THE STANDARD MODEL OF PARTICLE PHYSICS

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Preface

In these lectures I will mainly discuss the symmetries and concepts underlying the Standard Model that is so successful in describing the interactions of the elementary particles, the quarks and leptons. I will assume a basic knowledge of field theory. As an additonal help in understanding these notes, I suggest students to use the *Introductory quantum field theory* notes found under http://www.nat.vu.nl/ mulders/lectures.html#master or consult text books such as those given below.

1. L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 1985.
2. M.E. Peshkin and D.V. Schroeder, *An introduction to Quantum Field Theory*, Addison-Wesly, 1995.
3. M. Veltman, *Diagrammatica*, Cambridge University Press, 1994.
4. S. Weinberg, *The quantum theory of fields*; Vol. I: Foundations, Cambridge University Press, 1995; Vol. II: Modern Applications, Cambridge University Press, 1996.
5. C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill, 1980.

Contents

1 Gauge theories

Abelian gauge theories

Consider a theory that is invariant under *global gauge transformations* or gauge transformations of the first kind, e.g. in the Klein-Gordon theory for a scalar field, describing a spinless particle the transformation

$$\phi(x) \rightarrow e^{i\,e\Lambda}\phi(x), \tag{1}$$

in which the $U(1)$ phase involves an angle $e\,\Lambda$, independent of x. Gauge transformations of the second kind or *local gauge transformations* are transformations of the type

$$\phi(x) \rightarrow e^{i\,e\Lambda(x)}\phi(x), \tag{2}$$

i.e. the angle of the transformation depends on the space-time point x. The lagrangians for free particles (e.g. Klein-Gordon, Dirac) are invariant under global gauge transformations and corresponding to this there exist a conserved Noether current. Any lagrangian containing derivatives, however, is not invariant under *local* gauge transformations,

$$\begin{aligned}
\phi(x) &\rightarrow e^{i\,e\Lambda(x)}\phi(x), && (3)\\
\phi^*(x) &\rightarrow e^{-i\,e\Lambda(x)}\phi^*(x), && (4)\\
\partial_\mu\phi(x) &\rightarrow e^{i\,e\Lambda(x)}\partial_\mu\phi(x) + i\,e\,\partial_\mu\Lambda(x)\,e^{i\,e\Lambda(x)}\phi(x), && (5)
\end{aligned}$$

where it is the last term that spoils gauge invariance.

A solution is the one known as *minimal substitution* in which the derivative ∂_μ is replaced by a *covariant derivative* D_μ which satisfies

$$D_\mu\phi(x) \rightarrow e^{i\,e\Lambda(x)}D_\mu\phi(x). \tag{6}$$

To achieve invariance it is necessary to introduce a vector field A_μ,

$$D_\mu\phi(x) \equiv (\partial_\mu + i\,eA_\mu(x))\phi(x), \tag{7}$$

The required transformation for D_μ then demands

$$\begin{aligned}
D_\mu\phi(x) &= (\partial_\mu + i\,eA_\mu(x))\phi(x)\\
&\rightarrow e^{i\,e\Lambda}\partial_\mu\phi + i\,e\,(\partial_\mu\Lambda)\,e^{i\,e\Lambda}\phi + i\,eA'_\mu\,e^{i\,e\Lambda}\phi\\
&= e^{i\,e\Lambda}\left(\partial_\mu + i\,e(A'_\mu + \partial_\mu\Lambda)\right)\phi\\
&\equiv e^{i\,e\Lambda}\left(\partial_\mu + i\,eA_\mu\right)\phi. && (8)
\end{aligned}$$

Thus the covariant derivative has the correct transformation behavior provided

$$A_\mu \rightarrow A_\mu - \partial_\mu\Lambda, \tag{9}$$

the behavior which is familar as the gauge freedom in electromagnetism described via for massless vector fields and the (free) lagrangian density $\mathcal{L} = -(1/4)F_{\mu\nu}F^{\mu\nu}$. Replacing derivatives by covariant derivatives and adding the (free) part for the vector fields to the original lagrangian produces a gauge invariant lagrangian,

$$\mathcal{L}(\phi, \partial_\mu\phi) \implies \mathcal{L}(\phi, D_\mu\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{10}$$

The field ϕ is used here in a general sense standing for any possible field. As an example consider the Dirac lagrangian,

$$\mathcal{L} = \frac{i}{2}\left(\overline{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\overline{\psi})\gamma^\mu\psi\right) - M\,\overline{\psi}\psi.$$

Minimal substitution $\partial_\mu\psi \to (\partial_\mu + i\,eA_\mu)\psi$ leads to the gauge invariant lagrangian

$$\mathcal{L} = \frac{i}{2}\,\overline{\psi}\,\overleftrightarrow{\not\partial}\,\psi - M\,\overline{\psi}\psi - e\,\overline{\psi}\gamma^\mu\psi\,A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{11}$$

We note first of all that the coupling of the Dirac field (electron) to the vector field (photon) can be written in the familiar interaction form

$$\mathcal{L}_{int} = -e\,\overline{\psi}\gamma^\mu\psi\,A_\mu = -e\,j^\mu A_\mu, \tag{12}$$

involving the interaction of the charge ($\rho = j^0$) and three-current density ($\boldsymbol{j}$) with the electric potential ($\phi = A^0$) and the vector potential ($\boldsymbol{A}$), $-e\,j^\mu A_\mu = -e\,\rho\phi + e\,\boldsymbol{j}\cdot\boldsymbol{A}$. The equation of motion for the fermion follow from

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta(\partial_\mu\overline{\psi})} &= -\frac{i}{2}\gamma^\mu\psi, \\ \frac{\delta\mathcal{L}}{\delta\overline{\psi}} &= \frac{i}{2}\not\partial\psi - M\,\psi - e\not{A}\,\psi \end{aligned}$$

giving the Dirac equation in an electromagnetic field,

$$(i\not{D} - M)\,\psi = (i\not\partial - e\not{A} - M)\,\psi = 0. \tag{13}$$

For the photon the equations of motion follow from

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)} &= -F^{\mu\nu}, \\ \frac{\delta\mathcal{L}}{\delta A_\nu} &= -e\overline{\psi}\gamma^\nu\psi, \end{aligned}$$

giving the Maxwell equation coupling to the electromagnetic current,

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{14}$$

where $j^\mu = e\,\overline{\psi}\gamma^\mu\psi$. This latter current is the conserved current that is obtained for the Dirac lagrangian using Noether's theorem.

Non-abelian gauge theories

Quantum electrodynamics is an example of a very successful gauge theory. The photon field A_μ was introduced as to render the lagrangian invariant under local gauge transformations. The extension to non-abelian gauge theories is straightforward. The symmetry group is a Lie-group G generated by generators T_a, which satisfy commutation relations

$$[T_a, T_b] = i\,c_{abc}\,T_c, \tag{15}$$

with c_{abc} known as the *structure constants* of the group. For a compact Lie-group they are antisymmetric in the three indices. In an abelian group the structure constants would be zero (for instance the trivial example of $U(1)$). Consider a field transforming under the group,

$$\phi(x) \;\longrightarrow\; e^{i\,\theta^a(x)L_a}\phi(x) \;\stackrel{\text{inf.}}{=}\; (1 + i\,\theta^a(x)L_a)\,\phi(x) \tag{16}$$

where L_a is a representation matrix for the representation to which ϕ belongs, i.e. for a three-component field $\vec{\phi}$ under an $SO(3)$ or $SU(2)$ symmetry transformation,

$$\vec{\phi} \;\longrightarrow\; e^{i\,\vec{\theta}\cdot\vec{L}}\vec{\phi} \;\approx\; \vec{\phi} - \vec{\theta}\times\vec{\phi}. \tag{17}$$

The complication arises (as in the abelian case) when one considers for a lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ the behavior of $\partial_\mu \phi$ under a local gauge transformation, $\underline{U}(\theta) = e^{i\,\theta^a(x) L_a}$,

$$\phi(x) \quad\longrightarrow\quad \underline{U}(\theta)\phi(x), \tag{18}$$

$$\partial_\mu\phi(x) \quad\longrightarrow\quad \underline{U}(\theta)\partial_\mu\phi(x) + \left(\partial_\mu \underline{U}(\theta)\right)\phi(x). \tag{19}$$

Introducing as many gauge fields as there are generators in the group, which are conveniently combined in the matrix valued field $\underline{W}_\mu = W_\mu^a L_a$, one defines

$$\underline{D}_\mu \phi(x) \equiv \left(\partial_\mu - ig\,\underline{W}_\mu\right)\phi(x), \tag{20}$$

and one obtains after transformation

$$\underline{D}_\mu \phi(x) \;\longrightarrow\; \underline{U}(\theta)\partial_\mu\phi(x) + \left(\partial_\mu \underline{U}(\theta)\right)\phi(x) - ig\,\underline{W}'_\mu\,\underline{U}(\theta)\phi(x).$$

Requiring that $\underline{D}_\mu\phi$ transforms as $\underline{D}_\mu\phi \to \underline{U}(\theta)\,\underline{D}_\mu\phi$, i.e.

$$\underline{D}_\mu \phi(x) \;\longrightarrow\; \underline{U}(\theta)\partial_\mu\phi(x) - ig\,\underline{U}(\theta)\,\underline{W}_\mu\phi(x),$$

one obtains

$$\underline{W}'_\mu = \underline{U}(\theta)\,\underline{W}_\mu\,\underline{U}^{-1}(\theta) - \frac{i}{g}\,\left(\partial_\mu \underline{U}(\theta)\right)\underline{U}^{-1}(\theta), \tag{21}$$

or infinitesimal

$$W'^a_\mu = W^a_\mu - c_{abc}\theta^b W^c_\mu + \frac{1}{g}\,\partial_\mu\theta^a = W^a_\mu + \frac{1}{g}\,D_\mu\theta^a.$$

It is necessary to introduce the free lagrangian density for the gauge fields just like the term $-(1/4)F_{\mu\nu}F^{\mu\nu}$ in QED. For abelian fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (i/g)[D_\mu, D_\nu]$ is gauge invariant. In the nonabelian case $\partial_\mu \underline{W}^a_\nu - \partial_\nu \underline{W}^a_\mu$ does not provide a gauge invariant candidate for $\underline{G}_{\mu\nu} = G^a_{\mu\nu} L_a$, as can be checked easily. Expressing $\underline{G}_{\mu\nu}$ in terms of the covariant derivatives provides a gauge invariant definition for $\underline{G}_{\mu\nu}$ with

$$\underline{G}_{\mu\nu} = \frac{i}{g}\,[\underline{D}_\mu, \underline{D}_\nu] = \partial_\mu \underline{W}_\nu - \partial_\nu \underline{W}_\mu - ig\,[\underline{W}_\mu, \underline{W}_\nu], \tag{22}$$

and thus

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,c_{abc} W^b_\mu W^c_\nu, \tag{23}$$

It transforms like

$$\underline{G}_{\mu\nu} \;\to\; \underline{U}(\theta)\,\underline{G}_{\mu\nu}\,\underline{U}^{-1}(\theta). \tag{24}$$

The gauge-invariant lagrangian density is now constructed as

$$\mathcal{L}(\phi, \partial_\mu\phi) \longrightarrow \mathcal{L}(\phi, D_\mu\phi) - \frac{1}{2}\mathrm{Tr}\,\underline{G}_{\mu\nu}\underline{G}^{\mu\nu} = \mathcal{L}(\phi, D_\mu\phi) - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu\,a} \tag{25}$$

with the standard normalization $\mathrm{Tr}(L_a L_b) = (1/2)\delta_{ab}$. Note that the gauge fields must be massless, as a mass term $\propto M_W^2 W^a_\mu W^{\mu\,a}$ would break gauge invariance.

QCD, an example of a nonabelian gauge theory

As an example of a nonabelian gauge theory consider *quantum chromodynamics* (QCD), the theory describing the interactions of the colored quarks. The existence of an extra degree of freedom for each species of quarks is evident for several reasons, e.g. the necessity to have an antisymmetric wave function for the Δ^{++} particle consisting of three *up* quarks (each with charge $+(2/3)e$). With the quarks belonging to the fundamental (three-dimensional) representation of $SU(3)_C$, i.e. having three components in color space

$$\psi = \begin{pmatrix} \psi_r \\ \psi_g \\ \psi_b \end{pmatrix},$$

the wave function of the baryons (such as nucleons and deltas) form a singlet under $SU(3)_C$,

$$|\text{color}\rangle = \frac{1}{\sqrt{6}} \left(|rgb\rangle - |grb\rangle + |gbr\rangle - |bgr\rangle + |brg\rangle - |rbg\rangle\right). \tag{26}$$

The nonabelian gauge theory that is obtained by making the 'free' quark lagrangian, for one specific species (flavor) of quarks just the Dirac lagrangian for an elementary fermion,

$$\mathcal{L} = i\,\overline{\psi}\partial\!\!\!/\psi - m\,\overline{\psi}\psi,$$

invariant under local $SU(3)_C$ transformations has proven to be a good candidate for the microscopic theory of the strong interactions. The representation matrices for the quarks and antiquarks in the fundamental representation are given by

$$\begin{aligned} F_a &= \frac{\lambda_a}{2} \quad \text{for quarks,} \\ F_a &= -\frac{\lambda_a^*}{2} \quad \text{for antiquarks,} \end{aligned}$$

which satisfy commutation relations $[F_a, F_b] = i\,f_{abc}F_c$ in which f_{abc} are the (completely antisymmetric) structure constants of $SU(3)$ and where the matrices λ_a are the eight Gell-Mann matrices[1]. The (locally) gauge invariant lagrangian density is

$$\mathcal{L} = -\frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu\,a} + i\,\overline{\psi}D\!\!\!\!/\,\psi - m\,\overline{\psi}\psi, \tag{27}$$

with

$$\begin{aligned} D_\mu\psi &= \partial_\mu\psi - ig\,A^a_\mu F_a\psi, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,c_{abc}A^b_\mu A^c_\nu. \end{aligned}$$

Note that the term $i\,\overline{\psi}D\!\!\!\!/\,\psi = i\,\overline{\psi}\partial\!\!\!/\psi + g\,\overline{\psi}A\!\!\!/^a F_a\psi = i\,\overline{\psi}\partial\!\!\!/\psi + j^{\mu\,a}A^a_\mu$ with $j^{\mu\,a} = \overline{\psi}\gamma^\mu F_a\psi$ describes the interactions of the gauge bosons A^a_μ (gluons) with the color current of the quarks (this is again precisely the Noether current corresponding to color symmetry transformations). Note furthermore that the lagrangian terms for the gluons contain interaction terms corresponding to vertices with three gluons and four gluons due to the nonabelian character of the theory.

[1] The Gell-Mann matrices are the eight traceless hermitean matrices generating $SU(3)$ transformations,

$$\lambda_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

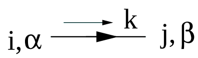
$$\lambda_7 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

Feynman rules for QCD

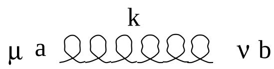
For writing down the complete set of Feynman rules it is necessary to account for the gauge symmetry in the quantization procedure. This will lead (depending on the choice of gauge conditions) to the presence of ghost fields. In the axial gauge $n^\mu A_\mu^a = 0$ gauge fields are not needed. From the Lagrangina of QCD,

$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu}(x)F^{\mu\nu a}(x) + \overline{\psi}(x)(\not{D} - m)\psi(x) - \frac{\lambda}{2}(n^\mu A^a_\mu)^2, \tag{28}$$

including a gauge fixing term that assures $n^\mu A_\mu^a = 0$ one reads off the Feynman rules. The propagators are derived from the quadratic terms

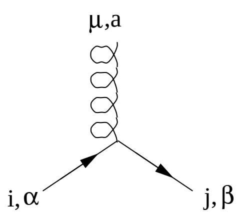


$$\left(\frac{i\,\delta_{ij}}{\not{k} - m + i\epsilon}\right)_{\beta\alpha} = \frac{i\,\delta_{ij}\,(\not{k} + m)_{\beta\alpha}}{k^2 - m^2 + i\epsilon}$$

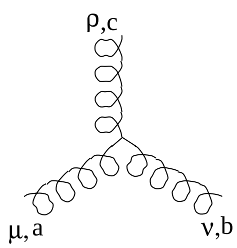


$$\frac{-i\,\delta_{ab}}{k^2 + i\epsilon}\left[g^{\mu\nu} + \left(n^2 + \frac{1}{\lambda}\,k^2\right)\frac{k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right].$$

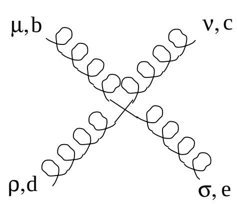
The vertices are derived from the interaction terms in the lagrangian,



$$i\,g\,(\gamma_\mu)_{\beta\alpha}(F^a)_{ji}$$



$$g\,c_{abc}[(p-q)^\rho\,g^{\mu\nu} + (q-r)^\mu\,g^{\nu\rho} + (r-p)^\rho\,g^{\rho\mu}]$$



$$\begin{aligned} &i\,g^2\,c_{abc}c_{ade}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &+i\,g^2\,c_{abd}c_{ace}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})\ . \\ &+i\,g^2\,c_{abe}c_{acd}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma}) \end{aligned}$$

2 Spontaneous symmetry breaking

In this section we consider the situation that the groundstate of a physical system is degenerate. Consider as an example a ferromagnet with an interaction hamiltonian of the form

$$H = -\sum_{i>j} J_{ij}\,\mathbf{S}_i\cdot\mathbf{S}_j,$$

which is rotationally invariant. If the temperature is high enough the spins are oriented randomly and the (macroscopic) ground state is spherically symmetric. If the temperature is below a certain critical temperature ($T < T_c$) the kinetic energy is no longer dominant and the above hamiltonian prefers a lowest energy configuration in which all spins are parallel. In this case there are many possible groundstates (determined by a fixed direction in space). This characterizes spontaneous symmetry breaking, the groundstate itself appears degenerate. As there can be one and only one groundstate, this means that there is more than one possibility for the groundstate. Nature will choose one, usually being (slightly) prejudiced by impurities, external magnetic fields, i.e. in reality a not perfectly symmetric situation.

Nevertheless, we can disregard those 'perturbations' and look at the ideal situation, e.g. a theory for a scalar degree of freedom (a scalar field) having three (real) components,

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$

with a lagrangian density of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\partial^\mu\vec{\phi} \underbrace{-\frac{1}{2}\,m^2\,\vec{\phi}\cdot\vec{\phi} - \frac{1}{4}\,\lambda(\vec{\phi}\cdot\vec{\phi})^2}_{-V(\vec{\phi})}. \tag{29}$$

The potential $V(\vec{\phi})$ is shown in fig. 1. Classically the (time-independent) ground state is found for a constant field ($\nabla\vec{\phi} = 0$) and the condition

$$\left.\frac{\partial V}{\partial\vec{\phi}}\right|_{\varphi_c} = 0 \qquad \longrightarrow \qquad \vec{\varphi}_c\cdot\vec{\varphi}_c = 0 \quad \text{or} \quad \vec{\varphi}_c\cdot\vec{\varphi}_c = -\frac{m^2}{\lambda} \equiv F^2,$$

the latter only forming a minimum for $m^2 < 0$. In this situation one speaks of *spontaneous symmetry breaking*. The classical groundstate appears degenerate. Any constant field φ_c with 'length' $|\vec{\phi}| = F$ is a possible groundstate. The presence of a nonzero value for the classical groundstate value of the field will have an effect when the field is quantized. A quantum field theory has only *one* nondegenerate groundstate $|0\rangle$. Writing the field $\vec{\phi}$ as a sum of a classical and a quantum field, $\vec{\phi} = \vec{\varphi}_c + \vec{\phi}_{\text{quantum}}$ where for the (operator-valued) coefficients in the quantum field one wants $\langle 0|c^\dagger = c|0\rangle = 0$ one has

$$\langle 0|\phi_{\text{quantum}}|0\rangle = 0 \quad \text{and} \quad \langle 0|\vec{\phi}|0\rangle = \vec{\varphi}_c. \tag{30}$$

Stability of the action requires the classical groundstate $\vec{\varphi}_c$ to have a well-defined value (which can be nonzero), while the quadratic terms must correspond with non-negative masses. In the case of degeneracy, therefore a choice must be made, say

$$\langle 0|\vec{\phi}|0\rangle = \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix}. \tag{31}$$

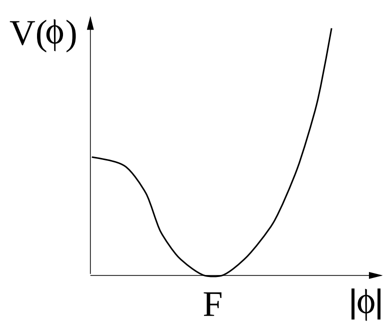


Figure 1: The symmetry-breaking 'potential' in the lagrangian for the case that $m^2 < 0$.

The situation now is the following. The original lagrangian contained an SO(3) invariance under (length conserving) rotations among the three fields, while the lagrangian including the nonzero groundstate expectation value chosen by nature, has less symmetry. It is only invariant under rotations around the 3-axis.

It is appropriate to redefine the field as

$$\vec{\phi} = \left(\begin{array}{c} \varphi_1 \\ \varphi_2 \\ F + \eta \end{array} \right), \tag{32}$$

such that $\langle 0|\varphi_1|0\rangle = \langle 0|\varphi_2|0\rangle = \langle 0|\eta|0\rangle = 0$. The field along the third axis plays a special role because of the choice of the vacuum expectation value. In order to see the consequences for the particle spectrum of the theory we construct the lagrangian in terms of the fields φ_1, φ_2 and η. It is sufficient to do this to second order in the fields as the higher (cubic, etc.) terms constitute interaction terms. The result is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\,(\partial_\mu\varphi_1)^2 + \frac{1}{2}\,(\partial_\mu\varphi_2)^2 + \frac{1}{2}\,(\partial_\mu\eta)^2 - \frac{1}{2}\,m^2\,(\varphi_1^2 + \varphi_2^2) \\ &\quad -\frac{1}{2}\,m^2\,(F+\eta)^2 - \frac{1}{4}\,\lambda\,(\varphi_1^2 + \varphi_2^2 + F^2 + \eta^2 + 2\,F\eta)^2 && (33) \\ &= \frac{1}{2}\,(\partial_\mu\varphi_1)^2 + \frac{1}{2}\,(\partial_\mu\varphi_2)^2 + \frac{1}{2}\,(\partial_\mu\eta)^2 + m^2\,\eta^2 + \ldots\,. && (34) \end{aligned}$$

Therefore there are *2 massless scalar particles*, corresponding to the number of broken generators (in this case rotations around 1 and 2 axis) and *1 massive scalar particle* with mass $m_\eta^2 = -2m^2$. The massless particles are called *Goldstone bosons*.

Realization of symmetries

In this section we want to discuss a bit more formal the two possible ways that a symmetry can be implemented. They are known as the *Weyl mode* or the *Goldstone mode*:

Weyl mode. In this mode the lagrangian *and* the vacuum are both invariant under a set of symmetry transformations generated by Q^a, i.e. for the vacuum $Q^a|0\rangle = 0$. In this case the spectrum is described by degenerate representations of the symmetry group. Known examples are rotational symmetry and the fact that the the spectrum shows multiplets labeled by angular momentum ℓ (with members labeled by m). The generators Q^a (in that case the rotation operators L_z, L_x and L_y or instead of the latter two L_+ and L_-) are used to label the multiplet members or transform them into one another. A bit more formal, if the generators Q^a generate a symmetry, i.e. $[Q^a, H] = 0$, and $|a\rangle$ and $|a'\rangle$ belong to the same multiplet (there is a Q^a such that $|a'\rangle = Q^a|a\rangle$) then $H|a\rangle = E_a|a\rangle$ implies that $H|a'\rangle = E_a|a'\rangle$, i.e. a and a' are degenerate states.

Goldstone mode. In this mode the lagrangian is invariant but $Q^a|0\rangle \neq 0$ for a number of generators. This means that they are operators that create states from the vacuum, denoted $|\pi^a(k)\rangle$. As the generators for a symmetry are precisely the zero-components of a conserved current $J_\mu^a(x)$ integrated over space, there must be a nonzero expectation value $\langle 0|J_\mu^a(x)|\pi^a(k)\rangle$. Using translation invariance and as k_μ is the only four vector on which this matrix element could depend one may write

$$\langle 0|J_\mu^a(x)|\pi^b(k)\rangle = f_\pi\,k_\mu\,e^{i\,k\cdot x}\,\delta_{ab} \qquad (f_\pi \neq 0) \tag{35}$$

for all the states labeled by a corresponding to 'broken' generators. Taking the derivative,

$$\langle 0|\partial^\mu J_\mu^a(x)|\pi^b(k)\rangle = f_\pi\,k^2\,e^{i\,k\cdot x}\,\delta_{ab} = f_\pi\,m_{\pi^a}^2\,e^{i\,k\cdot x}\,\delta_{ab}. \tag{36}$$

If the transformations in the lagrangian give rise to a symmetry the Noether currents are conserved, $\partial^\mu J^a_\mu = 0$, irrespective of the fact if they annihilate the vacuum, and one must have $m_{\pi^a} = 0$, i.e. a massless Goldstone boson for each 'broken' generator. Note that for the fields $\pi^a(x)$ one would have the relation $\langle 0|\pi^a(x)|\pi^a(k)\rangle = e^{i\,k\cdot x}$, suggesting the stronger relation $\partial^\mu J^a_\mu(x) = f_\pi\, m^2_{\pi^a}\,\pi^a(x)$.

Chiral symmetry

An example of spontaneous symmetry breaking is chiral symmetry breaking in QCD. Neglecting at this point the local color symmetry, the lagrangian for the quarks consists of the free Dirac lagrangian for each of the types of quarks, called flavors. Including a sum over the different flavors (up, down, strange, etc.) one can write

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - M)\psi, \tag{37}$$

where ψ is extended to a vector in *flavor space* and M is a diagonal matrix,

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \\ \vdots \end{pmatrix}, \qquad M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & \ddots \end{pmatrix} \tag{38}$$

(Note that each of the entries in the vector for ψ is a 4-component Dirac spinor). This lagrangian density then is invariant under unitary (vector) transformations in the flavor space,

$$\psi \;\longrightarrow\; e^{i\,\vec{\alpha}\cdot\vec{T}}\psi, \tag{39}$$

which for instance including only two flavors form an $SU(2)_V$ symmetry (isospin symmetry) generated by the Pauli matrices, $\vec{T} = \vec{\tau}/2$. The conserved currents corresponding to this symmetry transformation are found directly using Noether's theorem (see chapter 6),

$$\vec{V}^\mu = \overline{\psi}\gamma^\mu\vec{T}\psi. \tag{40}$$

Using the Dirac equation, it is easy to see that one gets

$$\partial_\mu\vec{V}^\mu = i\,\overline{\psi}\,[M,\vec{T}]\,\psi. \tag{41}$$

Furthermore $\partial_\mu\vec{V}^\mu = 0 \Longleftrightarrow [M,\vec{T}] = 0$. From group theory (Schur's theorem) one knows that the latter can only be true, if in flavor space M is proportional to the unit matrix, $M = m\cdot 1$. I.e. $SU(2)_V$ (isospin) symmetry is good if the up and down quark masses are identical. This situation, both are very small, is what happens in the real world. This symmetry is realized in the Weyl mode with the spectrum of QCD showing an almost perfect isospin symmetry, e.g. a doublet (isospin 1/2) of nucleons, proton and neutron, with almost degenerate masses (M_p = 938.3 MeV/c^2 and M_n =939.6 MeV/c^2), but also a triplet (isospin 1) of pions, etc.

There exists another set of symmetry transformations, socalled axial vector transformations,

$$\psi \;\longrightarrow\; e^{i\,\vec{\alpha}\cdot\vec{T}\gamma_5}\psi, \tag{42}$$

which for instance including only two flavors form $SU(2)_A$ transformations generated by the Pauli matrices, $\vec{T}\gamma_5 = \vec{\tau}\gamma_5/2$. Note that these transformations also work on the spinor indices. The currents corresponding to this symmetry transformation are again found using Noether's theorem,

$$\vec{A}^\mu = \overline{\psi}\gamma^\mu\vec{T}\gamma_5\psi. \tag{43}$$

Using the Dirac equation, it is easy to see that one gets

$$\partial_\mu\vec{A}^\mu = i\,\overline{\psi}\,\{M,\vec{T}\}\,\gamma_5\,\psi. \tag{44}$$

In this case $\partial_\mu \vec{A}^\mu = 0$ will be true if the quarks have zero mass, which is approximately true for the up and down quarks. Therefore the world of up and down quarks describing pions, nucleons and atomic nuclei has not only an isospin or vector symmetry $SU(2)_V$ but also an axial vector symmetry $SU(2)_A$. This combined symmetry is what one calls *chiral symmetry.*

That the massless theory has this symmetry can also be seen by writing it down for the socalled lefthanded and righthanded fermions, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$, in terms of which the Dirac lagrangian density looks like

$$\mathcal{L} = i\,\overline{\psi_L} \partial\!\!\!/ \psi_L + i\,\overline{\psi_R} \partial\!\!\!/ \psi_R - \overline{\psi_R} M \psi_L - \overline{\psi_L} M \psi_R. \tag{45}$$

If the mass is zero the lagrangian is split into two disjunct parts for L and R showing that there is a direct product $SU(2)_L \otimes SU(2)_R$ symmetry, generated by $\vec{T}_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\vec{T}$, which is equivalent to the V-A symmetry. This symmetry, however, is by nature not realized in the Weyl mode. How can we see this. The chiral fields ψ_R and ψ_L are transformed into each other under parity. Therefore realization in the Weyl mode would require that all particles come double with positive and negative parity, or, stated equivalently, parity would not play a role in the world. We know that mesons and baryons (such as the nucleons) have a well-defined parity that is conserved.

The conclusion is that the original symmetry of the lagrangian is spontaneously broken and as the vector part of the symmetry is the well-known isospin symmetry, nature has choosen the path

$$SU(2)_L \otimes SU(2)_R \quad \Longrightarrow \quad SU(2)_V,$$

i.e. the lagrangian density is invariant under left (L) and right (R) rotations independently, while the groundstate is only invariant under isospin rotations ($R = L$). From the number of broken generators it is clear that one expects three massless Goldstone bosons, for which the field (according to the discussion above) has the same behavior under parity, etc. as the quantity $\partial_\mu A^\mu(x)$, i.e. (leaving out the flavor structure) the same as $\overline{\psi}\gamma_5\psi$, i.e. behaves as a pseudoscalar particle (spin zero, parity minus). In the real world, where the quark masses are not completely zero, chiral symmetry is not perfect. Still the basic fact that the generators acting on the vacuum give a nonzero result (i.e. $f_\pi \neq 0$ remains, but the fact that the symmetry is not perfect and the right hand side of Eq. 44 is nonzero, gives also rise to a nonzero mass for the Goldstone bosons according to Eq. 36. The Goldstone bosons of QCD are the pions for which $f_\pi = 93$ MeV and which have a mass of $m_\pi \approx 138$ MeV/c^2, much smaller than any of the other mesons or baryons.

3 The Higgs mechanism

The Higgs mechanism occurs when spontaneous symmetry breaking happens in a gauge theory where gauge bosons have been introduced in order to assure the local symmetry. Considering the same example with rotational symmetry ($SO(3)$) as for spontaneous symmetry breaking of a scalar field (Higgs field) with three components, made into a gauge theory,

$$\mathcal{L} = -\frac{1}{4}\,\vec{G}_{\mu\nu} \cdot \vec{G}^{\mu\nu} + \frac{1}{2}\,D_\mu\vec{\phi} \cdot D^\mu\vec{\phi} - V(\vec{\phi}), \tag{46}$$

where

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} - ig\,W_\mu^a L_a \vec{\phi}. \tag{47}$$

Since the explicit (conjugate, in this case three-dimensional) representation $(L_a)_{ij} = -i\,\epsilon_{aij}$ one sees that the fields $\vec{W}_\mu$ and $\vec{G}_{\mu\nu}$ also can be represented as three-component fields,

$$\begin{aligned} D_\mu\vec{\phi} &= \partial_\mu\vec{\phi} + g\,\vec{W}_\mu \times \vec{\phi}, && (48) \\ \vec{G}_{\mu\nu} &= \partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + g\vec{W}_\mu \times \vec{W}_\nu. && (49) \end{aligned}$$

The symmetry is broken in the same way as before and the same choice for the vacuum,

$$\vec{\varphi}_c = \langle 0|\vec{\phi}|0\rangle = \left(\begin{array}{c} 0 \\ 0 \\ F \end{array}\right).$$

is made. The difference comes when we reparametrize the field $\vec{\phi}$. We have the possibility to perform local gauge transformations. Therefore we can always rotate the field ϕ into the z-direction in order to make the calculation simple, i.e.

$$\vec{\phi} = \left(\begin{array}{c} 0 \\ 0 \\ \phi_3 \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ F+\eta \end{array}\right). \tag{50}$$

Explicitly one then has

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + g\,\vec{W}_\mu \times \vec{\phi} = \left(\begin{array}{c} gF\,W_\mu^2 + g\,W_\mu^2\,\eta \\ -gF\,W_\mu^1 - g\,W_\mu^1\,\eta \\ \partial_\mu\eta \end{array}\right),$$

which gives for the lagrangian density up to quadratic terms

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\,\vec{G}_{\mu\nu}\cdot\vec{G}^{\mu\nu} + \frac{1}{2}\,D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - \frac{1}{2}\,m^2\vec{\phi}\cdot\vec{\phi} - \frac{\lambda}{4}\,(\vec{\phi}\cdot\vec{\phi})^2 \\
&= -\frac{1}{4}\,(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu) - \frac{1}{2}\,g^2F^2\left(W_\mu^1W^{\mu\,1} + W_\mu^2W^{\mu\,2}\right) \\
&\quad +\frac{1}{2}\,(\partial_\mu\eta)^2 + m^2\,\eta^2 + \ldots,
\end{aligned} \tag{51}$$

from which one reads off that the particle content of the theory consists of one massless gauge boson (W_μ^3), two massive bosons (W_μ^1 and W_μ^2 with $M_W = gF$) and a massive scalar particle (η with $m_\eta^2 = -2\,m^2$. The latter is a spin 0 particle (real scalar field) called a Higgs particle. Note that the number of massless gauge bosons (in this case one) coincides with the number of generators corresponding to the remaining symmetry (in this case rotations around the 3-axis), while the number of massive gauge bosons coincides with the number of 'broken' generators.

One may wonder about the degrees of freedom, as in this case there are no massless Goldstone bosons. Initially there are 3 massless gauge fields (each, like a photon, having two independent spin components) and three scalar fields (one degree of freedom each), thus 9 independent degrees of freedom. After symmetry breaking the same number (as expected) comes out, but one has 1 massless gauge field (2), 2 massive vector fields or spin 1 bosons (2×3) and one scalar field (1), again 9 degrees of freedom.

4 The standard model $SU(2)_W \otimes U(1)_Y$

The symmetry ideas discussed before play an essential role in the standard model that describes the elementary particles, the quarks (up, down, etc.), the leptons (elektrons, muons, neutrinos, etc.) and the gauge bosons responsible for the strong, electromagnetic and weak forces. In the standard model one starts with a very simple basic lagrangian for (massless) fermions which exhibits more symmetry than observed in nature. By introducing gauge fields and breaking the symmetry a more complex lagrangian is obtained, that gives a good description of the physical world. The procedure, however, implies certain nontrivial relations between masses and mixing angles that can be tested experimentally and sofar are in excellent agreement with experiment.

The lagrangian for the leptons consists of three families each containing an elementary fermion (electron e^-, muon μ^- or tau τ^-), its corresponding neutrino (ν_e, ν_μ and ν_τ) and their antiparticles.

As they are massless, left- and righthanded particles, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$ decouple. For the neutrino only a lefthanded particle (and righthanded antiparticle) exist. Thus

$$\mathcal{L}^{(f)} = i\,\overline{e_R}\partial\!\!\!/ e_R + i\,\overline{e_L}\partial\!\!\!/ e_L + i\,\overline{\nu_{eL}}\partial\!\!\!/ \nu_{eL} + (\mu,\tau). \tag{52}$$

One introduces a (weak) $SU(2)_W$ symmetry under which e_R forms a singlet, while the lefthanded particles form a doublet, i.e.

$$L = \left(\begin{array}{c} \nu_e \\ e_L \end{array} \right) \qquad \text{with } I_W = \frac{1}{2} \text{ and } I_W^3 = \left\{ \begin{array}{l} +1/2 \\ -1/2 \end{array} \right.$$

and

$$R = e_R \qquad\qquad \text{with } I_W = 0 \text{ and } I_W^3 = 0.$$

Thus the lagrangian density is

$$\mathcal{L}^{(f)} = i\,\overline{L}\partial\!\!\!/ L + i\,\overline{R}\partial\!\!\!/ R, \tag{53}$$

which has an $SU(2)_W$ symmetry under transformations $e^{i\vec{\alpha}\cdot\vec{T}}$, explicitly

$$L \xrightarrow{SU(2)_W} e^{i\,\vec{\alpha}\cdot\vec{\tau}/2} L, \tag{54}$$

$$R \xrightarrow{SU(2)_W} R. \tag{55}$$

One notes that the charges of the leptons can be obtained as $Q = I_W^3 - 1/2$ for lefthanded particles and $Q = I_W^3 - 1$ for righthanded particles. This is written as

$$Q = I_W^3 + \frac{Y_W}{2}, \tag{56}$$

and Y_W is considered as an operator that generates a $U(1)_Y$ symmetry, under which the lefthanded and righthanded particles with $Y_W(L) = -1$ and $Y_W(R) = -2$ transform with $e^{i\beta Y_W/2}$, explicitly

$$L \xrightarrow{U(1)_Y} e^{-i\,\beta/2} L, \tag{57}$$

$$R \xrightarrow{U(1)_Y} e^{-i\,\beta} R. \tag{58}$$

Next the $SU(2)_W \otimes U(1)_Y$ symmetry is made into a local symmetry introducing gauge fields $\vec{W}_\mu$ and B_μ,

$$D_\mu L = \partial_\mu L + \frac{i}{2}\,g\,\vec{W}_\mu \cdot \vec{\tau}\,L - \frac{i}{2}\,g'\,B_\mu\,L, \tag{59}$$

$$D_\mu R = \partial_\mu R - i\,g'\,B_\mu\,R, \tag{60}$$

where $\vec{W}_\mu$ is a triplet of gauge bosons with $I_W = 1$, $I_W^3 = \pm 1$ or 0 and $Y_W = 0$ (thus $Q = I_W^3$) and B_μ is a singlet under $SU(2)_W$ ($I_W = I_W^3 = 0$) and also has $Y_W = 0$. Putting this in leads to

$$\mathcal{L}^{(f)} = \mathcal{L}^{(f1)} + \mathcal{L}^{(f2)}, \tag{61}$$

$$\begin{aligned} \mathcal{L}^{(f1)} &= i\,\overline{R}\gamma^\mu(\partial_\mu - ig'B_\mu)R + i\,\overline{L}\gamma^\mu(\partial_\mu - \frac{i}{2}\,g'B_\mu + \frac{i}{2}\,g\vec{W}_\mu \cdot \vec{\tau})L \\ \mathcal{L}^{(f2)} &= -\frac{1}{4}(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + g\,\vec{W}_\mu \times \vec{W}_\nu)^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2. \end{aligned}$$

In order to break the symmetry to the symmetry of the physical world, the $U(1)_Q$ symmetry (generated by the charge operator), a complex Higgs field

$$\phi = \left(\begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right) = \left(\begin{array}{c} \frac{1}{\sqrt{2}}(\theta_2 + i\theta_1) \\ \frac{1}{\sqrt{2}}(\theta_4 - i\theta_3) \end{array} \right) \tag{62}$$

with $I_W = 1/2$ and $Y_W = 1$ is introduced, with the following lagrangian density consisting of a symmetry breaking piece and a coupling to the fermions,

$$\mathcal{L}^{(h)} = \mathcal{L}^{(h1)} + \mathcal{L}^{(h2)}, \tag{63}$$

where

$$\begin{aligned}
\mathcal{L}^{(h1)} &= (D_\mu\phi)^\dagger(D^\mu\phi)\underbrace{-m^2\,\phi^\dagger\phi - \lambda\,(\phi^\dagger\phi)^2}_{-V(\phi)},\\
\mathcal{L}^{(h2)} &= -G_e(\overline{L}\phi R + \overline{R}\phi^\dagger L),
\end{aligned}$$

and

$$D_\mu\phi = (\partial_\mu + \frac{i}{2}\,g\vec{W}_\mu\cdot\vec{\tau} + \frac{i}{2}\,g'\,B_\mu)\phi. \tag{64}$$

The Higgs potential $V(\phi)$ is choosen such that it gives rise to spontaneous symmetry breaking with $\varphi_c^\dagger\varphi = -m^2/2\lambda \equiv v^2/2$. For the classical field the choice $\theta_4 = v$ is made. Using *local* gauge invariance θ_i for $i = 1$, 2 and 3 may be eliminated (the necessary $SU(2)_W$ rotation is precisely $e^{-i\vec{\theta}(x)\cdot\tau}$), leading to the parametrization

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x)\end{pmatrix} \tag{65}$$

and

$$D_\mu\phi = \begin{pmatrix} \frac{ig}{2}\left(\frac{W^1_\mu - iW^2_\mu}{\sqrt{2}}\right)(v+h) \\ \partial_\mu h - \frac{i}{2}\left(\frac{gW^3_\mu - g'B_\mu}{\sqrt{2}}\right)(v+h)\end{pmatrix}. \tag{66}$$

Up to cubic terms, this leads to the lagrangian

$$\begin{aligned}
\mathcal{L}^{(h1)} &= \frac{1}{2}(\partial_\mu h)^2 + m^2\,h^2 + \frac{g^2\,v^2}{8}\left[(W^1_\mu)^2 + (W^2_\mu)^2\right]\\
&\quad + \frac{v^2}{8}\left(gW^3_\mu - g'B_\mu\right)^2 + \ldots &(67)\\
&= \frac{1}{2}(\partial_\mu h)^2 + m^2\,h^2 + \frac{g^2\,v^2}{8}\left[(W^+_\mu)^2 + (W^-_\mu)^2\right]\\
&\quad + \frac{(g^2 + g'^2)\,v^2}{8}(Z_\mu)^2 + \ldots, &(68)
\end{aligned}$$

where the quadratically appearing gauge fields that are furthermore eigenstates of the charge operator are

$$\begin{aligned}
W^\pm_\mu &= \frac{1}{\sqrt{2}}\left(W^1_\mu \pm i\,W^2_\mu\right), &(69)\\
Z_\mu &= \frac{g\,W^3_\mu - g'\,B_\mu}{\sqrt{g^2 + g'^2}} \equiv \cos\theta_W\,W^3_\mu - \sin\theta_W\,B_\mu, &(70)\\
A_\mu &= \frac{g'\,W^3_\mu + g\,B_\mu}{\sqrt{g^2 + g'^2}} \equiv \sin\theta_W\,W^3_\mu + \cos\theta_W\,B_\mu, &(71)
\end{aligned}$$

and correspond to three massive particle fields ($W^\pm$ and Z^0) and one massless field (photon γ) with

$$\begin{aligned}
M_W^2 &= \frac{g^2\,v^2}{4}, &(72)\\
M_Z^2 &= \frac{g^2\,v^2}{4\,\cos^2\theta_W} = \frac{M_W^2}{\cos^2\theta_W}, &(73)\\
M_\gamma^2 &= 0. &(74)
\end{aligned}$$

The weak mixing angle is related to the ratio of coupling constants, $g'/g = \tan\theta_W$.

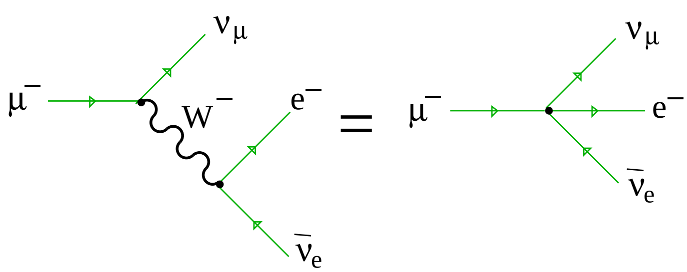
The coupling of the fermions to the physical gauge bosons are contained in $\mathcal{L}^{(f1)}$ giving

$$\begin{aligned}\mathcal{L}^{(f1)} &= i\,\overline{e}\gamma^\mu\partial_\mu e + i\,\overline{\nu_e}\gamma^\mu\partial_\mu\nu_e - g\sin\theta_W\,\overline{e}\gamma^\mu e\,A_\mu \\ &\quad + \frac{g}{\cos\theta_W}\left(\sin^2\theta_W\,\overline{e_R}\gamma^\mu e_R - \frac{1}{2}\cos 2\theta_W\,\overline{e_L}\gamma^\mu e_L + \frac{1}{2}\,\overline{\nu_e}\gamma^\mu\nu_e\right) Z_\mu \\ &\quad + \frac{g}{\sqrt{2}}\left(\overline{\nu_e}\gamma^\mu e_L\,W_\mu^- + \overline{e_L}\gamma^\mu\nu_e\,W_\mu^+\right). \end{aligned} \tag{75}$$

From the coupling to the photon, we can read off

$$e = g\,\sin\theta_W = g'\cos\theta_W. \tag{76}$$

The coupling of electrons or muons to their respective neutrinos, for instance in the amplitude for the decay of the muon



is given by

$$\begin{aligned}-i\,\mathcal{M} &= -\frac{g^2}{2}(\overline{\nu_\mu}\gamma^\rho\mu_L)\frac{-i\,g_{\rho\sigma}+\ldots}{k^2+M_W^2}(\overline{e_L}\gamma^\sigma\nu_e) \\ &\approx i\,\frac{g^2}{8\,M_W^2}\underbrace{(\overline{\nu_\mu}\gamma^\rho(1-\gamma_5)\mu)}_{(j_L^{(\mu)\dagger})_\rho}\underbrace{(\overline{e}\gamma^\sigma(1-\gamma^5)\nu_e)}_{(j_L^{(e)})^\rho} \qquad (77) \\ &\equiv i\,\frac{G_F}{\sqrt{2}}\,(j_L^{(\mu)\dagger})_\rho\,(j_L^{(e)})^\rho, \qquad (78)\end{aligned}$$

the good old four-point interaction introduced by Fermi to explain the weak interactions, i.e. one has the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\,M_W^2} = \frac{e^2}{8\,M_W^2\,\sin^2\theta_W} = \frac{1}{2\,v^2}. \tag{79}$$

In this way the parameters g, g' and v determine a number of experimentally measurable quantities, such as

$$\begin{aligned} e^2/4\pi &\approx 1/137, & (80)\\ G_F &= 1.166\,4\times 10^{-5}\ \mathrm{GeV}^{-2}, & (81)\\ \sin^2\theta_W &= 0.231\,1, & (82)\\ M_W &= 80.42\ \mathrm{GeV}, & (83)\\ M_Z &= 91.198\ \mathrm{GeV}. & (84)\end{aligned}$$

The coupling of the Z^0 to fermions is given by $g/\cos\theta_W$ multiplied with

$$I_W^3\,\frac{1}{2}(1-\gamma_5) - \sin^2\theta_W\,Q \equiv \frac{1}{2}\,C_V - \frac{1}{2}\,C_A\,\gamma_5, \tag{85}$$

with

$$
\begin{aligned}
C_V &= I_W^3 - 2\sin^2\theta_W\, Q, & (86)\\
C_A &= I_W^3. & (87)
\end{aligned}
$$

From this coupling it is straightforward to calculate the partial width for Z^0 into a fermion-antifermion pair,

$$
\Gamma(Z^0 \to f\overline{f}) = \frac{M_Z}{48\pi}\,\frac{g^2}{\cos^2\theta_W}\,(C_V^2 + C_A^2). \tag{88}
$$

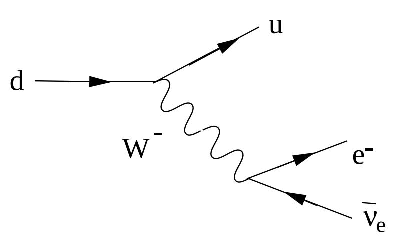
For the electron, muon or tau, leptons with $C_V = -1/2 + 2\sin^2\theta_W \approx -0.05$ and $C_A = -1/2$ we calculate $\Gamma(e^+e^-) \approx 78.5$ MeV (exp. $\Gamma_e \approx \Gamma_\mu \approx \Gamma_\tau \approx 83$ MeV). For each neutrino species (with $C_V = 1/2$ and $C_A = 1/2$ one expects $\Gamma(\overline{\nu}\nu) \approx 155$ MeV. Comparing this with the total width into (invisible!) channels, $\Gamma_{invisible} = 480$ MeV one sees that three families of (light) neutrinos are allowed. Actually including corrections corresponding to higher order diagrams the agreement for the decay width into electrons can be calculated much more accurately and the number of allowed (light) neutrinos turns to be even closer to three.

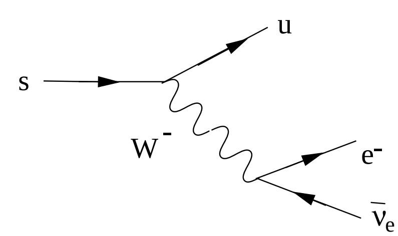
The masses of the fermions and the coupling to the Higgs particle are contained in $\mathcal{L}^{(h2)}$. With the choosen vacuum expectation value for the Higgs field, one obtains

$$
\begin{aligned}
\mathcal{L}^{(h2)} &= -\frac{G_e\, v}{\sqrt{2}}\,(\overline{e_L}e_R + \overline{e_R}e_L) - \frac{G_e}{\sqrt{2}}\,(\overline{e_L}e_R + \overline{e_R}e_L)\,h \\
&= -m_e\,\overline{e}e - \frac{m_e}{v}\,\overline{e}eh. & (89)
\end{aligned}
$$

First, the mass of the electron comes from the spontaneous symmetry breaking but is not predicted (it is in the coupling G_e). The coupling to the Higgs particle is weak as the value for v calculated e.g. from the M_W mass is about 250 GeV, i.e. m_e/v is extremely small.

Finally we want to say something about the weak properties of the quarks, as appear for instance in the decay of the neutron or the decay of the Λ (quark content uds),





$n \longrightarrow pe^-\overline{\nu_e} \quad \Longleftrightarrow \quad d \longrightarrow ue^-\overline{\nu_e}$, $\qquad \Lambda \longrightarrow pe^-\overline{\nu_e} \quad \Longleftrightarrow \quad s \longrightarrow ue^-\overline{\nu_e}$.

The quarks also turn out to fit into doublets of $SU(2)_W$ for the lefthanded species and into singlets for the righthanded quarks. A complication arises as it are not the 'mass' eigenstates that appear in the weak isospin doublets but linear combinations of them,

$$
\begin{pmatrix} u \\ d' \end{pmatrix}_L \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L,
$$

where

$$
\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \tag{90}
$$

This mixing allows all quarks with $I_W^3 = -1/2$ to decay into an up quark, but with different strength. Comparing neutron decay and Λ decay one can get an estimate of the mixing parameter V_{us} in the socalled Cabibbo-Kobayashi-Maskawa mixing matrix. Decay of B-mesons containing b-quarks allow estimate of V_{ub}, etc. In principle one complex phase is allowed in the most general form of the CKM matrix, which can account for the (observed) CP violation of the weak interactions. This is only true if the mixing matrix is at least three-dimensional, i.e. CP violation requires three generations. The magnitudes of the entries in the CKM matrix are nicely represented in a socalled Wolfenstein parametrization

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & \lambda^3 A(\rho - i\,\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & \lambda^2 A \\ \lambda^3 A(1-\rho-i\,\eta) & -\lambda^2 A & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

with $\lambda \approx 0.23$, $A \approx 0.81$ and $\rho \approx 0.23$ and $\eta \approx 0.35$. The imaginary part $i\,\eta$ gives rise to CP violation in decays of $\overline{K}$ and B-mesons (containing s and b quarks, respectively).

5 Family mixing in the Higgs sector and neutrino masses

The quark sector

Allowing for the most general (Dirac) mass generating term in the lagrangian one starts with

$$\mathcal{L}^{\,(\mathrm{h2,q})} = -\overline{Q_L}\phi\Lambda_d D_R - \overline{D_R}\Lambda_d^\dagger\phi^\dagger Q_L, -\overline{Q_L}\phi^c\Lambda_u U_R - \overline{U_R}\Lambda_u^\dagger\phi^{c\dagger}Q_L \tag{91}$$

where we include now the three lefthanded quark doublets in Q_L, the three righthanded quarks with charge $+2/3$ in U_R and the three righthanded quarks with charges $-1/3$ in D_R, each of these containing the three families, e.g. $\overline{U_R} = \begin{pmatrix}\overline{u_R} & \overline{c_R} & \overline{t_R}\end{pmatrix}$. The Λ_u and Λ_d are complex matrices in the 3×3 family space. The Higgs field is still limited to one complex doublet. Note that we need the conjugate Higgs field to get a $U(1)_Y$ singlet in the case of the charge $+2/3$ quarks, for which we need the appropriate weak isospin doublet

$$\phi^c = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} v+h \\ 0 \end{pmatrix}.$$

For the (squared) complex matrices we can find positive eigenvalues,

$$\Lambda_u\,\Lambda_u^\dagger = V_u\,G_u^2\,V_u^\dagger, \quad \text{and} \quad \Lambda_d\,\Lambda_d^\dagger = W_d\,G_d^2\,W_d^\dagger, \tag{92}$$

where V_u and W_d are unitary matrices, allowing us to write

$$\Lambda_u = V_u\,G_u\,W_u^\dagger \quad \text{and} \quad \Lambda_d = V_d\,G_d\,W_d^\dagger, \tag{93}$$

with G_u and G_d being real and positive and W_u and V_d being different unitary matrices. Thus one has

$$\mathcal{L}^{\,(\mathrm{h2,q})} \Longrightarrow -\overline{D_L}V_d\,M_d\,W_d^\dagger D_R - \overline{D_R}W_d\,M_d\,V_d^\dagger D_L, -\overline{U_L}V_u\,M_u\,W_u^\dagger U_R - \overline{U_R}W_u\,M_u\,V_u^\dagger U_L \tag{94}$$

with $M_u = G_u v/\sqrt{2}$ (diagonal matrix containing m_u, m_c and m_t) and $M_d = G_d v/\sqrt{2}$ (diagonal matrix containing m_d, m_s and m_b). One then reads off that starting with the family basis as defined via the left doublets that the mass eigenstates (and states coupling to the Higgs field) involve the righthanded states $U_R^{\rm mass} = W_u^\dagger U_R$ and $D_R^{\rm mass} = W_d^\dagger D_R$ and the lefthanded states $U_L^{\rm mass} = V_u^\dagger U_L$ and $D_L^{\rm mass} = V_d^\dagger D_L$. Working with the mass eigenstates one simply sees that the weak current coupling to the $W^\pm$ becomes $\overline{U}_L\,\gamma^\mu\gamma_5\,D_L$, $\overline{U}_L^{\rm mass}\,\gamma^\mu\gamma_5\,V_u^\dagger V_d\,D_L^{\rm mass}$, i.e. the weak mass eigenstates are

$$D_L' = D_L^{\rm weak} = V_u^\dagger V_d\,D_L^{\rm mass} = V_{\rm CKM}\,D_L^{\rm mass}, \tag{95}$$

the unitary CKM-matrix introduced above in an ad hoc way.

The lepton sector (massless neutrinos)

For a lepton sector with a lagrangian density of the form

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} = -\overline{L}\phi\Lambda_e E_R - \overline{E_R}\Lambda_e^\dagger\phi^\dagger L, \tag{96}$$

in which

$$L = \begin{pmatrix} N_L \\ E_L \end{pmatrix},$$

is a weak doublet containing the three families of neutrinos (N_L) and charged leptons (E_L) and E_R is a three-family weak singlet, we find massless neutrinos. As before, one can write $\Lambda_e = V_e\,G_e\,W_e^\dagger$ and we find

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} \Longrightarrow -M_e\left(\overline{E_L}V_e W_e^\dagger E_R - \overline{E_R}W_e V_e^\dagger E_L\right), \tag{97}$$

with $M_e = G_e v/\sqrt{2}$ the diagonal mass matrix with masses m_e, m_μ and m_τ. The mass fields $E_R^{\mathrm{mass}} = W_e^\dagger E_R$, $E_L^{\mathrm{mass}} = V_e^\dagger E_L$ and the neutrino fields $N_L^{\mathrm{mass}} = V_e^\dagger N_L$ are also the states appearing in the W-current, i.e. there is no family mixing and the neutrinos are massless.

The lepton sector (massive Dirac neutrinos)

In principle a massive Dirac neutrino could be accounted for by a lagrangian of the type

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} = -\overline{L}\phi\Lambda_e E_R - \overline{E_R}\Lambda_e^\dagger\phi^\dagger L, -\overline{L}\phi^c\Lambda_n N_R - \overline{N_R}\Lambda_n^\dagger\phi^{c\dagger} L \tag{98}$$

with three righthanded neutrinos added to the previous case, decoupling from all known interactions. Again we continue as before now with matrices $\Lambda_e = V_e\,G_e\,W_e^\dagger$ and $\Lambda_n = V_n\,G_n\,W_n^\dagger$, and obtain

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} \Longrightarrow -\overline{E_L}V_e\,M_e\,W_e^\dagger E_R - \overline{E_R}W_e\,M_e\,V_e^\dagger E_L, -\overline{N_L}V_n\,M_n\,W_n^\dagger N_R - \overline{N_R}W_n\,M_n\,V_n^\dagger N_L. \tag{99}$$

We note that there are mass fields $E_R^{\mathrm{mass}} = W_e^\dagger E_R$, $E_L^{\mathrm{mass}} = V_e^\dagger E_L$, $N_L^{\mathrm{mass}} = V_n^\dagger N_L$ and $N_R^{\mathrm{mass}} = W_n^\dagger N_R$ and the weak current becomes $\overline{E_L}\,\gamma^\mu\gamma_5\,N_L = \overline{E_L^{\mathrm{mass}}}\,\gamma^\mu\gamma_5\,V_e^\dagger V_n\,N_L^{\mathrm{mass}}$. Working with the mass eigenstates for the charged leptons we see that the weak eigenstates for the neutrinos are $N_L^{\mathrm{weak}} = V_e^\dagger\,N_L$ with the relation to the mass eigenstates for the lefthanded neutrinos given by

$$N_L^\prime = N_L^{\mathrm{weak}} = V_e^\dagger\,V_n\,N_L^{\mathrm{mass}}, \tag{100}$$

or $N_L^{\mathrm{mass}} = U\,N_L^{\mathrm{weak}}$ with $U = V_n^\dagger\,V_e$.

The lepton sector (massive Majorana fields)

The simplest option is to add in Eq. 97 a Majorana mass term for (lefthanded) neutrino mass eigenstates,

$$\mathcal{L}^{\,\mathrm{mass},\nu} = -\frac{1}{2}\left(M_L\,\overline{N_L^c}\,N_L + M_L^*\,\overline{N_L}\,N_L^c\right), \tag{101}$$

but this option is not attractive as it violates the electroweak symmetry. The way to circumvent this is to introduce as in the previous section righthanded neutrinos with for the righthanded sector a mass term M_R,

$$\mathcal{L}^{\,\mathrm{mass},\nu} = -\frac{1}{2}\left(M_R\,\overline{N_R}\,N_R^c + M_R^*\,\overline{N_R^c}\,N_R\right). \tag{102}$$

For neutrinos as well as charged leptons, the right- and lefthanded species are coupled through Dirac mass terms as in the previous section. Thus (disregarding family structure) one has two Majorana neutrinos, one being massive. For the charged leptons there are no Majorana mass terms (it would break the U(1) electromagnetic symmetry) and left- and righthanded species combine to a Dirac fermion. Moreover, if the Majorana mass $M_R \gg M_D$ one obtains in a natural way tiny neutrino masses. This is called the seesaw mechanism.

The seesaw mechanism

Consider (for one family $N = n$) the most general Lorentz invariant mass term for two independent Majorana spinors, Υ_1' and Υ_2' (satisfying $\Upsilon^c = \Upsilon$ and as discussed in chapter 6, $\Upsilon_L^c \equiv (\Upsilon_L)^c = \Upsilon_R$ and $\Upsilon_R^c = \Upsilon_L$). We use here the primes starting with the weak eigenstates. Actually, it is easy to see that this incorporates the Dirac case by considering the lefthanded part of Υ_1' and the righthanded part of Υ_2' as a Dirac spinor ψ. Thus

$$\Upsilon_1' = n_L^c + n_L, \qquad \Upsilon_2' = n_R + n_R^c, \qquad \psi = n_R + n_L. \tag{103}$$

As the most general mass term in the lagrangian density we have

$$\begin{aligned}
\mathcal{L}^{\,\text{mass}} &= -\frac{1}{2}\left(M_L\,\overline{n_L^c}\,n_L + M_L^*\,\overline{n_L}\,n_L^c\right) - \frac{1}{2}\left(M_R\,\overline{n_R}\,n_R^c + M_R^*\,\overline{n_R^c}\,n_R\right) \\
&\quad - \frac{1}{2}\left(M_D\,\overline{n_L^c}\,n_R^c + M_D^*\,\overline{n_L}\,n_R\right) - \frac{1}{2}\left(M_D\,\overline{n_R}\,n_L + M_D^*\,\overline{n_R^c}\,n_L^c\right) \qquad (104)\\
&= -\frac{1}{2}\begin{pmatrix}\overline{n_L^c} & \overline{n_R}\end{pmatrix}\begin{pmatrix} M_L & M_D \\ M_D & M_R\end{pmatrix}\begin{pmatrix} n_L \\ n_R^c\end{pmatrix} + \text{ h.c.} \qquad (105)
\end{aligned}$$

which for $M_D = 0$ is a pure Majorana lagrangian and for $M_L = M_R = 0$ and real M_D represents the Dirac case. The mass matrix can be written as

$$M = \begin{pmatrix} M_L & |M_D|\,e^{i\phi} \\ |M_D|\,e^{i\phi} & M_R\end{pmatrix} \tag{106}$$

taking M_L and M_R real and non-negative. This choice is possible without loss of generality because the phases can be absorbed into Υ_1' and Υ_2' (real must be replaced by hermitean if one includes families). This is a mixing problem with a symmetric (complex) mass matrix leading to two (real) mass eigenstates. The diagonalization is analogous to what was done for the Λ-matrices and one finds $U\,M\,U^T = M_0$ with a (unitary) matrix U, which implies $U^*\,M^\dagger\,U^\dagger = U^*\,M^*\,U^\dagger = M_0$ and a 'normal' diagonalization of the (hermitean) matrix $MM^\dagger$,

$$U\,(MM^\dagger)\,U^\dagger = M_0^2, \tag{107}$$

Thus one obtains from

$$MM^\dagger = \begin{pmatrix} M_L^2 + |M_D|^2 & |M_D|\,\left(M_L\,e^{-i\phi} + M_R\,e^{+i\phi}\right) \\ |M_D|\,\left(M_L\,e^{+i\phi} + M_R\,e^{-i\phi}\right) & M_R^2 + |M_D|^2\end{pmatrix}, \tag{108}$$

the eigenvalues

$$\begin{aligned}
M_{1/2}^2 &= \frac{1}{2}\Big[M_L^2 + M_R^2 + 2|M_D|^2 \\
&\qquad \pm\sqrt{(M_L^2 - M_R^2)^2 + 4|M_D|^2\,\left(M_L^2 + M_R^2 + 2M_LM_R\,\cos(2\phi)\right)}\Big],
\end{aligned} \tag{109}$$

and we are left with two decoupled Majorana fields Υ_1 and Υ_2, related via

$$\begin{pmatrix}\Upsilon_{1L}\\ \Upsilon_{2L}\end{pmatrix} = U^*\begin{pmatrix} n_L \\ n_R^c\end{pmatrix}, \qquad \begin{pmatrix}\Upsilon_{1R}\\ \Upsilon_{2R}\end{pmatrix} = U\begin{pmatrix} n_L^c \\ n_R\end{pmatrix}. \tag{110}$$

for each of which one finds the lagrangians

$$\mathcal{L} = \frac{1}{4}\,\overline{\Upsilon_i}\,i\,\overleftrightarrow{\partial\!\!\!/}\,\Upsilon_i - \frac{1}{2}\,M_i\,\overline{\Upsilon_i}\,\Upsilon_i \tag{111}$$

for $i = 1,\,2$ with real masses M_i. For the situation $M_L = 0$ and $M_R \gg M_D$ (taking M_D real) one finds $M_1 \approx M_D^2/M_R$ and $M_2 \approx M_R$.

Exercises

Exercise 1

Consider the case of the Weyl mode for symmetries. Prove that if the generators Q^a generate a symmetry, i.e. $[Q^a, H] = 0$, and $|a\rangle$ and $|a'\rangle$ belong to the same multiplet (there is a Q^a such that $|a'\rangle = Q^a|a\rangle$) then $H|a\rangle = E_a|a\rangle$ implies that $H|a'\rangle = E_a|a'\rangle$, i.e. a and a' are degenerate states.

Exercise 2

Derive for the vector and axial vector currents, $\vec{V}^\mu = \overline{\psi}\gamma^\mu\vec{T}\psi$ and $\vec{A}^\mu = \overline{\psi}\gamma^\mu\gamma_5\vec{T}\psi$

$$\begin{aligned} \partial_\mu \vec{V}^\mu &= i\,\overline{\psi}\,[M,\vec{T}]\,\psi, \\ \partial_\mu \vec{A}^\mu &= i\,\overline{\psi}\,\{M,\vec{T}\}\,\gamma_5\,\psi. \end{aligned}$$

Exercise 3

(a) The coupling of the Z^0 particle to fermions is described by the vertex

$$-i\,\frac{g}{2\cos\theta_W}\,\left(c_V^f\,\gamma^\mu - c_A^f\,\gamma^\mu\gamma_5\right),$$

with

$$\begin{aligned} C_V &= I_W^3 - 2Q\,\sin^2\theta_W, \\ C_A &= I_W^3. \end{aligned}$$

Write down the matrix element squared (averaged over initial spins and summed over final spins) for the decay of the Z^0. Neglect the masses of fermions and use the fact that the sum over polarizations is

$$\sum_{\lambda=1}^{3}\epsilon_\mu^{(\lambda)}(p)\epsilon_\nu^{(\lambda)*}(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}.$$

to calculate the width $\Gamma(Z^0 \to f\overline{f})$,

$$\Gamma(Z^0 \to f\overline{f}) = \frac{M_Z}{48\pi}\,\frac{g^2}{\cos^2\theta_W}\,\left(C_V^{f\,2} + C_A^{f\,2}\right).$$

(b) Calculate the width to electron-positron pair, $\Gamma(Z^0 \to e^+e^-)$, and the width to a pair of neutrino's, $\Gamma(Z^0 \to \nu_e\overline{\nu_e})$. The mass of the Z^0 is $M_Z = 91$ GeV, the weak mixing angle is given by $\sin^2\theta_W = 0.231$.

Exercise 4

Calculate the lifetime $\tau = 1/\Gamma$ for the top quark (t) given that the dominant decay mode is

$$\mathrm{t} \to \mathrm{b} + W^+.$$

In the standard model this coupling is described by the vertex

$$\frac{-i\,g}{2\sqrt{2}}\,(\gamma^\mu - \gamma^\mu\gamma_5)\,.$$

The masses are $m_t \approx 175$ GeV, $m_b \approx 5$ GeV and $M_W \approx 80$ GeV.

Exercise 5

Show that the coupling to the Higgs (W^+W^-h, ZZh, hhh and e^+e^-h) are proportional to the mass squared (bosons) or mass (fermions) of the particles. Note that you can find the answer without explicit construction of the interaction terms lagrangian.

Exercise 6

Check that the Wolfenstein parametrization of the CKM matrix respects unitarity up to the required (which?) order in λ.

Exercise 7

In this exercise two limits are investigated for the two-Majorana case.

(a) Calculate for the special choice $M_L = M_R = 0$ and M_D real, the mass eigenvalues and show that the mixing matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

which enables one to rewrite the Dirac field in terms of Majorana spinors. Give the explicit expressions that relate ψ and ψ^c with Υ_1 and Υ_2.
(solution)
One finds $M_1 = M_2 = M_D$. For both left- and righthanded fields the relations between ψ, ψ^c and Υ_1 and Υ_2 are the same,

$$\psi = \frac{1}{\sqrt{2}} (\Upsilon_1 + i\,\Upsilon_2), \qquad \psi^c = \frac{1}{\sqrt{2}} (\Upsilon_1 - i\,\Upsilon_2).$$

(b) A more interesting situation is $0 = M_L < |M_D| \ll M_R$, which leads to the socalled seesaw mechanism. Calculate the eigenvalues $M_L = 0$ and $M_R = M_X$. Given that neutrino masses are of the order of 1/20 eV, what is the mass M_X if we take for M_D the electroweak symmetry breaking scale v (about 250 GeV).
(solution)
The eigenvalues are $M_1 \approx M_D^2/M_X\sqrt{2}$ and $M_2 \approx M$. For a neutrino mass of the order of 1/20 eV, and a fermion mass of the order of the electroweak breaking scaling 250 GeV, this leads to $M_X \sim 10^{15}$ GeV. The recoupling matrix in this case is

$$U = \begin{pmatrix} i\,\cos\theta_S & -i\,\sin\theta_S \\ \sin\theta_S & \cos\theta_S \end{pmatrix},$$

with $\sin\theta_S \approx M_D/M_X$. The weak current couples to $n_L = \sin\theta_S\,\Upsilon_2 - i\,\cos\theta_S\,\Upsilon_1$, where Υ_1 is the light neutrino (mass) eigenstate.

A kinematics in scattering processes

Phase space

The 1-particle state is denoted $|p\rangle$. It is determined by the energy-momentum four vector $p = (E, \boldsymbol{p})$ which satisfies $p^2 = E^2 - \boldsymbol{p}^2 = m^2$. A physical state has positive energy. The phase space is determined by the weight factors assigned to each state in the summation or integration over states, i.e. the 1-particle phase space is

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3\, 2E} = \int \frac{d^4\boldsymbol{p}}{(2\pi)^4}\, \theta(p^0)\, (2\pi)\delta(p^2 - m^2). \tag{112}$$

This is generalized to the multi-particle phase space

$$d\mathcal{R}(p_1, \ldots, p_n) = \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_i}, \tag{113}$$

and the *reduced phase space element* by

$$d\mathcal{R}(s, p_1, \ldots, p_n) = (2\pi)^4\, \delta^4(P - \sum_i p_i)\, d\mathcal{R}(p_1, \ldots, p_n), \tag{114}$$

which is useful because the total 4-momentum of the final state usually is fixed by overall momentum conservation. Here s is the invariant mass of the n-particle system, $s = (p_1 + \ldots + p_n)^2$. It is a useful quantity, for instance for determining the threshold energy for the production of a final state $1 + 2 + \ldots + n$. In the CM frame the threshold value for s obviously is

$$s_{\text{threshold}} = \left(\sum_{i=1}^{n} m_i \right)^2 . \tag{115}$$

For two particle states $|p_a, p_b\rangle$ we start with the four vectors $p_a = (E_a, \boldsymbol{p}_a)$ and $p_b = (E_b, \boldsymbol{p}_b)$ satisfying $p_a^2 = m_a^2$ and $p_b^2 = m_b^2$, and the total momentum four-vector $P = p_a + p_b$. For two particles, the quantity

$$s = P^2 = (p_a + p_b)^2, \tag{116}$$

is referred to as the invariant mass squared. Its square root, $\sqrt{s}$ is for obvious reasons known as the center of mass (CM) energy.

To be specific let us consider two frequently used frames. The first is the CM system. In that case

$$p_a = (E_a^{\text{cm}}, \boldsymbol{q}), \tag{117}$$

$$p_b = (E_b^{\text{cm}}, -\boldsymbol{q}), \tag{118}$$

It is straightforward to prove that the unknowns in the particular system can be expressed in the invariants (m_a, m_b and s). Prove that

$$|\boldsymbol{q}| = \sqrt{\frac{(s - m_a^2 - m_b^2)^2 - 4\, m_a^2 m_b^2}{4s}} = \sqrt{\frac{\lambda(s, m_a^2, m_b^2)}{4s}}, \tag{119}$$

$$E_a^{\text{cm}} = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \tag{120}$$

$$E_b^{\text{cm}} = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}. \tag{121}$$

The function $\lambda(s, m_a^2, m_b^2)$ is a function symmetric in its three arguments, which in the specific case also can be expressed as $\lambda(s, m_a^2, m_b^2) = 4(p_a \cdot p_b)^2 - 4p_a^2\, p_b^2$.

The second frame considered explicitly is the socalled target rest frame in which one of the particles (called the target) is at rest. In that case

$$p_a = (E_a^{\rm trf}, \boldsymbol{p}_a^{\rm trf}), \tag{122}$$
$$p_b = (m_b, \boldsymbol{0}), \tag{123}$$

Also in this case one can express the energy and momentum in the invariants. Prove that

$$E_a^{\rm trf} = \frac{s - m_a^2 - m_b^2}{2m_b}, \tag{124}$$
$$|\boldsymbol{p}_a^{\rm trf}| = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2m_b}. \tag{125}$$

One can, for instance, use the first relation and the abovementioned threshold value for s to calculate the threshold for a specific n-particle final state in the target rest frame,

$$E_a^{\rm trf}(\text{threshold}) = \frac{1}{2\,\mathrm{m_b}}\left((\sum_{\mathrm{i}} \mathrm{m_i})^2 - \mathrm{m_a^2} - \mathrm{m_b^2}\right). \tag{126}$$

Explicit calculation of the reduced two-body phase space element gives

$$\begin{aligned} d\mathcal{R}(s, p_1, p_2) &= \frac{1}{(2\pi)^2}\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\delta^4(P - p_1 - p_2) \\ &\stackrel{\rm CM}{=} \frac{1}{(2\pi)^2}\frac{d^3q}{4\,E_1\,E_2}\delta(\sqrt{s} - E_1 - E_2) \\ &= \frac{1}{(2\pi)^2}\,d\Omega(\hat{q})\frac{\boldsymbol{q}^2\,d|\boldsymbol{q}|}{4\,E_1\,E_2}\delta(\sqrt{s} - E_1 - E_2) \end{aligned}$$

which using $|\boldsymbol{q}|\,d|\boldsymbol{q}| = E_1\,dE_1 = E_2\,dE_2$ gives

$$\begin{aligned} d\mathcal{R}(s, p_1, p_2) &= \frac{|\boldsymbol{q}|}{(2\pi)^2}\,d\Omega(\hat{q})\frac{d(E_1 + E_2)}{4(E_1 + E_2)}\delta(\sqrt{s} - E_1 - E_2) \\ &= \frac{|\boldsymbol{q}|}{4\pi\sqrt{s}}\,\frac{d\Omega(\hat{q})}{4\pi} = \frac{\sqrt{\lambda_{12}}}{8\pi\,s}\,\frac{d\Omega(\hat{q})}{4\pi}, \end{aligned} \tag{127}$$

where λ_{12} denotes $\lambda(s, m_1^2, m_2^2)$.

Kinematics of 2 → 2 scattering processes

The simplest scattering process is 2 particles in and 2 particles out. Examples appear in

$$\pi^- + p \quad \rightarrow \quad \pi^- + p \tag{128}$$
$$\rightarrow \quad \pi^0 + n \tag{129}$$
$$\rightarrow \quad \pi^+ + \pi^- + n \tag{130}$$
$$\rightarrow \quad \ldots\,. \tag{131}$$

The various possibilities are referred to as different reaction channels, where the first is referred to as elastic channel and the set of all other channels as the inelastic channels. Of course there are not only 2-particle channels. The initial state, however, usually is a 2-particle state, while the final state often arises from a series of 2-particle processes combined with the decay of an intermediate particle (resonance).

Consider the process $a+b \to c+d$. An often used set of invariants are the Mandelstam variables,

$$s = (p_a + p_b)^2 = (p_c + p_d)^2 \tag{132}$$
$$t = (p_a - p_c)^2 = (p_b - p_d)^2 \tag{133}$$
$$u = (p_a - p_d)^2 = (p_b - p_c)^2 \tag{134}$$

which are not independent as $s+t+u = m_a^2 + m_b^2 + m_c^2 + m_d^2$. The variable s is always larger than the minimal value $(m_a + m_b)^2$. A specific reaction channel starts contributing at the threshold value (Eq. 115). Instead of the scattering angle, which for the above $2 \to 2$ process in the case of azimuthal symmetry is defined as $\hat{\boldsymbol{p}}_a \cdot \hat{\boldsymbol{p}}_c = \cos\theta$ one can use in the CM the invariant

$$t \equiv (p_a - p_c)^2 \stackrel{\rm CM}{=} m_a^2 + m_c^2 - 2\,E_a E_c + 2\,q q'\,\cos\theta_{\rm cm},$$

with $q = \sqrt{\lambda_{ab}}/2\sqrt{s}$ and $q' = \sqrt{\lambda_{cd}}/2\sqrt{s}$. The minimum and maximum values for t correspond to $\theta_{\rm cm}$ being 0 or 180 degrees,

$$\begin{aligned} t_{\rm min}^{\rm max} &= m_a^2 + m_c^2 - 2\,E_a E_c \pm 2\,q q' \\ &= m_a^2 + m_c^2 - \frac{(s + m_a^2 - m_b^2)(s + m_c^2 - m_d^2)}{2\,s} \pm \frac{\sqrt{\lambda\lambda'}}{2\,s}. \end{aligned} \tag{135}$$

Using the relation between t and $\cos\theta_{\rm cm}$ it is straightforward to express $d\Omega_{\rm cm}$ in dt, $dt = 2\,qq'\,d\cos\theta_{\rm cm}$ and obtain for the two-body phase space element

$$d\mathcal{R}(s, p_c, p_d) = \frac{q'}{4\pi\sqrt{s}}\,\frac{d\Omega_{\rm cm}}{4\pi} = \frac{\sqrt{\lambda_{cd}}}{8\pi\,s}\,\frac{d\Omega_{\rm cm}}{4\pi} \tag{136}$$
$$= \frac{dt}{8\pi\,\sqrt{\lambda_{ab}}} = \frac{dt}{16\pi\,q\,\sqrt{s}}. \tag{137}$$

B Cross sections and lifetimes

Scattering process

For a scattering process $a+b \to c+\ldots$ (consider for convenience the rest frame for the target, say b) the cross section $\sigma(a+b \to c+\ldots)$ is defined as the proportionality factor in

$$\frac{N_c}{T} = \sigma(a+b \to c+\ldots)\cdot N_b \cdot \mathrm{flux}(a),$$

where V and T indicate the volume and the time in which the experiment is performed, N_c/T indicates the number of particles c detected in the scattering process, N_b indicates the number of (target) particles b, which for a density ρ_b is given by $N_b = \rho_b \cdot V$, while the flux of the beam particles a is $\mathrm{flux}(a) = \rho_a \cdot v_a^{\rm trf}$. The proportionality factor has the dimension of area and is called the cross section, i.e.

$$\sigma = \frac{N}{T\cdot V}\,\frac{1}{\rho_a \rho_b\, v_a^{\rm trf}}. \tag{138}$$

Although this at first sight does not look covariant, it is. N and $T\cdot V$ are covariant. Using $\rho_a^{\rm trf} = \rho_a^{(0)}\cdot\gamma_a = \rho_a^{(0)}\cdot E_a^{lab}/m_a$ (where $\rho_a^{(0)}$ is the rest frame density) and $v_a^{lab} = p_a^{lab}/E_a^{lab}$ we have

$$\rho_a \rho_b\, v_a^{\rm trf} = \frac{\rho_a^{(0)}\rho_b^{(0)}}{4\,m_a m_b}\,2\sqrt{\lambda_{ab}}$$

or with $\rho_a^{(0)} = 2\,m_a$,

$$\sigma = \frac{1}{2\sqrt{\lambda_{ab}}}\,\frac{N}{T\cdot V}. \tag{139}$$

Decay of particles

For the decay of particle a one has macroscopically

$$\frac{dN}{dt} = -\Gamma\, N, \tag{140}$$

i.e. the amount of decaying particles is proportional to the number of particles with proportionality factor the em decay width Γ. From the solution

$$N(t) = N(0)\, e^{-\Gamma\, t} \tag{141}$$

one knows that the decay time $\tau = 1/\Gamma$. Microscopically one has

$$\frac{N_{\text{decay}}}{T} = N_a \cdot \Gamma$$

or

$$\Gamma = \frac{N}{T \cdot V}\,\frac{1}{\rho_a}. \tag{142}$$

This quantity is not covariant, as expected. The decay time for moving particles τ is related to the decay time in the rest frame of that particle (the proper decay time τ_0) by $\tau = \gamma\,\tau_0$. For the (proper) decay width one thus has

$$\Gamma_0 = \frac{1}{2\,m_a}\,\frac{N}{T \cdot V}. \tag{143}$$

Fermi's Golden Rule

In both the scattering cross section and the decay constant the quantity N/TV appears. For this we employ in essence *Fermi's Golden rule* stating that when the S-matrix element is written as

$$S_{fi} = \delta_{fi} - (2\pi)^4\,\delta^4(P_i - P_f)\, i\mathcal{M}_{fi} \tag{144}$$

(in which we can calculate $-i\mathcal{M}_{fi}$ using Feynman diagrams), the number of scattered or decayed particles is given by

$$N = \left|(2\pi)^4\,\delta^4(P_i - P_f)\, i\mathcal{M}_{fi}\right|^2\, d\mathcal{R}(p_1, \ldots, p_n). \tag{145}$$

One of the δ functions can be rewritten as $T \cdot V$ (remember the normalization of plane waves),

$$\begin{aligned}
&\left|(2\pi)^4\,\delta^4(P_i - P_f)\right|^2 \\
&\qquad = (2\pi)^4\,\delta^4(P_i - P_f) \int_{V,T} d^4x\, e^{i(P_i - P_f)\cdot x} \\
&\qquad = (2\pi)^4\,\delta^4(P_i - P_f) \int_{V,T} d^4x = V \cdot T\,(2\pi)^4\,\delta^4(P_i - P_f).
\end{aligned}$$

(Using normalized wave packets these somewhat ill-defined manipulations can be made more rigorous). The result is

$$\frac{N}{T \cdot V} = |\mathcal{M}_{fi}|^2\, d\mathcal{R}(s, p_1, \ldots, p_n). \tag{146}$$

Combining this with the expressions for the width or the cross section one obtains for the decay width

$$\Gamma \quad = \frac{1}{2m} \int d\mathcal{R}(m^2, p_1, \ldots, p_n)\, |\mathcal{M}|^2 \tag{147}$$

$$\stackrel{\text{2−body decay}}{=} \frac{q}{32\pi^2\, m^2} \int d\Omega\, |M|^2. \tag{148}$$

The *differential cross section* (final state not integrated over) is given by

$$d\sigma = \frac{1}{2\sqrt{\lambda_{ab}}}\,|\mathcal{M}_{fi}|^2\,d\mathcal{R}(s,p_1,\ldots,p_n), \tag{149}$$

and for instance for two particles

$$d\sigma = \frac{q'}{q}\left|\frac{\mathcal{M}(s,\theta_{\rm cm})}{8\pi\sqrt{s}}\right|^2 d\Omega_{\rm cm} = \frac{\pi}{\lambda_{ab}}\left|\frac{\mathcal{M}(s,t)}{4\pi}\right|^2 dt. \tag{150}$$

This can be used to get the full expression for $d\sigma/dt$ for $e\mu$ and e^+e^- scattering, for which the amplitudes squared have been calculated in the previous chapter. The amplitude $-\mathcal{M}/8\pi\sqrt{s}$ is the one to be compared with the quantum mechanical scattering amplitude $f(E,\theta)$, for which one has $d\sigma/d\Omega = |f(E,\theta)|^2$. The sign difference comes from the (conventional) sign in relation between S and quantummechanical and relativistic scattering amplitude, respectively.

C Unitarity condition

The unitarity of the S-matrix, i.e.

$$(S^\dagger)_{fn}\,S_{ni} = \delta_{fi}$$

implies for the scattering matrix $\mathcal{M}$,

$$\left[\delta_{fn} + i(2\pi)^4\,\delta^4(P_f - P_n)\,(\mathcal{M}^\dagger)_{fn}\right]\left[\delta_{ni} - i(2\pi)^4\,\delta^4(P_i - P_n)\,\mathcal{M}_{ni}\right] = \delta_{fi},$$

or

$$-i\left[\mathcal{M}_{fi} - (\mathcal{M}^\dagger)_{fi}\right] = -\sum_n (\mathcal{M}^\dagger)_{fn}\,(2\pi)^4\,\delta^4(P_i - P_n)\,\mathcal{M}_{ni}. \tag{151}$$

Since the amplitudes also depend on all momenta the full result for two-particle intermediate states is (in CM, see 136)

$$-i\left[\mathcal{M}_{fi} - (\mathcal{M}^\dagger)_{fi}\right] = -\sum_n \int d\Omega(\hat{q}_n)\mathcal{M}^*_{nf}(\boldsymbol{q}_f,\boldsymbol{q}_n)\,\frac{q_n}{16\pi^2\sqrt{s}}\mathcal{M}_{ni}(\boldsymbol{q}_i,\boldsymbol{q}_n). \tag{152}$$

Partial wave expansion

Often it is useful to make a partial wave expansion for the amplitude $\mathcal{M}(s,\theta)$ or $\mathcal{M}(\boldsymbol{q}_i,\boldsymbol{q}_f)$,

$$\mathcal{M}(s,\theta) = -8\pi\sqrt{s}\sum_\ell (2\ell+1)M_\ell(s)\,P_\ell(\cos\theta), \tag{153}$$

(in analogy with the expansion for $f(E,\theta)$ in quantum mechanics; note the sign and $\cos\theta = \hat{q}_i\cdot\hat{q}_f$). Inserted in the unitarity condition for $\mathcal{M}$,

$$i\left[\frac{\mathcal{M}}{8\pi\sqrt{s}} - \frac{\mathcal{M}^\dagger}{8\pi\sqrt{s}}\right]_{fi} = \sum_n \int d\Omega_n\,\frac{\mathcal{M}^*_{nf}}{8\pi\sqrt{s}}\,\frac{q_n}{2\pi}\,\frac{\mathcal{M}_{ni}}{8\pi\sqrt{s}},$$

we obtain

$$\text{LHS} = -i\sum_\ell (2\ell+1)\,P_\ell(\hat{q}_i\cdot\hat{q}_f)\,\left((M_\ell)_{fi} - (M_\ell^\dagger)_{fi}\right),$$

while for the RHS use is made of

$$P_\ell(\hat{q}\cdot\hat{q}') = \sum_m \frac{4\pi}{2\ell+1}\,Y^{(\ell)}_m(\hat{q})\,Y^{(\ell)*}_m(\hat{q}')$$

and the orthogonality of the $Y_m^{(\ell)}$ functions to prove that

$$\mathrm{RHS} = 2\sum_n \sum_\ell (2\ell+1) P_\ell(\hat{q}_i \cdot \hat{q}_f)\,(M_\ell^\dagger)_{fn}\, q_n\,(M_\ell)_{ni},$$

i.e.

$$-i\left((M_\ell)_{fi} - (M_\ell^\dagger)_{fi}\right) = 2\sum_n (M_\ell^\dagger)_{fn}\, q_n\,(M_\ell)_{ni}. \tag{154}$$

If only one channel is present this simplifies to

$$-i\,(M_\ell - M_\ell^*) = 2\,q M_\ell^*\, M_\ell, \tag{155}$$

or $Im\, M_\ell = q\,|M_\ell|^2$, which allows writing

$$M_\ell(s) = \frac{S_\ell(s) - 1}{2i\,q} = \frac{e^{2i\,\delta_\ell(s)} - 1}{2i\,q}, \tag{156}$$

where $S_\ell(s)$ satisfies $|S_\ell(s)| = 1$ and $\delta_\ell(s)$ is the phase shift.

In general a given channel has $|S_\ell(s)| \le 1$, parametrized as $S_\ell(s) = \eta_\ell(s)\,\exp(2i\,\delta_\ell(s))$. Using

$$\sigma = \int d\Omega\;\frac{q'}{q}\left|\frac{\mathcal{M}(s,\theta_{\mathrm{cm}}}{)}8\pi\,\sqrt{s}\right|^2,$$

in combination with the partial wave expansion for the amplitudes $\mathcal{M}$ and the orthogonality of the Legendre polynomials immediately gives for the elastic channel,

$$\begin{aligned}\sigma_{\mathrm{el}} &= 4\pi\sum_\ell (2\ell+1)|M^{(\ell)}(s)|^2 \\ &= \frac{4\pi}{q^2}\sum_\ell (2\ell+1)\left|\frac{\eta_\ell\, e^{\,2i\,\delta_\ell} - 1}{2i}\right|^2,\end{aligned} \tag{157}$$

and for the case that this is the only channel (purely elastic scattering, $\eta = 1$) the result

$$\sigma_{\mathrm{el}} = \frac{4\pi}{q^2}\sum_\ell (2\ell+1)\,\sin^2\delta_\ell. \tag{158}$$

From the imaginary part of $\mathcal{M}(s,0)$, the total cross section can be determined. Show that

$$\begin{aligned}\sigma_T &= \frac{4\pi}{q}\sum_\ell (2\ell+1)\,Im\,M^{(\ell)}(s) \\ &= \frac{2\pi}{q^2}\sum_\ell (2\ell+1)\,(1 - \eta_\ell\,\cos 2\delta_\ell).\end{aligned} \tag{159}$$

The difference is the inelastic cross section,

$$\sigma_{\mathrm{inel}} = \frac{\pi}{q^2}\sum_\ell (2\ell+1)\,(1-\eta_\ell^2). \tag{160}$$

Note that the total cross section is maximal in the case of full absorption, $\eta = 0$, in which case, however, $\sigma_{\mathrm{el}} = \sigma_{\mathrm{inel}}$.

We note that unitarity is generally broken in a finite order calculation. Relating Im$\mathcal{M}$ and $|\mathcal{M}|^2$ we obtain relations between terms at different order in the coupling constant.

D Unstable particles

For a stable particle the propagator is

$$i\Delta(k) = \frac{i}{k^2 - M^2 + i\epsilon} \tag{161}$$

(note that we have disregarded spin). The prescription for the pole structure, i.e. one has poles at k^0 $= \pm(E - i\epsilon)$ where $E = +\sqrt{\boldsymbol{k}^2 + M^2}$ guarantees the correct behavior, specifically one has for $t > 0$ that the Fourier transform is

$$\int dk^0\; e^{-ik^0 t}\,\Delta(k) \propto \int dk^0\; \frac{e^{-ik^0 t}}{(k^0 - E + i\epsilon)(k^0 + E - i\epsilon)} \stackrel{(t>0)}{\propto} e^{-iEt},$$

i.e. $\propto U(t,0)$, the time-evolution operator. For an unstable particle one expects that

$$U(t,0) \propto e^{-i(E - i\Gamma/2)t},$$

such that $|U(t,0)|^2 = e^{-\Gamma t}$. This is achieved with a propagator

$$i\Delta_R(k) = \frac{i}{k^2 - M^2 + iM\Gamma} \tag{162}$$

(again disregarding spin). The quantity Γ is precisely the width for unstable particles. This is (somewhat sloppy!) seen by considering the (amputated) 1PI two-point vertex

$$\Gamma^{(2)} = \frac{-i}{\Delta}$$

as the amplitude $-i\mathcal{M}$ for scattering a particle into itself through the decay channels as intermediate states. The unitarity condition then states

$$\begin{aligned} 2\,Im\,\Delta_R^{-1}(k) &= \sum_n \int d\mathcal{R}(p_1,\ldots,p_n)\mathcal{M}^\dagger_{Rn}\,(2\pi)^4\,\delta^4(k - P_n)\,\mathcal{M}_{nR} \\ &= 2M\sum_n \Gamma_n = 2M\,\Gamma. \end{aligned} \tag{163}$$

This shows that Γ is the width of the resonance, which is given by a sum of the partial widths into the different channels. It is important to note that the physical width of a particle is the imaginary part of the two-point vertex at $s = M^2$.

For the amplitude in a scattering process going through a resonance, it is straightforward to write down the partial wave amplitude,

$$(q\,M_\ell)_{ij}(s) = \frac{-M\sqrt{\Gamma_i\Gamma_j}}{s - M^2 + iM\Gamma}. \tag{164}$$

(Prove this using the unitarity condition for partial waves). From this one sees that a resonance has the same shape in all channels but different strength. Limiting ourselves to a resonance in one channel, it is furthermore easy to prove that the cross section is given by

$$\sigma_{\rm el} = \frac{4\pi}{q^2}\,(2\ell + 1)\,\frac{M^2\Gamma^2}{(s - M^2)^2 + M^2\Gamma^2}, \tag{165}$$

reaching the unitarity limit for $s = M^2$, where furthermore $\sigma_{\rm inel} = 0$. This characteristic shape of a resonance is called the Breit-Wigner shape. The half-width of the resonance is $M\Gamma$. The phase shift in the resonating channel near the resonance is given by

$$\tan\delta_\ell(s) = \frac{M\Gamma}{M^2 - s}, \tag{166}$$

showing that the phase shift at resonance rises through $\delta = \pi/2$ with a 'velocity' $\partial\delta/\partial s = 1/M\Gamma$, i.e. a fast change in the phase shift for a narrow resonance. Note that because of the presence of a background the phase shift at resonance may actually be shifted.
Three famous resonances are:

- The Δ-resonance seen in pion-nucleon scattering. Its mass is $M = 1232$ MeV, its width $\Gamma = 120$ MeV. At resonance the cross section $\sigma_T(\pi^+ p)$ is about 210 mb. The cross section $\sigma_T(\pi^- p)$ also shows a resonance with the same width with a value of about 70 mb. This implies that the resonance has spin $J = 3/2$ (decaying in a P-wave ($\ell = 1$) pion-nucleon state) and isospin $I = 3/2$ (the latter under the assumption that isospin is conserved for the strong interactions).

- The J/ψ resonance in e^+e^- scattering. This is a narrow resonance discovered in 1974. Its mass is $M = 3096.88$ MeV, the full width is $\Gamma = 88$ keV, the partial width into e^+e^- is $\Gamma_{ee} = 5.26$ keV.

- The Z^0 resonance in e^+e^- scattering with $M = 91.2$ GeV, $\Gamma = 2.49$ GeV. Essentially this resonance can decay into quark-antiquark pairs or into pairs of charged leptons. All these decays can be seen and leave an 'invisible' width of 498 MeV, which is attributed to neutrinos. Knowing that each neutrino contributes about 160 MeV, one can reconstruct the resonance shape for different numbers of neutrino species. Three neutrinos explain the resonance shape. The cross section at resonance is about 30 nb.