)$, $d_p(x)$, etc.,

$$\frac{F_2^{ep\,(\gamma\gamma)}(x)}{x} = 2\,F_1^{ep\,(\gamma\gamma)}(x) = \frac{4}{9}\,(u_p(x)+\overline{u}_p(x)) + \frac{1}{9}\,(d_p(x)+\overline{d}_p(x)) + \frac{1}{9}\,(s_p(x)+\overline{s}_p(x)) + \ldots$$
$$\equiv \frac{4}{9}\,(u(x)+\overline{u}(x)) + \frac{1}{9}\,(d(x)+\overline{d}(x)) + \frac{1}{9}\,(s(x)+\overline{s}(x)) + \ldots. \quad (386)$$

Using isospin summetry, $u_p = d_n$, $d_p = u_n$, $s_p = s_n$ one has for en scattering

$$\frac{F_2^{en\,(\gamma\gamma)}(x)}{x} = 2\,F_1^{en\,(\gamma\gamma)}(x) = \frac{1}{9}\,(u(x)+\overline{u}(x)) + \frac{4}{9}\,(d(x)+\overline{d}(x)) + \frac{1}{9}\,(s(x)+\overline{s}(x)) + \ldots. \quad (387)$$

As the difference between quarks and antiquarks contributes to the quantum numbers, it is convenient to divide the quark distribution in a valence part and a sea part

$$q(x) = q_v(x) + q_s(x), \quad (388)$$

where

$$q_v(x) \equiv q(x) - \overline{q}(x). \quad (389)$$

The quark distributions are positive definite, so for instance

$$\frac{1}{4} \le \frac{F_2^{ep}}{F_2^{ep}} = \frac{(u+\overline{u}) + 4(d+\overline{d}) + (s+\overline{s}) + \ldots}{4(u+\overline{u}) + (d+\overline{d}) + (s+\overline{s}) + \ldots} \le 4. \quad (390)$$

If one looks at the experimental result one sees near $x \approx 0$ a ratio that is about 1, indicating dominance of sea quarks with $u\overline{u}$ and $d\overline{d}$ pairs in equal amounts. Near $x \approx 1$ the valence quarks dominate. Naively one might expect $u = 2\,d$ and all others zero, i.e. a ratio of 2/3. The experimentally observed limit for $x \to 1$ tends to 1/4, the lower limit, which is reached for $d \ll u$, i.e. dominance of the u-quark in the proton (and therefore the d-quark in the neutron).

Including the weak interactions one can in general write for the most important structure functions in deep inelastic scattering

$$2\,F_1^{(V_1V_2)} = \frac{F_2^{(V_1V_2)}}{x} = \sum_q A_q^{(V_1V_2)}\,[q(x)+\overline{q}(x)]\,, \quad (391)$$
$$F_3^{(V_1V_2)} = \sum_q B_q^{(V_1V_2)}\,[q(x)-\overline{q}(x)]\,, \quad (392)$$

with

$$A_q^{(V_1V_2)} = \frac{1}{2}\left(g_{Lq}^{(V_1)}\,g_{Lq}^{(V_2)} + g_{Rq}^{(V_1)}\,g_{Rq}^{(V_2)}\right), \quad (393)$$
$$B_q^{(V_1V_2)} = \frac{1}{2}\left(g_{Lq}^{(V_1)}\,g_{Lq}^{(V_2)} - g_{Rq}^{(V_1)}\,g_{Rq}^{(V_2)}\right). \quad (394)$$

Some explicit examples for weak interactions through the charged current,

$$J_\mu^{(W)} = \overline{u}\gamma_\mu(1-\gamma_5)[d\,\cos\theta_c + s\,\sin\theta_c] + \overline{c}\gamma_\mu(1-\gamma_5)[s\,\cos\theta_c - d\,\sin\theta_c], \quad (395)$$

are

$$\frac{F_2^{\nu p\,(WW)}(x)}{x} = 2\,F_1^{\nu p\,(WW)}(x) = 2\,[d(x)+s(x)+\overline{u}(x)+\overline{c}(x)]\,, \quad (396)$$
$$F_3^{\nu p\,(WW)}(x) = 2\,[d(x)+s(x)-\overline{u}(x)-\overline{c}(x)] \quad (397)$$

(similarly $F_2^{e^+p\,(WW)}$) and

$$\frac{F_2^{\overline{\nu}p\,(WW)}(x)}{x} = 2\,F_1^{\overline{\nu}p\,(WW)}(x) = 2\left[u(x)+c(x)+\overline{d}(x)+\overline{s}(x)\right], \tag{398}$$

$$F_3^{\overline{\nu}p\,(WW)}(x) = 2\left[u(x)+c(x)-\overline{d}(x)-\overline{s}(x)\right] \tag{399}$$

(similarly $F_2^{e^-p\,(WW)}$). Note that the quarks that are eigenstates of I_W^3 are linear combinations of the strong interaction eigenstates with mixing angle θ_c. Including all quarks of both (lowest) families no dependence on θ_c appears in the inclusive result. Below the charm threshold factors $\cos^2\theta_c$ and $\sin^2\theta_c$ will appear. As noted, these same structure functions can also be measured in $e^+p \to \overline{\nu}X$ and $e^-p \to \nu X$ respectively if the kinematics is fixed from the direction of the produced jet.

A complete table of coefficients is:

	$(\gamma\gamma)$	(γZ)	(ZZ)	(WW)
A_u	$\frac{4}{9}$	$\frac{1}{3}\left(1-\frac{8}{3}\,s^2\right)$	$\frac{1}{2}\left(1-\frac{8}{3}\,s^2+\frac{32}{9}\,s^4\right)$	2
B_u	0	$\frac{1}{3}$	$\frac{1}{2}\left(1-\frac{8}{3}\,s^2\right)$	2
A_d	$\frac{1}{9}$	$\frac{1}{6}\left(1-\frac{4}{3}\,s^2\right)$	$\frac{1}{2}\left(1-\frac{4}{3}\,s^2+\frac{8}{9}\,s^4\right)$	2
B_d	0	$\frac{1}{6}$	$\frac{1}{2}\left(1-\frac{4}{3}\,s^2\right)$	2

(Note: $s^2 = \sin^2\theta_W$). The above explicit partonic expressions for the structure functions allow one to disentangle the quark distributions from electroweak scattering data.

39 Sum rules

As probability distributions the quark distribution functions satisfy a number of obvious sum rules, such as

$$\int_0^1 dx\ \left[u(x)-\overline{u}(x)\right] = n_u = 2, \tag{400}$$

$$\int_0^1 dx\ \left[d(x)-\overline{d}(x)\right] = n_d = 1, \tag{401}$$

$$\int_0^1 dx\ \left[s(x)-\overline{s}(x)\right] = n_s = 2, \tag{402}$$

corresponding to the (net) number of each of these quark species in the proton. On the basis of this one finds a number of sum rules for the structure functions, such as the *Adler sum rule*,

$$\int_0^1 \frac{dx}{x}\ \left[F_2^{\overline{\nu}p}(x) - F_2^{\nu p}(x)\right] = 2\left[n_u - n_d + n_c - n_s\right] = 2, \tag{403}$$

the *Bjorken sum rule*,

$$\int_0^1 dx\ \left[F_1^{\overline{\nu}p}(x) - F_1^{\nu p}(x)\right] = n_u - n_d + n_c - n_s = 1, \tag{404}$$

the *Gross-Llewellyn Smith sum rule*,

$$\int_0^1 dx\ \left[F_3^{\overline{\nu}p}(x) + F_3^{\nu p}(x)\right] = 2\left[n_u + n_d + n_s + n_c\right] = 6, \tag{405}$$

the *Gottfried sum rule* which is based on a symmetric sea distribution, $\overline{u}(x) = \overline{d}(x)$,

$$\int_0^1 \frac{dx}{x}\ [F_2^{ep}(x) - F_2^{en}(x)] = \frac{1}{3}\int_0^1 dx[u(x) - d(x) + \overline{u}(x) - \overline{d}(x)] = \frac{1}{3}. \tag{406}$$

The experimental result (NMC), giving 0.240 ± 0.016 indicates possibly that $\overline{U}(x) \neq \overline{d}(x)$. Most of these sum rules will acquire a QCD correction. So also, the quantity

$$\int_0^1 dx\ x[u(x) + \overline{u}(x) + d(x) + \overline{d}(x) + s(x) + \overline{s}(x) + \ldots] \equiv \int_0^1 dx\, x\, \Sigma(x) = \epsilon_q \leq 1, \tag{407}$$

represents the total momentum fraction of the proton carried by quarks. It must obviously be smaller than one.

For the polarized structure functions one can obtain similar estimates using the naive flavor-spin structure of the proton based on $SU(6)$ symmetry,

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\Bigg(2\ u_\uparrow u_\uparrow d_\downarrow - u_\uparrow u_\downarrow d_\uparrow - u_\downarrow u_\uparrow d_\uparrow +\ [d \text{ at places 1 and 2}]\Bigg).\Bigg), \tag{408}$$

From this wave function one finds in terms of a normalized one-quark distribution $q(x)$ the 'naive' results

$$\begin{aligned} u_\uparrow(x) &= \frac{5}{3}\,q(x), & u_\downarrow(x) &= \frac{1}{3}\,q(x), \\ d_\uparrow(x) &= \frac{1}{3}\,q(x), & d_\downarrow(x) &= \frac{2}{3}\,q(x), \end{aligned} \tag{409}$$

or

$$\begin{aligned} u(x) &= 2\,q(x), & \Delta u(x) &= \frac{4}{3}\,q(x), \\ d(x) &= q(x), & \Delta d(x) &= -\frac{1}{3}\,q(x), \end{aligned} \tag{410}$$

and all other distributions (strange quarks or antiquarks) are zero. In this case one obtains naive sum rule results for the polarized distributions,

$$\int_0^1 dx\ [\Delta u(x) + \Delta\overline{u}(x)] = \Delta u = \frac{4}{3}, \tag{411}$$

$$\int_0^1 dx\ [\Delta d(x) + \Delta\overline{d}(x)] = \Delta d = -\frac{1}{3}. \tag{412}$$

Note that the sum $\Delta\Sigma = \Delta u + \Delta d + \Delta s + \ldots$ represents (as probability distributions) the total number of quarks parallel to the proton spin. If the proton spin comes from the quark spins, as is the case for the above $SU(6)$ wave function multiplying a spherically symmetric spatial wave function, one expects this to be one. It leads to

$$\Gamma_1^p = \int_0^1 dx\ g_1^p(x) = \frac{1}{2}\left[\frac{4}{9}\,\Delta u + \frac{1}{9}\,\Delta d + \frac{1}{9}\,\Delta s\right] = \frac{5}{18} \approx 0.28, \tag{413}$$

$$\Gamma_1^n = \int_0^1 dx\ g_1^n(x) = \frac{1}{2}\left[\frac{4}{9}\,\Delta d + \frac{1}{9}\,\Delta u + \frac{1}{9}\,\Delta s\right] = 0, \tag{414}$$

which is in disagreement with the experimental result with the experimental result, $\Gamma_1^p \approx 0.15$ and the result $\Gamma_1^n \approx -0.04$, obtained from the deuteron sum rule $\Gamma_1^d = (\Gamma_1^p + \Gamma_1^n)\ \left(1 + \frac{3}{2}\omega_D\right)$ with $\omega_D \approx 0.05$ being the D-wave probability in the deuteron.

A different view of these sum rules comes from the expressions in terms of correlation functions. One has

$$\begin{aligned}\int_0^1 dx\ [q(x)-\overline{q}(x)] &= \int_{-1}^1 dx\ q(x) \\ &= \frac{\langle P|\overline{\psi}(0)\gamma^+\psi(0)|P\rangle}{2P^+} = n_q,\end{aligned} \tag{415}$$

where n_q is the coefficient in the expectation value $\langle P|\bar{\psi}(x)\gamma^\mu\psi(0)|P\rangle = 2\,n_q\,P^\mu$. Using the vector currents to obtain the quantum numbers for flavor (upness, downness, strangeness, etc.) n_q is precisely the quark number. In general one obtains

$$\begin{aligned}\int_{-1}^1 dx\ x^{n-1}\ q(x) &= \int_0^1 dx\ x^{n-1}\,[q(x)+(-)^n\,\overline{q}(x)] \\ &= \frac{1}{2P^+}\langle P|\overline{\psi}(0)\,\gamma^+\left(\frac{i\partial^+}{P^+}\right)^{n-1}\psi(0)|P\rangle.\end{aligned} \tag{416}$$

The moments of the structure functions are related to expectation values of particular quark operators. One problem that occurs here is that these expectation values are in fact precisely the Green's functions with operator insertions, discussed in chapter 1. In general these depend on the renormalization scale μ^2 and thus a similar renormalization scale dependence must be present for the structure functions. In the next sections these QCD corrections will be discussed in more detail. Note that the sum rule in Eq. 415 is scale independent, as it concerns a conserved current. The situation is different for the second moment,

$$\begin{aligned}\int_{-1}^1 dx\ x\ q(x,\ln\mu^2) &= \int_0^1 dx\ x\,[q(x,\ln\mu^2)+\overline{q}(x,\ln\mu^2)] \\ &= \frac{\langle P|\overline{\psi}(0)\,i\gamma^+\partial^+\psi(0)|P\rangle_{(\mu^2)}}{2\,(P^+)^2} = \epsilon_q(\mu^2),\end{aligned} \tag{417}$$

where ϵ_q is defined in $\langle P|\overline{\psi}(0)\,i\gamma_\mu\partial_\nu\psi(0)|P\rangle = 2\,\epsilon_q\,P_\mu P_\nu + \ldots$ and is the relative contribution of quarks to the energy momentum tensor of the proton (the dots indicate trace terms $\propto M^2\,g_{\mu\nu}$).

For polarized structure functions the lowest moment is given by

$$\begin{aligned}\int_{-1}^1 dx\ \Delta q(x,\ln\mu^2) &= \int_0^1 dx\ [\Delta q(x,\ln\mu^2)+\Delta\overline{q}(x,\ln\mu^2)] \\ &= \frac{\langle P|\overline{\psi}(0)\gamma^+\gamma_5\psi(0)|P\rangle_{(\mu^2)}}{2\,P^+} = \Delta_q(\mu^2)\end{aligned} \tag{418}$$

where Δq appears in the expectation value for the axial current, $\langle P|\overline{\psi}(0)\gamma^+\gamma_5\psi(0)|P\rangle = 2M\,\Delta q\,S^\mu$ and S^μ is the spin vector for the nucleon (S^2 = -1; $S\cdot p = 0$). Also this is in general not independent of the renormalization scale as the flavor singlet axial current is not conserved (because of the ABJ-anomaly). For the axial currents that are important in the weak decays of baryons, e.g. for the neutron $\propto \tau_+\,\gamma^\mu\gamma_5$, the current is conserved (up to mass effects) and the corresponding matrix elements are scale independent. From the neutron decay one deduces the (scale independent) flavor nonsinglet combinations

$$\Delta q^3 = \Delta u(\mu^2) - \Delta d(\mu^2) = G_A = 1.259, \tag{419}$$

while from hyperon decays one finds (using $SU(3)$ symmetry),

$$\Delta q^8 = \Delta u(\mu^2) + \Delta d(\mu^2) - 2\,\Delta s(\mu^2) \approx 0.6, \tag{420}$$

A sum rule involving only flavor nonsinglet combinations is for example the *polarized Bjorken sum rule*,

$$\int_0^1 dx\ [g_1^p(x) - g_1^n(x)] = \frac{1}{6}\Delta q^3 = \frac{G_A}{6}, \tag{421}$$

in reasonable agreement with experiment.

Assuming that in the proton $\Delta s = 0$ one can solve $\Delta u \approx 0.93$ and $\Delta d \approx -0.33$. Although these values deviate from the naive values of $\Delta u = 4/3$ and $\Delta d = -1/3$, one can understand this for instance from the consideration of spinor densities $\overline{\psi}\gamma_5\gamma_3\psi = \psi^\dagger\sigma_z\psi$ in a model, such as the bag model. The lower component of the solution gives a reduction of this density compared to the vector density $\overline{\psi}\gamma^0\psi$ $= \psi^\dagger\psi$. Using the phenomenological reduction factor $G_A/(5/3) \approx 0.75$, corrections one finds

$$\int dx\ g_1^p(x) = \frac{1}{2}\left(\frac{4}{9}\Delta u + \frac{1}{9}\Delta d + \frac{1}{9}\Delta s\right) = \frac{1}{9}\,\Delta\Sigma(\mu^2) + \frac{1}{12}\,\Delta q^3 + \frac{1}{36}\,\Delta q^8 \ \approx\ 0.21\,, \tag{422}$$

which still is quite somewhat larger than the experimental result $\Gamma_1^p \approx 0.15$. We will come back to this sum rule in a later section.

40 The operator product expansion

The basic idea of the operator product expansion (OPE) is that the product of two operators simplifies in the limit that their arguments coincide,

$$\lim_{x\rightarrow 0} A(x)B(0) \propto \sum_\beta C^{(\beta)}(x^2;g)O^{(\beta)}(0), \tag{423}$$

where $C^{(\beta)}$ are c-number functions that also depend on the parameters of the theory, and $O^{(\beta)}$ are local operators. After taking care of all Lorentz indices the functions are invariant functions depending on x^2, which in the (well-defined) Euclidean formulation corresponds to a short-distance expansion.

An explicit example is the $g_{\mu\nu}$ contribution in the current-current commutator for free quark currents,

$$\begin{aligned}
[J_\mu(x), J_\nu(0)] &= [:\overline{\psi}(x)\gamma_\mu\psi(x):, :\overline{\psi}(0)\gamma_\nu\psi(0):] \\
&= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(x^2)\,\epsilon(x^0)\right] :\overline{\psi}(x)\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\psi(x): \qquad (424)\\
&= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(x^2)\,\epsilon(x^0)\right]\sum_{n=0}^{\infty}\frac{x^{\mu_1}\ldots x^{\mu_n}}{n!} \\
&\qquad \times :[\partial_{\mu_1}\ldots\partial_{\mu_n}\overline{\psi}(0)]\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\,\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(0):. \qquad (425)
\end{aligned}$$

In calculating matrix elements one encounters on the right hand side the matrix elements of local operators. In momentum space, the contribution from the T-ordered product of currents corresponds to

$$\mathcal{T}\,J_\mu J_\nu = \gamma_\mu\,\frac{1}{\not{p} + \not{q} + i\epsilon}\,\gamma_\nu. \tag{426}$$

One can expand this in the following way (restricting us to the $g_{\mu\nu}$ part,

$$g_{\mu\nu}\sum_{n=0}^{\infty}\not{q}\left(\frac{\not{p}\not{q}}{-q^2}\right)^n,$$

which correspond precisely to the contributions of the operators $\overline{\psi}\gamma^\mu\partial^{\mu_1}\ldots\partial^{\mu_n}\psi$ multiplied with a coefficient in momentum space, in our case

$$C_n^{\mu_1\cdots\mu_n}(q) = \frac{q^{\mu_1}\ldots q^{\mu_n}}{(-q^2)^n}. \tag{427}$$

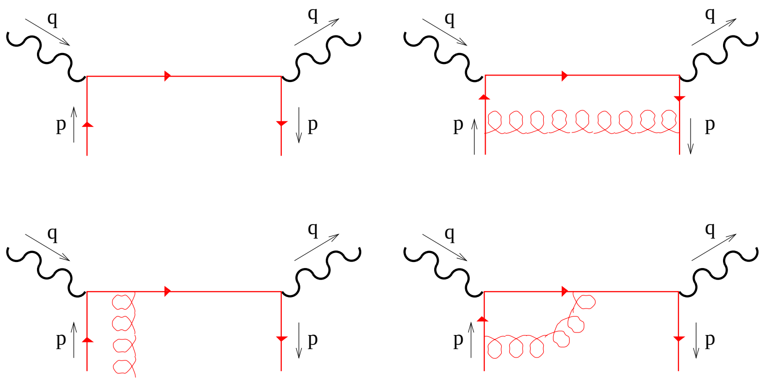


Figure 8: The $\mathcal{O}(g^2)$ contributions to the time-ordered product of currents.

From a similar collinear expansion of the result of the diagrams in Fig. 8 one can find the momentum space coefficients $C(q^2, g)$ for QCD including interactions.

The important features of the OPE are:

- The OPE yields an expansion in which the target dependence is in the matrix elements. The coefficient functions are independent of the target (factorization!). Their singularity structure plays a crucial role.
- The OPE gives a systematic expansion of the operators in terms of twist and spin, that corresponds to an expansion in powers of $1/Q^2$ for the matrix elements.
- In the operator expectation values (Green's functions) the renormalization scale μ enters. It is precisely cancelled by the scale dependence in $C^{(\beta)}(x^2; g, \mu)$. The latter scale dependence can be calculated using perturbative QCD.

Now let us make things somewhat more precise by considering the expansion for the Compton amplitude, in terms of a set of operators (suppressing the indices on the currents),

$$\mathcal{T} J(x) J(0) \propto \sum_{\beta,n} C_n^{(\beta)}(x^2; g)\, x^{\mu_1} \ldots x^{\mu_n}\, O^{(\beta)}_{\mu_1 \ldots \mu_n}(0), \tag{428}$$

where $O^{(\beta)}_{\mu_1 \ldots \mu_n}$ represents a set of (irreducible) symmetric and traceless operators. Their expectation value is given as

$$\langle P | O^{(\beta)}_{\mu_1 \ldots \mu_n}(0) | P \rangle_{(\mu^2)} = \Theta^{(\beta)}(\mu^2) [P_{\mu_1} \ldots P_{\mu_n} + \ldots], \tag{429}$$

where the dots are dictated by the tracelessness of O. After Fourier transforming one has for the forward Compton amplitude or the hadronic tensor W, after Fourier transforming $C(x^2)$,

$$\begin{aligned} 2M\, T(q^2, \nu) &\propto \sum_{\beta,n} \tilde{C}_n^{(\beta)}(q^2; g, \mu^2)\, (-)^n\, \frac{q^{\mu_1} \ldots q^{\mu_n}}{(q^2)^n}\, \langle P | O^{(\beta)}_{\mu_1 \ldots \mu_n}(0) | P \rangle_{(\mu^2)} \\ &= \sum_{\beta,n} \tilde{C}_n^{(\beta)}(q^2; g, \mu^2)\, \Theta_n^{(\beta)}(\mu^2) \left[\left(\frac{1}{2x} \right)^n + \mathcal{O}\left(\frac{1}{Q^2} \right) \right]. \end{aligned} \tag{430}$$

Dimensional analysis gives

$$d\left[C_n^{(\beta)}(x^2) \right] = 6 - d\left[O_n^{(\beta)} \right] + n, \tag{431}$$

$$d\left[\tilde{C}_n^{(\beta)}(q^2) \right] = 2 - d\left[O_n^{(\beta)} \right] + n \equiv 2 - t, \tag{432}$$

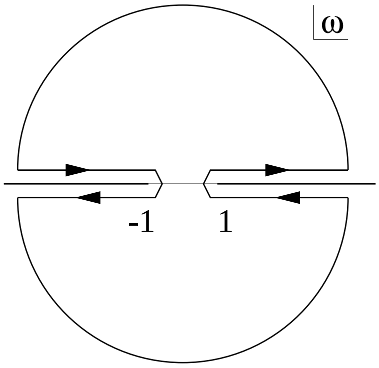
where

$$t = d\left[O_n^{(\beta)}\right] - n \tag{433}$$

is the *twist* of the operator O. The dimension of $C(q^2; g, \mu^2)$ must be provided by q^2, with logarithmic behavior arising from the anomalous dimensions of the operator (as we will see below),

$$\tilde{C}_n^{(\beta)}(q^2; g, \mu^2) = \frac{c_n^{(\beta)}\left(\ln(q^2/\mu^2)\right)}{(q^2)^{t-2}}. \tag{434}$$

In order to make contact with the deep inelastic structure functions we want to connect the expansion coefficients in $\omega \equiv 1/x$ with observable cross sections. For this we use the analytic properties of the forward Compton amplitude (considered in one of the previous chapter), $T_1(\omega) = T_1(-\omega)$ and the real-analyticity $T_1^*(\omega^*) = T_1(\omega)$. Considering the following contour in the complex ω plane with branch cuts in the physical regions (where $\mathcal{I}m\ T_1 \neq 0$)



one obtains

$$\begin{aligned}
T_1(\omega) &= \frac{1}{2\pi\, i}\left[\int_{-\infty}^{-1} d\omega'\ \frac{T(\omega'+i\epsilon)}{\omega'-\omega} + \int_{-1}^{-\infty} d\omega'\ \frac{T(\omega'-i\epsilon)}{\omega'-\omega}\right.\\
&\qquad \left. + \int_{\infty}^{1} d\omega'\ \frac{T(\omega'-i\epsilon)}{\omega'-\omega} + \int_{1}^{\infty} d\omega'\ \frac{T(\omega'+i\epsilon)}{\omega'-\omega}\right]\\
&= \frac{1}{\pi}\int_1^\infty d\omega'\ \left[\frac{\mathcal{I}m\ T(\omega'+i\epsilon)}{\omega'+\omega} + \frac{\mathcal{I}m\ T(\omega'+i\epsilon)}{\omega'-\omega}\right]\\
&= \frac{2}{\pi}\int_1^\infty d\omega'\ \frac{\omega'\,\mathcal{I}m\ T(\omega')}{\omega'^2-\omega^2}.
\end{aligned} \tag{435}$$

In case of bad convergence one can use a subtracted relation,

$$T_1(\omega) - T_1(0) = \frac{2\omega^2}{\pi}\int_1^\infty d\omega'\ \frac{\mathcal{I}m\,T(\omega')}{\omega'\,(\omega'^2-\omega^2)}, \tag{436}$$

etc. Expanding around zero one finds

$$T_1(\omega) = \frac{2}{\pi}\sum_{\substack{n=0\\ n\ \text{even}}}^{\infty}\int_1^\infty \frac{d\omega'}{\omega'}\ \left(\frac{\omega}{\omega'}\right)^n\ \mathcal{I}m\ T_1(\omega'). \tag{437}$$

Note that this expansion around $\omega = 1/x = 0$ is in fact a short distance expansion since in deep inelastic processes, the relevant contributions in the current-current matrix elements came from $|\boldsymbol{x}| \approx$

$t \approx 1/Mx$. Recalling that the structure functions $W_i(q^2,\nu)$ or $F_i(x,q^2)$ are found as the imaginary part of the forward Compton amplitude, to be precise $(1/\pi)\,\mathcal{I}m\;T(q^2,\nu)$. Thus

$$2M\,T(q^2,\nu) \propto 2i \sum_{n \text{ even}} M_n(q^2)\left(\frac{1}{x}\right)^n, \tag{438}$$

where

$$M_n(q^2) = \frac{1}{\pi}\int_0^1 dx\,x^{n-1}\,\mathcal{I}m\;T(q^2,\nu). \tag{439}$$

Thus one sees that the moments of the structure functions $F(x,Q^2) = \mathcal{I}m\,T(x,Q^2)/\pi$ are given by

$$M_n(Q^2) = \int_0^1 dx\,x^{n-1}\,F(x,Q^2)\;\propto\;\frac{c_n^{(\beta)}\left(\ln(q^2/\mu^2)\right)\;\Theta_n^{(\beta)}(\mu^2)}{(q^2)^{t-2}}. \tag{440}$$

The dominant contributions in the moments and thus also the structure functions come from twist two operators. They come in two classes, quark operators

$$O^F_{\mu_1\ldots\mu_n} = i^{n-1}\mathcal{S}\,\overline{\psi}\,\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_n}\,\psi + \ldots, \tag{441}$$

where $\mathcal{S}$ indicates symmetrization in the indices, D_μ are covariant derivatives; furthermore, there are also gluon operators

$$O^G_{\mu_1\ldots\mu_n} = (2\,i)^{n-2}\mathcal{S}\,F_{\mu_1\lambda}D_{\mu_2}\ldots D_{\mu_{n-1}}\,F_{\mu_n}{}^{\lambda} + \ldots\;. \tag{442}$$

The quark operators can of course be supplemented with flavor operators. Note that the sum rules derived for the parton distributions written as correlation functions and the expansion of the field operators in the correlation functions are precisely the (free) quark operators from the above set.

Twist 4 operators contributing to the unpolarized structure functions are for instance of the form $\mathcal{S}\,\overline{\psi}\gamma_{\mu_1}\psi\,\overline{\psi}\gamma_{\mu_2}\psi$ or $\overline{\psi}[D^\rho, F_{\rho\mu_1}]\gamma_{\mu_2}\psi$. They specifically contribute to the lowest moments of the contributions in deep inelastic scattering arising from diagrams in which more than two partons emerge from the soft scattering part.

41 QCD corrections in deep inelastic scattering

From the operator product expansion, it is easy to obtain the $\ln Q^2$ dependence of the moments. Consider for the operator expansion $\mathcal{T}J_A(x)J_B(0) = \sum_i C_i(x^2;g,\mu)\,O_i(0)$ the insertions in any Green's function,

$$G^{(n)}_{JJ} \overset{x\to 0}{\approx} \sum_i C_i(x^2;g)\,G^{(n)}_{O_i}, \tag{443}$$

where

$$G^{(n)}_{JJ}(x,x_1,\ldots,x_n;g,\mu) = \langle 0|\mathcal{T}J_A(x)J_B(0)\,\phi(x_1)\ldots\phi(x_n)|0\rangle, \tag{444}$$

$$G^{(n)}_{O_i}(x,x_1,\ldots,x_n;g,\mu) = \langle 0|\mathcal{T}O_i(0)\,\phi(x_1)\ldots\phi(x_n)|0\rangle. \tag{445}$$

The renormalization group equations read

$$[D - n\,\gamma(g) + \gamma_A(g) + \gamma_B(g)]\,G^{(n)}_{JJ} = 0, \tag{446}$$

$$[D - n\,\gamma(g) + \gamma_i(g)]\,G^{(n)}_{O_i} = 0, \tag{447}$$

where $D = \mu(\partial/\partial\mu) + \beta(g)\,(\partial/\partial g)$. Inserting in the expansion in Eq. 443 the RGE's for G_{JJ} and G_{O_i} one obtains an RGE for the coefficient function,

$$[D + \gamma_A(g) + \gamma_B(g) - \gamma_i(g)]\,C_i(x^2;g,\mu) = 0, \tag{448}$$

or for the Fourier-transform

$$[D + \gamma_A(g) + \gamma_B(g) - \gamma_i(g)]\, \tilde{C}_i(q^2; g, \mu) = 0. \tag{449}$$

The equation is independent of the specific Green's function in which the operators have been inserted, i.e. the $\ln Q^2$ behavior is independent of the target. If the currents in $J(x)J(0)$ are conserved currents the corresponding gamma functions are zero. We now can use the results in chapter 1 and obtain the QCD-corrections in leading order. After extraction of the $(q^2)^{2-t}$ factor one finds for the coefficients c_i

$$\begin{aligned} c_i(Q^2; g, \mu) &= c_i(\mu^2; \overline{g}(\tau), \mu)\ \exp\left(-\int_0^\tau d\tau' \gamma_i(\overline{g}(\tau'))\right) && (450)\\ &= c_i(\mu^2; \overline{g}(\tau), \mu)\ \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{\gamma_{0i}/2b_0}, && (451)\\ &= c_i(1; \overline{g}(\tau))\ \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{d_i}, && (452) \end{aligned}$$

where $\tau = \ln(Q^2/\mu^2)$ and γ_{0i} is the coefficient in the leading term in $\gamma_i(g) \approx (\gamma_{0i}/4\pi)\alpha_s$, while b_0 multiplies the leading term in the β function, $b_0 = (33 - 2f)/3$, and $d_i \equiv \gamma_{0i}/2b_0$.

The next to leading order (NLO) is a matter of calculating the γ function for the appropriate operators and calculating the coefficient functions,

$$\gamma_i(g) = \gamma_{0i}\ \left(\frac{\alpha_s}{4\pi}\right) + \gamma_{1i}\ \left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots, \tag{453}$$

$$c_i(1; g) = 1 + B_i\ \left(\frac{\alpha_s}{4\pi}\right) + \ldots\ . \tag{454}$$

Many of these coefficients are now know to this or even higher orders, which has been an enormous task.

For sum rules connected themselves to a conserved current the anomalous dimension $\gamma_i(g) = 0$ and the results for the coefficients B_i immediately give the NLO corrections to sum rules. There is no correction to the Adler sum rule, but one has for the Bjorken sum rule,

$$\int_0^1 dx\ \left[F_1^{\overline{\nu}p}(x) - F_1^{\nu p}(x)\right] = 1 - \frac{2}{3}\frac{\alpha_s}{\pi}, \tag{455}$$

and for the Gross-Llewellyn Smith sum rule,

$$\int_0^1 dx\ \left[F_3^{\overline{\nu}p}(x) + F_3^{\nu p}(x)\right] = 6\ \left(1 - \frac{\alpha_s}{\pi}\right). \tag{456}$$

The Gottfried sum rule gets a very small correction. The polarized Bjorken sum rule gets a correction $(1 - \alpha_s/\pi)$, or explicitly (including also the next order),

$$\int_0^1 dx\ \left[g_1^p(x, Q^2) - g_1^n(x, Q^2)\right] = \frac{G_A}{6}\ \left(1 - \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2}\,(-4.5833 + f/3)\right), \tag{457}$$

where f is the number of (active) flavors. This gives an excellent explanation of the experimental result $\Gamma_1^p - \Gamma_1^n \approx 0.19$ being somewhat smaller than $G_A/6 = 0.21$.

Exercise. As we have seen, the g_1^p sum rule by itself involves the singlet combination $\Delta\Sigma$ connected to the singlet axial current, i.e. the combination which is not conserved because of the anomaly. Thus, we need to include the scale dependence for which we need the anomalous dimension that is given by

$$\gamma_1^\Sigma(\alpha_s) = \gamma_1^\Sigma\ \left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots = 16f\ \left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots,$$

Derive the leading order solution

$$\Delta\Sigma(Q^2) = \exp\left(\frac{\gamma_1^\Sigma}{8\pi\, b_0}(\alpha_s(Q^2) - \alpha_s(Q_0^2))\right)\, \Delta\Sigma(Q_0^2). \tag{458}$$

Together with the coefficient B^Σ this gives the corrections for the singlet contribution in the spin sum rules

$$\Gamma_1^S(Q^2) = \left(1 - \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2}\,(-4.5833 + 1.16248 f)\right)\, \Delta\Sigma(Q^2). \tag{459}$$

Including this the deviation of the proton sumrule from the naive result can be pretty well understood.

42 Evolution equations

It is possible to extend the intuitive folding picture that we have used to derive the parton model to obtain the QCD correction in a different approach, which provides also a practical way to calculate the coefficient functions $c_i(g)$ and gamma functions $\gamma_i(g)$.

Extending the result for the transverse structure function written as a delta function contribution,

$$\sigma_T = \sum_i \int_0^1 dz\, f_i(z)\left[e_i^2\,\frac{4\pi^2\alpha}{Q^2}\,x\,\delta(x-z) + \delta\hat{\sigma}_i(z,Q^2) + \ldots\right], \tag{460}$$

one obtains contributions from the process $\gamma^* q \to Gq$ process, which is no longer proportional to $\delta(x-z)$ because of the fact that there are two particles in the final state. The the amplitude is

$$|\mathcal{M}|^2 = 32\pi^2\, e_i^2\, \alpha\alpha_s\, \frac{4}{3}\,\left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}}\right) \tag{461}$$

and contributes

$$\delta\sigma(\hat{s}) = e_i^2\,\frac{2\pi\alpha\alpha_s}{(\hat{s}+Q^2)^2}\,\frac{4}{3}\left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} - \frac{2\hat{t}Q^2 + 2\hat{s}Q^2 + 2Q^4}{\hat{s}\hat{t}}\right) d\hat{t} \tag{462}$$

We now express the momenta (using lightcone components) for $\gamma^*(q)$ + quark $(k) \to$ gluon (p_G) + quark (p_q) as

$$\begin{aligned}
q &\equiv \left[\frac{Q^2}{xA\sqrt{2}}, -\frac{xA}{\sqrt{2}}, \mathbf{0}_\perp\right] \qquad \text{(photon)}, &(463)\\
k &\equiv \left[\frac{m^2}{zA\sqrt{2}}, \frac{zA}{\sqrt{2}}, \mathbf{0}_\perp\right] \approx \left[0, \frac{zA}{\sqrt{2}}, \mathbf{0}_\perp\right] \qquad \text{(parton)}, &(464)\\
p_G &= \left[\zeta q^-, \frac{\boldsymbol{p}_\perp^2}{2\zeta q^-}, \boldsymbol{p}_\perp\right] \approx \left[\frac{\zeta Q^2}{xA\sqrt{2}}, 0, \boldsymbol{p}_\perp\right], &(465)\\
p_q &= \left[(1-\zeta)q^-, \frac{\boldsymbol{p}_\perp^2}{2(1-\zeta)q^-}, -\boldsymbol{p}_\perp\right] \approx \left[\frac{(1-\zeta)Q^2}{xA\sqrt{2}}, 0, -\boldsymbol{p}_\perp\right]. &(466)
\end{aligned}$$

Note that z can be written as x/ξ, where $\xi \equiv -q^+/k^+ = x/z$ is the Bjorken scaling variable for the subprocess. We will neglect all particles masses, in which case $p_q^2 = 0$ gives

$$\boldsymbol{p}_\perp^2 = \frac{\zeta(1-\zeta)(1-\xi)}{\xi}\,Q^2. \tag{467}$$

The invariants for the subprocess become

$$\begin{aligned}
\hat{s} &= (k+q)^2 = \frac{1-\xi}{\xi}\,Q^2 = \frac{1}{\zeta(1-\zeta)}\,\boldsymbol{p}_\perp^2, &(468)\\
\hat{t} &= (p_G - q)^2 = -\frac{1-\zeta}{\xi}\,Q^2 = -\frac{1}{\zeta(1-\xi)}\,\boldsymbol{p}_\perp^2, &(469)
\end{aligned}$$

The kinematic range of the process $\hat{s} \geq 0$, $-(\hat{s}+Q^2) \leq t \leq 0$ in principle restricts the ranges of (ξ,ζ) to $0 \leq \xi \leq 1$ and $0 \leq \zeta \leq 1$. The cross section becomes

$$\delta\hat{\sigma}_i(\xi,Q^2) = e_i^2\,\frac{2\pi\alpha\alpha_s}{Q^2}\,\frac{4}{3}\,\xi\left[\frac{1-\zeta-2\,\xi}{1-\xi} + \frac{1+\xi^2}{(1-\xi)(1-\zeta)}\right]d\zeta. \tag{470}$$

Integrating for inclusive scattering over the final state, i.e. integrating over ζ one sees that there are singular points, specifically for $\zeta = 1$, which corresponds to a gluon radiated with $\boldsymbol{p}_\perp = 0$. These divergences are therefore referred to as *collinear divergences*. There are again several ways of regularizing, either by giving quarks and gluons masses or by dimensional regularization. In this case a $\boldsymbol{p}_\perp$ cutoff also provides a regularization. The restriction $\boldsymbol{p}_\perp^2 \geq \mu^2$ modifies the allowed region in the (ξ,ζ) plane. For a given ξ (not to close to unity) the integration is limited to

$$\frac{\xi}{1-\xi}\,\frac{\mu^2}{Q^2} \leq \zeta \leq 1 - \frac{\xi}{1-\xi}\,\frac{\mu^2}{Q^2}. \tag{471}$$

With this regularization the result becomes

$$\begin{aligned}
\delta\hat{\sigma}_i(\xi,Q^2) &= e_i^2\,\frac{2\pi\alpha\alpha_s}{Q^2}\,\frac{4}{3}\,\xi\left[\frac{1-4\,\xi}{2(1-\xi)} + \frac{1+\xi^2}{(1-\xi)}\,\ln\left(\frac{1-\xi}{\xi}\right) + \frac{1+\xi^2}{(1-\xi)}\,\ln\left(\frac{Q^2}{\mu^2}\right)\right] && (472)\\
&= e_i^2\,\frac{2\pi\alpha\alpha_s}{Q^2}\,\xi\left[B(\xi) + P_{qq}(\xi)\,\ln\left(\frac{Q^2}{\mu^2}\right)\right]. && (473)
\end{aligned}$$

where

$$\begin{aligned}
B(\xi) &= \frac{4}{3}\left[\frac{1-4\,\xi}{2(1-\xi)} + \frac{1+\xi^2}{(1-\xi)}\,\ln\left(\frac{1-\xi}{\xi}\right)\right], && (474)\\
P_{qq}(\xi) &= \frac{4}{3}\,\frac{1+\xi^2}{(1-\xi)}. && (475)
\end{aligned}$$

The functions P_{qq} are called *splitting functions.* Combining the results of the (leading) contribution (omitting for now the term $B(\xi)$) one has

$$\sigma_T = \frac{4\pi^2\alpha}{Q^2}\,x\sum_i e_i^2\int_x^1 \frac{dz}{z}\,f_i(z)\left[\delta\left(1-\frac{x}{z}\right) + \frac{\alpha_s}{2\pi}\,P_{qq}\left(\frac{x}{z}\right)\,\ln\left(\frac{Q^2}{\mu^2}\right)\right]. \tag{476}$$

This can be rewritten in parton form

$$\sigma_T = \frac{4\pi^2\alpha}{Q^2}\,x\sum_i e_i^2\,f_i(x,\ln Q^2), \tag{477}$$

where the functions $f_i(x,\ln Q^2)$ satisfy

$$\frac{\partial f_i(x,\ln Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dz}{z}\,f_i(z)\,P_{qq}\left(\frac{x}{z}\right). \tag{478}$$

As in the case of integrating over ζ above one still encounters divergences, in this case for $\xi = x/z \to 1$. This also gives rise to $\ln(Q^2/\mu^2)$ corrections. They are considered separately by considering the $1/(1-\xi)$ appearing in the splitting functions as functionals,

$$\begin{aligned}
\int_0^{1-\delta} dx\,\frac{f(x)}{1-x} &= -\int f(x)\,d\ln(1-x)\\
&= \int_0^{1-\delta} dx\,\frac{f(x)-f(1)}{1-x} \;-\; f(1)\,\ln\delta\\
&= \int_0^1 dx\,\frac{f(x)-f(1)}{1-x} \;-\; \ln\delta\int dx\,f(x)\,\delta(1-x)\\
&\equiv \int_0^1 dx\,\frac{f(x)}{(1-x)_+} \;-\; \ln\delta\int dx\,f(x)\,\delta(1-x)
\end{aligned}$$

or

$$\frac{1}{1-x} = \frac{1}{(1-x)_+} - \delta(1-x)\,\ln\delta. \tag{479}$$

Including these and other singular contributions (vertex corrections to the process $\gamma^* + q \to q$) one then has

$$f_i(x,Q^2) = \int_x^1 \frac{dz}{z}\,f_i(z)\left\{\delta\left(1-\frac{x}{z}\right) + \frac{\alpha_s}{2\pi}\,\ln\left(\frac{Q^2}{\mu^2}\right)\left[a\,\delta\left(1-\frac{x}{z}\right) + P_{qq}\left(\frac{x}{z}\right)\right]\right\}, \tag{480}$$

The piece between curly brackets can be seen as the probability density $\mathcal{P}_{qq}$ of finding a quark inside a quark with fraction $\xi = x/z$ of the parent quark to first order in α_s. Instead of calculating all contributions and checking all cancellations, it is easier to see what the final result must be from

$$\int d\xi\,\mathcal{P}_{qq}(\xi) = 1. \tag{481}$$

Keeping the form in Eq. 476 one can write the full piece between square brackets in Eq. 480 as

$$P_{qq}(\xi) = \frac{4}{3}\,\frac{1+\xi^2}{(1-\xi)_+} + 2\,\delta(1-\xi). \tag{482}$$

In a similar way other splitting functions can be calculated, e.g. P_{qG} from the proces $\gamma^* G \to q\overline{q}$. Summarizing one has

$$\frac{dq(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right)\,q(z,\tau) + P_{qG}\left(\frac{x}{z}\right)\,G(z,\tau)\right], \tag{483}$$

$$\frac{d\overline{q}(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right)\,\overline{q}(z,\tau) + P_{qG}\left(\frac{x}{z}\right)\,G(z,\tau)\right], \tag{484}$$

$$\frac{dG(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{Gq}\left(\frac{x}{z}\right)\sum_f\left(q_f(z,\tau) + \overline{q}_f(z,\tau)\right) + P_{GG}\left(\frac{x}{z}\right)\,G(z,\tau)\right]. \tag{485}$$

It is convenient to distinguish singlet and nonsinglet distribution functions at this point. The singlet distribution is the sum over all (relevant) flavors,

$$\Sigma(x,\tau) = \sum_f\left(q_f(x,\tau) + \overline{q}_f(x,\tau)\right), \tag{486}$$

everything else is 'nonsinglet', e.g. for three flavors

$$\begin{aligned} q_v &= q - \overline{q}, \qquad \text{for } q = u,d,s \\ q^3 &= (u+\overline{u}) - (d+\overline{d}), \\ q^8 &= (u+\overline{u}) + (d+\overline{d}) - 2\,(s+\overline{s}). \end{aligned}$$

For the nonsinglet structure function we have

$$\frac{dq(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\,P_{qq}\left(\frac{x}{z}\right)\,q(z,\tau), \tag{487}$$

for the singlet

$$\frac{d\Sigma(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right)\,\Sigma(z,\tau) + 2f\,P_{qG}\left(\frac{x}{z}\right)\,G(z,\tau)\right], \tag{488}$$

$$\frac{dG(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{Gq}\left(\frac{x}{z}\right)\,\Sigma(z,\tau) + P_{GG}\left(\frac{x}{z}\right)\,G(z,\tau)\right], \tag{489}$$

i.e. the nonsinglet has a simple evolution equation, while the singlet distribution mixes with the gluon distribution.

The splitting functions are given by

$$
\begin{aligned}
P_{qq}(x) &= \frac{4}{3}\,\frac{1+x^2}{(1-x)_+} + 2\,\delta(1-x), && (490)\\
P_{Gq}(x) &= \frac{4}{3}\,\frac{1+(1-x)^2}{x}, && (491)\\
P_{qG}(x) &= \frac{1}{2}\left(x^2+(1-x)^2\right), && (492)\\
P_{GG}(x) &= 6\left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \frac{33-2f}{6}\,\delta(1-x). && (493)
\end{aligned}
$$

(Note that the coefficient for the δ-function in P_{GG} is determined from momentum conservation.

For the polarized distribution functions a similar set of splitting functions can be derived,

$$
\begin{aligned}
\Delta P_{qq}(x) &= P_{qq}(x), && (494)\\
\Delta P_{Gq}(x) &= \frac{4}{3}\,\frac{1-(1-x)^2}{x}, && (495)\\
\Delta P_{qG}(x) &= \frac{1}{2}\left(x^2-(1-x)^2\right), && (496)\\
\Delta P_{GG}(x) &= 6\left[(1+x^4)\left(\frac{1}{x}+\frac{1}{(1-x)_+}\right) - \frac{(1-x)^3}{x}\right] + \frac{33-2f}{6}\,\delta(1-x). && (497)
\end{aligned}
$$

43 Solutions of the evolution equations

The evolution equations can be solved most easily by considering moments. Starting from

$$
\begin{aligned}
\frac{dq(x,\tau)}{d\tau} &= \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\,P_{qq}\left(\frac{x}{z}\right)\,q(z,\tau),\\
&= \frac{\alpha_s(\tau)}{2\pi}\int_0^1 dz\int_0^1 dy\delta(x-yz)\,P_{qq}(y)\,q(z,\tau), && (498)
\end{aligned}
$$

and defining

$$
M_n(\tau) \equiv \int_0^1 dx\,x^{n-1}\,q(x,\tau), \tag{499}
$$

$$
A_n \equiv \int_0^1 dy\,y^{n-1}\,P_{qq}(y,\tau) = \frac{4}{3}\left[-\frac{1}{2}+\frac{1}{n(n+1)}-2\sum_{j=2}^{n}\frac{1}{j}\right], \tag{500}
$$

one has

$$
\frac{dM_n(\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\,A_n\,M_n(\tau). \tag{501}
$$

Using the leading order QCD result for $\alpha_s(\tau)$ (which reads $d\alpha_x/d\tau = -b_0(\alpha_s/4\pi)$) this is easily solved, giving

$$
\frac{M_n(Q^2)}{M_n(\mu^2)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{d_n}, \tag{502}
$$

where $d_n = -2\,A_n/b_0$. Comparison with Eqs 451 and 452 show that the moments of the splitting functions are up to a factor equal to the coefficients γ_0,

$$
A_n = -\frac{\gamma_0}{4}. \tag{503}
$$

ΛAs an example, consider the second moment, for which $A_2 = -16/9$. The result for d_2 is

$$d_2 = \frac{32}{9\,b_0} = \frac{32}{99 - 6\,f)} \stackrel{f=3}{=} \frac{32}{81},$$

and the second moment of the nonsinglet distributions, e.g. the fraction of momentum carried by the valence quarks, vanishes for $Q^2 \to \infty$, e.g.

$$M_2(Q^2) = \int dx\, x[q(x, \ln Q^2) - \overline{q}(x, \ln Q^2)] \;\propto\; (\alpha_s(Q^2))^{d_2}. \tag{504}$$

For sea quarks and gluons mixing occurs and one has

$$\frac{d}{d\tau}\begin{pmatrix} \Sigma_n(\tau) \\ G_n(\tau) \end{pmatrix} = \frac{\alpha_s(\tau)}{2\pi}\begin{pmatrix} A_n(qq) & 2f\,A_n(qG) \\ A_n(Gq) & A_n(GG) \end{pmatrix}\begin{pmatrix} \Sigma_n(\tau) \\ G_n(\tau) \end{pmatrix}. \tag{505}$$

For the second moment the anomalous dimensions can be calculated from the splitting functions or through the relation with the anomalous dimensions for the relevant operators. As can be seen from Eq. 40 or 442, the relevant operators are those for which the anomalous dimensions are given in Eq. 290. One has

$$\begin{pmatrix} A_2(qq) & 2f\,A_2(qG) \\ A_2(Gq) & A_2(GG) \end{pmatrix} = \begin{pmatrix} -\frac{16}{9} & \frac{f}{3} \\ \frac{16}{9} & -\frac{f}{3} \end{pmatrix}. \tag{506}$$

The eigenvalues of the A_2 matrix are *zero* and $-(16+3f)/9$. Thus one linear combination of Σ_2 and G_2 (the sum!) does not vanish. This corresponds to momentum conservation. One can write

$$\begin{pmatrix} \Sigma_2(Q^2) \\ G_2(Q^2) \end{pmatrix} = \begin{pmatrix} \frac{3f}{16+3f} \\ \frac{16}{16+3f} \end{pmatrix} + \begin{pmatrix} S(Q^2) \\ -S(Q^2) \end{pmatrix}, \tag{507}$$

where

$$\frac{S(Q^2)}{S(\mu^2)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{(32+6f)/9b_0}. \tag{508}$$

Inverse Mellin transforms of moments

In principle the evolution of the structure functions can be done using the evolution equations. Using the moments is actually quite convenient, although one needs all of them. A very useful method in practice is to parametrize the distributions at one Q^2 with a function for which the moments can be easily calculated, evolve the moments and apply an inverse Mellin transform. One has

$$M_n(Q^2) = \int_0^1 dx\, x^{n-1}\, q(x, Q^2), \tag{509}$$

$$q(x, Q^2) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, M_n(Q^2), \tag{510}$$

where c must be such that M_n has no singularities in the complex n-plane right of the line $\mathcal{R}e\, n = c$. A similar relation exists for the moments of the splitting functions, or if we use the relation between A_n and γ_{0n},

$$\gamma_{0n} = -4\int_0^1 dx\, x^{n-1}\, P_{qq}(x), \tag{511}$$

$$P_{qq}(x) = \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, \gamma_{0n}, \tag{512}$$

We have discussed the extensions to higher orders in Eqs 453 and 454 involving γ_1 and B. The γ_1 coefficients are precisely moments of the next orders in the splitting functions. If

$$P_{qq}(x) = P_{qq}^{(0)}(x) + \frac{\alpha_s(Q^2)}{2\pi}\,P_{qq}^{(1)}(x), \tag{513}$$

then

$$\begin{aligned} \gamma_{1n} &= -8\int_0^1 dx\,x^{n-1}\,P_{qq}^{(1)}(x), &(514)\\ P_{qq}^{(1)}(x) &= \frac{1}{16\pi i}\int_{c-i\infty}^{c+i\infty} dn\,x^{-n}\,\gamma_{1n}. &(515)\end{aligned}$$

The coefficients B correspond to the moments of the additional pieces which appear in the expressions for the structure functions, such as the $B(x)$ appearing in the calculation of F_1 in Eq. 473. If one extends the delta-function contribution,

$$C(x) = \delta(1-x) + \frac{\alpha_s(Q^2)}{4\pi}B(x), \tag{516}$$

then

$$\begin{aligned} B_n &= \int_0^1 dx\,x^{n-1}\,B(x), &(517)\\ B(x) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\,x^{-n}\,B_n. &(518)\end{aligned}$$

While the splitting functions are universal (process independent), the contributions $C(x)$ are in general process dependent. For instance the longitudinal structure function (for which the dominant parton model result is zero) picks up a contribution of which the result in Eq. 473 is a part that for electroproduction is given by

$$\begin{aligned} F_L(x,Q^2) &= \frac{\alpha_s(Q^2)}{\pi}\Bigg[\frac{4}{3}\int_x^1 \frac{dz}{z}\,\left(\frac{x}{z}\right)^2 F_2(z,Q^2)\\ &\qquad + \left(2\sum_q e_q^2\right)\int_x^1 \frac{dz}{z}\,\left(\frac{x}{z}\right)^2\left(1-\frac{x}{z}\right) z\,G(z,Q^2)\Bigg]. &(519)\end{aligned}$$

44 Quark correlation functions in 1PI leptoproduction

We now consider the case in which one particle is detected in coincidence with the scattered lepton, *one-particle-inclusive* or 1PI leptoproduction. This is an example (like Drell-Yan scattering) where two hadrons are involved and the operator product expansion cannot be used. Within the framework of QCD and knowing that the photon or Z^0 current couples to the quarks, it is possible to write down a diagrammatic expansion for leptoproduction, with in the deep inelastic limit ($Q^2 \to \infty$) as relevant diagrams only the ones given in Fig. 9 for 1-particle inclusive scattering. The expression for $\mathcal{W}_{\mu\nu}$ can be rewritten as a nonlocal product of currents and it is a straightforward exercise to show by inserting the currents $j_\mu(x) =: \overline{\psi}(x)\gamma_\mu\psi(x)$: that for 1-particle inclusive scattering one obtains in tree approximation

$$\begin{aligned} &2M\mathcal{W}_{\mu\nu}(q;PS;P_hS_h)\\ &\quad= \frac{1}{(2\pi)^4}\int d^4x\;e^{iq\cdot x}\,\langle PS|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x):\sum_X|X;P_hS_h\rangle\end{aligned}$$

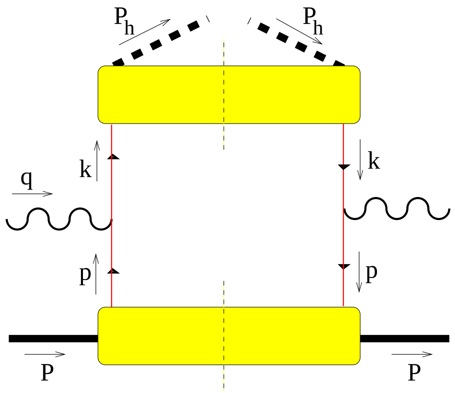


Figure 9: *The simplest (parton-level) diagrams for semi-inclusive scattering, of which we consider one-particle inclusive leptoproduction. Note that also the diagram with opposite fermion flow has to be added.*

$$
\begin{aligned}
&\times\langle X;P_hS_h|:\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|PS\rangle \\
&= \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle(\gamma_\mu)_{jk} \\
&\qquad \langle 0|\psi_k(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li} \\
&\quad + \frac{1}{(2\pi)^4}\int d^4x\; e^{iq\cdot x}\,\langle PS|\psi_k(x)\overline{\psi}_l(0)|PS\rangle(\gamma_\nu)_{li} \\
&\qquad \langle 0|\overline{\psi}_j(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\psi_i(0)|0\rangle(\gamma_\mu)_{jk}, \\
&= \int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\left(\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right) + \left\{\begin{array}{c} q\leftrightarrow -q \\ \mu\leftrightarrow\nu\end{array}\right\},
\end{aligned}
\tag{520}
$$

where

$$
\begin{aligned}
\Phi_{ij}(p) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{ip\cdot\xi}\,\langle PS|\overline{\psi}_j(0)\psi_i(\xi)|PS\rangle, \\
\Delta_{kl}(k) &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{ik\cdot\xi}\,\langle 0|\psi_k(\xi)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle.
\end{aligned}
$$

Note that in Φ (quark production) a summation over colors is assumed, while in Δ (quark decay) an averaging over colors is assumed. The quantities Φ and Δ correspond to the blobs in Fig. 9 and parametrize the soft physics. Soft refers to all invariants of momenta being small as compared to the hard scale, i.e. for $\Phi(p)$ one has $p^2 \sim p\cdot P \sim P^2 = M^2 \ll Q^2$.

In general many more diagrams have to be considered in evaluating the hadron tensors, but in the deep inelastic limit they can be neglected or considered as corrections to the soft blobs. We return to this later. One situation we want to mention here are target fragmentation parts involving matrix elements of the form $\langle PS|\psi_j(x)\sum_X|X;P_hS_h\rangle\,\langle X;P_hS_h|\psi_i(0)|PS\rangle$, known as *fracture functions*. They are relevant in the situation where $P\cdot P_h \sim M^2$ (target fragmentation region), which can be distinguished experimentally from the region we are interested in, $P\cdot P_h \sim Q^2$ (current fragmentation region).

Lorentz structure of soft parts

As mentioned above, the relevant structural information for the hadrons is contained in soft parts (the blobs in Fig. 9) which represent specific matrix elements of quark fields. The form of Φ is constrained by hermiticity, parity and time-reversal invariance. The quantity depends besides the quark momentum p on the target momentum P and the spin vector S and one must have

$$\begin{aligned}
&[\text{Hermiticity}] \quad \Rightarrow \quad \Phi^\dagger(p,P,S) = \gamma_0\,\Phi(p,P,S)\,\gamma_0, &(521)\\
&[\text{Parity}] \quad \Rightarrow \quad \Phi(p,P,S) = \gamma_0\,\Phi(\bar p,\bar P,-\bar S)\,\gamma_0, &(522)\\
&[\text{Time reversal}] \;\Rightarrow\; \Phi^*(p,P,S) = (-i\gamma_5 C)\,\Phi(\bar p,\bar P,\bar S)\,(-i\gamma_5 C), &(523)
\end{aligned}$$

where $C = i\gamma^2\gamma_0$, $-i\gamma_5 C = i\gamma^1\gamma^3$ and $\bar p = (p^0, -\boldsymbol{p})$. The most general way to parametrize Φ using only the constraints from hermiticity and parity invariance, is

$$\begin{aligned}
\Phi(p,P,S) \;=\;& M\,A_1 + A_2\,\not{P} + A_3\not{p} + i\,A_4\,\frac{[\not{P},\not{p}]}{2M}\\
&+ i\,A_5\,(p\cdot S)\gamma_5 + M\,A_6\,\not{S}\gamma_5 + A_7\,\frac{(p\cdot S)}{M}\,\not{P}\gamma_5\\
&+ A_8\,\frac{(p\cdot S)}{M}\,\not{p}\gamma_5 + A_9\,\frac{[\not{P},\not{S}]}{2}\,\gamma_5 + A_{10}\,\frac{[\not{p},\not{S}]}{2}\,\gamma_5\\
&+ A_{11}\,\frac{(p\cdot S)}{M}\,\frac{[\not{P},\not{p}]}{2M}\,\gamma_5 + A_{12}\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu p^\rho S^\sigma}{M},
\end{aligned} \tag{524}$$

where the first four terms do not involve the hadron polarization vector. Hermiticity requires all the amplitudes $A_i = A_i(p\cdot P, p^2)$ to be real. The amplitudes A_4, A_5 and A_{12} vanish when also time reversal invariance applies.

Redoing the parton distributions in inclusive scattering

In inclusive deep inelastic scattering one integrates over the component $p^- \equiv p\cdot n_+$ and transverse components, which restricts the nonlocality in $\Phi(p)$. The relevant soft part then is some Dirac trace of the quantity

$$\begin{aligned}
\Phi_{ij}(x) \;&=\; \int dp^-\,d^2p_T\;\Phi_{ij}(p,P,S)\\
&=\; \int \frac{d\xi^-}{2\pi}\;e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0},
\end{aligned} \tag{525}$$

depending on the lightcone fraction $x = p^+/P^+$. To be precise one puts in the full form for the quark momentum,

$$p = x\,P^+ n_+ + \frac{p^2 + \boldsymbol{p}_T^2}{2x\,P^+}\,n_- + p_T, \tag{526}$$

and performs the integration over $\Phi(p)$ using

$$\int dp^-\,d^2p_T \ldots = \frac{\pi}{P^+}\int d(p\cdot P)\,dp^2 \ldots\;. \tag{527}$$

When one wants to calculate the leading order in $1/Q$ for a hard process, one looks for leading parts in M/P^+ because $P^+ \propto Q$. The leading contribution turns out to be proportional to $(M/P^+)^0$,

$$\Phi(x) = \frac{1}{2}\left\{ f_1(x)\;\not{n}_+ + \lambda\,g_1(x)\,\gamma_5\;\not{n}_+ + h_1(x)\,\frac{\gamma_5\,[\,\not{S}_\perp,\,\not{n}_+]}{2}\right\} + \mathcal{O}\left(\frac{M}{P^+}\right) \tag{528}$$

The precise expression of the functions $f_1(x)$, etc. as integrals over the amplitudes can be easily written down.

The functions f_1, g_1 and h_1 can be obtained from the correlator $\Phi(x)$ after tracing with the appropriate Dirac matrix,

$$
\begin{aligned}
f_1(x) &= \int \frac{d\xi^-}{4\pi}\, e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, && (529)\\
\lambda\, g_1(x) &= \int \frac{d\xi^-}{4\pi}\, e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, && (530)\\
S_T^i\, h_1(x) &= \int \frac{d\xi^-}{4\pi}\, e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\, i\sigma^{i+}\gamma_5\,\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, && (531)
\end{aligned}
$$

Including flavor indices, the functions $f_1^q(x) = q(x)$ and $g_1^q(x) = \Delta q(x)$ are precisely the functions that we have seen before.

Just as we have seen before for $f_1(x)$ and $g_1(x)$, the function h_1 can be interpreted as a density, but one needs instead of the projectors on quark chirality states, $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$, and on quark transverse spin states, $P_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^i\gamma_5)$ to see that

$$
\begin{aligned}
f_1(x) &= f_{1R}(x) + f_{1L}(x) = f_{1\uparrow}(x) + f_{1\downarrow}(x), && (532)\\
g_1(x) &= f_{1R}(x) - f_{1L}(x), && (533)\\
h_1(x) &= f_{1\uparrow}(x) - f_{1\downarrow}(x). && (534)
\end{aligned}
$$

One sees some trivial bounds such as $f_1(x) \ge 0$ and $|g_1(x)| \le f_1(x)$. Since $P_n^+ \le 0$ and sees $x \le 1$. From the antiquark distribution $\bar{f}_1(x)$ and its relation to $f_1(x)$ one obtains $x \ge -1$, thus the support of the functions is $-1 \le x \le 1$.

Exercise. Show that the Dirac structure for h_1 in terms of chirality states is $\overline{\psi}_R\psi_L$ and $\overline{\psi}_L\psi_R$. Such functions are referred to as *chiral-odd*. Explain why chiral-odd functions cannot be measured in inclusive deep inelastic scattering.

45 Bounds on the distribution functions

The trivial bounds on the distribution functions ($|h_1(x)| \le f_1(x)$ and $|g_1(x)| \le f_1(x)$) can be sharpened. For instance one can look explicitly at the structure in Dirac space of the correlation function Φ_{ij}. Actually, we will look at the correlation functions $(\Phi\,\gamma_0)_{ij}$, which involves at leading order matrix elements $\psi_{+j}^\dagger(0)\psi_{+i}(\xi)$. One has in Weyl representation ($\gamma^0 = \rho^1$, $\gamma^i = -i\rho^2\sigma^i$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \rho^3$) the matrices

$$
\begin{aligned}
P_+ &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},\\
P_+\gamma_5 &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad P_+\gamma^1\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

The good projector only leaves two (independent) Dirac spinors, one righthanded (R), one lefthanded (L). On this basis of good R and L spinors the for hard scattering processes relevant matrix $(\Phi\rlap{/}n_-)$ is given by

$$
(\Phi\,\rlap{/}n_-)_{ij}(x) = \begin{pmatrix} f_1 + \lambda\, g_1 & (S_T^1 - i\,S_T^2)\,h_1 \\ (S_T^1 + i\,S_T^2)\,h_1 & f_1 - \lambda\, g_1 \end{pmatrix} \qquad (535)
$$

One can also turn the S-dependent correlation function Φ into a matrix in the nucleon spin space via the standard spin 1/2 density matrix $\rho(P,S)$. The relation is $\Phi(x;P,S) = \mathrm{Tr}\,[Phi(x;P)\,\rho(P,S)]$. Writing

$$\Phi(x;P,S) \quad = \quad \Phi_O + \lambda\,\Phi_L + S_T^1\,\Phi_T^1 + S_T^2\,\Phi_T^2, \tag{536}$$

one has on the basis of spin 1/2 target states with $\lambda = +1$ and $\lambda = -1$ respectively

$$\Phi_{ss'}(x) = \begin{pmatrix} \Phi_O + \Phi_L & \Phi_T^1 - i\,\Phi_T^2 \\ \Phi_T^1 + i\,\Phi_T^2 & \Phi_O - \Phi_L \end{pmatrix} \tag{537}$$

Exercise. Show by generalizing $\Phi(p)$ to a matrix elements between states $\langle P,s|$ and $|P,s'$ that for the matrix $M = (\Phi\not{n}_-)^T$ (transposed in Dirac space) one has $v^\dagger M v \geq 0$ for any direction v in Dirac space.

On the basis $+R$, $-R$, $+L$ and $-L$ the matrix in quark $\otimes$ nucleon spin-space becomes

$$(\Phi(x)\,\not{n}_-)^T \quad = \quad \begin{bmatrix} f_1+g_1 & 0 & 0 & 2\,h_1 \\ 0 & f_1-g_1 & 0 & 0 \\ 0 & 0 & f_1-g_1 & 0 \\ 2\,h_1 & 0 & 0 & f_1+g_1 \end{bmatrix} \tag{538}$$

Of this matrix any diagonal matrix element must always be positive, hence the eigenvalues must be positive, which gives a bound on the distribution functions stronger than the trivial bounds, namely

$$|h_1(x)| \leq \frac{1}{2}\,(f_1(x) + g_1(x)) \tag{539}$$

known as the Soffer bound.

Exercise. Show that a change to the transverse quark spin basis gives the quark prodcution matrix

$$(\Phi(x)\,\not{n}_-)^T \quad = \quad \begin{bmatrix} f_1+h_1 & 0 & 0 & g_1+h_1 \\ 0 & f_1-h_1 & g_1-h_1 & 0 \\ 0 & g_1-h_1 & f_1-h_1 & 0 \\ g_1+h_1 & 0 & 0 & f_1+h_1 \end{bmatrix} \tag{540}$$

46 Transverse momentum dependent correlation functions

For 1-particle inclusive scattering one parametrizes the momenta

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ P_h^2 = M_h^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \\ 2\,P_h\cdot q = -z_h\,Q^2 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} P_h = \frac{z_h\,Q}{\sqrt{2}}\,n_- + \frac{M_h^2}{z_h\,Q\sqrt{2}}\,n_+ \\ \\ q = \frac{Q}{\sqrt{2}}\,n_- \;-\; \frac{Q}{\sqrt{2}}\,n_+ \;+\; q_T \\ \\ P = \frac{x_B M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right.$$

Note that this works for socalled current fragmentation, in which case the produced hadron is *hard* with respect to the target momentum, i.e. $P\cdot P_h \sim Q^2$. The minus component p^- is irrelevant in the lower soft part, while the plus component k^+ is irrelevant in the upper soft part. Note that after the choice of P and P_h one can no longer omit a transverse component in the other vector, in the consideration above put in the momentum transfer q. One sees that one has (up to mass effects) the relation

$$q_T^\mu = q^\mu + x_B\,P^\mu - \frac{P_h^\mu}{z_h}. \tag{541}$$

This relation allows the experimental determination of the 'transverse momentum' effect from the external vectors q, P and P_h which are in general not collinear. A consequence is that one can no longer simply integrate over the transverse component of the quark momentum, defined in Eq. 526.

In this case the (theoretically) most convenient way to describe the spin vector of the target is via an expansion of the form

$$S^\mu = -\lambda\,\frac{M x_B}{Q\sqrt{2}}\,n_- + \lambda\,\frac{Q}{M x_B\sqrt{2}}\,n_+ + S_T. \tag{542}$$

One has up to $\mathcal{O}(1/Q^2)$ corrections $\lambda \approx M\,(S\cdot q)/(P\cdot q)$ and $S_T \approx S_\perp$, where the subscript $\perp$ still refers to perpendicular to q and P. For a pure state one has $\lambda^2 + \boldsymbol{S}_T^2 = 1$, in general this quantity being less or equal than one.

Without integration over p_T, the soft part is

$$\Phi(x,\boldsymbol{p}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\psi(\xi)|P,S\rangle\bigg|_{\xi^+=0}. \tag{543}$$

For the leading order results, it is parametrized as

$$\Phi(x,\boldsymbol{p}_T) = \Phi_O(x,\boldsymbol{p}_T) + \Phi_L(x,\boldsymbol{p}_T) + \Phi_T(x,\boldsymbol{p}_T), \tag{544}$$

with the parts involving unpolarized targets (O), longitudinally polarized targets (L) and transversely polarized targets (T) up to parts proportional to M/P^+ given by

$$\Phi_O(x,\boldsymbol{p}_T) \;=\; \frac{1}{2}\left\{f_1(x,\boldsymbol{p}_T)\;\not{n}_+ + h_1^\perp(x,\boldsymbol{p}_T)\,\frac{i\,[\not{p}_T,\,\not{n}_+]}{2M}\right\} \tag{545}$$

$$\Phi_L(x,\boldsymbol{p}_T) \;=\; \frac{1}{2}\left\{\lambda\,g_{1L}(x,\boldsymbol{p}_T)\,\gamma_5\;\not{n}_+ + \lambda\,h_{1L}^\perp(x,\boldsymbol{p}_T)\frac{\gamma_5\,[\not{p}_T,\,\not{n}_+]}{2M}\right\} \tag{546}$$

$$\begin{aligned} \Phi_T(x,\boldsymbol{p}_T) \;=\; & \frac{1}{2}\Bigg\{f_{1T}^\perp(x,\boldsymbol{p}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_+^\nu p_T^\rho S_T^\sigma}{M} \\ & + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\,g_{1T}(x,\boldsymbol{p}_T)\,\gamma_5\;\not{n}_+ + h_{1T}(x,\boldsymbol{p}_T)\,\frac{\gamma_5\,[\not{S}_T,\,\not{n}_+]}{2} \\ & + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\,h_{1T}^\perp(x,\boldsymbol{p}_T)\,\frac{\gamma_5\,[\not{p}_T,\,\not{n}_+]}{2M}\Bigg\}. \end{aligned} \tag{547}$$

All functions appearing here have a natural interpretation as densities. This is seen as discussed before for the $\boldsymbol{p}_T$-integrated functions. Now it includes densities such as the density of longitudinally polarized quarks in a transversely polarized nucleon (g_{1T}) and the density of transversely polarized quarks in a longitudinally polarized nucleon ($h_{1L}^\perp$).

Several functions vanish from the soft part upon integration over p_T. We are then left with Eq. 528 with $f_1(x) = \int d^2p_T\, f_1(x,p_T)$, $g_1(x) = \int d^2p_T\, g_{1L}(x,p_T)$ and $h_1(x) = \int d^2p_T \left[h_{1T}(x) + \frac{\boldsymbol{p}_T^2}{2M^2}\, h_{1T}^\perp(x,p_T)\right]$. Actually we find that different functions survive when one integrates over $\boldsymbol{p}_T$ weighting with p_T^α, e.g.

$$\begin{aligned}
\Phi_\partial^\alpha(x) &\equiv \int d^2p_T\, \frac{p_T^\alpha}{M}\,\Phi(x,\boldsymbol{p}_T) \\
&= \frac{1}{2}\Bigg\{-g_{1T}^{(1)}(x)\, S_T^\alpha\, \not{n}_+\gamma_5 - \lambda\, h_{1L}^{\perp(1)}(x)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2} \\
&\qquad - f_{1T}^{\perp(1)}\,\epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_-^\nu S_T^\rho - h_1^{\perp(1)}\,\frac{i[\gamma^\alpha,\not{n}_+]}{2}\Bigg\},
\end{aligned} \tag{548}$$

where we define $\boldsymbol{p}_T^2/2M^2$-moments as

$$g_{1T}^{(1)}(x) = \int d^2p_T\, \frac{\boldsymbol{p}_T^2}{2M^2}\, g_{1T}(x,\boldsymbol{p}_T), \tag{549}$$

and similarly the other functions. The functions $h_1^\perp$ and $f_{1T}^\perp$ are *T-odd* and vanish for distribution functions because one can use T-reversal invariance in the matrix element. We have kept them in order to apply the formalism to fragmentation functions. The T-odd functions correspond to unpolarized quarks in a transversely polarized nucleon ($f_{1T}^\perp$) or transversely polarized quarks in an unpolarized hadron ($h_1^\perp$). The easiest way to interpret the functions is by considering their place in the quark production matrix, giving now

$$(\Phi(x,p_T)\,\not{n}_-)^T = \begin{pmatrix}
f_1 + g_{1L} & \frac{|p_T|}{M}\,e^{i\phi}\,g_{1T} & \frac{|p_T|}{M}\,e^{-i\phi}\,h_{1L}^\perp & 2\,h_1 \\
\frac{|p_T|}{M}\,e^{-i\phi}\,g_{1T} & f_1 - g_{1L} & \frac{|p_T|^2}{M^2}\,e^{-2i\phi}\,h_{1T}^\perp & -\frac{|p_T|}{M}\,e^{-i\phi}\,h_{1L}^\perp \\
\frac{|p_T|}{M}\,e^{i\phi}\,h_{1L}^\perp & \frac{|p_T|^2}{M^2}\,e^{2i\phi}\,h_{1T}^\perp & f_1 - g_{1L} & -\frac{|p_T|}{M}\,e^{i\phi}\,g_{1T} \\
2\,h_1 & -\frac{|p_T|}{M}\,e^{i\phi}\,h_{1L}^\perp & -\frac{|p_T|}{M}\,e^{-i\phi}\,g_{1T} & f_1 + g_{1L}
\end{pmatrix}. \tag{550}$$

In this representation T-odd functions appear as imaginary parts, $f_{1T}^\perp = \mathrm{Im}\, g_{1T}$ and $h_1^\perp = \mathrm{Im}\, h_{1L}^\perp$.

47 Fragmentation functions

Just as for the distribution functions one can perform an analysis of the soft part describing the quark fragmentation. One needs

$$\Delta_{ij}(z,\boldsymbol{k}_T) = \sum_X \int \frac{d\xi^+ d^2\boldsymbol{\xi}_T}{(2\pi)^3}\, e^{ik\cdot\xi}\, Tr\langle 0|\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)|0\rangle\Big|_{\xi^-=0}. \tag{551}$$

For the production of unpolarized (or spin 0) hadrons h in hard processes one needs to leading order in $1/Q$ the correlation function,

$$\Delta_O(z,\boldsymbol{k}_T) = z\,D_1(z,\boldsymbol{k}_T')\;\not{n}_- + z\,H_1^\perp(z,\boldsymbol{k}_T')\,\frac{i\,[\not{k}_T,\not{n}_-]}{2M_h} + \mathcal{O}\left(\frac{M_h}{P_h^-}\right). \tag{552}$$

The arguments of the fragmentation functions D_1 and $H_1^\perp$ are $z = P_h^-/k^-$ and $\boldsymbol{k}'_T = -z\boldsymbol{k}_T$. The first is the (lightcone) momentum fraction of the produced hadron, the second is the transverse momentum of the produced hadron with respect to the quark. The fragmentation function D_1 is the equivalent of the distribution function f_1. It can be interpreted as a quark decay function, giving the probability of finding a hadron h in a quark. The quantity $n_h = \int dz\; D_1(z)$ is the number of hadrons.

Exercise. Show that the normalization of the fragmentation functions is given by $\sum_h \int dz\; z\, D_1^{q\to h}(z)$ by relating this quantity to a local operator. One needs to eliminate the hadrons in the intermediate state via the momentum operator

$$P^\mu = \sum_{h,X} |P_h, X\rangle\, P_h^\mu\, \langle P_h, X|.$$

The function $H_1^\perp$, interpretable as the difference between the numbers of unpolarized hadrons produced from a transversely polarized quark depending on the hadron's transverse momentum, is allowed because of the non-applicability of time reversal invariance. This is natural for the fragmentation functions because of the appearance of out-states $|P_h, X\rangle$ in the definition of Δ, in contrast to the plane wave states appearing in Φ. The function $H_1^\perp$ is of interest because it is chiral-odd. This means that it can be used to probe the chiral-odd quark distribution function h_1, which can be achieved e.g. by measuring a particular azimuthal asymmetry of produced pions in the current fragmentation region.

The spin structure of fragmentation functions is also conveniently summarized by explicitly giving it on a R and L chiral quark basis, for which we find for decay into spin zero hadrons,

$$(\Delta(z,k_T)\not{n}_+)^T \;=\; \begin{array}{cc} \left(\begin{array}{cc} D_1 & i\frac{|k_T|\,e^{-i\phi}}{M_h}\,H_1^\perp \\[2ex] -i\frac{|k_T|\,e^{+i\phi}}{M_h}\,H_1^\perp & D_1 \end{array}\right) & \begin{array}{c} R \\[2ex] L \end{array} \\ \begin{array}{cc} \quad R \quad & \quad\quad L \quad \end{array} & \end{array} \tag{553}$$

48 Inclusion of subleading contributions

If one proceeds up to order $1/Q$ one also needs terms in the parametrization of the soft part proportional to M/P^+. Limiting ourselves to the $\boldsymbol{p}_T$-integrated correlations one needs

$$\begin{aligned} \Phi(x) \;=\; & \frac{1}{2}\left\{ f_1(x)\;\not{n}_+ + \lambda\, g_1(x)\,\gamma_5\;\not{n}_+ + h_1(x)\,\frac{\gamma_5\,[\,\not{S}_T,\,\not{n}_+]}{2}\right\} \\ & + \frac{M}{2P^+}\left\{ e(x) + g_T(x)\,\gamma_5\;\not{S}_T + \lambda\, h_L(x)\,\frac{\gamma_5\,[\,\not{n}_+,\,\not{n}_-]}{2}\right\} \\ & + \frac{M}{2P^+}\left\{ -\lambda\, e_L(x)\, i\gamma_5 - f_T(x)\,\epsilon_T^{\rho\sigma}\,\gamma_\rho S_{T\sigma} + h(x)\,\frac{i\,[\,\not{n}_+,\,\not{n}_-]}{2}\right\}. \end{aligned} \tag{554}$$

The last set of three terms proportional to M/P^+ vanish when time-reversal invariance applies. The moments of the functions in the second and third set involve local operators in which gluon fields appear.

We will use inclusive scattering off a transversely polarized nucleon ($|\boldsymbol{S}_\perp| = 1$) as an example to show how higher twist effects can be incorporated in the cross section. The hadronic tensor for a transversely polarized nucleon is zero in leading order in $1/Q$. At order $1/Q$ one obtains from the

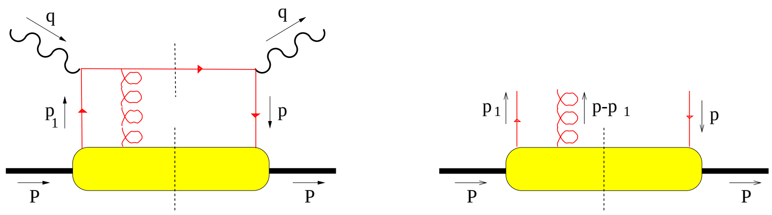


Figure 10: Examples of gluonic diagrams that must be included at subleading order in lepton hadron inclusive scattering (left) and the soft part entering this process (right).

handbag diagram a contribution

$$2M\,W^{\mu\nu}_{A(a)}(q,P,S_T) = i\,\frac{2M}{Q}\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}\,\left(g_{1T}^{(1)}(x_B) - \frac{m}{M}\,h_1(x_B)\right). \tag{555}$$

One sees that p_T-dependence cannot be neglected in this stage. There is a second contribution at order $1/Q$ coming from diagrams as the one shown in Fig. 10. For these gluon diagrams one needs matrix elements containing $\overline{\psi}(0)\,gA_T^\alpha(\eta)\,\psi(\xi)$. At order $1/Q$ one only needs the matrix element of the bilocal combinations $\overline{\psi}(0)\,gA_T^\alpha(\xi)\,\psi(\xi)$ and $\overline{\psi}(0)\,gA_T^\alpha(0)\,\psi(\xi)$ These soft parts have a structure quite similar to Φ_∂^α and are parametrized as

$$\begin{aligned}\Phi_A^\alpha(x) \;=\; \frac{M}{2}\Bigg\{&-x\,\tilde{g}_T(x)\,S_T^\alpha\,\not{n}_+\gamma_5 - \lambda\,x\,\tilde{h}_L(x)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2}\\ &- x\,\tilde{f}_T(x)\,\epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_-^\nu S_T^\rho - x\,\tilde{h}(x)\,\frac{i[\gamma^\alpha,\not{n}_+]}{2}\Bigg\}.\end{aligned} \tag{556}$$

This contributes also to $W_A^{\mu\nu}$,

$$2M\,W^{\mu\nu}_{A(b)}(q,P,S_T) = i\,\frac{2Mx_B}{Q}\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}\,\tilde{g}_T(x_B). \tag{557}$$

Using the QCD equations of motion, however, these functions can be related to the functions appearing in Φ. To be precise one combines $i\partial$ in Φ_∂ (see Eq. 548) and A_μ in Φ_A to Φ_D containing $iD_\mu = i\partial_\mu + g\,A_\mu$ for which one has via the equations of motion

$$\begin{aligned}\Phi_D^\alpha(x) \;=\; \frac{M}{2}\Bigg\{&-\left(x\,g_T - \frac{m}{M}\,h_1\right)\,S_T^\alpha\,\not{n}_+\gamma_5\\ &-\lambda\left(x\,h_L - \frac{m}{M}\,g_1\right)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2}\\ &- x\,f_T(x)\,\epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_-^\nu S_T^\rho - x\,\tilde{h}(x)\,\frac{i[\gamma^\alpha,\not{n}_+]}{2}\Bigg\}.\end{aligned} \tag{558}$$

Hence one obtains

$$x\,\tilde{g}_T = x\,g_T - g_{1T}^{(1)} - \frac{m}{M}\,h_1, \tag{559}$$

$$x\,\tilde{h}_L = x\,h_L - h_{1L}^{\perp(1)} - \frac{m}{M}\,g_1, \tag{560}$$

$$x\,\tilde{f}_T = x\,f_T + f_{1T}^{(1)}, \tag{561}$$

$$x\,\tilde{h} = x\,h + 2\,h_1^{\perp(1)}. \tag{562}$$

and in the case of polarized inclusive deep inelastic scattering one obtains the full contribution

$$2M\, W_A^{\mu\nu}(q, P, S_T) = i\,\frac{2M x_B}{Q}\,\hat{t}^{[\mu}\epsilon_\perp^{\nu]\rho} S_{\perp\rho}\, g_T(x_B), \tag{563}$$

leading for the structure function $g_T(x_B, Q^2)$ defined in Eq. 341 to the result

$$g_T(x_B, Q^2) = \frac{1}{2}\sum_a e_a^2\left(g_T^a(x_B) + g_T^{\bar{a}}(x_B)\right). \tag{564}$$

From Lorentz invariance one obtains, furthermore, some interesting relations between the subleading functions and the transverse momentum dependent leading functions. Just by using the expressions for the functions in terms of the amplitudes A_i in Eq. 524 one finds

$$g_T = g_1 + \frac{d}{dx}\, g_{1T}^{(1)}, \tag{565}$$

$$h_L = h_1 - \frac{d}{dx}\, h_{1L}^{\perp(1)}, \tag{566}$$

$$f_T = -\frac{d}{dx}\, f_{1T}^{\perp(1)}, \tag{567}$$

$$h = -\frac{d}{dx}\, h_1^{\perp(1)}. \tag{568}$$

Exercise. As an application of the relations between twist-three functions and transverse momentum dependent functions, one can eliminate $g_{1T}^{(1)}$ using Eq. 565 and obtain a relation between g_T, g_1 and $\tilde{g}_T$ (assuming sufficient neat behavior of the functions). Show that this relation for for $g_2 = g_T - g_1$ takes the form

$$\begin{aligned} g_2(x) \quad = \quad & -\left[g_1(x) - \int_x^1 dy\, \frac{g_1(y)}{y}\right] + \frac{m}{M}\left[\frac{h_1(x)}{x} - \int_x^1 dy\, \frac{h_1(y)}{y^2}\right] \\ & + \left[\tilde{g}_T(x) - \int_x^1 dy\, \frac{\tilde{g}_T(y)}{y}\right]. \end{aligned} \tag{569}$$

One can use this to obtain for each quark flavor $\int dx\, g_2^a(x) = 0$, known as the Burkhardt-Cottingham sumrule. Neglecting the interaction-dependent part one obtains the Wandzura-Wilczek approximation for g_2, which in particular when one neglects the quark mass term provides a simple and often used estimate for g_2. It has become the standard with which experimentalists compare the results for g_2.

49 Color gauge invariance

We have sofar neglected two problems. The first problem is that the correlation function Φ discussed in previous sections involve two quark fields at different space-time points and hence are not color gauge invariant. The second problem comes from the gluonic diagrams similar as the ones we have discussed in the previous section (see Fig. 10) We note that diagrams involving matrix elements with longitudinal (A^+) gluon fields,

$$\overline{\psi}_j(0)\, gA^+(\eta)\, \psi_i(\xi)$$

do not lead to any suppression. The reason is that because of the +-index in the gluon field the matrix element is proportional to P^+, p^+ or $M\,S^+$ rather than the proportionality to $M\,S_T^\alpha$ or p_T^α that we have seen in Eq. 556 for a gluonic matrix element with transverse gluons.

A straightforward calculation, however, shows that the gluonic diagrams with one or more longitudinal gluons involve matrix elements (soft parts) of operators $\overline{\psi}\psi$, $\overline{\psi}\,A^+\,\psi$, $\overline{\psi}\,A^+A^+\,\psi$, etc. that can be resummed into a correlation function

$$\Phi_{ij}(x) = \int \frac{d\xi^-}{2\pi}\; e^{ip\cdot\xi}\, \langle P,S\vert\overline{\psi}_j(0)\,\mathcal{U}(0,\xi)\,\psi_i(\xi)\vert P,S\rangle\Big\vert_{\xi^+=\xi_T=0}, \tag{570}$$

where $\mathcal{U}$ is a gauge link operator

$$\mathcal{U}(0,\xi) = \mathcal{P}\exp\left(-i\int_0^{\xi^-} d\zeta^-\, A^+(\zeta)\right) \tag{571}$$

(path-ordered exponential with path along $-$-direction). Et voila, the unsuppressed gluonic diagrams combine into a color gauge invariant correlation function. We note that at the level of operators, one expands

$$\overline{\psi}(0)\psi(\xi) = \sum_n \frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(0), \tag{572}$$

in a set of local operators, but only the expansion of the nonlocal combination with a gauge link

$$\overline{\psi}(0)\psi(\xi) = \sum_n \frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)D_{\mu_1}\ldots D_{\mu_n}\psi(0), \tag{573}$$

is an expansion in terms of local gauge invariant operators. The latter operators are precisely the local (quark) operators that appear in the operator product expansion applied to inclusive deep inelastic scattering.

For the p_T-dependent functions, one finds that inclusion of A^+ gluonic diagrams leads to a color gauge invariant matrix element with links running via $\xi^- = \pm\infty$. For instance in lepton-hadron scattering one finds

$$\Phi(x,\boldsymbol{p}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\; e^{ip\cdot\xi}\,\langle P,S\vert\overline{\psi}(0)\,\mathcal{U}(0,\infty)\,\mathcal{U}(\infty,\xi)\,\psi(\xi)\vert P,S\rangle\Big\vert_{\xi^+=0}, \tag{574}$$

where the gauge links are at constant ξ_T. One can multiply this correlator with p_T^α and make this into a derivative ∂_α. Including the links one finds the color gauge invariant result

$$\begin{aligned} p_T^\alpha\,\Phi_{ij}(x,\boldsymbol{p}_T) &= (\Phi_\partial^\alpha)_{ij}(x,\boldsymbol{p}_T) \\ &= \int \frac{d\xi^-\, d^2\boldsymbol{\xi}_T}{(2\pi)^3}\; e^{ip\cdot\xi}\Bigg\{\langle P,S\vert\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\,iD_T^\alpha\psi_i(\xi)\vert P,S\rangle\Big\vert_{\xi^+=0} \\ &\qquad - \langle P,S\vert\overline{\psi}_j(0)\,\mathcal{U}(0,\infty)\int_\infty^{\xi^-} d\eta^-\,\mathcal{U}(\infty,\eta) \\ &\qquad\qquad \times g\,G^{+\alpha}(\eta)\,\mathcal{U}(\eta,\xi)\,\psi_i(\xi)\vert P,S\rangle\Big\vert_{\xi^+=0}\Bigg\}, \end{aligned} \tag{575}$$

which gives after integration over p_T the expected result $\Phi_\partial^\alpha(x) = \Phi_D^\alpha(x) - \Phi_A^\alpha(x)$. Note that in $A^+ = 0$ gauge all the gauge links disappear, while one has $G^{+\alpha} = \partial^+ A^\alpha$, but there presence is essential to perform the above differentiations.

50 Evolution and the diagrammatic approach

The explicit treatment of transverse momenta provides also a transparent way to include the evolution equations for quark distribution and fragmentation functions. Remember that we have assumed that soft parts vanish sufficiently fast as a function of the invariants $p\cdot P$ and p^2, which at constant x implies

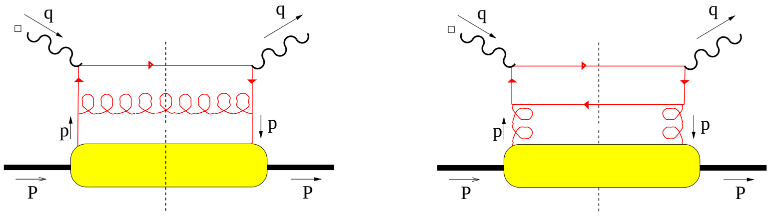


Figure 11: Ladder diagrams used to calculate the asymptotic behavior of the correlation functions.

a sufficiently fast vanishing as a function of $\boldsymbol{p}_T^2$. This simply turns out not to be true. Assuming that the result for $\boldsymbol{p}_T^2 \geq \mu^2$ is given by the diagram shown in Fig. 11 one finds

$$\begin{aligned} f_1(x, \boldsymbol{p}_T^2) &= \theta(\mu^2 - \boldsymbol{p}_T^2)\, f_1(x, \boldsymbol{p}_T^2) \\ &+ \theta(\boldsymbol{p}_T^2 - \mu^2)\, \frac{1}{\pi\, \boldsymbol{p}_T^2}\, \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dy}{y}\, P_{qq}\left(\frac{x}{y}\right)\, f_1(y; \mu^2), \end{aligned} \tag{576}$$

where $f_1(x; \mu^2) = \pi \int_0^{\mu^2} d\boldsymbol{p}_T^2\; f_1(x, \boldsymbol{p}_T^2)$ and the splitting function is given by

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(1-z)\right], \tag{577}$$

with $\int dz\, f(z)/(1-z)_+ \equiv \int dz\, (f(z) - f(1))/(1-z)$ and the color factor $C_F = 4/3$ for $SU(3)$. With the introduction of the scale in $f_1(x; \mu^2)$ one sees that the scale dependence satisfies

$$\frac{\partial f_1(x; \mu^2)}{\partial \ln \mu^2} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y}\, P_{qq}\left(\frac{x}{y}\right)\, f_1(y; \mu^2). \tag{578}$$

This is the standard nonsinglet evolution equation for the valence quark distribution function. For the flavor singlet combination of quark distributions or the sea distributions one also needs to take into account contributions as shown in Fig. 11 (right) involving the gluon distribution functions related to matrix elements with gluon fields $F_{\mu\nu}(\xi)$ but otherwise proceeding along analogous lines. Even the δ-function contribution can be explicitly calculated by including vertex corrections on the quark legs of the quark-quark soft part (virtual diagrams), but it is again easier to derive them by requiring that the sum rules for f_1 remain valid under evolution, which requires that $\int_0^1 dz\; P_{qq}(z) = 0$.