Quantum Chromodynamics and Hard Scattering Processes

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Contents

1 Introduction **101**

2 Quantum Chromodynamics **201**
2.1 Quantum Field Theories . . . 201
2.1.1 Action and Lagrangian density . . . 201
2.1.2 Field Quantization . . . 202
2.1.3 The Generating Functional . . . 203
2.1.4 Feynman rules . . . 205
2.1.5 Scattering amplitudes . . . 207
2.2 Scattering theory . . . 208
2.2.1 kinematics in scattering processes . . . 208
2.2.2 Cross sections and lifetimes . . . 210
2.3 Quantization of QCD . . . 212
2.3.1 Quantization of nonabelian gauge theories . . . 212
2.3.2 Feynman rules for QCD . . . 214
2.3.3 Playing with color . . . 216
2.4 Renormalization . . . 218
2.4.1 Divergencies . . . 218
2.4.2 Regularization . . . 219
2.4.3 Renormalization schemes . . . 220
2.5 The renormalization group . . . 222
2.5.1 The renormalization group equations (RGE) . . . 222
2.5.2 Asymptotic solutions of the RGE. . . . 224
2.5.3 The running coupling constant . . . 226
2.6 Composite operators . . . 227

3 Models for hadrons **301**
3.1 Introduction . . . 301
3.2 Flavor and spin structure of hadrons . . . 301
3.3 Properties of hadrons and form factors . . . 303
3.4 Quark potential models . . . 305
3.5 The bag model . . . 306
3.6 Chiral lagrangians and the Skyrme model . . . 309

4 ELECTROWEAK SCATTERING PROCESSES **401**
4.1 The hadronic tensor in lepton-hadron scattering . . . 402
4.2 Scaling variables in lepton-hadron scattering . . . 403
4.3 Elastic lepton-hadron scattering . . . 405
4.3.1 Elastic scattering off point particles . . . 407
4.3.2 Elastic scattering off composite particles . . . 407

4.4 Inclusive inelastic lepton scattering processes . . . 408
4.5 Symmetry properties of the structure functions . . . 410
4.6 Electroweak lepton - hadron interactions . . . 412
4.6.1 Charged current cross sections . . . 413
4.6.2 Neutral current cross sections . . . 414
4.7 Polarized inclusive lepton - hadron scattering . . . 415
4.7.1 Asymmetry in $\vec{e}$ - hadron scattering . . . 415
4.7.2 Inclusive $\vec{e}$ - $\vec{p}$ scattering . . . 416
4.8 Generalization of the expansion of structure functions . . . 417
4.9 The meaning of spin vectors . . . 418
4.10 Scaling variables in lepton-hadron scattering . . . 419
4.11 Electron-positron annihilation . . . 420
4.11.1 The hadronic tensor . . . 420
4.11.2 Scaling variables for e^+e^- annihilation . . . 421
4.12 The Drell-Yan process . . . 422
4.12.1 The hadronic tensor . . . 422
4.12.2 Scaling variables for Drell-Yan scattering . . . 423

5 THEORETICAL ANALYSIS OF LEPTON-HADRON SCATTERING 501
5.1 Introduction . . . 501
5.2 The principle of scaling . . . 501
5.3 The parton model . . . 502
5.3.1 Photon-parton cross sections . . . 502
5.3.2 Diagrammatic approach . . . 504
5.4 Parton distributions in electroweak processes . . . 505
5.5 Sum rules . . . 508
5.6 The operator product expansion . . . 510
5.7 QCD corrections in deep inelastic scattering . . . 513
5.8 Evolution equations for the structure functions . . . 514
5.9 Solutions of the evolution equations . . . 518
5.9.1 Inverse Mellin transforms of moments . . . 519

Chapter 1

Introduction

Quantum Chromodynamics (QCD) is the field theory based on a nonabelian gauge group $SU(N_c)$ with $N_c = 3$. This theory underlies the strong interactions of which the existence is known since the thirties. At that time the goal was to understand the forces in the atomic nucleus. The experimental results of many experiments revealed the existence of a rich spectrum of hadrons, baryons and mesons. The presence of these excitation spectra and also explicit measurements of the charge and current distributions of the hadrons revealed the composite nature of hadrons. The quark model brought some order into this situation, describing baryons as composite systems of three valence quarks and mesons as a composite system of a valence quark and antiquark. Valence refers here to the fact that these quarks are the contributors to the quantum numbers (upness, downness, strangeness, etc.) of the hadrons. The symmetry considerations based on flavor $SU(3)$ and flavor-spin $SU(6)$ are the basis of the success of the quark model. Color was introduced in a rather early stage to solve a number of problems such as the fermion statistics of the quarks. The experimental results of the SLAC-MIT deep inelastic electron scattering experiments, which indicated the existence of hard 'point-like' constituents in the nucleon and the field-theoretical developments in nonabelian gauge theories, specifically the notion of asymptotic freedom and the proof of renormalizability, led to the natural emergence of QCD as the theory for the strong interactions between the quarks and gluons. The important feature of QCD is the fact that the force becomes weaker at short distances. This anti-screening behavior or asymptotic freedom is a unique feature of non-abelian gauge theories.

At present Quantum Chromodynamics is well-established as the theory of the strong interactions, describing the interactions between the colored quarks and gluons. As a consequence, this quantum field theoretical framework with all of its complexities such as renormalization, anomalies and nonperturbative aspects should be taken seriously.

QCD and the Standard Model.
QCD is part of the Standard Model that describes the strong and electroweak forces. All forces in the Standard Model are described within the framework of nonabelian gauge theories, based on a gauge symmetry. It is tempting to unify the gauge group to a simpler group than the present direct product of $SU(3)_C \otimes SU(2)_W \otimes U(1)_{Y_W}$, with $SU(5)$ and $SO(10)$ offering interesting possibilities. For instance, the present accurate knowledge of the strong coupling constant, $\alpha_s(M_Z) \approx 0.115(5)$, puts constraints on the possible physics scenarios between the 100 GeV scale (10^{-17} m) and physics at the unification scale for strong and electroweak interactions around $10^{15\pm1}$ GeV (10^{-30} m).

Tests of QCD.
For testing QCD one preferably avoids as much as possible the presence of hadrons, as these constitute complex bound state systems. Useful processes therefore are

$$\begin{aligned} e^+e^- &\longrightarrow X\ (inclusive), \\ e^+e^- &\longrightarrow jets\ (semi-inclusive), \end{aligned}$$

$$\ell H \quad \longrightarrow \quad \ell' X \ (inclusive\ leptoproduction).$$

Analyzing jets produced in e^+e^- scattering represent, modulo complications arising from hadronization, hard scattering processes $e^+e^- \to q\bar{q}$, $e^+e^- \to q\bar{q}g$, etc. For these processes perturbative QCD can be used to do reliable calculations. Similarly one can use perturbative QCD to study the $\ln Q^2$ dependence of the elementary $\ell q \to \ell' q$ cross sections that are important in leptoproduction. These elementary cross sections are target independent.

Structure of Hadrons.
The properties of hadrons are poorly described directly starting from QCD. They require a nonperturbative approach in quantum field theory. This has led to the use of models that incorporate some features like confinement and asymptotic freedom and symmetries of the underlying theory, such as quark potential models, bag models or chiral models. Most promising from a fundamental point of view are lattice gauge approaches.

Controlling and selectively probing the nonperturbative regime in high energy scattering processes is the key to study the structure of hadrons in the context of QCD. The control parameters for the target and the probe are the spin and flavor, which in combination with the kinematical flexibility in scattering processes is used to select the observable, its gluonic or quarkic nature. Examples are

$$\begin{aligned}
\ell H \quad &\longrightarrow \quad \ell' H \ (elastic) && form\ factors,\\
\ell H \quad &\longrightarrow \quad \ell' X \ (inclusive) && distribution\ functions,\\
\ell H \quad &\longrightarrow \quad \ell' h X \ (semi-inclusive) && distribution + fragmentation\ functions,\\
e^+e^- \quad &\longrightarrow \quad h X \ (semi-inclusive) && fragmentation functions,\\
H_1 H_2 \quad &\longrightarrow \quad \mu^+\mu^- X (Drell-Yan) && distribution\ functions.
\end{aligned}$$

These notes consist of four parts. The first part deals with quantum field theories aimed specifically at non-abelian gauge theories. The write-up of this part is not intended to be complete, but merely serves to give background information The write-up may also be useful to adjust to different notation, metric, etc. The second part reviews a few models used to attack the structure of hadrons. Next an introduction to electroweak scattering processes, cross sections, form factors, and structure functions is given. The last part deals with deep inelastic lepton-hadron scattering and aspects of QCD. Some general references are [1] - [18].

At the Dutch Research School for Theoretical Physics (9 hours) sections 4.6 - 4.8 and sections 5.4, 5.7 - 5.9 have not been covered.

At the Dutch Research School for Subatomic Physics (4 hours) a selection has been made consisting of sections 3.1 - 3.3, 4.1 - 4.5, 5.1 - 5.3 and section 5.5.

Chapter 2

Quantum Chromodynamics

2.1 Quantum Field Theories

2.1.1 Action and Lagrangian density

The central role is played by the action S [1],

$$S[\phi] = \int dx \;\; \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)), \tag{2.1}$$

where we use in this section dx as the shorthand for d^4x or more general d^Dx in four- or D-dimensional space-time. Specific examples of the lagrangian density $\mathcal{L}$ are,
for a **real scalar field** ϕ (neutral spin 0 particle):

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\,M^2\,\phi^2 = -\frac{1}{2}\,\phi\left(\partial_\mu\partial^\mu + M^2\right)\phi, \tag{2.2}$$

for a **complex scalar field** ϕ, ϕ^* (charged spin 0 particle):

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - M^2\,\phi^*\phi = -\phi^*\left(\partial_\mu\partial^\mu + M^2\right)\phi, \tag{2.3}$$

for a **Dirac field** ψ, $\overline{\psi} = \psi^\dagger\gamma^0$ (spin 1/2 particle),

$$\mathcal{L} = \frac{i}{2}\overline{\psi}\overset{\leftrightarrow}{\not\partial}\psi - M\,\overline{\psi}\psi = \overline{\psi}\,(i\not\partial - M)\,\psi, \tag{2.4}$$

where the γ-matrices satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\,g_{\mu\nu}$ and
for **vector field** V_μ (spin 1 particle),

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,M^2\,V_\mu V^\mu - \frac{\lambda}{2}\,(\partial_\mu V^\mu)^2 = \frac{1}{2}V^\mu\left[(\partial^2 + M^2)g_{\mu\nu} - (1-\lambda)\partial_\mu\partial_\nu\right]V^\nu, \tag{2.5}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. Finally we give the result for a general nonabelian gauge theory. If $\mathcal{L}_0(\phi, \partial_\mu\phi)$ is the lagrangian invariant under global transformations generated by the hermitean Lie-algebra with generators satisfying $[L_a, L_b] = ic_{abc}L_c$ [2],

$$\mathcal{L}_{\,GT} = \mathcal{L}_{\,0}(\phi, \underline{D}_\mu\phi) - \frac{1}{4}\,G^a_{\mu\nu}G^{\mu\nu\,a}, \tag{2.6}$$

[1] Note that we use units where $\hbar = c = 1$.
[2] We consider a semi-simple Lie group for which the structure constants c_{abc} are completely antisymmetric

with

$$\underline{D}_\mu\phi = \left(\partial_\mu \underline{1} - ig\underline{W}_\mu\right)\phi, \tag{2.7}$$

$$\underline{G}_{\mu\nu} = \partial_\mu \underline{W}_\nu - \partial_\nu \underline{W}_\mu - ig[\underline{W}_\mu, \underline{W}_\nu] = \frac{i}{g}[\underline{D}_\mu, \underline{D}_\nu]. \tag{2.8}$$

where the notation $\underline{W}_\mu \equiv W^a_\mu F_a$ with F_a being appropriate representation matrices of the Lie-group acting on the fields ϕ.

2.1.2 Field Quantization

The fields are quantized after introducing creation and annihilation operators acting in the space of particle momentum eigenstates, satisfying the well-known commutation (bosons) or anticommutation relations,
for **real scalar field** ϕ,

$$\phi(x) = \int d\tilde{k}\ \left[a(k)\, e^{-ik\cdot x} + a^\dagger(k)\, e^{ik\cdot x}\right], \tag{2.9}$$

where $d\tilde{k}$ is used as shorthand notation for $d^3k/((2\pi)^3\, 2E)$;
for **complex scalar field** ϕ, ϕ^*,

$$\phi(x) = \int d\tilde{k}\ \left[a(k)\, e^{-ik\cdot x} + b^\dagger(k)\, e^{ik\cdot x}\right]; \tag{2.10}$$

for **Dirac field** ψ, $\overline{\psi} = \psi^\dagger\gamma^0$,

$$\psi(x) = \sum_s \int d\tilde{k}\ \left[b_s(k)\, u(k;s)\, e^{-ik\cdot x} + d^\dagger_s(k)\, v(k;s)\, e^{ik\cdot x}\right]; \tag{2.11}$$

for **vector field** V_μ,

$$V_\mu(x) = \int d\tilde{k}\ \left[a_\lambda(k)\, \epsilon_\mu(k;\lambda)\, e^{-ik\cdot x} + a^\dagger_\lambda(k)\, \epsilon^*_\mu(k;\lambda)\, e^{ik\cdot x}\right]. \tag{2.12}$$

The wave functions multiplying the creation and annihilation operators are the solutions of the field equations,

$$(\partial^2 + M^2)\phi = 0, \tag{2.13}$$

$$(i\not{\partial} - M)\psi = 0, \tag{2.14}$$

$$\partial_\mu F^{\mu\nu} = 0 \quad (\text{and } \partial_\mu V^\mu = 0). \tag{2.15}$$

These are plane waves multiplied with momentum dependent wave functions exhibiting the Lorentz structure, being 1 for spin 0, $u(k,s)$ satisfying $(\not{k} - M)u = 0$ and $v(k,s)$ satisfying $(\not{k} + M)v = 0$ for spin 1/2, and $\epsilon(k,\lambda)$ satisfying $k^\mu\epsilon_\mu = 0$ for spin 1. Useful completeness relations are

$$\sum_s u(k,s)\bar{u}(k,s) = (\not{k} + M), \tag{2.16}$$

$$\sum_s v(k,s)\bar{v}(k,s) = (\not{k} - M), \tag{2.17}$$

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon^*_\nu(k,\lambda) = -g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}. \tag{2.18}$$

2.1.3 The Generating Functional

The basic quantity from which the scattering amplitude for physical processes will be derived is the vacuum to vacuum amplitude in the presence of a source term. It is given by a functional integration over all possible field configurations, $\mathcal{D}\phi = \prod_x d\phi(x)$,

$$\langle 0_{out}|0_{in}\rangle_J = Z[J] = \int \mathcal{D}\phi \ \exp\left(i\int_{T\to i\infty}^{T'\to -i\infty} dx\ [\mathcal{L}(\phi,\partial_\mu\phi) + J\phi]\right) \tag{2.19}$$

$$= \int \mathcal{D}\phi \ \exp\left(-\int_{t_{_E}=-\infty}^{t_{_E}=\infty} dx_{_E}\ [\mathcal{L}_{_E}(\phi, \frac{\partial\phi}{\partial x^\mu_{_E}}) - J\phi]\right), \tag{2.20}$$

where $\mathcal{L}_{_E}(\phi,\partial^E_\mu\phi) \equiv -\mathcal{L}(\phi,\partial_\mu\phi)$. The quantities indicated with the subscript E are in Euclidean space (see table 2.1), where the functional integral is well-behaved. In ordinary space-time the time integration in the action has to be analytically continued. The matrix element $\langle 0_{in}|\Phi_i\rangle \propto \exp(iE_0t)$ becomes well-behaved if one analytically continues the point $t \to -\infty$ to the point $T \to i\infty$. This corresponds to $t_{_E} = -\infty$.

Table 2.1: *Euclidean space coordinates*

$x^4 = ix^0$	$k^4 = -ik^0$
$x^2 = (x^0)^2 - \boldsymbol{x}^2 = -\sum_{\mu=1}^4 (x^\mu)^2 = -x^2_{_E}$	$k^2 = (k^0)^2 - \boldsymbol{k}^2 = -\sum_{\mu=1}^4 (k^\mu)^2 = -k^2_{_E}$
$d^4x_{_E} = i\, d^4x$	$d^4k_{_E} = -i\, d^4k$

The functional derivatives of the generating functional $Z[J] = Z_{_E}[J]$ are equal to the time ordered products of the fields (Green's functions). I.e., if

$$Z[J] = \sum_n \frac{i^n}{n!}\int dx_1\ldots dx_n\ G^{(n)}(x_1,\ldots,x_n)\, J(x_1)\ldots J(x_n) \tag{2.21}$$

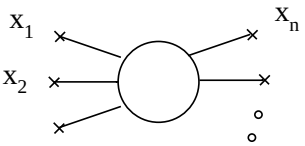
$$= \sum_n \frac{1}{n!}\int dx_{_E1}\ldots dx_{_En}\ G^{(n)}(x_1,\ldots,x_n)\, J(x_1)\ldots J(x_n), \tag{2.22}$$

the *Green's functions* are given by

$$G^{(n)}(x_1,\ldots,x_n) = (-i)^n\ \frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} = \frac{\delta^n Z[J]}{\delta J(x_{_E1})\ldots\delta J(x_{_En})}\bigg|_{J=0}$$

$$= \langle 0|\mathcal{T}\phi(x_1)\ldots\phi(x_n)|0\rangle \tag{2.23}$$

or pictorially

$G^{(n)}(x_1,\ldots,x_n) =$ 

Dividing by the result in the absence of sources one obtains $Z[J]/Z[0]$, which is expanded in terms of *source conncected* Green's functions $G_{sc}(x_1,\ldots x_n)$.

Taking the logarithm of $Z[J]$, one obtains the connected Green's functions, which appear in the expansion of

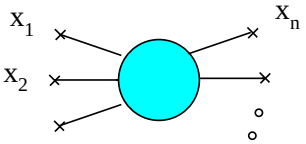
$$\ln\left(\frac{Z[J]}{Z[0]}\right) = \sum_n \frac{i^n}{n!}\int dx_1\ldots dx_n\; G_c^{(n)}(x_1,\ldots,x_n)\,J(x_1)\ldots J(x_n) \tag{2.24}$$

$$= \sum_n \frac{1}{n!}\int dx_{_E 1}\ldots dx_{_E n}\; G_c^{(n)}(x_1,\ldots,x_n)\,J(x_1)\ldots J(x_n), \tag{2.25}$$

with the *connected Green's functions*

$$G_c^{(n)}(x_1,\ldots,x_n) = \frac{(-i)^n}{Z[0]}\;\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} = \frac{1}{Z[0]}\;\frac{\delta^n Z[J]}{\delta J(x_{_E 1})\ldots\delta J(x_{_E n})}\bigg|_{J=0}$$

$$\equiv \langle 0|\mathcal{T}\phi(x_1)\ldots\phi(x_n)|0\rangle_c \equiv \tag{2.26}$$



For the free field theories one can calculate the generating functional. For the scalar field,

$$S[\phi] = -\frac{1}{2}\int dx\,dy\;\phi(x)\,K(x,y)\,\phi(y), \tag{2.27}$$

with $K(x,y) = (\partial^2 + M^2)\,\delta(x-y)$. Then

$$Z_0[J] = \int \mathcal{D}\phi\;\exp\left(iS[\phi] + i\int dx\,J\phi\right)$$

$$= Z_0[0]\,\exp\left(-\frac{1}{2}\int dx\,dy\;J(x)\,i\Delta_F(x-y)\,J(y)\right), \tag{2.28}$$

where $K(x,y)\Delta_F(x-y) = -\delta(x-y)$.

In the well-defined Euclidean space this becomes

$$S_{_E}[\phi] = \frac{1}{2}\int dx\,dy\;\phi(x)\,K(x,y)\,\phi(y), \tag{2.29}$$

with $K_{_E}(x_{_E},y_{_E}) = (\partial_{_E}^2 - M^2)\,\delta(x_{_E}-y_{_E})$. Then

$$Z_0[J] = \int \mathcal{D}\phi\;\exp\left(-S_{_E}[\phi] + \int dx_{_E}\;J\phi\right)$$

$$= Z_0[0]\,\exp\left(\frac{1}{2}\int dx_{_E}\,dy_{_E}\;J(x_{_E})\,\Delta_F^E(x_{_E}-y_{_E})\,J(y_{_E})\right), \tag{2.30}$$

where $K_{_E}(x_{_E},y_{_E})\Delta_F^E(x_{_E}-y_{_E}) = -\delta(x_{_E}-y_{_E})$. From this relation in Euclidean space one can obtain the connected Green's functions

$$\Delta_F^E(x_{_E}-y_{_E}) = \frac{1}{Z_0[J]}\;\frac{\delta^2 Z_0[J]}{\delta J(x_{_E})\delta J(y_{_E})}\bigg|_{J=0} = \int\frac{d^4k_{_E}}{(2\pi)^4}\,\frac{e^{-i\,k_{_E}\cdot(x_{_E}-x_{_E})}}{k_{_E}^2+M^2} =$$

$$i\Delta_F(x-y) = \frac{(-i)^2}{Z_0[J]}\;\frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(y)}\bigg|_{J=0} = \int\frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-i\,k\cdot(x-y)}}{k^2-M^2+i\epsilon}$$

$$= G_c^{(2)}(x,y) = \langle 0|\mathcal{T}\,\phi(x)\phi(y)|0\rangle_c. \tag{2.31}$$

When interactions are present, i.e. $\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi)$, the generating functional can be written,

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \ \exp\left(i \int dx \ [\mathcal{L}_{\ 0}(\phi) + \mathcal{L}_{\ I}(\phi) + J\phi]\right) && (2.32) \\ &= \exp\left(i \int dz \ \mathcal{L}_{\ I}\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)\right) \\ &\qquad \int \mathcal{D}\phi \ \exp\left(i \int dx \ [\mathcal{L}_{\ 0}(\phi) + J\phi]\right) && (2.33) \\ &= \exp\left(i \int dz \ \mathcal{L}_{\ I}\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)\right) \\ &\qquad \exp\left(-\frac{1}{2}\int dx\, dy\, J(x)\, i\Delta_F(x-y)\, J(y)\right) && (2.34) \\ &= \exp\left(\frac{1}{2}\int dx\, dy \frac{\delta}{\delta\phi(x)}\, i\,\Delta_F(x-y)\, \frac{\delta}{\delta\phi(y)}\right) \\ &\qquad \exp\left(i \int dz \ [\mathcal{L}_{\ I}(\phi) + J(z)\phi(z)]\right)\Big|_{\phi=0}. && (2.35) \end{aligned}$$

This expression will be the one from which Feynman rules will be derived. For complex scalar fields or fermions one has a similar expression, e.g. for fermions

$$\begin{aligned} Z[\eta,\bar\eta] &= \int \mathcal{D}\phi \ \exp\left(i \int dx \ [\overline{\psi}\,(-i\not\partial + M)\,\psi + \mathcal{L}_{\ I}(\psi) + \eta\psi + \overline{\psi}\eta]\right) \\ &= \exp\left(i \int dz \ \mathcal{L}_{\ I}\left(\frac{1}{i}\frac{\delta}{\delta\eta(z)}, \frac{1}{i}\frac{\delta}{\delta\bar\eta(z)}\right)\right) \\ &\qquad \exp\left(-\int dx\, dy\, \bar\eta(x)\, iS_F(x-y)\, \eta(y)\right), && (2.36) \end{aligned}$$

where iS_F is the Feynman propagator for fermions, which is the solution of $(i\not\partial - M)S_F(x) = \delta(x)$ (i.e. minus the inverse of the operator appearing in the quadratic piece) and is given by

$$\begin{aligned} iS_F(x-y) &= \langle 0|\mathcal{T}\,\overline{\psi}(x)\psi(y)|0\rangle = \\ &= (i\not\partial_x + M)\, i\Delta_F(x-y) && (2.37) \\ &= \int \frac{d^4k}{(2\pi)^4}\, e^{-i\,k\cdot(x-y)}\, \frac{i\,(\not k + M)}{k^2 - M^2 + i\epsilon}, && (2.38) \end{aligned}$$

2.1.4 Feynman rules

The generating functional defines a set of Green's functions that are calculated with the help of Feynman diagrams. It is most common to calculate the Green's functions in momentum space with an overall momentum conserving δ-function taken out,

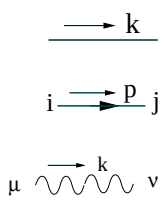
$$(2\pi)^4\, \delta^4(p_1 + \ldots + p_n)\, G^{(n)}(p_1, \ldots, p_n) = \int \prod_{i=1}^{n} dx_i \ e^{i\,p_i\cdot x_i} G^{(n)}(x_1, \ldots, x_n). \tag{2.39}$$

Correspondingly the sources are $J(k) = \int dx e^{-ik\cdot x} J(x)$.

In order to calculate the momentum space Green's functions $G^{(n)}(p_1, \ldots, p_n)$ one needs as basic objects the propagators $(i\,\Delta_F)$ and the vertices $(i\ \mathcal{L}_I)$ that appear in the expression for the functional $Z[J]$.

The propagators are the connected Green's functions $G_0^{(2)}(k)$ obtained from the quadratic pieces of

the free lagrangians. Note that for fermions (or complex scalar fields) the propagator is oriented, as the end can be connected to a source η or $\bar{\eta}$. They are diagrammatically depicted:



$$\frac{i}{k^2 - M^2 + i\epsilon}$$

$$\left(\frac{i}{\not{p} - M + i\epsilon}\right)_{ji} = \frac{i(\not{p} + M)_{ji}}{p^2 - M^2 + i\epsilon}$$

$$-i\left[\frac{g_{\mu\nu}}{k^2 - M^2 + i\epsilon} - \left(1 - \frac{1}{\lambda}\right)\frac{k_\mu k_\nu}{(k^2 - M^2 + i\epsilon)(k^2 - \frac{1}{\lambda}M^2 + i\epsilon)}\right].$$

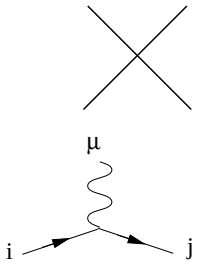
The vertices are derived from the interaction terms in the lagrangian, stripped from the fields and combinatorial factors that are conveniently introduced when one has identical fields. Derivatives are replaced by momenta, $i\partial_\mu \to k_\mu$, if momenta are defined inward. Momentum conservation is understood at these vertices. Examples for ϕ^4 theory,

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 - \frac{g}{4!}\phi^4, \tag{2.40}$$

or quantum electrodynamics with $D_\mu \equiv \partial_\mu + ie\,A_\mu$ (QED),

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 + \overline{\psi}\,(i\not{\partial} - e\not{A} - M)\,\psi \tag{2.41}$$

are

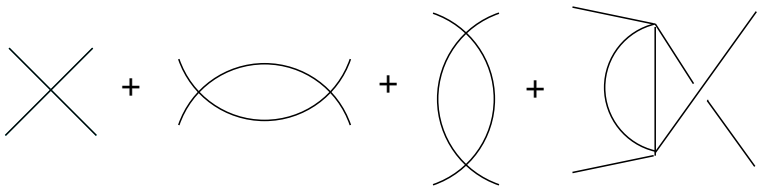


$$-i\,g$$

$$-i\,e\,(\gamma_\mu)_{ji}$$

The Greens function is then represented by a set of (topologically different) Feynman diagrams, corresponding to the terms appearing when the exponentials in $Z[J]$ are expanded and representing a mathematical expression. It becomes a useful way of calculation if the expansion of the exponentials is a perturbative expansion, i.e. if it makes sense to cut the expansion in some order of the coupling constant g that multiplies the interaction term in the lagrangian. A Feynman diagram is a collection of lines joined together by vertices. The lines in the Feynman diagrams can be external (leaving or entering the diagram) or internal (connecting vertices). For a given Green's function the external lines and the momenta entering the diagram are fixed. The rules are:

1. Start with external legs, corresponding to certain types of fields (particles) and draw all possible topologically different (for connected Green's functions only connected) diagrams, using the available vertices, e.g. up to order g^2 the four-point connected Green's function in ϕ^4 theory is given by



2. The contribution of each diagram is obtained by multiplying the contributions from the propagators, and vertices in that diagram

3. Carry out all integrations over all internal momenta (keeping track of momentum conservation at the vertices!).

4. Add a symmetry factor $1/S$ corresponding to permutation of internal lines and vertices (keeping external lines fixed). If problems arise go back to the defining expression for the generating functional. In order to determine the number of permutations, write down the vertices, count in how many ways the lines can be connected with the same topological result. Divide by the permutational factors of the vertices, which have been included in the definition in $\mathcal{L}_I$ and finally divide by the number of permutations of the identical vertices.

5. Each closed fermion loop gets a factor -1.

6. Feynman diagrams which only differ by exchanging identical fermion lines in have a relative minus sign.

These rules give the Green's functions in momentum space. Leaving out the Green's functions corresponding to the external lines, gives the amputated or truncated one-particle irreducible (1PI) Green's functions, for bosons

$$\Gamma^{(n)}(p_1,\ldots,p_n) = \left(\prod_{j=1}^{n} \frac{-i}{\Delta_F(p_j)}\right) G^{(n)}(p_1,\ldots,p_n), \tag{2.42}$$

and similarly for fermions with $\Delta_F \to S_F$.

2.1.5 Scattering amplitudes

The truncated Green's functions come in useful for the calculation of S-matrix elements. The (unitary) S-matrix known from scattering theory connects *in* and *out* states and (see text books like Ryder for the derivation) is given by

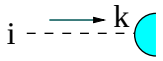
$$S =: \exp\left(\frac{1}{\sqrt{Z}} \int d^4x\, d^4y\, \phi_{in}(x)\, K(x,y)\, \frac{\delta}{\delta J(z)}\right) : \left.\frac{Z[J]}{Z[0]}\right|_{J=0}, \tag{2.43}$$

where ϕ_{in} is a free field. The colons indicate normal ordering of the operator acting on the generating functional. Therefore, an S-matrix element between momentum eigenstates in initial and final states is found by considering those source-connected Green's functions (action of $\delta/\delta J$ on $Z[J]/Z[0]$) where the external sources $J(x_i)$ are replaced by the particle wave functions (which are the result of acting with $\phi_{in}(x_i)$ on momentum eigenstates). Note that the Green's function connecting the external point x_i with the bubble is annihilated by $K(x,y)$.

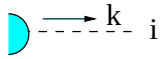
Usually we are interested in the part of the S-matrix describing the scattering,

$$S_{fi} = \delta_{fi} - i\,(2\pi)^4\,\delta^4(P_i - P_f)\,\mathcal{M}_{fi}, \tag{2.44}$$

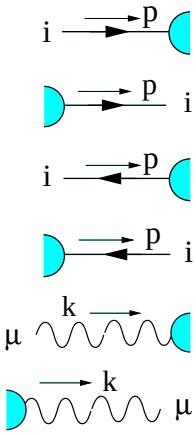
which is obtained considering only connected diagrams. Explicitly, the matrix element of $-i\mathcal{M}$ is given by the amputated Green's function multiplied with the momentum space wave functions of the particles in initial and final state:



1 incoming spin 0 particle



1 outgoing spin 0 particle



$u_i(p)$ incoming fermion

$\bar{u}_i(p)$ outgoing fermion

$\bar{v}_i(p)$ incoming antifermion

$v_i(p)$ outgoing antifermion

$\epsilon_\mu(k)$ incoming spin 1 particle

$\epsilon^*_\mu(k)$ outgoing spin 1 particle

2.2 Scattering theory

2.2.1 kinematics in scattering processes

Phase space

The 1-particle state is denoted $|p\rangle$. It is determined by the energy-momentum four vector $p = (E, \boldsymbol{p})$ which satisfies $p^2 = E^2 - \boldsymbol{p}^2 = m^2$. A physical state has positive energy. The phase space is determined by the weight factors assigned to each state in the summation or integration over states, i.e. the 1-particle phase space is

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3\, 2E} = \int \frac{d^4\boldsymbol{p}}{(2\pi)^4}\, \theta(p^0)\, (2\pi)\delta(p^2 - m^2), \tag{2.45}$$

(proof this!). This is generalized to the multi-particle phase space

$$d\mathcal{R}(p_1, \ldots, p_n) = \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3\, 2E_i}, \tag{2.46}$$

and the *reduced phase space element* by

$$d\mathcal{R}(s, p_1, \ldots, p_n) = (2\pi)^4\, \delta^4(P - \sum_i p_i)\, d\mathcal{R}(p_1, \ldots, p_n), \tag{2.47}$$

which is useful because the total 4-momentum of the final state usually is fixed by overall momentum conservation. Here s is the invariant mass of the n-particle system, $s = (p_1 + \ldots + p_n)^2$. It is a useful quantity, for instance for determining the threshold energy for the production of a final state $1 + 2 + \ldots + n$. In the CM frame the threshold value for s obviously is

$$s_{threshold} = \left(\sum_{i=1}^{n} m_i \right)^2. \tag{2.48}$$

For two particle states $|p_a, p_b\rangle$ we start with the four vectors $p_a = (E_a, \boldsymbol{p}_a)$ and $p_b = (E_b, \boldsymbol{p}_b)$ satisfying $p_a^2 = m_a^2$ and $p_b^2 = m_b^2$, and the total momentum four-vector $P = p_a + p_b$. The quantity

$$s = P^2 = (p_a + p_b)^2, \tag{2.49}$$

is referred to as the invariant mass squared. Its square root, $\sqrt{s}$ is for obvious reasons known as the center of mass (CM) energy.

To be specific let us consider two frequently used frames. The first is the CM system. In that case

$$p_a = (E_a^{cm}, \boldsymbol{q}), \tag{2.50}$$
$$p_b = (E_b^{cm}, -\boldsymbol{q}), \tag{2.51}$$

It is straightforward to proof that the unknowns in the particular system can be expressed in the invariants (m_a, m_b and s). Proof that

$$|\mathbf{q}| = \sqrt{\frac{(s - m_a^2 - m_b^2)^2 - 4\, m_a^2 m_b^2}{4s}} = \sqrt{\frac{\lambda(s, m_a^2, m_b^2)}{4s}}, \tag{2.52}$$
$$E_a^{cm} = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}, \tag{2.53}$$
$$E_b^{cm} = \frac{s - m_a^2 + m_b^2}{2\sqrt{s}}. \tag{2.54}$$

The function $\lambda(s, m_a^2, m_b^2)$ is a function symmetric in its three arguments, which in the specific case also can be expressed as $\lambda(s, m_a^2, m_b^2) = 4(p_a \cdot p_b)^2 - 4p_a^2\, p_b^2$.

The second frame considered explicitly is the socalled target rest frame in which one of the particles (called the target) is at rest. In that case

$$p_a = (E_a^{trf}, \boldsymbol{p}_a^{trf}), \tag{2.55}$$
$$p_b = (m_b, \mathbf{0}), \tag{2.56}$$

Also in this case one can express the energy and momentum in the invariants. Proof that

$$E_a^{trf} = \frac{s - m_a^2 - m_b^2}{2m_b}, \tag{2.57}$$
$$|\boldsymbol{p}_a^{trf}| = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2m_b}. \tag{2.58}$$

One can, for instance, use the first relation and the abovementioned threshold value for s to calculate the threshold for a specific n-particle final state in the target rest frame,

$$E_a^{lab}(threshold) = \frac{1}{2\, m_b}\left((\sum_i m_i)^2 - m_a^2 - m_b^2\right). \tag{2.59}$$

Explicit calculation of the reduced two-body phase space element gives

$$\begin{aligned} d\mathcal{R}(s, p_1, p_2) &= \frac{1}{(2\pi)^2}\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\delta^4(P - p_1 - p_2) \\ &\overset{CM}{=} \frac{1}{(2\pi)^2}\frac{d^3q}{4\, E_1\, E_2}\delta(\sqrt{s} - E_1 - E_2) \\ &= \frac{1}{(2\pi)^2}\, d\Omega(\hat{q})\frac{\boldsymbol{q}^2\, d|\boldsymbol{q}|}{4\, E_1\, E_2}\delta(\sqrt{s} - E_1 - E_2) \end{aligned}$$

which using $|\boldsymbol{q}|\, d|\boldsymbol{q}| = E_1\, dE_1 = E_2\, dE_2$ gives

$$\begin{aligned} d\mathcal{R}(s, p_1, p_2) &= \frac{|\boldsymbol{q}|}{(2\pi)^2}\, d\Omega(\hat{q})\frac{d(E_1 + E_2)}{4(E_1 + E_2)}\delta(\sqrt{s} - E_1 - E_2) \\ &= \frac{|\boldsymbol{q}|}{4\pi\sqrt{s}}\frac{d\Omega\,(\hat{q})}{4\pi} = \frac{\sqrt{\lambda_{12}}}{8\pi\, s}\frac{d\Omega(\hat{q})}{4\pi}, \end{aligned} \tag{2.60}$$

where λ_{12} denotes $\lambda(s, m_1^2, m_2^2)$.

Kinematics of 2 → 2 scattering processes

The simplest scattering process is 2 particles in and 2 particles out. Examples appear in

$$\begin{aligned} \pi^- + p &\rightarrow \pi^- + p && (2.61)\\ &\rightarrow \pi^0 + n && (2.62)\\ &\rightarrow \pi^+ + \pi^- + n && (2.63)\\ &\rightarrow \ldots\ . && (2.64) \end{aligned}$$

The various possibilities are referred to as different reaction channels, where the first is referred to as elastic channel and the set of all other channels as the inelastic channels. Of course there are not only 2-particle channels. The initial state, however, usually is a 2-particle state, while the final state often arises from a series of 2-particle processes combined with the decay of an intermediate particle (resonance).

Consider the process $a + b \rightarrow c + d$. An often used set of invariants are the Mandelstam variables,

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_c + p_d)^2 && (2.65)\\ t &= (p_a - p_c)^2 = (p_b - p_d)^2 && (2.66)\\ t &= (p_a - p_d)^2 = (p_b - p_c)^2 && (2.67) \end{aligned}$$

which are not independent as $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2$. The variable s is always larger than the minimal value $(m_a + m_b)^2$. A specific reaction channel starts contributing at the threshold value $s^{thr} = (\sum_i m_i)^2$. Instead of the scattering angle, which for the above $2 \rightarrow 2$ process in the case of azimuthal symmetry is defined as $\hat{\boldsymbol{p}}_a \cdot \hat{\boldsymbol{p}}_c = \cos\theta$ one can use in the CM the invariant

$$t \equiv (p_a - p_c)^2 \stackrel{CM}{=} m_a^2 + m_c^2 - 2\,E_a E_c + 2\,qq'\,\cos\theta^{cm},$$

with $q = \sqrt{\lambda_{ab}}/2\sqrt{s}$ and $q' = \sqrt{\lambda_{cd}}/2\sqrt{s}$. The minimum and maximum values for t correspond to θ^{cm} being 0 or 180 degrees,

$$\begin{aligned} t_{min}^{max} &= m_a^2 + m_c^2 - 2\,E_a E_c \pm 2\,qq' \\ &= m_a^2 + m_c^2 - \frac{(s + m_a^2 - m_b^2)(s + m_c^2 - m_d^2)}{2\,s} \pm \frac{\sqrt{\lambda\lambda'}}{2\,s}. \end{aligned} \qquad (2.68)$$

Using the relation between t and $\cos\theta^{cm}$ it is straightforward to express $d\Omega^{cm}$ in dt, $dt = 2\,qq'\,d\cos\theta^{cm}$ and obtain for the two-body phase space element

$$\begin{aligned} d\mathcal{R}(s, p_c, p_d) &= \frac{q'}{4\pi\sqrt{s}}\,\frac{d\Omega^{cm}}{4\pi} = \frac{\sqrt{\lambda_{cd}}}{8\pi\,s}\,\frac{d\Omega^{cm}}{4\pi} && (2.69)\\ &= \frac{dt}{8\pi\,\sqrt{\lambda_{ab}}} = \frac{dt}{16\pi\,q\,\sqrt{s}}. && (2.70) \end{aligned}$$

2.2.2 Cross sections and lifetimes

Scattering process

For a scattering process $a + b \rightarrow c + \ldots$ (consider for convenience the target rest frame) the cross section $\sigma(a + b \rightarrow c + \ldots)$ is defined as the proportionality factor in

$$\frac{N_c}{T} = \sigma(a + b \rightarrow c + \ldots) \cdot N_b \cdot \text{flux}(a),$$

where V and T indicate the volume and the time in which the experiment is performed, N_c/T indicates the number of particles c detected in the scattering process, N_b indicates the number of (target)

particles b, which for a density ρ_b is given by $N_b = \rho_b \cdot V$, while the flux of the beam particles a is $\text{flux}(a) = \rho_a \cdot v_a^{trf}$. The proportionality factor has the dimension of area and is called the cross section, i.e.

$$\sigma = \frac{N}{T \cdot V} \frac{1}{\rho_a \rho_b \, v_a^{trf}}. \tag{2.71}$$

Although this at first sight does not look covariant, it is. N and $T \cdot V$ are covariant. Using $\rho_a^{trf} = \rho_a^{(0)} \cdot \gamma_a = \rho_a^{(0)} \cdot E_a^{lab}/m_a$ (where $\rho_a^{(0)}$ is the rest frame density) and $v_a^{lab} = p_a^{lab}/E_a^{lab}$ we have

$$\rho_a \rho_b \, v_a^{trf} = \frac{\rho_a^{(0)} \rho_b^{(0)}}{4 \, m_a m_b} \, 2\sqrt{\lambda_{ab}}$$

or with $\rho_a^{(0)} = 2 \, m_a$,

$$\sigma = \frac{1}{2\sqrt{\lambda_{ab}}} \frac{N}{T \cdot V}. \tag{2.72}$$

Decay of particles

For the decay of particle a one has macroscopically

$$\frac{dN}{dt} = -\Gamma \, N, \tag{2.73}$$

i.e. the amount of decaying particles is proportional to the number of particles with proportionality factor the em decay width Γ. From the solution

$$N(t) = N(0) \, e^{-\Gamma \, t} \tag{2.74}$$

one knows that the decay time $\tau = 1/\Gamma$. Microscopically one has

$$\frac{N_{decay}}{T} = N_a \cdot \Gamma$$

or

$$\Gamma = \frac{N}{T \cdot V} \frac{1}{\rho_a}. \tag{2.75}$$

This quantity is not covariant, as expected. The decay time for moving particles τ is related to the decay time in the rest frame of that particle (the proper decay time τ_0) by $\tau = \gamma \, \tau_0$. For the (proper) decay width one thus has

$$\Gamma_0 = \frac{1}{2 \, m_a} \frac{N}{T \cdot V}. \tag{2.76}$$

Fermi's golden rule

In both the scattering cross section and the decay constant the quantity N/TV appears. For this we employ in essence *Fermi's Golden rule* stating that when the S-matrix element is written as

$$S_{fi} = \delta_{fi} - (2\pi)^4 \, \delta^4(P_i - P_f) \, i\mathcal{M}_{fi} \tag{2.77}$$

(in which we can calculate $-i\mathcal{M}_{fi}$ using Feynman diagrams), the number of scattered or decayed particles is given by

$$N = \left|(2\pi)^4 \, \delta^4(P_i - P_f) \, i\mathcal{M}_{fi}\right|^2 \, d\mathcal{R}(p_1, \ldots, p_n). \tag{2.78}$$

One of the δ functions can be rewritten as $T \cdot V$ (remember the normalization of plane waves),

$$\begin{aligned}
&\left|(2\pi)^4\,\delta^4(P_i - P_f)\right|^2 \\
&\qquad = (2\pi)^4\,\delta^4(P_i - P_f) \int_{V,T} d^4x\, e^{i(P_i - P_f)\cdot x} \\
&\qquad = (2\pi)^4\,\delta^4(P_i - P_f) \int_{V,T} d^4x = V \cdot T\,(2\pi)^4\,\delta^4(P_i - P_f).
\end{aligned}$$

(Using normalized wave packets these somewhat ill-defined manipulations can be made more rigorous). The result is

$$\frac{N}{T \cdot V} = |\mathcal{M}_{fi}|^2\, d\mathcal{R}(s, p_1, \ldots, p_n). \tag{2.79}$$

Combining this with the expressions for the width or the cross section one obtains for the decay width

$$\Gamma \qquad = \frac{1}{2m} \int d\mathcal{R}(m^2, p_1, \ldots, p_n)\, |\mathcal{M}|^2 \tag{2.80}$$

$$\stackrel{2-body\ decay}{=} \frac{q}{32\pi^2\, m^2} \int d\Omega\, |M|^2. \tag{2.81}$$

The *differential cross section* (final state not integrated over) is given by

$$d\sigma = \frac{1}{2\sqrt{\lambda_{ab}}}\, |\mathcal{M}_{fi}|^2\, d\mathcal{R}(s, p_1, \ldots, p_n), \tag{2.82}$$

and for instance for two particles

$$d\sigma = \frac{q'}{q} \left| \frac{\mathcal{M}(s, \theta_{cm})}{8\pi\sqrt{s}} \right|^2 d\Omega_{cm} = \frac{\pi}{\lambda_{ab}} \left| \frac{\mathcal{M}(s,t)}{4\pi} \right|^2 dt. \tag{2.83}$$

This can be used to get the full expression for $d\sigma/dt$ for $e\mu$ and e^+e^- scattering, for which the amplitudes squared have been calculated in the previous chapter. The amplitude $-\mathcal{M}/8\pi\sqrt{s}$ is the one to be compared with the quantum mechanical scattering amplitude $f(E,\theta)$, for which one has $d\sigma/d\Omega = |f(E,\theta)|^2$. The sign difference comes from the (conventional) sign in relation between S and quantummechanical and relativistic scattering amplitude, respectively.

2.3 Quantization of QCD

2.3.1 Quantization of nonabelian gauge theories

In principle generating functionals are very appropriate to get an intuitive approach to the quantization of gauge field theories. We will summarize the procedure in the language of path integrals (see text books for details). A problem with gauge theories is the fact that the integration over the gauge fields $\underline{A}_\mu$ includes specifically an integration over subsets $\{{}^g\underline{A}_\mu\}$, which are obtained through a gauge transformation g on $\underline{A}_\mu$,

$$\underline{A}_\mu \stackrel{g(x)}{\longrightarrow} {}^g\underline{A}_\mu = U(g)\underline{A}_\mu U^{-1}(g) - \frac{i}{g}[\partial_\mu U(g)]U^{-1}(g). \tag{2.84}$$

The action is constant on these gauge orbits and as a consequence the functional integral diverges. One must restrict $\underline{A}_\mu$ using constraints

$$\mathcal{F}^a(\underline{A}_\mu(x)) = B^a(x), \tag{2.85}$$

for example $\partial_\mu A^{\mu a} = 0$ (Lorentz gauge) or $\nabla \boldsymbol{A}^a = 0$ (Coulomb gauge). The constraints must provide a unique solution g of the equation $\mathcal{F}^a({}^g\underline{A}_\mu) = B^a$. Next introduce the quantity $\Delta_{\mathcal{F}}[\underline{A}_\mu]$,

$$\Delta_{\mathcal{F}}^{-1}[\underline{A}_\mu] \equiv \int \mathcal{D}g \prod_{x,a} \delta\left(\mathcal{F}^a({}^g\underline{A}_\mu(x)) - B^a(x)\right), \tag{2.86}$$

where $\mathcal{D}g$ is a functional integration over the group. Near the identity where $g(x) \approx 1 + d\theta^a(x) L_a$ one has $\mathcal{D}g \equiv \prod_{x,a} d\theta^a(x)$. The quantity $\Delta_{\mathcal{F}}$ is invariant under group transformations, $\Delta_{\mathcal{F}}[\underline{A}_\mu] = \Delta_{\mathcal{F}}[{}^g\underline{A}_\mu]$ because of the invariance of the group measure, $dg = dgg'$. The problematic integration over the gauge orbits now can be removed by writing

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\underline{A}_\mu \ldots \exp\left(iS[\underline{A}_\mu, \ldots] + i\int dx\, J_a^\mu A_\mu^a\right) \\
&= \int \mathcal{D}g\, \mathcal{D}\underline{A}_\mu \ldots \Delta_{\mathcal{F}}[\underline{A}_\mu] \prod_{x,a} \delta(\mathcal{F}^a({}^g\underline{A}_\mu) - B^a) \exp\left(iS[\underline{A}_\mu, \ldots] + i\int dx\, J_a^\mu A_\mu^a\right) \\
&= \int \mathcal{D}g\, \mathcal{D}\underline{A}_\mu \ldots \Delta_{\mathcal{F}}[\underline{A}_\mu] \prod_{x,a} \delta(\mathcal{F}^a(\underline{A}_\mu) - B^a) \exp\left(iS[\underline{A}_\mu, \ldots] + i\int dx\, J_a^\mu A_\mu^a\right) \\
&\propto \int \mathcal{D}\underline{A}_\mu \prod_{x,a} \delta(\mathcal{F}^a(\underline{A}_\mu) - B^a) \ldots \Delta_{\mathcal{F}}[\underline{A}_\mu] \exp\left(iS[\underline{A}_\mu, \ldots] + i\int dx\, J_a^\mu A_\mu^a\right).
\end{aligned} \tag{2.87}$$

This functional is actually independent under an infinitesimal change of B^a as there exists a gauge transformation for which $\mathcal{F}^a({}^g\underline{A}_\mu) - B^a - \delta B^a = \mathcal{F}^a(\underline{A}_\mu) - B^a$. Therefore it is possible to write

$$\begin{aligned}
Z[J] &= \int \mathcal{D}B^a \exp\left(-\frac{1}{2} i \int dx\, (B^a(x))^2\right) \\
&\qquad \times \int \mathcal{D}\underline{A}_\mu \prod_{x,a} \delta(\mathcal{F}(\underline{A}_\mu) - B^a) \ldots \Delta_{\mathcal{F}}[\underline{A}_\mu] \exp\left(iS[\underline{A}_\mu, \ldots] + i\int dx\, J_a^\mu A_\mu^a\right) \\
&= \int \mathcal{D}\underline{A}_\mu \ldots \Delta_{\mathcal{F}}[\underline{A}_\mu] \exp\left(i\{S[\underline{A}_\mu, \ldots] - \frac{1}{2}\int dx\, (\mathcal{F}^a(\underline{A}_\mu))^2\} + i\int dx\, J_a^\mu A_\mu^a\right).
\end{aligned} \tag{2.88}$$

The quantity $\Delta_{\mathcal{F}}[\underline{A}_\mu]$ can be calculated by considering the action of an infinitesimal gauge transformation on the constraint,

$$\mathcal{F}^a({}^g\underline{A}_\mu(x)) = \mathcal{F}^a(\underline{A}_\mu(x)) + \int dy\ [M_{\mathcal{F}}(x,y)]_{ab} \theta^b(y), \tag{2.89}$$

leading to

$$\Delta_{\mathcal{F}}^{-1}[\underline{A}_\mu] = \int \prod_{x,a} d\theta^a(x)\ \delta\left(\int dy\, [M_{\mathcal{F}}(x,y)]_{ab}\theta^b(y)\right) \quad = \quad \frac{1}{det\, M_{\mathcal{F}}} = e^{-Tr\, \ln M_{\mathcal{F}}}. \tag{2.90}$$

As an example, consider the Lorentz gauge, $\mathcal{F}^a(\underline{A}_\mu) = \partial_\mu A_\mu^a = 0$. From the transformation of the gauge fields,

$${}^g A_\mu^a(x) = A_\mu^a(x) - c_{abc}\theta^b(x) A_\mu^c(x) + \frac{1}{g}\partial_\mu \theta^a(x) = A_\mu^a(x) + \frac{1}{g} D_\mu \theta^a(x) \tag{2.91}$$

one obtains

$$\partial^\mu\, {}^g A_\mu^a(x) = \partial^\mu A_\mu^a(x) - c_{abc}[\partial^\mu \theta^b(x)] A_\mu^c(x) + \frac{1}{g}\partial^\mu \partial_\mu \theta^a(x), \tag{2.92}$$

or

$$[M_L(x,y)]_{ab} = [\frac{1}{g}\partial^\mu \partial_\mu\, \delta_{ab} - c_{abc} A_\mu^c \partial^\mu]\, \delta(x-y). \tag{2.93}$$

We use a functional integration property to rewrite $\Delta_{\mathcal{F}}[\underline{A}_\mu] = det\, M_{\mathcal{F}}$ as a functional integral involving anticommuting complex scalar fields (ghost fields),

$$det\, M_{\mathcal{F}} = \int \mathcal{D}c\,\mathcal{D}c^* \;\; \exp(-\int dx\, c^{a\,*}[M_{\mathcal{F}}]_{ab} c^b). \tag{2.94}$$

This equality holds if these ghost fields do not appear in initial or final state. One then ends up with a functional integral derived from the following action,

$$S = S[\underline{A}_\mu, \ldots] + S_{GF}[\underline{A}_\mu] + S[c, c^*] \tag{2.95}$$

where the first term is the (classical) lagrangian density, S_{GF} contains the gauge-fixing constraint and the last term contains the ghost fields.

As a second example, consider the axial gauge, $\mathcal{F}^a(\underline{A}_\mu) = n^\mu A^a_\mu = 0$ with $n^2 = 1$ (temporal gauge $A^0 = 0$), $n^2 = -1$ (spacelike axial gauge $A^3 = 0$) or $n^2 = 0$ (lightcone gauge $A^+ = 0$) [3]. In this case we obtain

$$n^\mu\, {}^g A^a_\mu(x) = n^\mu A^a_\mu(x) + \frac{1}{g}\, n^\mu \partial_\mu \theta^a(x), \tag{2.96}$$

i.e. M_L is independent of $\underline{A}_\mu$. So the ghost fields decouple from the gauge fields and may be integrated out.

2.3.2 Feynman rules for QCD

Lorentz gauge

Using the Lorentz gauge and appropriately rescaling the ghost fields, one obtains the QCD lagrangian density for quarks and gluons

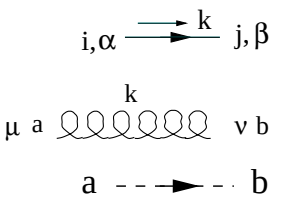
$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu}(x) F^{\mu\nu a}(x) + \overline{\psi}(x)(\not{D} - m)\psi(x) - \frac{\lambda}{2}(\partial^\mu A^a_\mu)^2 + c^{a*}(x)(\partial^2\, \delta_{ab} - g\, c_{abc} A^c_\mu \partial^\mu) c^b(x), \tag{2.97}$$

where

$$D_\mu \psi(x) = \partial_\mu \psi(x) - ig\, A^a_\mu(x) F_a\, \psi(x), \tag{2.98}$$

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g\, c_{abc} A^b_\mu(x) A^c_\nu(x). \tag{2.99}$$

The propagators are derived from the quadratic terms

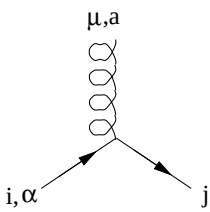


$$\left(\frac{i\,\delta_{ij}}{\not{k} - m + i\epsilon}\right)_{\beta\alpha} = \frac{i\,\delta_{ij}\,(\not{k} + m)_{\beta\alpha}}{k^2 - m^2 + i\epsilon}$$

$$-i\,\delta_{ab}\left[\frac{g_{\mu\nu}}{k^2 + i\epsilon} - \left(1 - \frac{1}{\lambda}\right)\frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2}\right].$$

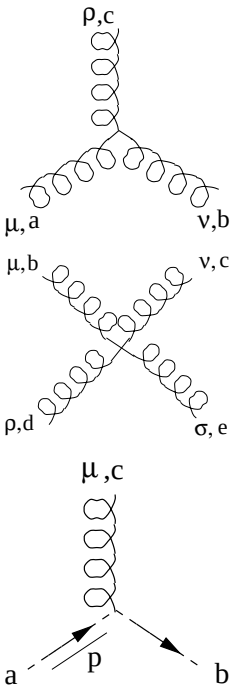
$$\frac{i\,\delta_{ab}}{k^2 + i\epsilon}$$

Two special cases for the gauge field propagator are the Feynman propagator with $\lambda = 1$ and the Landau propagator with $\lambda = \infty$. The vertices are derived from the interaction terms in the lagrangian,



$$i\, g\, (\gamma_\mu)_{\beta\alpha} (F^a)_{ji}$$

[3] The $\pm$ components of a four vector are $a^\pm = (a^0 \pm a^3)/\sqrt{2}$.



$$g\, c_{abc}[(p-q)^\rho\, g^{\mu\nu} + (q-r)^\mu\, g^{\nu\rho} + (r-p)^\rho\, g^{\rho\mu}]$$

$$\begin{aligned} &i\, g^2\, c_{abc} c_{ade}(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ &+i\, g^2\, c_{abd} c_{ace}(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ &+i\, g^2\, c_{abe} c_{acd}(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma}) \end{aligned}$$

$$-g\, c_{abc}\, p^\mu$$

Axial gauges

Except for the absence of ghost propagator and ghost-gauge field coupling, the only difference in the Feynman rules is in the gauge field propagator. It is derived from the quadratic term in

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}(x) F^{\mu\nu a}(x) + \overline{\psi}(x)(\not{D} - m)\psi(x) - \frac{\lambda}{2}(n^\mu A^a_\mu)^2, \tag{2.100}$$

yielding the propagator

$$iG^{\mu\nu}_{ab} = \frac{-i\,\delta_{ab}}{k^2 + i\epsilon}\left[g^{\mu\nu} + \left(n^2 + \frac{1}{\lambda}k^2\right)\frac{k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right]. \tag{2.101}$$

In the limit $\lambda \to \infty$, the propagator satisfies $n_\mu\, G^{\mu\nu}_{ab} = 0$, while moreover the contraction of the propagator with k^μ cancels the physical pole,

$$\lim_{k^2\to 0} k^2\, k_\mu G^{\mu\nu}_{ab} = 0. \tag{2.102}$$

This is referred to as a physical gauge, where there are only two (transverse) degrees of freedom.

Another form of the QCD Lagrangian that can be used is referred to as the planar gauge case with

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}(x) F^{\mu\nu a}(x) + \overline{\psi}(x)(\not{D} - m)\psi(x) - \frac{\lambda}{2\,n^2}(n^\mu A^a_\mu)\partial^2(n^\nu A^a_\nu), \tag{2.103}$$

yielding the propagator

$$iG^{\mu\nu}_{ab} = \frac{-i\,\delta_{ab}}{k^2 + i\epsilon}\left[g^{\mu\nu} + \left(1 - \frac{1}{\lambda}\right)\frac{n^2\, k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\right]. \tag{2.104}$$

In the limit $\lambda \to \infty$, the propagator is identical to the case above.

In the axial gauge spurious poles $(k\cdot n)^{-1}$ and $(k\cdot n)^{-2}$ appear. If one is to avoid the introduction of extra terms, one must choose a prescription that does not encounter poles while one performs the Wick rotation, e.g. the Mandelstam-Leibbrandt prescription for the lightcone gauge ($n^2 = 0$),

$$\frac{1}{k^+} = \lim_{\epsilon\to 0}\frac{1}{k^+ + i\epsilon k^-} = \lim_{\epsilon\to 0}\frac{k^-}{k^2 + i\epsilon} = \lim_{\epsilon\to 0}\frac{1}{k^+ + i\epsilon sgn(k^-)}, \tag{2.105}$$

which, however, does involve an additional lightcone vector.

2.3.3 Playing with color

The gauge group for QCD is the semi-simple Lie group $SU(N_c)$ with $N_G = N_c^2 - 1$ generators L_a that form the basis of the Lie algebra and satisfy

$$[L_a, L_b] = i\, c_{abc} L_c. \tag{2.106}$$

The quarks belong to the N_c-dimensional defining representation. For the representation matrices F_a Eq. 2.106 is realized as matrix commutation relations. Furthermore, for a semi-simple Lie group the representation matrices can be choosen such that the socalled Killing form is diagonal,

$$Tr\,(F_a F_b) = T_F\, \delta_{ab}, \tag{2.107}$$

which for $SU(N_c)$ is usually choosen to be $T_F = 1/2$. The representation matrices of the defining representation are for $SU(2)$ given in terms of the Pauli matrices, $F_a = \tau_a/2$ (a = 1,2,3) while for $SU(3)$ in the defining representation $\underline{3}$ they are usually expressed in terms of the Gell-Mann matrices, $(F_a)_{ij} = (\lambda_a)_{ij}/2$ (a = 1,...,8). Note that in SU(2) the defining representation is self-conjugate, which is not the case for $SU(3)$. The representation matrices for antiquarks are given by $F_a = -\lambda_a^*/2$ forming the conjugate representation $\underline{3}^*$. A useful relation for semi-simple Lie groups is further

$$ic_{abc} = \frac{1}{T_F} Tr\,(F_a F_b F_c - F_c F_b F_a). \tag{2.108}$$

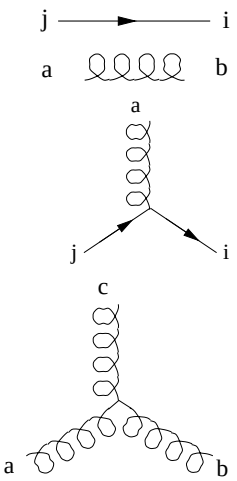
The gluons belong to the $N_c^2 - 1$ dimensional adjoint representation, for which the representation matrices are given by

$$(F_a)_{bc} = -i\, c_{abc}. \tag{2.109}$$

It is straightforward to prove that this is a representation using the Jacobi-identity for the commutator,

$$c_{abd}c_{dce} + c_{bcd}c_{dae} + c_{cad}c_{dbe} = 0. \tag{2.110}$$

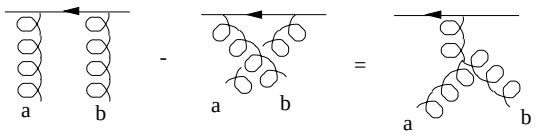
In practical calculations it is practical to separate the group theoretical factors from the dynamical parts. For the calculation the following factors are necessary



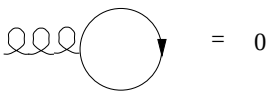
δ_{ij}	internal quark line
δ_{ab}	internal gluon or ghost line
$(F_a)_{ij}$	quark-quark-gluon vertex
$i\, c_{abc}$	three-gluon vertex

The group theoretic weight of the four-gluon vertex is the product of two three gluon vertex-factors.

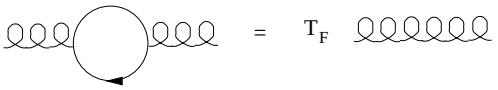
The properties of the Lie algebra are easily translated into diagrammatic language. For example (1) $[F_a, F_b] = i\, c_{abc} F_c$



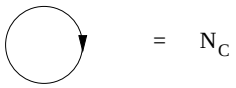
(2) $Tr\, F_a = 0$



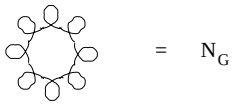
(3) $Tr\,(F_a F_b) = T_F\, \delta_{ab}$



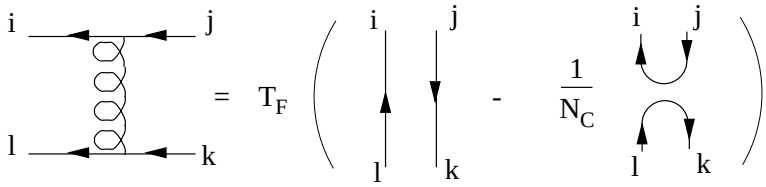
(4) $\delta_{ii} = N_c$



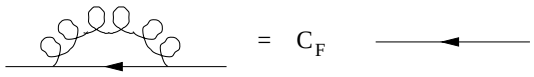
(5) $\delta_{aa} = N_G = N_c^2 - 1$



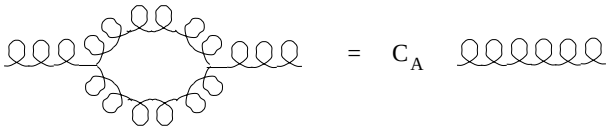
The color flow and the corresponding multiplicative factors appearing in diagrams with quarks and gluons are most easily proven starting with the following relation (proof!)
(6) $\sum_a (F_a)_{ij}\,(F_a)_{kl} = T_F\, \delta_{il}\, \delta_{jk} - \frac{T_F}{N_c}\, \delta_{ij}\, \delta_{kl}$,
diagrammatically



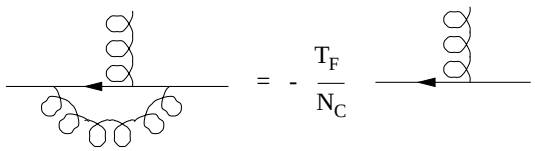
As next relations it is then straightforward to prove
(7) $\sum_a (F_a)_{ij}\,(F_a)_{jk} = C_F\, \delta_{ik}$ with $C_F = T_F\, N_G/N_c = (N_c^2 - 1)/2N_c$,



(8) $c_{adb}\, c_{adc} = C_A\, \delta_{bc}$ with $C_A = 2T_F(N_G + 1)/N_c = N_c$,



(9)



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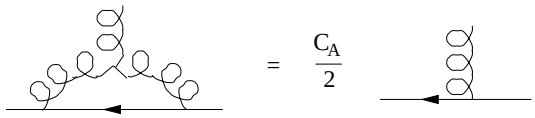
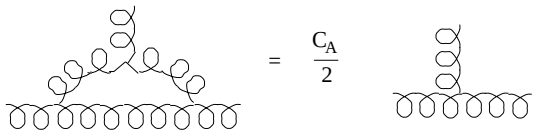


(11)



2.4 Renormalization

2.4.1 Divergencies

In calculating the Green's functions in a specific theory one usually encounters infinities. A first analysis of these infinities is obtained from dimensional analysis. Consider the Green's functions with e_b boson fields and e_f fermion fields. The canonical dimensions of the fields and the lagrangian density is

$$\begin{aligned} &d(\mathcal{L}) = 4, \\ &d(\phi) = d(A_\mu) = 1, \qquad\qquad d(\psi) = 3/2. \end{aligned}$$

From this one sees immediately that for the Green's function in coordinate space (G), those in momentum space ($\tilde{G}$) and the truncated Green's functions (Γ) one has

$$\begin{aligned} &d(G) = e_b + \frac{3}{2}\, e_f, \\ &d(\tilde{G}) = 4 - 3\, e_b - \frac{5}{2}\, e_f, \\ &d(\Delta_F) = 2, \qquad d(\tilde{\Delta}_F) = -2, \\ &d(S_F) = 3, \qquad d(\tilde{S}_F) = -1, \\ &d(\Gamma) = 4 - e_b - \frac{3}{2}\, e_f. \end{aligned}$$

In the calculation of the truncated Green's functions (in momentum space) one employs vertices, for which we define

$$\delta_i = b_i + \frac{3}{2}\, f_i + d_i - 4, \tag{2.111}$$

where b_i and f_i are the number of boson and fermion lines at the vertex and d_i is the number of derivatives appearing. Note that $-\delta_i$ indicates precisely the dimension of the coupling constant which has to multiply this vertex, in order for $\mathcal{L}$ to have the right dimension. Examples in 4 dimensions are $\delta(\phi^4) = 0$, $\delta(\bar{\psi} \not{A} \psi) = 0$ and $\delta(\phi^3) = $ -1. In the calculation of a specific diagram a degree of divergence (D) is introduced to 'measure' the divergence, e.g.

$$\begin{aligned} &\int dk\, k && D = 2 && \text{(quadratically divergent)} \\ &\int dk\, k^{-1} && D = 0 && \text{(logarithmically divergent)} \\ &\int dk\, k^{-2} && D = -1 && \text{(convergent)} \end{aligned}$$

These divergences arising from momentum integrations are called *ultraviolet divergences*. For a given diagram it is easy to find the degree of divergence adding the specific contributions from the integration in the loops (ℓ loops, each adding 4), the internal boson lines (i_b lines, each adding -2), the internal fermion lines (i_f lines, each adding -1) and the derivatives at the vertices (n_i vertices, each adding d_i). The result is

$$D = 4\,\ell - i_f - 2\,i_b + \sum_i n_i\,d_i, \tag{2.112}$$

which using the relation $e_b + 2\,i_b = \sum_i n_i\,b_i$, $e_f + 2\,i_f = \sum_i n_i\,f_i$ and $\ell = i - v + 1$ (here $v = \sum_i n_i$ is the number of vertices) is rewritten as

$$D = \sum_i n_i\,\delta_i + 4 - e_b - \frac{3}{2}\,e_f, \tag{2.113}$$

which indeed only differs from $d(\Gamma)$ by the dimensions in the coupling constants at the vertices.

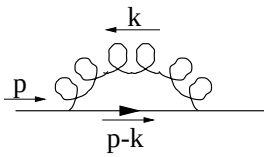
Theories with only vertices with negative δ_i are called *super-renormalizable*. Each truncated Green's function only contains a finite number of divergent graphs. Theories which do contain vertices with $\delta_i \leq 0$ are called *renormalizable*. The divergences are present in all diagrams, but only in a finite number of Green's functions (modulo some complications with divergent subdiagrams). The infinities will be absorbed in a redefinition of the parameters multiplying the various terms in the lagrangian. If there are vertices with $\delta_i > 0$ the theory is called *non-renormalizable*. The problem is that all Green's functions contain infinite terms, which requires in general the introduction of an infinite number of new terms in the lagrangian to absorb the infinities.

2.4.2 Regularization

We will demonstrate the first step in the renormalization procedure for the example of the self-energy contribution to $\Gamma^{(2)}(p)$. Writing the full Green's function

$$\Gamma^{(2)}_{ij}(p) = -i\,(S_F)^{-1}_{ij}(p) = -i[(\not{p} - m)\delta_{ij} + \Sigma_{ij}(p)], \tag{2.114}$$

the contribution of the one-loop diagram in the Feynman gauge ($\lambda = 1$ in the gluon propagator),



to the self-energy is given by

$$-i\,\Sigma(p) = -g^2\,C_F \int \frac{d^4k}{(2\pi)^4}\,\frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{k^2\,((p-k)^2 - m^2)}. \tag{2.115}$$

This diagram has therefore $D = 1$, i.e. it is linearly divergent.

Pauli-Villars regulator method

One way of dealing with the divergence is the replacement

$$\frac{1}{k^2 - m^2} \;\longrightarrow\; \frac{1}{k^2 - m^2} - \frac{1}{k^2 - M^2} = \frac{m^2 - M^2}{(k^2 - m^2)(k^2 - M^2)}. \tag{2.116}$$

This wil render a finite expression for the above integral.

Dimensional regularization

A second way to regularize the infinity is dimensional regularization. In this method one extends the dimension of space-time to $n = 4 - 2\epsilon$. The integration is taken

$$\frac{d^4k}{(2\pi)^4} \longrightarrow \frac{d^nk}{(2\pi)^n}, \tag{2.117}$$

We note that the dimensions of the fields and parameters change accordingly,

$$\begin{aligned} & d(\mathcal{L}) = n = 4 - 2\epsilon, \\ \text{from } (\partial_\mu\phi)^2 : \quad & d(\phi) = \frac{n-2}{2} = 1 - \epsilon, \\ \text{from } g\phi^4 : \quad & d(g) = 2\epsilon, \\ \text{from } \bar{\psi}\partial\!\!\!/\psi : \quad & d(\psi) = \frac{n-1}{2} = \frac{3}{2} - \epsilon, \\ \text{from } g\,\bar{\psi}A\!\!\!/\psi : \quad & d(g) = \epsilon. \end{aligned}$$

Also with γ-matrices we have to be careful, e.g.

$$\begin{aligned} & Tr\,(\gamma_\mu\gamma_\nu) = 4\,g_{\mu\nu}, \\ & \gamma_\mu\gamma^\mu = n, \\ & \gamma_\mu\gamma_\nu\gamma^\mu = (2-n)\gamma_\nu, \end{aligned}$$

Using this for massless quarks and introducing a mass scale M through $g \to g_0\,M^\epsilon$ in order to make the new g_0 dimensionless, the result for $\Sigma(p)$ becomes

$$\Sigma(p) = g_0^2\,C_F\,(2-n)\,M^{2\epsilon}\int \frac{d^nk}{(2\pi)^n\,i}\;\frac{p\!\!\!/ - k\!\!\!/}{k^2\,(p-k)^2}. \tag{2.118}$$

The result using standard techniques in dimensional integration, including a rotation into Euclidean space, becomes after expanding around $n = 4$

$$\Sigma(p) = \left(\frac{g_0^2}{4\pi}\right)^2 C_F\,p\!\!\!/\left(\frac{1}{\epsilon} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\,M^2}\right)\right) + \mathcal{O}(\epsilon^2), \tag{2.119}$$

where γ is Euler's constant.

2.4.3 Renormalization schemes

The complete result of the one-loop calculation for the inverse quark propagator thus is

$$S_F^{-1}(p) = i\Gamma^{(2)}(p) = p\!\!\!/\left[1 + \left(\frac{g_0}{4\pi}\right)^2 C_F\left(\frac{1}{\epsilon} - \gamma + 1 - \ln\left(\frac{-p^2}{4\pi\,M^2}\right)\right)\right]. \tag{2.120}$$

The presence of the finite (p^2-dependent) terms introduce ambiguities in the way one defines renormalized quantities, such as a renormalized mass or renormalized fields. In this case ($m = 0$) we only deal with the wave function renormalization. Writing

$$\psi = Z_2^{1/2}\,\psi_R, \tag{2.121}$$

one has for instance

$$G_R^{(2)}(x_1, x_2) = \langle 0|\mathcal{T}\bar{\psi}_R(x_1)\psi_R(x_2)|0\rangle = Z_2^{-1}\,G^{(2)}(x_1, x_2), \tag{2.122}$$

$$\Gamma_R^{(2)}(p) = Z_2\,\Gamma^{(2)}(p). \tag{2.123}$$

The infinity $\propto 1/\epsilon$ is absorbed in the renormalization constant such that the result for $\Gamma_R^{(2)}$ is finite. Possible schemes are

On-shell subtraction scheme.

In this cases one requires

$$i\Gamma_R^{(2)}(p) \stackrel{p^2=m^2}{=} \not{p} - m \stackrel{m=0}{=} \not{p}. \tag{2.124}$$

For massless quarks this gives trouble because of the mass singularity in $\ln(-p^2)$.

Off-shell subtraction scheme or momentum-space subtraction scheme (MOM).

In this case one requires at a value $p^2 = -\mu^2 < 0$ that $\Gamma_R^{(2)}$ has the form of the free (massless) propagator,

$$i\Gamma_R^{(2)}(p) \stackrel{p^2=-\mu^2}{=} \not{p}. \tag{2.125}$$

Then

$$Z_2^{-1} = 1 + \left(\frac{g_0}{4\pi}\right)^2 C_F \left(\frac{1}{\epsilon} - \gamma + 1 - \ln\left(\frac{\mu^2}{4\pi M^2}\right)\right), \tag{2.126}$$

and

$$i\Gamma_R^{(2)}(p) \;=\; \not{p}\left[1 - \left(\frac{g_0}{4\pi}\right)^2 C_F \ln\left(\frac{-p^2}{\mu^2}\right)\right]. \tag{2.127}$$

Minimal subtraction scheme (MS).

In this case only the pole term $\propto 1/\epsilon$ is removed,

$$Z_2^{-1} = 1 + \left(\frac{g_0}{4\pi}\right)^2 C_F \frac{1}{\epsilon}, \tag{2.128}$$

and

$$i\Gamma_R^{(2)}(p) \;=\; \not{p}\left[1 - \left(\frac{g_0}{4\pi}\right)^2 C_F \left(\gamma - 1 + \ln\left(\frac{-p^2}{4\pi M^2}\right)\right)\right]. \tag{2.129}$$

This is converted in the result in the MOM-scheme after setting $\mu^2 = 4\pi e^{1-\gamma} M^2$.

Modified minimal subtraction scheme ($\overline{\text{MS}}$).

In this case the pole term in combination with the constants γ and $\ln 4\pi$ are removed,

$$Z_2^{-1} = 1 + \left(\frac{g_0}{4\pi}\right)^2 C_F \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right), \tag{2.130}$$

and

$$i\Gamma_R^{(2)}(p) \;=\; \not{p}\left[1 - \left(\frac{g_0}{4\pi}\right)^2 C_F \left(-1 + \ln\left(\frac{-p^2}{M^2}\right)\right)\right]. \tag{2.131}$$

Again this can be converted to the results in the other schemes after redefining the momentum scales.

After the renormalization procedure one can obtain finite results for all Green's functions involving the (finite) renormalized fields by re-expressing the lagrangian in terms of renormalized fields and renormalized parameters. For instance, for the kinetic term in the Dirac Lagrangian,

$$\mathcal{L} = \bar{\psi}(i\not{\partial})\psi = Z_2\,\bar{\psi}_R(i\not{\partial})\psi_R = \bar{\psi}_R(i\not{\partial})\psi_R + \mathcal{L}_C, \tag{2.132}$$

where

$$\mathcal{L}_C = (Z_2 - 1)\,\bar{\psi}_R(i\not{\partial})\psi_R, \tag{2.133}$$

which gives rise to a new vertex

$$(Z_2 - 1)\,(\not{p})_{ij}.$$

For the full QCD lagrangian density one needs to redefine fields,

$$A_\mu^a = Z_3^{1/2}\,A_{R\mu}^a, \qquad c^a = \tilde{Z}_3^{1/2}\,c_R^a, \qquad \psi = Z_2^{1/2}\psi_R, \tag{2.134}$$

and redefine parameters,

$$g = Z_g\, g_R, \qquad \lambda = Z_3^{-1}\, \lambda_R, \qquad m = Z_m\, m_R. \tag{2.135}$$

The renormalization constant for λ is related to that of the gluon field in order to keep the gauge-fixing term in the same form. Note that there are in principle 8 terms in the lagrangian density of QCD. The fact that five renormalization constants are sufficient is a consequence of gauge invariance. This provides relations between different Green's functions, the socalled Ward-Takahashi identities.

2.5 The renormalization group

For an n-point Green's function in a theory with fields ϕ and parameters g and m we have seen

$$\Gamma_R^{(n)}(p; g_R, m_R, \mu) = Z_\phi^{n/2}(R)\, \Gamma^{(n)}(p; g, m), \tag{2.136}$$

where μ is the renormalization scale and the renormalization constants are defined

$$\phi = Z_\phi^{1/2}\, \phi_R, \qquad g = Z_g\, g_R, \qquad m = Z_m\, m_R \tag{2.137}$$

(Note that the precise definitions may vary depending on the theory, e.g. in a bosonic theory one conventionally has $m^2 = Z_m\, m_R^2$). The renormalization constants and the renormalized quantities depend on the bare parameters and the renormalization scale as we have also seen in the example in the previous section, for instance $g_R = g_R(g, m, \mu)$, $Z_g = Z_g(g, m, \mu)$. For another scheme (and/or renormalization scale) one has

$$\Gamma_{R'}^{(n)}(p; g_{R'}, m_{R'}, \mu) = Z_\phi^{n/2}(R')\, \Gamma^{(n)}(p; g, m). \tag{2.138}$$

Between the two schemes we have a finite renormalization constant

$$\begin{aligned}
\phi_{R'} &= Z_\phi^{1/2}(R', R)\, \phi_R = \frac{Z_\phi^{1/2}(R)}{Z_\phi^{1/2}(R')}\, \phi_R, \\
g_{R'} &= Z_g(R', R)\, g_R = \frac{Z_g(R)}{Z_g(R')}\, g_R, \\
m_{R'} &= Z_m(R', R)\, m_R = \frac{Z_m(R)}{Z_m(R')}\, m_R.
\end{aligned} \tag{2.139}$$

One can define a group of renormalization constants $\{Z_g(R', R)\}$ with a multiplication $Z_g(R'', R) = Z_g(R'', R')\, Z_g(R', R)$, an inversion $[Z_g(R', R)]^{-1} = Z_g(R, R')$ and an identity $Z_g(R, R) = 1$, which all belong to or remain in the group. This abelian group together with similar groups for the other renormalization constants is known as the renormalization group.

2.5.1 The renormalization group equations (RGE)

The renormalization schemes R are characterized by a scale μ. The bare parameters are independent of this scale, $dg/d\mu = dm/d\mu = 0$, and also

$$\frac{d}{d\mu}\, \Gamma^{(n)}(p; g, m) = 0. \tag{2.140}$$

Thus

$$Z_\phi^{-n/2} \left(\frac{\partial}{\partial \mu} + \frac{\partial g_R}{\partial \mu}\, \frac{\partial}{\partial g_R} + \frac{\partial m_R}{\partial \mu}\, \frac{\partial}{\partial m_R} \right) \Gamma_R^{(n)} + \left(\frac{\partial Z_\phi^{-n/2}}{\partial \mu} \right) \Gamma_R^{(n)} = 0, \tag{2.141}$$

or

$$\left(\mu\,\frac{\partial}{\partial\mu}+\beta\,\frac{\partial}{\partial g_R}-\gamma_m\,m_R\,\frac{\partial}{\partial m_R}-n\,\gamma\right)\Gamma_R^{(n)}=0, \tag{2.142}$$

where

$$\beta = \mu\,\frac{\partial g_R}{\partial\mu}=\beta(g_R,m_R/\mu), \tag{2.143}$$

$$\gamma_m = -\frac{\mu}{m_R}\,\frac{\partial m_R}{\partial\mu}=\gamma_m(g_R,m_R/\mu), \tag{2.144}$$

$$\gamma = \frac{\mu}{2}\,\frac{\partial\ln Z_\phi}{\partial\mu}=\mu^2\,\frac{\partial\ln Z_\phi}{\partial\mu^2}=\gamma(g_R,m_R/\mu). \tag{2.145}$$

The (finite) dimensionless functions β, γ_m and γ depend on the (renormalized) parameters and the renormalization scale in dimensionless combinations and the regulators. The dependence on the latter is either direct or indirectly through g_R. In the MS or $\overline{\rm MS}$ scheme where the dependence on μ enters through $g_0\,\mu^\epsilon$ the functions actually only depend on g_R. Finally note that through the relations in Eq. 2.135 the functions β and γ_m can also be obtained in terms of derivatives of $\ln Z_g$ and $\ln Z_m$ respectively. Similarly the function γ can be obtained from the scale-dependence of $\Gamma_R^{(2)}$.

We conclude this section by giving for QCD the renormalization group equations for the $1PI$ Green's functions with n_F fermions and n_G gluons,

$$\Big[\mu\frac{\partial}{\partial\mu}+\beta(g_R)\frac{\partial}{\partial g_R}-m_R\,\gamma_m(g_R)\frac{\partial}{\partial m_R}-2\lambda_R\,\delta(g_R)\,\frac{\partial}{\partial\lambda_R}$$
$$-n_G\,\gamma_G(g_R)-n_F\,\gamma_F(g_R)\Big]\Gamma_R^{(n_F,n_G)}(p_i;g_R,m_R,\lambda_R,\mu)=0 \tag{2.146}$$

with

$$\beta(g_R) = \mu\,\frac{\partial g_R}{\partial\mu}, \tag{2.147}$$

$$\gamma_m(g_R) = -\frac{\mu}{m_R}\,\frac{\partial m_R}{\partial\mu}, \tag{2.148}$$

$$\delta(g_R) = -\frac{\mu}{2\lambda_R}\,\frac{\partial\lambda_R}{\partial\mu}=\gamma_G(g_R), \tag{2.149}$$

$$\gamma_G(g_R) = -\frac{\mu}{2}\,\frac{\partial\ln Z_3}{\partial\mu}, \tag{2.150}$$

$$\gamma_F(g_R) = -\frac{\mu}{2}\,\frac{\partial\ln Z_2}{\partial\mu}, \tag{2.151}$$

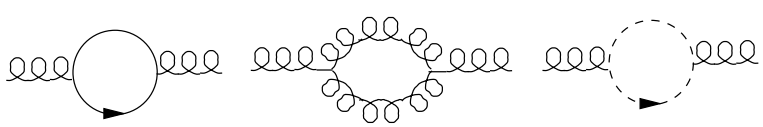
(Remember that the partial derivative here means, keeping parameters $g=g_0\,\mu^\epsilon$, m, and λ fixed).

The γ_F-function up to order g^2 can be immediately obtained from the calculation of Z_2 or $\Gamma_R^{(2)}$,

$$\gamma_F(g)=\left(\frac{g}{4\pi}\right)^2 C_F+\ldots. \tag{2.152}$$

(Because $\epsilon\,\ln\mu$ is fixed, the ϵ-dependence contains the μ-dependence.)

The β-function can for instance be obtained from the one-loop vacuum polarization corrections to the gluon propagator,



which gives for the regularized coupling squared for $|q^2| \gg \mu^2$ in the $\overline{\rm MS}$ scheme

$$g^2(q^2) \approx g_0^2 \left\{ 1 - \left(\frac{g_0}{4\pi}\right)^2 \left[\frac{11\,C_A - 4f\,T_F}{3}\right] \ln\left(\frac{|q^2|}{\mu^2}\right) \right\}, \tag{2.153}$$

where f is the number of flavors, i.e. the number of quarks contributing in the loops. From this one finds the first coefficient (b_0) in the β-function. Including also the next order one has

$$\beta(g) = -b_0 \frac{g^3}{32\pi^2} - b_1 \frac{g^5}{512\pi^4} + \dots \ , \tag{2.154}$$

with

$$b_0 = \frac{11\,C_A - 4f\,T_F}{3} \stackrel{N_c=3}{=} \frac{33-2f}{3}. \tag{2.155}$$

$$b_1 = \frac{34\,C_A^3 - 4f\,T_F\,(5\,C_A + 3\,C_F)}{3} \stackrel{N_c=3}{=} \frac{306-38f}{3}. \tag{2.156}$$

The result for QED is also easily obtained with $g \to e$, $C_A \to 0$, $f \to 1$ and $T_F \to 1$,

$$\beta(e) = \frac{e^3}{24\pi^2} + \dots \ . \tag{2.157}$$

(Note the sign change!)

The result for γ_m is

$$\gamma_m(g) = \gamma_{m0} \left(\frac{g}{4\pi}\right)^2 + \gamma_{m1} \left(\frac{g}{4\pi}\right)^4 + \dots \tag{2.158}$$

with

$$\gamma_{m0} = 6\,C_F \stackrel{N_c=3}{=} 8, \tag{2.159}$$

$$\gamma_{m1} = C_F \frac{9\,C_F + 97\,C_A - 20f\,T_F}{3} \stackrel{N_c=3}{=} \frac{4}{9}(303 - 10f). \tag{2.160}$$

2.5.2 Asymptotic solutions of the RGE.

Using the RGE it is possible to obtain the asymptotic behavior of the renormalized Green's functions, without knowing their precise form. Naive dimensional analysis of the Green's functions shows that

$$\Gamma^{(n)}(p_i; g, m, \mu) = \mu^{4-n} \overline{\Gamma}\left(\frac{p_i}{\mu}; g, \frac{m}{\mu}\right) \tag{2.161}$$

(Note: parameters and Green's functions are renormalized ones! The same equation also holds in coordinate space!). Consider the rescaled momenta σp_i in the Green's function for $\sigma \to \infty$. Then the mass dependence does not play a role anymore and

$$\Gamma^{(n)}(\sigma p_i; g, \mu) = \mu^{4-n} \overline{\Gamma}\left(\frac{\sigma p_i}{\mu}; g\right). \tag{2.162}$$

From

$$\left(\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma}\right) \overline{\Gamma}\left(\frac{\sigma p_i}{\mu}; g\right) = 0, \tag{2.163}$$

it follows that

$$\left(\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma} + (n-4)\right) \Gamma^{(n)}(\sigma p_i; g, \mu) = 0, \tag{2.164}$$

or

$$\left[\sigma\frac{\partial}{\partial\sigma} - \beta(g)\frac{\partial}{\partial g} + n\,\gamma(g) + (n-4)\right] \Gamma^{(n)}(\sigma p_i; g, \mu) = 0. \tag{2.165}$$

Write the Green's function as

$$\Gamma^{(n)}(\sigma p_i; g, \mu) \equiv \sigma^{4-n} \exp\left(n \int_0^g dg' \frac{\gamma(g')}{\beta(g')}\right) \; F^{(n)}(\sigma p_i; g, \mu) \tag{2.166}$$

and $\tau = \ln \sigma$. Then

$$\left(\frac{\partial}{\partial \tau} - \beta(g)\frac{\partial}{\partial g}\right) F^{(n)}(e^\tau p_i; g, \mu) = 0. \tag{2.167}$$

Consider the *running coupling* $\bar{g}(\tau, g)$ which satisfies

$$\bar{g}(0, g) \equiv g \quad \text{and} \quad \frac{\partial \bar{g}(\tau, g)}{\partial \tau} = \beta(\bar{g}). \tag{2.168}$$

This function satisfies

$$\left(\frac{\partial}{\partial \tau} - \beta(g)\frac{\partial}{\partial g}\right) \bar{g}(\tau, g) = 0. \tag{2.169}$$

(Proven by starting with $d\tau = d\bar{g}/\beta(\bar{g})$ for given g, integrating left- and right-hand side (from 0 to τ and g to $\bar{g}$ respectively), and finally differentiating with respect to g). The dependence on g and τ in $F^{(n)}$ is therefore entirely contained in $\bar{g}(\tau, g)$, i.e. the function $F^{(n)}(t, g)$ satisfying Eq. 2.167 is $F^{(n)}(0, \bar{g}(t, g))$. Applied to the renormalized Green's functions one has then

$$\Gamma^{(n)}(\sigma p_i; g, \mu) = \sigma^{4-n} \exp\left(n \int_0^g dg' \frac{\gamma(g')}{\beta(g')}\right) \; F^{(n)}(p_i; \bar{g}(\tau, g), \mu) \tag{2.170}$$

Finally rewrite

$$\begin{aligned} n \int_0^g dg' \frac{\gamma(g')}{\beta(g')} &= n \int_0^{\bar{g}} dg' \frac{\gamma(g')}{\beta(g')} - n \int_g^{\bar{g}} dg' \frac{\gamma(g')}{\beta(g')} \\ &= n \int_0^{\bar{g}} dg' \frac{\gamma(g')}{\beta(g')} - n \int_0^{\tau} d\tau' \, \gamma(\bar{g}(\tau', g)), \end{aligned}$$

leading to the final result

$$\Gamma^{(n)}(e^\tau p_i; g, \mu) = \exp\left((4-n)\tau - n \int_0^\tau d\tau' \, \gamma(\bar{g}(\tau', g))\right) \; \Gamma^{(n)}(p_i; \bar{g}(\tau, g), \mu). \tag{2.171}$$

The result tells us that rescaling the momenta ($\tau = 0 \leftrightarrow \sigma = 1$ and $\tau = \infty \leftrightarrow \sigma = \infty$) is equivalent to:

- rescaling with canonical dimension, σ^{4-n} factor.
- replacing the coupling constant by an effective coupling constant $g \to \bar{g}(\tau)$ with $\bar{g}(0) = g$, its behavior governed by the β-function.
- multiplication with an exponential factor, showing the *anomalous dimension* of the fields, which is the result of summing logarithms and is controlled by the γ-function.

We note that the mass parameter can be included in a similar way. The mass m must be replaced by a running mass $\overline{m}$, which for $\tau = 0$ coincides with the renormalized mass m and the running of which is given by

$$\frac{1}{\overline{m}} \frac{\partial \overline{m}}{\partial \tau} = -1 - \gamma_m(\bar{g}), \tag{2.172}$$

which knowing $\bar{g}(\tau)$ is easily solved

$$\overline{m}(\tau) = m \exp\left(-\tau - \int_0^\tau d\tau' \, \gamma_m(\bar{g}(\tau))\right). \tag{2.173}$$

2.5.3 The running coupling constant

The behavior of the running coupling constant is governed by the β-function,

$$\frac{d\bar{g}(\tau)}{d\tau} = \beta(\bar{g}) \tag{2.174}$$

Interesting is the behavior if $\beta(\bar{g})$ has a zero at $\bar{g} = g_0$. Depending on the behavior near this zero, the coupling constant will tend towards g_0 for $\tau \to \infty$ (UV stable point) or for $\tau \to -\infty$ (IR stable). Specifically if $\bar{g} = 0$ is a zero of the β function it depends on the sign of the first coefficient what is the behavior as a function of τ. In QED and QCD we have seen that $\beta(\bar{g}) \propto -b_0\,\bar{g}^3$. If $b_0 > 0$ the point $\bar{g} = 0$ is UV stable, i.e. for $\tau \to \infty$ $\bar{g}(\tau) \to 0$. This situation, which is the case for QCD (at least provided f is not too large) is known as *asymptotic freedom*. Note that QED is not asymptotically free, the coupling constant rises with Q^2.

For QCD it is straightforward to obtain the explicit result for the behavior of the coupling constant, $\alpha_s(\mu^2)$ from the explicit result for the β-function,

$$\frac{\partial \alpha_s(\mu^2)}{\partial \ln(\mu^2)} = -\frac{b_0}{4\pi}\,\alpha_s^2 - \frac{b_1}{(4\pi)^2}\,\alpha_s^3 + \dots \ . \tag{2.175}$$

The solution up to first order is given by

$$\frac{1}{\alpha_s(\mu^2)} = \frac{1}{\alpha_s(\mu_0^2)} + \frac{b_0}{4\pi}\,\ln\left(\frac{\mu^2}{\mu_0^2}\right), \tag{2.176}$$

or introducing the scale-invariant quantity

$$\frac{1}{\alpha_s(\mu_0^2)} - \frac{b_0}{4\pi}\,\ln\mu_0^2 \equiv -\frac{b_0}{4\pi}\,\ln\Lambda^2, \tag{2.177}$$

one can write the asymptotic solution

$$\alpha_s(\mu^2) = \frac{4\pi}{b_0\,\ln(\mu^2/\Lambda^2)}. \tag{2.178}$$

The parameter Λ is known as the *QCD scale parameter*. Note that its definition does not depend on the scale μ but it does depend on the number of flavors and the order of the calculation! Better is therefore, e.g. to fix the scale dependence of QCD by giving, say, $\alpha_s(M_Z^2)$. Experimentally this is about $\alpha_s(M_Z^2) = 0.115 \pm 0.005$.

One should take care in dealing with the running coupling constant, because the scale parameter Λ depends on the number of flavors and the order, i.e. one should properly use $\Lambda^{(f)}_{(order)}$ in conjunction with the equation for the running coupling constant. Putting the transition between $\Lambda^{(f-1)}$ and $\Lambda^{(f)}$ at the threshold at which the new quark flavor (f) must be included (μ_f^2), one has assuming continuity for $\alpha_s(Q^2)$ that

$$\frac{35-2f}{12\pi}\,ln\left(\frac{\Lambda^{(f-1)2}}{\mu_f^2}\right) \equiv \frac{33-2f}{12\pi}\,ln\left(\frac{\Lambda^{(f)2}}{\mu_f^2}\right). \tag{2.179}$$

The result for the scale parameter as a function of the number of flavors changes because of the passing of the various thresholds. Assuming these to be given roughly by the production thresholds for $K\bar{K}$, $D\bar{D}$ or $B\bar{B}$ mesons, which is roughly equal to $4m_q^2$ one finds the ($\overline{MS}$) results below.

f	μ_f^2 [GeV^2]	b_0	$\Lambda^{(f)}_{(1)}$ [MeV]	$\alpha_s(\mu_f^2)$
2		29/3	159	
3	$4M_K^2 \approx 1$	9	140	0.354
4	$4M_D^2 \approx 10$	25/3	110	0.224
5	$4M_B^2 \approx 100$	23/3	75	0.167
6	$4M_t^2 \approx 150{,}000$	7	34	0.096
	M_Z^2			0.115

Also for the second order result a scale invariant definition of Λ can be found,

$$\frac{1}{\alpha_s(\mu^2)} - \frac{b_1}{4\pi\, b_0}\,\ln\left(1+\frac{4\pi\, b_0}{b_1}\,\frac{1}{\alpha_s(\mu^2}\right) - \frac{b_0}{4\pi}\,\ln\mu_0^2 \equiv -\frac{b_0}{4\pi}\,\ln\Lambda^2. \tag{2.180}$$

The results corresponding to $\alpha_s(M_Z^2) = 0.115 \pm 0.005$ are $\Lambda^{(5)}_{(1)} = 73^{+26}_{-20}$ MeV and $\Lambda^{(5)}_{(2)} = 204^{+68}_{-55}$ MeV.

Knowing the behavior of $\bar{g}(\tau)$ it is possible to calculate explicitly the behavior of other quantities such as the running mass $\overline{m}(\tau)$. The anomalous behavior is (using first order results) governed by

$$\int_0^\tau d\tau' \gamma_m(\bar{g}(\tau')) = -\int_{\alpha_s(0)}^{\alpha_s(\tau)} \frac{d\alpha_s}{\alpha_s}\,\frac{\gamma_{m0}}{2b_0} = -\frac{\gamma_{m0}}{2b_0}\,\ln\left(\frac{\alpha_s(\tau)}{\alpha_s(0)}\right). \tag{2.181}$$

The scale dependence of $\overline{m}$ in going from scale μ_0 to μ is therefore

$$\begin{aligned}\overline{m}(\mu) &= \overline{m}(\mu_0)\,\exp\left(-\ln\frac{\mu}{\mu_0} + \frac{\gamma_{m0}}{2b_0}\,\ln\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)\right) \\ &= \frac{\overline{m}(\mu_0)\mu_0}{\mu}\,\left(\frac{\ln(\mu^2/\Lambda^2)}{\ln(\mu_0^2/\Lambda^2)}\right)^{12/(33-2f)}.\end{aligned} \tag{2.182}$$

Also the asymptotic behavior of the Green's functions to first order can be written down explicitly. One obtains by evaluating the integral over $\gamma(\bar{g}) \approx (\gamma_0/4\pi)\,\alpha_s$ the result

$$\Gamma^{(n)}\left(\frac{\mu}{\mu_0}\,p;\overline{g}(\mu_0),\frac{\overline{m}(\mu_0)}{\mu_0},\mu_0\right) = \left(\frac{\mu}{\mu_0}\right)^{4-n}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{n\gamma_0/2b_0}\Gamma^{(n)}\left(p;\overline{g}(\mu),\frac{\overline{m}(\mu)}{\mu_0},\mu_0\right) \tag{2.183}$$

or using the full rescaling properties of the Green's function (Eq. 2.161)

$$\Gamma^{(n)}\left(p;\overline{g}(\mu_0),\frac{\overline{m}(\mu_0)}{\mu},\mu\right) = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{n\gamma_0/2b_0}\Gamma^{(n)}\left(p;\overline{g}(\mu),\frac{\overline{m}(\mu)}{\mu_0},\mu_0\right). \tag{2.184}$$

2.6 Composite operators

We want to conclude this general chapter with the treatment of composite operators such as $\phi^2(x)$, $\overline{\psi}(x)\psi(x)$, $\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$ (symmetrized in μ and ν) and $F_{\mu\alpha}F^{\alpha}_{\ \nu}$. These composite operators contain more fields with the same space-time argument. Adding for a composite operator Ω a source term to the lagrangian density,

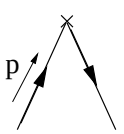
$$\mathcal{L} = \mathcal{L}_{\ 0} + \chi(x)\Omega(x), \tag{2.185}$$

e.g. $\chi\,\overline{\psi}\psi$ or $\frac{1}{2}\,\chi\,\phi^2$ one can obtain Green's functions

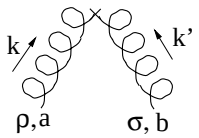
$$G^{(n)}_{\Omega}(x,x_1,\ldots,x_n) = \langle 0|\mathcal{T}\Omega(x)\phi(x_1)\ldots\phi(x_N)|0\rangle. \tag{2.186}$$

from additional vertices for the composite operators introduced along the same line as interaction terms in the lagrangian. Examples are

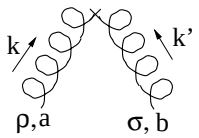
ϕ^2 1,

$i\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$ 

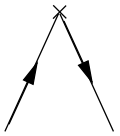
$\frac{1}{2}(\gamma_\mu p_\nu + \gamma_\nu p_\mu)$,

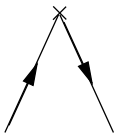
$F_{\mu\alpha}F^{\alpha}_{\ \nu}$ 

$\delta_{ab}[q_\rho k_\mu g_{\nu\sigma} + q_\sigma k'_\nu g_{\mu\rho} + k_\mu k'_\nu g_{\rho\sigma} + (k\cdot k')g_{\mu\rho}g_{\nu\sigma}]$,

$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ 

$2\,\delta_{ab}\,\epsilon_{\mu\nu\rho\sigma}k^\mu k'^\nu$,

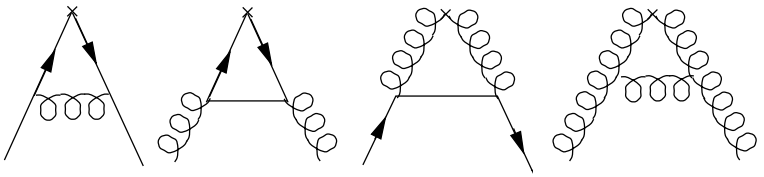
$\overline{\psi}\gamma_\mu\psi$ 
γ_μ,

$\overline{\psi}\gamma_\mu\gamma_5\psi$ 
$\gamma_\mu\gamma_5$,

In general the truncated Green's functions will contain divergences, which need to be regulated and renormalized. One has

$$\Gamma^{(n)}_{\Omega,R}(p,p_i) = Z_\Omega\, Z_\phi^{n/2}\,\Gamma^{(n)}_{\Omega}(p,p_i), \tag{2.187}$$

where Z_Ω can be calculated in a renormalizable theory. In many cases operators with the same dimension d and the same 'quantum numbers' will mix, e.g. $\Omega_F = i\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$ and $\Omega_G = F^a_{\mu\lambda}F^{a\,\lambda}_{\ \ \ \nu}$ mix as should be clear from diagrams like



In this case one has

$$\Gamma^{(n)}_{\Omega_i,R}(p,p_i) = (Z_\Omega)_{ij}\, Z_\phi^{n/2}\,\Gamma^{(n)}_{\Omega_j}(p,p_i). \tag{2.188}$$

It is straightforward to write down the renormalization group equations for the renormalized Green's functions involving composite operators,

$$\left[\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\,\frac{\partial}{\partial g} - \gamma_m(g)\,m\,\frac{\partial}{\partial m} - n\,\gamma(g)\right)\delta_{ij} + (\gamma_\Omega)_{ij}\right]\Gamma^{(n)}_{\Omega_i,R} = 0, \tag{2.189}$$

with the anomalous dimension (matrix) function

$$\gamma_\Omega = \mu\, Z_\Omega\,\frac{\partial Z_\Omega^{-1}}{\partial\mu} = -\mu\,\frac{\partial Z_\Omega}{\partial\mu}\,Z_\Omega^{-1}. \tag{2.190}$$

As an explicit example we give the result for the anomalous dimension (matrix) function $\gamma(g) = \gamma_0 g^2/(4\pi)^2$ for the quark and gluon operators $\Omega_F = i\overline{\psi}\gamma_{\{\mu}\partial_{\nu\}}\psi$ and $\Omega_G = F^a_{\mu\lambda}F^{a\,\lambda}_{\ \ \ \nu} - \frac{1}{4}\,g_{\mu\nu}\,F^a_{\rho\sigma}F^a_{\rho\sigma}$.

$$\begin{bmatrix} \gamma_0^{FF} & \gamma_0^{FG} \\ \gamma_0^{GF} & \gamma_0^{GG} \end{bmatrix} = \begin{bmatrix} \frac{16}{3}\,C_F & -\frac{8}{3}\,f\,T_F \\ -\frac{16}{3}\,C_F & \frac{8}{3}\,f\,T_F \end{bmatrix}. \tag{2.191}$$

The anomalous dimension determines the asymptotic behavior. Naively one has in this case (assuming vertices with $\delta = 0$) for the momentum space truncated Green's function the naive dimension

$$d(\Gamma_\Omega) = d(\Omega) - e_b - \frac{3}{2}\, e_f, \tag{2.192}$$

where $d(\Omega)$ is the canonical dimension of Ω, e.g. $d(\phi^2) = 2$. Completely analogous to the behavior of the ordinary Green's functions one obtains in this case

$$\Gamma_\Omega^{(n)}(e^\tau p, e^\tau p_i; g, \mu) = \exp\left((d(\Omega) - n)\tau + \int_0^\tau d\tau'\, \gamma_\Omega(\bar{g}(\tau', g))\right)\ \Gamma_\Omega^{(n)}(p, p_i; \bar{g}(\tau, g), \mu). \tag{2.193}$$

The first order solution of this is

$$\Gamma_\Omega^{(n)}(p, p_i; g(\mu_0), \mu) = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{(n\gamma_0 - \gamma_{\Omega 0})/2b_0}\ \Gamma_\Omega^{(n)}(p, p_i; g(\mu), \mu_0). \tag{2.194}$$

A special case are the Green's functions involving conserved currents, such as the vector currents

$$\underline{V}^\mu(x) = \overline{\psi}(x)\, \gamma^\mu\, \underline{T}\, \psi(x), \tag{2.195}$$

or the axial vector currents

$$\underline{A}^\mu(x) = \overline{\psi}(x)\, \gamma^\mu \gamma_5\, \underline{T}\, \psi(x), \tag{2.196}$$

where $\underline{T}$ is an operator acting in color and/or flavor space of the quarks. These currents are also the Noether currents for some symmetry group with

$$\underline{Q} = \int d^3x\, \underline{V}^0(x) \tag{2.197}$$

being the generator of the symmetries.

The vector currents satisfy

$$\partial_\mu \underline{V}^\mu = i\, \overline{\psi}\, [\underline{M}, \underline{T}]\, \psi, \tag{2.198}$$

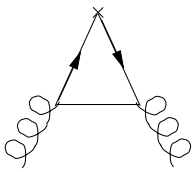
where $\underline{M}$ is the quark mass matrix. We see that when $\underline{M}$ is proportional to the unit matrix in flavor space, one has exact flavor $SU(f)$ symmetry, e.g. isospin symmetry for the two lightest flavors (u and d quarks). There is also a $U(1)_V$ symmetry (corresponding to $\underline{T} = \underline{1}$ in flavor space). Note also that as $\underline{M}$ is a diagonal matrix, any vector current, diagonal in flavor space, e.g. the electromagnetic current, is conserved.
For the axial currents one has

$$\partial_\mu \underline{A}^\mu = i\, \overline{\psi}\, \{\underline{M}, \underline{T}\}\, \gamma_5\, \psi, \tag{2.199}$$

i.e. they are only conserved in the chiral limit, i.e. when $\underline{M} \to 0$, which is reasonably well satisfied for the two lightest flavors.

The presence of the symmetries (commutation relations) and conserved currents gives rise to identities between Green's functions, which among others ensure that Green's functions with currents (e.g. γ_μ for fermions) inserted have anomalous dimension zero. This behavior is not destroyed by the presence of mass terms that break the symmetry. A continuous change in the mass parameter does not modify the anomalous dimensions.

A special case is the singlet axial current with $\underline{T} = \underline{1}$. In this case the above conservation equation is not complete. The result above follows from the equations of motion already at the classical level. At the quantum level, the one loop contribution,



does not vanish, but it is proportional to $\epsilon_{\mu\nu\rho\sigma} k^\mu k'^\nu$, which stands for the insertion of $F_{\mu\nu}\tilde{F}^{\mu\nu}$. To be precise an additional term proportional to this operator is required on the right hand side of the singlet axial operator,

$$\partial_\mu(\overline{\psi}\,\gamma^\mu\gamma_5\,\psi) = 2i\,m\,\overline{\psi}\,\gamma_5\,\psi + \frac{\alpha_s}{2\pi}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}. \tag{2.200}$$

Note that the right-hand side can be written as a total derivative,

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu k^\mu \tag{2.201}$$

with

$$k_\mu = \epsilon_{\mu\nu\rho\sigma}\,A^{\nu a}\,(\partial^\rho A^{\sigma a} - \partial^\sigma A^{\rho a} - \frac{2}{3}g c_{abc} A^{\rho b} A^{\sigma c}). \tag{2.202}$$

A term like this, which arises only at the quantum level, is known as an anomaly term, in this case it is known as the Adler-Bell-Jackiw anomaly or $U(1)_A$ anomaly.

Other examples of conserved currents are those connected to the Lorentz group, such as the energy momentum tensor. The energy momentum tensor actually is proportional to the sum of the operators Ω_F and Ω_G discussed above. Indeed, as can be seen from the γ-function, the anomalous dimension vanishes for this combination.

Chapter 3

Models for hadrons

3.1 Introduction

Many models for hadrons exist. Quark potential models [23] are the earliest models treating quarks explicitly and subjecting them to an interquark potential that confines them. Specifically for the heavy quarks, potential models are useful because they can be treated nonrelativistically using the Schrödinger equation. The latter is one of the main objections agains the use of potential models for the light quarks. Also the bag models [24, 25] treat quarks explicitly. They build in confinement in an ad hoc way, by imposing appropriate boundary conditions that ensure confinement of the quark currents to a region of space called the bag. The advantage is that within this bag quarks can be relativistic. In fact they are usually treated as free particles in accordance with the asymptotic freedom of QCD. Chiral symmetry requires the addition of mesons [26], in order to ensure continuity of axial currents in and outside the bag.

Effective field theories are also employed for the study of hadrons. Most well-known are the Skyrme models, based on an $1/N_c$ expansion of QCD, where N_c is the number of colors [27]. In the limit $N_c \to \infty$, QCD only contains stable mesons with finite masses. Simple counting arguments show that the three-meson (MMM) coupling is proportional to $1/\sqrt{N_c}$ and the MMMM coupling is proportional to $1/N_c$. Baryons in this limit have infinite mass ($\propto N_c$) and finite size ($\propto 1$). The latter behavior is also found for solitons, topologically stable classical solutions arising in field theories with a nontrivial topological structure. An example of such a model is the nonlinear sigma model, with pions as the only degrees of freedom. Provided a higher order coupling term between pions is present, the in principle present topological structure leads to stable soliton solutions, that can be identified with the baryons of QCD.

3.2 Flavor and spin structure of hadrons

The flavor and spin structure of the experimentally observed hadrons can be accounted for by quarks. Baryons are in essence built from three quarks and mesons are built from a quark and an antiquark. The quarks can be considered as belonging to the basic representation of an $SU(N_f)$ flavor symmetry group. Only for $SU(2)$ this symmetry (isospin) is a very good symmetry as the bare quark masses m_u and m_d are much smaller than the QCD scale parameter. Also for three flavors (u, d and s) the symmetry is useful, specifically because it is often only the quark masses which break the symmetry. For the heavy quarks, another $SU(3)$ symmetry involving the c, b and t quarks can be useful when one considers quantities that are independent of the quark masses [21]. The quarks have spin $j = 1/2$, belonging to the basic representation of $SU(2)$. For symmetry considerations it is often useful to consider the three quark flavors with the two spin degrees of freedom as the basic representation of an $SU_{FS}(6)$ flavor-spin symmetry. Antiquarks belong to the conjugate representations of the various

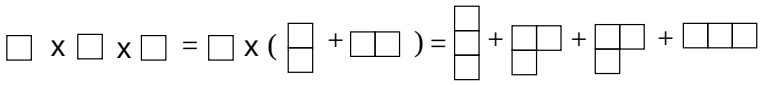


Figure 3.1: determination of the product of basic representations in $SU(N)$ using Young diagrams

symmetry groups.

The spin and flavor structure of baryons and mesons can be found by combining the basic representations [3]. Starting with the q^3 baryons one has for the spin of three quarks

$$SU_J(2): \quad \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}, \tag{3.1}$$

where the spin (j) values are used to denote the (2j+1)-dimensional representations. In this case the decomposition can be read off from the Young diagram rule in Fig. 3.1, noting that a representation in which three basic entities are coupled antisymmetrically is impossible for $SU(2)$, i.e. all columns with more than two boxes are omitted. Furthermore columns of two boxes, corresponding to an antisymmetrized combinations of spin up and spin down, represent the trivial one-dimensional representation. In combination with other boxes they also can be omitted. For the full rules of combining representations using Young diagrams I refer to the review by Itzykson and Nauenberg [22].

For flavor one has

$$SU_F(3): \quad \underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10}, \tag{3.2}$$

where the representations are labeled by their dimensions. For flavor $\otimes$ spin one has

$$SU_{FS}(6): \quad [6] \otimes [6] \otimes [6] = [20] \oplus [70] \oplus [70] \oplus [56], \tag{3.3}$$

where the representations are also labeled by their dimensions. Besides the above quantum numbers color has to be taken into account. Requiring the hadrons to be color singlets, i.e. belonging to the completely antisymmetric $SU_C(3)$ representation, one deduces that the flavor $\otimes$ spin wave function must be completely symmetric, i.e. belong to the [56] representation. The $SU(3) \otimes SU(2)$ content, $(\underline{f}, s)$, of this representation is

$$[56] \supset (\underline{8}, 1/2) + (\underline{10}, 3/2). \tag{3.4}$$

The explicit flavor-spin wave functions for the proton can be constructed by symmetrizing the direct product of flavor and spin wave functions. Only the combinations in the [56] representation of $SU(6)$ will give a nonvanishing result. For the proton one gets

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\left(2\ u_\uparrow u_\uparrow d_\downarrow - u_\uparrow u_\downarrow d_\uparrow - u_\downarrow u_\uparrow d_\uparrow + \ [d \text{ at places 1 and 2}]\ \right).\Big), \tag{3.5}$$

and for the lambda

$$|\Lambda\uparrow\rangle = \frac{1}{\sqrt{12}}\left(u_\uparrow d_\downarrow s_\uparrow - u_\downarrow d_\uparrow s_\uparrow - d_\uparrow u_\downarrow s_\uparrow + d_\downarrow u_\uparrow s_\uparrow + \ [s_\uparrow \text{ at places 1 and 2}]\ \right). \tag{3.6}$$

Other wave functions can be obtained directly by replacements $u \to d$ (e.g. neutron from proton) or $u \to s$ (Σ^- from n). Combining spatial wave functions (accounting for total antisymmetry of the wave function) one gets in a simple quark shell-model the allowed baryon spectrum with $([fs], L^P)$ states,

$$([56], 0^+) \oplus ([70], 1^-) \oplus ([56], 0^+) \oplus \ldots, \tag{3.7}$$

combining to states with $J = L + s, \ldots, L - s$.

For the spin of $q\bar{q}$ mesons all combinations

$$SU_J(2): \quad \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1, \tag{3.8}$$

$$SU_F(3): \quad \underline{3} \otimes \underline{3^*} = \underline{1} \oplus \underline{8}, \tag{3.9}$$

are possible. Including orbital excitations the allowed J^{PC} values are, for $L = 0, 1, 2, \ldots$, states with $J = L - S, \ldots, L + S$, $P = (-)^{L+1}$, and for neutral mesons $C = L + S$. C-parity is useful as it determines the number of photons to which the meson couples, $C = (-)^{n_\gamma}$. For mesons $G-parity$, $G = (-)^{L+S+I}$ is useful, which determines the number of pions to which the meson couples, $C = (-)^{n_\pi}$.

3.3 Properties of hadrons and form factors

In this section we will study the currents of the quarks (j^μ), which play an important role as they are connected to symmetries in the theory (Noether's theorem) and give rise to a conserved quantity,

$$\partial_\mu j^\mu(x) = 0 \quad \leftrightarrow \quad Q = \int d^3x \; j^0(x) \qquad \text{is conserved.} \tag{3.10}$$

Examples are the electromagnetic current (conserved electric charge Q) and the baryon current (conserved baryon number B) corresponding to $U(1)$ transformations generated by the charge operator or the unit operator, respectively. Another example form the flavor currents such as the isovector currents (conserved isospin) and the strangeness currents (conserved strangeness S or hypercharge $Y = B + S$.

For the quarks we will consider in more detail the vector and axial vector currents, V_μ^k and A_μ^k,

$$V_\mu^k(x) = \overline{\psi}_i(x)\gamma_\mu(T^k)_{ij}\psi_j(x), \tag{3.11}$$

$$A_\mu^k(x) = \overline{\psi}_i(x)\gamma_\mu\gamma_5(T^k)_{ij}\psi_j(x), \tag{3.12}$$

where the T^k can be various operators acting on the quark quantum numbers, color, flavor, charge. For instance the electromagnetic current is given by

$$Q = \begin{pmatrix} 2/3 & & & \\ & -1/3 & & \\ & & -1/3 & \\ & & & \ddots \end{pmatrix} \tag{3.13}$$

or explicitly

$$J_\mu^{em} = (2/3)\,\overline{\psi^u}\gamma_\mu\psi^u - (1/3)\,\overline{\psi^d}\gamma_\mu\psi^d - (1/3)\,\overline{\psi^s}\gamma_\mu\psi^s + \ldots, \tag{3.14}$$

where ψ^q indicates the Dirac fields for flavor q.

For an elementary fermion, e.g. an electron the expectation value of the vector and axial vector currents are trivial,

$$\langle p', s'|V_\mu(x)|p, s\rangle = \bar{u}(p', s')\gamma_\mu u(p, s)\, e^{i(p'-p)\cdot x}, \tag{3.15}$$

$$\langle p', s'|A_\mu(x)|p, s\rangle = \bar{u}(p', s')\gamma_\mu u(p, s)\, e^{i(p'-p)\cdot x}. \tag{3.16}$$

For composite systems the expectation values of the current operator in Eq. 4.45 is in general unknown. One can however write down a most general form and use translation invariance. The latter states

$$\langle P', S'|J_\mu(x)|P, S\rangle = e^{i(P'-P)\cdot x}\langle P', S'|J_\mu(0)|P, S\rangle. \tag{3.17}$$

The most general form consistent with the requirements of Poincaré invariance, gauge invariance and invariance under parity and time reversal yields for a spin 0 system (e.g. a pion or a ^{12}C nucleus) the form

$$\langle P', S'|V_\mu(0)|P, S\rangle = (P_\mu + P'_\mu)\, F(Q^2), \tag{3.18}$$

involving one invariant function (*form factor*) depending on the (only) invariant, $q^2 \equiv -Q^2$, where $q = P' - P$. For spin 1/2 systems (such as a nucleon, ^3H or ^3He) the most general expression for the expectation value of the current can be written using free Dirac spinors $U(P,S)$,

$$\langle P',S'|V_\mu(0)|P,S\rangle = \overline{U}(P',S')\left[\gamma_\mu F_1(Q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}F_2(Q^2)\right]U(P,S), \tag{3.19}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu]$ and $q = P' - P$. The functions F_1 and F_2 are the Dirac and Pauli form factors. For the forward matrix element one obtains using $\overline{U}(P,S')\gamma_\mu U(P,S) = 2P_\mu\,\delta_{SS'}$,

$$\langle P,S'|V_\mu(0)|P,S\rangle = 2P_\mu\,\delta_{SS'}\,F_1(0). \tag{3.20}$$

For the (diagonal) vector currents $V^q_\mu = \overline{\psi}^q\gamma_\mu\psi^q$ the quantity $F^q_1(0)$ is precisely the quark numbers $F^q_1(0) = n_q$, in accordance with the matrix element of the electromagnetic current J^{em}_μ (given above) satisfying $F^{em}_1(0) = Q$ (the charge of the hadron state).

Form factors for spacelike q probe the spatial distribution of the charges. Note that there always exist a frame in which q is purely spacelike ($q^0 = 0$). For the interpretation of the significance of the form factors this frame (brick-wall frame) is therefore particularly useful. Working out the current expression for the nucleon in the Breit frame ($P'^0 = P^0$, $\boldsymbol{P} = -\boldsymbol{q}/2$, $\boldsymbol{P'} = +\boldsymbol{q}/2$), gives

$$\langle P',S'|J^{em}_0|P,S\rangle = 2M\left[F_1 - \frac{Q^2}{4M^2}F_2\right] \equiv 2M\,G_E(Q^2)e^{iq\cdot x}, \tag{3.21}$$

$$\langle P',S'|\boldsymbol{J}^{em}|P,S\rangle = [F_1+F_2]\;(i\boldsymbol{\sigma}_N\times\boldsymbol{q})e^{iq\cdot x} \equiv G_M(Q^2)\,(i\boldsymbol{\sigma}_N\times\boldsymbol{q})e^{iq\cdot x}, \tag{3.22}$$

where $\chi^\dagger_{S'}\,\boldsymbol{\sigma}\,\chi_S \equiv \boldsymbol{\sigma}_N$. These expressions show the relevance of the Sachs form factors G_E and G_M. The quantity $e\,G_E(0)$ is the charge Q of the nucleon, $e\,G_M(0)/M$ is the magnetic moment of the nucleon. The quantity $\kappa = G_M(0) - Q = G_M(0) - G_E(0)$ is the anomalous magnetic moment.

For an elementary fermion one has $F_1(Q^2) = 1$ and $F_2(Q^2) = 0$. For the nucleon one has for the proton $F^p_1(0) = Q_p = 1$ and $F^p_2(0) = \kappa_p = 1.793$ and for the neutron $F^n_1(0) = Q_n = 1$ and $F^n_2(0) =$ $\kappa_n =$ -1.913. Roughly one has for nucleons the dipole-like behavior

$$G^p_E(Q^2) \approx \frac{G^p_M}{\mu_p} \approx \frac{G^n_M}{\mu_n} \approx \frac{1}{(1+Q^2/\Lambda^2)^2}, \tag{3.23}$$

with $\Lambda^2 \approx 0.71$ GeV^2 and $\mu_N = G^N_M(0) = Q_N + \kappa_N$ is the nucleon magnetic moment.

For the axial current the expectation value for a spin 1/2 state (e.g. nucleon) can be written as

$$\langle P',S'|A_\mu(x)|P,S\rangle = \overline{U}(P',S')\left[\gamma_\mu\;G_A(Q^2) + q_\mu\;H_A(Q^2)\right]\gamma_5\,U(P,S)\,e^{-iq\cdot x}. \tag{3.24}$$

The value at $Q^2 = 0$ which using the spin vector $\overline{U}(P,S)\,\gamma_\mu\gamma_5\,U(P,S) = 2M\,S_\mu(P)$ (satisfying $S^2 =$ -1, $P\cdot S = 0$) appears in the forward matrix element,

$$\langle P,S|A_\mu(0)|P,S\rangle = 2M\,S_\mu(P)\,G_A(0). \tag{3.25}$$

For the diagonal axial vector currents $A^q_\mu = \overline{\psi^q}\gamma_\mu\gamma_5\psi^q$ the quantity $G^q_A(0) \equiv \Delta q$. Again using the brick wall frame and taking $\boldsymbol{q} = 0$ one sees

$$\langle P,S|\boldsymbol{A}(0)|P,S\rangle = 2M\,\boldsymbol{\sigma}_N\,G_A(0). \tag{3.26}$$

The quantity $G_A(0)$ is the nucleon axial charge. Its value,$G_A(0) = 1.26$, is obtained from neutron β-decay where one measures a flavor-changing axial current, $\langle p|\mathbf{A}_+|n\rangle = 2M\,G_A(0)\,\langle p|\boldsymbol{\sigma}_N\,\tau_{N+}|n\rangle$. Using isospin invariance one sees that $G_A(0)$ can be expressed as $G_A(0) = \Delta u$ - Δd.

3.4 Quark potential models

Quark potential models are the oldest models in the study of hadron structure. In their simplest form quarks move in a background potential, leading to a shell model structure. The groundstate baryons, the $1/2^+$ octet and $3/2^+$ decouplet, and the groundstate mesons, the 0^- and 1^- nonets, only contain the lowest s-wave solution. An often used potential is the harmonic oscillator, which also allows treatment of excited baryons and mesons, because it is easy to separate center of mass and relative coordinates. Clearly quark potential models may be expected to work best for heavy quark systems. For such systems more realistic potentials than the abovementioned harmonic oscillator have been used, behaving as $\alpha_s(r)/r$ for short distances in accordance with asymptotic freedom and as $T_0\, r$ corresponding to a linear confining potential for large distances. The typical value of the linear energy density (or string density) is typically $T_0 \approx 0.18$ GeV$^2 \approx 0.9$ GeV/fm. This relates to the slope of meson Regge trajectories, $J = \alpha_0 + \alpha'\, M^2$ via the relation $1/\alpha' = 2\pi\, T_0 \approx 1.1$ GeV^2.

The relation between the slope of Regge trajectories and the string tension can be obtained with the following semi-classical argument (noting that this is the limit of the quantum mechanical treatment for large angular momenta). Consider a rotating string, rotating around an axis perpendicular on the string, the ends (with massless quarks) moving with $v = \pm 1$. In the rotating string the energy-density along the string is given by $T_0\, \gamma = T_0/\sqrt{1-v^2}$. This leads (using $dr = dv/\omega$ with ω the angular velocity) to the following expressions for mass and angular momentum,

$$\begin{aligned} M &= \frac{T_0}{\omega}\int_{-1}^{1} dv\, \frac{1}{\sqrt{1-v^2}} = \frac{\pi T_0}{\omega}, \\ J &= \frac{T_0}{\omega^2}\int_{-1}^{1} dv\, \frac{v^2}{\sqrt{1-v^2}} = \frac{\pi T_0}{2\omega^2}. \end{aligned}$$

Thus one finds

$$M^2 = \frac{\pi^2\, T_0^2}{\omega^2} = 2\pi T_0\, J,$$

the linear relation between mass squared of orbitally excited states and the angular momentum.

Except for the confinement mechanism another important feature in models with explicit quark degrees of freedom is the hyperfine structure due to the color-magnetic part of the one-gluon exchange interaction. It is of the form

$$H_{hf} = \alpha_s\, M\, \Delta \tag{3.27}$$

with

$$\Delta \;=\; -\sum_{i>j} \langle (\boldsymbol{\sigma} F^a)_i \cdot (\boldsymbol{\sigma} F^a)_j \rangle, \tag{3.28}$$

where F^a indicate the color charges, for the quarks represented by $F_q^a = \lambda^a/2$, for the antiquarks by $F_{\bar{q}}^a = -\lambda^{a*}/2$. The quantity M depend on the model. In quark potential models it could be a hyperfine interaction $\propto 1/r^3$

The operator Δ can be calculated straightforwardly. For instance, for a meson we must form a color singlet. Since $\langle F^2 \rangle = \langle (F_q + F_{\bar{q}})^2 \rangle = 0$ and $\langle F_q^2 \rangle = \langle F_{\bar{q}}^2 \rangle = 4/3$, one obtains $\langle F_q \cdot F_{\bar{q}} \rangle = -4/3$. Similarly manipulations are well-known for the spin, yielding $\langle \boldsymbol{\sigma}_q \cdot \boldsymbol{\sigma}_{\bar{q}} \rangle = -3$ for a spin 0 meson and $\langle \boldsymbol{\sigma}_q \cdot \boldsymbol{\sigma}_{\bar{q}} \rangle = +1$ for a spin 1 meson. Similarly one can find for the baryons $\langle F_1 \cdot F_2 \rangle = -2/3$ for any pair. For the spin the explicit wave function can be used to find the result for Δ. For the case of two quarks (coupling to a color triplet c = $\underline{3}$), a quark and an antiquark and three quarks (baryons) the expectation value of Δ for some specific configurations, namely a diquark (system of two quarks), mesons and baryons are as follows.

quark configuration	spin	flavor (isospin)	Δ
qq	0	$\underline{3}^*$ (0)	-2
	1	$\underline{6}$ (1)	$+2/3$
$q\bar{q}$	0	$\underline{1} \oplus \underline{8}$ $(0 \oplus 1)$	-4
	1	$\underline{1} \oplus \underline{8}$ $(0 \oplus 1)$	$+4/3$
qqq	1/2	$\underline{8}$ (1/2)	-2
	3/2	$\underline{10}$ (3/2)	$+2$

3.5 The bag model

As an example of a model which incorporates asymptotic freedom and confinement is the (MIT) bag model [28]. In essence confinement is obtained by ensuring that quarks cannot flow out of a certain volume, called *the bag*. For this purpose the quark current, $j_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$, is replaced by $j_{B\mu} \equiv j_\mu(x).\theta_B(x)$, where θ_B is a theta-function with the value one within the bag volume. The surface normal is indicated as n_μ, satisfying $\partial_\mu\theta_B(x) = n_\mu\delta_B(x)$. Current conservation in the bag model therefore requires that

$$\partial_\mu j_B^\mu(x) = (\partial_\mu j^\mu(x))\,\theta_B(x) + n_\mu j^\mu(x)\,\delta_B(x). \tag{3.29}$$

The boundary condition thus is $n_\mu j^\mu = 0$ at the surface of the bag. For the quark current a sufficient condition is $\not{n}\psi = \psi$. Applied to a free s-wave Dirac spinor,

$$\psi = \frac{N}{\sqrt{4\pi}} \begin{bmatrix} i\,F(r) \\ -\,\boldsymbol{\sigma}\cdot\hat{r}\,G(r) \end{bmatrix} e^{-iEt}, \tag{3.30}$$

with

$$F(r) = \sqrt{\frac{E+m}{E}}\,j_0(kr), \tag{3.31}$$

$$G(r) = \sqrt{\frac{E-m}{E}}\,j_1(kr). \tag{3.32}$$

The confinement condition on this spinor leads for a spherical bag with radius R to the condition

$$\tan x = \frac{x}{1-\lambda-\alpha}, \tag{3.33}$$

where $x = kR$, $\alpha = ER$ and $\lambda = mR$. The lowest energy solution for a massless quark yields

$$m = 0; \qquad x = \alpha = 2.043, \tag{3.34}$$

and for what turns out to be an appropriate value for the strange quark ($m \approx 280$ MeV and $R \approx 5\ \mathrm{GeV}^{-1} \approx 1$ fm)

$$\lambda = 1.4; \qquad x = 2.489; \qquad \alpha = 2.856. \tag{3.35}$$

For a specific quark mode in the bag model the normalization becomes

$$\frac{N^2}{4\pi} = \frac{\alpha(\alpha-\lambda)}{2\alpha(\alpha-1)+\lambda}\,\frac{1}{4\pi\,R^3\,j_0^2(x)}. \tag{3.36}$$

The equations of motion for the bag model including the boundary conditions can also be obtained in a lagrangian formulation. One needs

$$\mathcal{L} = (i\,\overline{\psi}\not{\partial}\psi - m\,\overline{\psi}\psi - B)\theta_B(x) - \frac{1}{2}\,\overline{\psi}\psi\,\delta_B(x). \tag{3.37}$$

The bag pressure B is essential to obtain stable solutions. It gives rise to a volume energy $\propto BR^3$ preventing the bag to blow up and balancing the quark kinetic energies $\propto N\,\alpha/R$. Its value is something like $B^{1/4} = 0.15$ GeV.

The hyperfine interaction in the bag model is of the form discussed in the previous paragraph. The quantity M in this case simply is a function of the mode energy which depends on $\lambda_i = m_i R$ for the two quarks involved, $M(m_i, m_j, R) = M(\lambda_i, \lambda_j)$ with $M(0,0) = 0.177$, $M(0, 1.4) = 0.143$ and $M(1.4, 1.4) = 0.116$.

In principle the structure of the nucleon in terms of the quarks combined with the electromagnetic current in terms of the quarks determines the form factors. For example in the bag model the effects of the quarks is additive with the expectation value of the zero and three-vector components given by the elementary quark currents

$$\begin{aligned}
\langle \mathrm{Bag}|J^0(x)|\mathrm{Bag}\rangle &= \sum_i e_i\ \rho_i(x) && (3.38)\\
\rho(x) &= \overline{\psi}(x)\gamma^0\psi(x) = \psi^\dagger(x)\psi(x) = \frac{N^2}{4\pi}(F^2+G^2), && (3.39)\\
\langle \mathrm{Bag}|\mathbf{J}(x)|\mathrm{Bag}\rangle &= \sum_i e_i\ \mathbf{j}_i(x) && (3.40)\\
\mathbf{j}(x) &= \overline{\psi}(x)\gamma\psi(x) = \psi^\dagger(x)\rho_1\sigma\psi(x)\\
&= \frac{N^2}{4\pi}\,F.G\,(\boldsymbol{\sigma}\times\hat{r}) \equiv \frac{f(r)}{r^2}\,(\boldsymbol{\sigma}\times\mathbf{r}). && (3.41)
\end{aligned}$$

Taking the Fourier transform, e.g. of $\mathbf{j}(\mathbf{r})$ we obtain

$$\mathbf{j}(\boldsymbol{q}) = \sum_i e_i\,(i\boldsymbol{\sigma}_i\times\boldsymbol{q})G_M^{bag}(\boldsymbol{q}), \qquad (3.42)$$

with for $\mu_i^{bag} = G_M^{bag}(0)$ [25]

$$\begin{aligned}
\mu_i^{bag} = \int d^3r\ f(r)/3 &= \frac{R}{6}\,\frac{4\alpha+2\lambda-3}{2\alpha(\alpha-1)+\lambda} && (3.43)\\
&= \frac{1}{2m}\,\frac{\lambda}{3}\,\frac{4\alpha+2\lambda-3}{2\alpha(\alpha-1)+\lambda} && (3.44)\\
&\overset{m=0}{=} \frac{4x-3}{12x(x-1)}\,R = 0.202\,R. && (3.45)
\end{aligned}$$

The latter value, $\mu_n^{bag} = 0.202\,R$ is appropriate for the (almost) massless up and down quarks, for a strange quark with $\lambda = mR = 1.4$ the result is $\mu_s^{bag} = 0.156\,R = 0.436/2m_s$. The flavor dependence of the magnetic moments are as in the nonrelativistic quark model further determined by the expectation values of $\sum e_i\,\mu_i^{bag}\,\sigma_{zi}$, i.e.

$$\begin{aligned}
\frac{\mu_p}{2M} = \frac{G_M^p(0)}{2M} &= \langle p|\sum e_i\,\mu_i^{bag}\,\sigma_{zi}|p\rangle\\
&= \frac{4}{3}\,e_u\,\mu_u^{bag} - \frac{1}{3}\,e_d\,\mu_d^{bag} = \mu_n^{bag}, && (3.46)
\end{aligned}$$

and in a similar way obtain

$$\begin{aligned}
\frac{\mu_n}{2M} &= -\frac{1}{3}\,e_u\,\mu_u^{bag} + \frac{4}{3}\,e_d\,\mu_d^{bag} = -\frac{2}{3}\,\mu_n^{bag}, && (3.47)\\
\frac{\mu_\Lambda}{2M} &= e_s\,\mu_s^{bag} = -\frac{1}{3}\mu_s^{bag}. && (3.48)
\end{aligned}$$

With a characteristic bag radius $R \approx 1$ fm the result for the proton is $\mu_p = 2M\,\mu_n^{bag} \approx 1.9$. The experimental result for the proton is $\mu_p = 2.79$. One problem the bag model - or in any other static

Table 3.1: The quark model G_A/G_V ratios. One can see that the $SU(6)$ results are reproduced for $F = 2/3$ and $D = 1$. For the bag model result only an F and D can be calculated for the $s \to u$ and $d \to u$ transitions separately.

transition	$(G_A/G_V)_{QM}$	$(G_A/G_V)_{SU(3)}$	$(G_A/G_V)_{bag}$	experiment
$n \to p$	$5/3$	$F + D$	1.09	1.259 ± 0.004
$\Sigma^- \to n$	$-1/3$	$F - D$	-0.24	-0.362 ± 0.043
$\Lambda \to p$	1	$F + \frac{1}{3}D$	0.72	-0.694 ± 0.025
$\Xi^- \to \Lambda$	$1/3$	$F - \frac{1}{3}D$	0.24	0.25 ± 0.05
$\Xi^- \to \Sigma^0$	$5/3$	$F + D$	1.20	

extended hadron model - is the treatment of the center of mass motion [29]. The calculation has been performed using a static bag, which is not an momentum eigenstate, such as appears in the nucleon current expectation value. At best a bag can be considered as a superposition of plane waves,

$$|Bag\rangle = \int \frac{d^3P}{(2\pi)^3}\ \Phi(\boldsymbol{P})\,|\boldsymbol{P}\rangle. \tag{3.49}$$

For such a superposition one has $\langle \boldsymbol{P}\rangle = 0$ but $\langle \boldsymbol{P}^2\rangle \neq 0$. One can estimate the corrections for the magnetic moments yielding a correction of about +10 %. Note the successful calculation of the ratios of the magnetic moments, $\mu_n/\mu_p = -2/3$ (exp: -0.685) and $\mu_\Lambda/\mu_p = -0.26$ (exp: -0.22), which are a success for the naive additive quark model based on $SU(6)$ symmetry.

The axial current density can also be calculated in the bag model. For a quark one has $A^0 = \psi^\dagger\gamma_5\psi = 0$ and

$$\mathbf{A} = \psi^\dagger\boldsymbol{\sigma}\psi = \frac{N^2}{2\pi}\left[\left(F^2+G^2\right)\boldsymbol{\sigma} + 2G^2\,(\boldsymbol{\sigma}\cdot\hat{r})\hat{r}\right]. \tag{3.50}$$

For $G_A(0)$ we obtain for a transition of quark mode i to quark mode j

$$G^{bag}_{A,i-j} = \frac{N^2}{4\pi}\int d^3r\ \left(F^2 - \frac{1}{3}G^2\right) \tag{3.51}$$

$$= \frac{2x_ix_j}{3(x_i^2-x_j^2)}\ \frac{2(\alpha_i-\alpha_j)+(\lambda_i-\lambda_j)}{\sqrt{2\alpha_i(\alpha_i-1)+\lambda_i}\,\sqrt{2\alpha_j(\alpha_j-1)+\lambda_j}} \tag{3.52}$$

$$\overset{\lambda_i=\lambda_j}{=} \frac{2\alpha^2+4\lambda\alpha-3\lambda}{3\left[2\alpha(\alpha-1)+\lambda\right]} \tag{3.53}$$

$$\overset{\lambda=0}{=} \frac{x}{3(x-1)}. \tag{3.54}$$

Explicitly one has for massless nonstrange quarks $G^{bag}_{A,n-n} = 0.653$. For the strange quark one finds $G^{bag}_{A,s-n} = 0.710$ and $G^{bag}_{A,s-s} = 0.781$. In order to get the results in the bag model, one further has to calculate the group theoretical contribution. The result is

$$G_A^{n\to p} = G^{bag}_{A,d\to u}\langle p|\sum_i F_i^{d\to u}\sigma_{zi}|n\rangle = \frac{5}{3}G^{bag}_{A,d\to u}, \tag{3.55}$$

where the flavor operator $F^{d\to u}$ denotes $(\lambda_1 + i\lambda_2)/2$ or for two flavors $(\tau_1 + i\tau_2)/2$. It is common to compare the group-theoretical result with the (vector current) matrix element $\langle p|\sum_i F_i^{d\to u}|n\rangle$, which equals one, i.e. the quark model result is $(G_A/G_V)_{QM} = 5/3$. The full result, $G_A = \frac{5}{3}G^{bag}_{A,d\to u} = 1.09$ is in reasonable agreement with the experimental result, specifically if one includes center of mass

corrections which in this case can be estimated to be about +6 %. For other cases, like the $\Sigma^- \to ne^-\bar{\nu}$ decay we have to use the $G^{bag}_{A,s\to u}$ result. In fact the overlap between modes corresponding to different masses also deviates from one, to be precise

$$G^{bag}_{V,i-j} = \frac{N^2}{4\pi}\int d^3r \ \left(F^2+G^2\right) \tag{3.56}$$

$$= \frac{2x_ix_j}{(x_i^2-x_j^2)}\frac{(\lambda_i-\lambda_j)}{\sqrt{2\alpha_i(\alpha_i-1)+\lambda_i}\,\sqrt{2\alpha_j(\alpha_j-1)+\lambda_j}}, \tag{3.57}$$

for instance $G^{bag}_{V,n\to s} = 0.985$. The results for G_A/G_V in the quark model can be found explicitly using the spin-isospin wave functions. The results for the various decays are in fact related because they are proportional to Clebsch-Gordan coefficients for $\underline{8}\otimes\underline{8}\to\underline{8}$. Since there occur two octet representations in the product, the ratios can be expressed in terms of two reduced matrix elements, usually referred to as F and D. Using the explicit calculations or Clebsch-Gordan coefficients [30, 31] we get the result in tabel 3.1. Note that for example for $\Lambda\to p$ one has $(G_A)_{QM} = (G_V)_{QM} = \sqrt{3/2}$.

3.6 Chiral lagrangians and the Skyrme model

From Noethers theorem or by using explicitly the Dirac equation it is easy to see that for the vector currents,

$$\partial_\mu \underline{V}^\mu = i\overline{\psi}\,[\underline{M},\underline{Q}]\,\psi, \tag{3.58}$$

where the underlining indicates operators in flavor space. Since $\underline{M}$ is the diagonal mass matrix for the quarks, one sees immediately that diagonal flavor operators like the baryon current, $\underline{B} = \frac{1}{3}\underline{I}$, the charge current or the strangeness current are conserved. If the quark mass operator would be of the form $\underline{M} = m\underline{I}$ for instance for three flavors one would have an exact flavor $SU(3)$ symmetry. For the axial currents,

$$\partial_\mu \underline{A}^\mu = i\overline{\psi}\,\{\underline{M},\underline{Q}\}\,\psi, \tag{3.59}$$

which only vanishes if the quark masses vanish, i.e. is approximately good for the up and down quarks. In that case one has in QCD an $SU_L(2)\otimes SU_R(2)$ symmetry corresponding to the left and righthanded currents $\underline{J}^\mu_{R/L} = \frac{1}{2}\overline{\psi}\gamma_\mu(1\pm\gamma_5)\underline{Q}\psi$. For the flavor singlet case (unit matrix) the vector current is conserved and connected to the conserved fermion number. The axial current is not conserved. The axial $U_A(1)$ symmetry is broken at the quantum level by the Adler-Bell-Jackiw anomaly [32],

$$\partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi) = \frac{\alpha_s}{4\pi}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}. \tag{3.60}$$

Presence of the symmetry $SU_L(2)\otimes SU_R(2)\otimes U_V(1)$ in nature in the Wigner mode (i.e. $Q_i|0\rangle = 0$ for all generators of the symmetry) would imply among others that isospin multiplets would appear also in parity doublets. This is clearly not the case. Only the isospin generators annihilate the vacuum, leading to isospin multiplets (e.g. nucleon doublet, pion triplet etc.). This means that only the diagonal symmetry is realized, and the full symmetry is spontaneously broken,

$$SU_L(2)\otimes SU_R(2)\otimes U_V(1) \ \longrightarrow \ SU_V(2)\otimes U_V(1).$$

This means that there is a (parity changing) triplet of generators for which $Q_i|0\rangle \neq 0$, creating an isotriplet of 0^- massless Goldstone bosons.

Consider explicitly QCD with two flavors, omitting all gluons which do not care about the flavor symmetry,

$$\mathcal{L} = i\,\overline{\psi}\partial\!\!\!/\psi - \overline{\psi}M\,\psi, \tag{3.61}$$

where ψ and M now have also an isospin structure,

$$\psi = \begin{pmatrix}\psi^u\\ \psi^d\end{pmatrix}, \qquad M = \begin{pmatrix}m_u & 0\\ 0 & m_d\end{pmatrix}. \tag{3.62}$$

Projecting on righthanded and lefthanded fermions $\psi_{R/L} = P_{R/L}\,\psi$ with $P_{R/L} = (1 \pm \gamma_5)/2$ one obtains

$$\mathcal{L} = i\,\overline{\psi}_R \not{\partial} \psi_R + i\,\overline{\psi}_L \not{\partial} \psi_L - \overline{\psi}_R M \psi_L - \overline{\psi}_L M \psi_R, \tag{3.63}$$

which shows immediately that if $M = 0$ the lagrangian is invariant under $SU_L(2) \otimes SU_R(2)$ (*chiral symmetry*),

$$SU_R(2): \ \psi_R \to V_R\,\psi_R \qquad \text{with} \qquad V_R = \exp\left(i\boldsymbol{\alpha}_R \cdot \boldsymbol{T}\right), \tag{3.64}$$

$$SU_L(2): \ \psi_L \to V_L\,\psi_L \qquad \text{with} \qquad V_L = \exp\left(i\boldsymbol{\alpha}_L \cdot \boldsymbol{T}\right), \tag{3.65}$$

with for two flavors $\boldsymbol{T} = \boldsymbol{\tau}/2$. In nature only the symmetry $V_R = V_L$ is realized, i.e. $\psi \to V\,\psi$. The mechanism underlying the spontaneous symmetry breaking may be *instantons* in QCD. The fact that $M \neq 0$ exactly, implies that $m_\pi \neq 0$.

The implications of chiral symmetry in the hadronic world can be studied in socalled effective theories. For instance one expects that in the mesonic sector (which in the large N_c limit is the most important) the Goldstone bosons will be the lightest ones which dominate the physics at low energies. Representing meson fields by a unitary $N_f \times N_f$ matrix,

$$U = \exp\left(i\,\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{f_\pi}\right) = \frac{1}{f_\pi}\left(\sigma + i\,\boldsymbol{\tau}\cdot\boldsymbol{\phi}\right), \tag{3.66}$$

with for the latter representation $\sigma^2 + i\boldsymbol{\phi}^2 = f_\pi^2$ the simplest term in the lagrangian is

$$\mathcal{L} = \frac{f_\pi^2}{4}\,Tr\left(\partial_\mu U\,\partial^\mu U^\dagger\right) \approx \frac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} + \ldots, \tag{3.67}$$

which is invariant under

$$U \ \longrightarrow \ V_L\,U\,V_R^\dagger. \tag{3.68}$$

The symmetry is spontaneously broken, that is the groundstate, which is a constant unitary matrix, say $U_0 \equiv 1$, is only invariant under a combined left-right multiplication, $V_L = V_R$. The symmetry, therefore, is spontaneously broken to $SU_V(2)$, the isospin (or in general flavor) symmetry group. Connected with such a spontaneous symmetry breaking there are 3 (massless) Goldstone modes. With the choice of groundstate ($U_0 = 1$) they are precisely represented as the 2×2 unitary matrices U, so the lagrangian above also represents the result after breaking the symmetry.

Already early in the sixties, Skyrme noted that the possibility of topological excitations exists in this theory. In order to find out if there are classical (time-independent) solutions with finite energy that approach the vacuum, $U(\boldsymbol{r}) \to 1$ for $|\boldsymbol{r}| \to \infty$ one must consider the mappings from R^3 with all points at infinity identified ($\sim S^3$) to the parameter-space of the fields, i.e. $SU(2) \sim S^3$. The topological structure of these mappings is $\Pi_3(SU(2)) = \Pi_3(S^3) = Z$. A representative solution in each class is the hedgehog solution,

$$U_c(\boldsymbol{r}) = \exp\left(i\boldsymbol{\tau}\cdot\boldsymbol{e}_\pi\,F(r)\right) = \cos F(r) + i\boldsymbol{\tau}\cdot\boldsymbol{e}_\pi(\boldsymbol{r})\,\sin F(r), \tag{3.69}$$

with e.g. for the simplest nontrivial solution $\boldsymbol{e}_\pi(\boldsymbol{r}) = \boldsymbol{r}/r$. The angle F is referred to as the chiral angle and satisfies $F(r) \to 0$ for $r \to \infty$. To find univalued solutions at the origin one most have $F(0) = n\pi$ with n integer, exhibiting the topological structure Z. The mappings can be characterized by a topological quantum number that has no dynamical origin. Nevertheless it can be connected to a conserved (topological) current, in this case

$$j^\mu = \frac{1}{24\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,Tr(L_\nu L_\rho L_\sigma), \tag{3.70}$$

where $L_m u \equiv U^\dagger\partial_\mu U$ (the Maurer-Cartan forms). The normalization of the current is chosen such that n corresponds to the topological charge,

$$\int d^3x\,j^0(x) = -\int_0^\infty dr\,\frac{2\,F'(r)\,\sin^2 F(r)}{\pi} = \frac{F(0) - F(\infty)}{\pi} = n, \tag{3.71}$$

where $F' = dF/dr$. The solitons in this model are in fact the groundstates in different sectors of the Hilbert space, already by Skyrme identified with sectors with nonzero baryon number. The topological current is identified as the baryon current. We will not discuss this here in further detail, nor the study of the excitation spectrum in the baryon sector. We only mention the success of the model for the description of groundstate baryon excitations and the nucleon-pion phase shifts, all of this achieved by studying and (semi-classically) quantizing the fluctuations of the classical skyrmion solutions. We note that the simple first order term indicated in the effective lagrangian suffices to discuss the possibility of nontrivial classical solutions, but to obtain stable solutions from the equations of motion requires higher order terms, which by the way are quite natural in an effective theory.

A nice example to illustrate how symmetries can be used to constrain the terms in the effective lagrangian is the mass term. We note first that the mass term in QCD breaks chiral symmetry. This is best seen by writing it in terms of left- and righthanded fermion fields,

$$\mathcal{L}_m^{QCD} = -\overline{\psi} M \psi = -\overline{\psi}_R M \psi_L - \overline{\psi}_L M \psi_R \tag{3.72}$$

($M = M^\dagger$ is the diagonal quark mass matrix). Secondly we note that when one would allow M to transform under chiral transformations as $M \to V_L M V_R^\dagger$ (and thus$M^\dagger \to V_R M V_L^\dagger$) chiral invariance is restored. The simplest realization of the above behavior in the effective lagrangian is a term linear in U of the form

$$\mathcal{L}_m^{eff} = \frac{B f_\pi^2}{2} \,\mathrm{Tr}\left(M\,U + M\,U^\dagger\right) = (m_u + m_d)\,B\,\left(f_\pi^2 - \frac{1}{2}\,\boldsymbol{\pi}^2 + \ldots\right). \tag{3.73}$$

Equating the vacuum expectation value of the QCD and effective mass term leads to

$$B\,f_\pi^2(m_u + m_d) = -m_u\langle 0|\overline{\psi^u}\psi^u|0\rangle - m_d\langle 0|\overline{\psi^d}\psi^d|0\rangle, \tag{3.74}$$

or since $B\,(m_u + m_d) = M_\pi^2$ one gets the *Gell-Mann - Oakes - Renner relation*

$$f_\pi^2\,M_\pi^2 = -(m_u + m_d)\langle 0|\overline{\psi^u}\psi^u|0\rangle, \tag{3.75}$$

(assuming isospin symmetry). From it one sees that chiral symmetry breaking implies $M_\pi^2 \neq 0$ for the hadronic world, corresponding to $\langle 0|\overline{\psi^u}\psi^u|0\rangle \neq 0$ for the (nonstrange) quark fields.

Chapter 4

ELECTROWEAK SCATTERING PROCESSES

In this chapter the use of electroweak scattering processes to study properties of hadrons is discussed. Only one specific process, namely lepton-hadron scattering will be dealt with in detail. A signal for an electroweak process is the presence of leptons which do not feel strong interactions. Consider for instance the following three types of processes, The *annihilation* process

$$\ell\bar{\ell} \to X,$$

the *lepton-hadron* scattering process

$$\ell H \to \ell' X,$$

and the *lepton-pair production* process

$$AB \to \ell\bar{\ell}X$$

(Drell-Yan process). All of these processes involve electroweak currents, coupling to the leptons in a known way. The basic advantage of electromagnetic processes lies in the fact that the process is accurately described in terms of the exchange of one photon, as the coupling, $\alpha = e^2/4\pi \approx 1/137$ is weak. The same is true for the weak vector bosons. On the hadronic side, the coupling to the quarks is known, but the structure of hadrons in terms of quarks and gluons is the unknown part. The fact that the coupling to the quarks is known, however, enables the study of hadron structure. For the above three processes, one has

$$q^2 = (k + k')^2 \equiv Q^2 \geq 0 \tag{4.1}$$

In the case of production of hadrons with a timelike (virtual) photon one can consider the rest-frame of the virtual photon, in which case it is clear that Q is a measure of the excitation energy.

$$q^2 = (k - k')^2 \equiv -Q^2 \leq 0 \tag{4.2}$$

$$2P \cdot q \equiv 2M\,\nu \equiv \frac{Q^2}{x_{_B}} \tag{4.3}$$

The variable $x_{_B}$ is the Bjorken scaling variable. Since the invariant mass squared of the hadronic final state satisfies

$$W_X^2 = (P + q)^2 = \frac{1 - x_{_B}}{x_{_B}}\,Q^2 + M^2 \geq M^2, \tag{4.4}$$

one has $0 \leq x_{_B} \leq 1$, with $x_{_B} = 1$ corresponding to elastic scattering, i.e. $W_X^2 = M^2$. In this case a hadron is probed with a spacelike (virtual) photon, for which one can consider a frame in which

the momentum only has a spatial component, from which it is clear that the resolving power of the probing photon is of the order $\lambda \approx 1/Q$. Roughly spoken one probes a nucleus (1 - 10 fm) with $Q \approx 10 - 100$ MeV, baryon or meson structure (with sizes in the order of 1 fm) with $Q \approx 0.1 - 1$ MeV and one probes deep into the nucleon (< 0.1 fm) with $Q > 2$ GeV.

$$q^2 = (k + k')^2 \equiv Q^2 \geq 0 \tag{4.5}$$

$$2P_A \cdot q \equiv \frac{Q^2}{x_A} \tag{4.6}$$

$$2P_B \cdot q \equiv \frac{Q^2}{x_B} \tag{4.7}$$

4.1 The hadronic tensor in lepton-hadron scattering

Consider the process $\ell + H \to \ell' + h + X$, in which a lepton with momentum k scatters off a hadron H with momentum P and one hadron h with momentum P_h is measured in coincidence with the scattered lepton with momentum k'. The lepton emits a highly virtual photon with momentum $q = k - k'$ with $q^2 \equiv -Q^2 \leq 0$. The unobserved outstate will be denoted by $|P_X\rangle$, having a total momentum P_X. We will consider the most general case of a pure incoming spin state, characterized by the spin vectors S^μ, an observed hadronic spin state characterized by the spin vector S_h (see appendix) and lepton helicities λ and λ'.

We have the following relations

$$P^2 = M^2, \quad P_h^2 = M_h^2, \tag{4.8}$$

$$k^2 = m^2 \approx 0, \quad k'^2 = m'^2 \approx 0, \tag{4.9}$$

$$(k + P)^2 = s, \tag{4.10}$$

$$S^2 = S_h^2 = -1, \tag{4.11}$$

$$P \cdot S = P_h \cdot S_h = 0. \tag{4.12}$$

We will work in the limit where Q^2, $P \cdot q$ and $P_h \cdot q$ are large keeping the ratios $x_B = Q^2/2P \cdot q$ and $z_h = 2P_h \cdot q/Q^2$ finite. The invariant ampitude for the process is given by

$$\mathcal{M} = \bar{u}(k', \lambda')\gamma^\mu u(k, \lambda)\frac{e^2}{Q^2}\langle P_X; P_h S_h|J_\mu(0)|PS\rangle. \tag{4.13}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathcal{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(\ell H)} H^{\mu\nu(\ell H)}, \tag{4.14}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(\ell H)}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda_e\, \epsilon_{\mu\nu\rho\sigma} q^\rho k^\sigma\right). \tag{4.15}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(\ell H)}(P_X; PS; P_h S_h) = \langle PS|J_\mu(0)|P_X; P_h, S_h\rangle\langle P_X; P_h S_h|J_\nu(0)|PS\rangle, \tag{4.16}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\mathcal{R}, \tag{4.17}$$

with the flux factor

$$F = 4\sqrt{(P \cdot k)^2 - M^2 m^2} \overset{TRF}{=} 2M\,E, \tag{4.18}$$

and the Lorentz invariant phase space

$$d\mathcal{R} = (2\pi)^4\delta^4(k+P-k'-P_X-P_h)\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3k'}{(2\pi)^3 2k'^0}\frac{d^3P_h}{(2\pi)^3 2P_h^0}. \tag{4.19}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$2M\mathcal{W}_{\mu\nu}^{(\ell H)}(q;PS;P_hS_h) = \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q+P-P_X-P_h)H_{\mu\nu}^{(\ell H)}(P_X;PS;P_hS_h). \tag{4.20}$$

The phase space for the scattered lepton can be rewritten as

$$\frac{d^3k'}{(2\pi)^3 2k'^0} = \frac{E'\,dE'\,d\Omega}{16\pi^3}, \tag{4.21}$$

where Ω is the lepton scattering angle and E' the energy of the scattered lepton. Thus one gets

$$\frac{2E_h\,d\sigma^{(\ell H)}}{d^3P_h\,d\Omega dE'} = \frac{E'}{E}\frac{\alpha^2}{Q^4}L_{\mu\nu}^{(\ell H)}\mathcal{W}^{\mu\nu(\ell H)}, \tag{4.22}$$

where the lepton tensor is given by the expression between brackets in Eq. (4.15) and the hadron tensor by Eq. (4.20).

The cross section for inclusive measurements (in which nothing of the final state is detected) is

$$\frac{d\sigma^{(\ell H)}}{d\Omega dE'} = \frac{E'}{E}\frac{\alpha^2}{Q^4}L_{\mu\nu}^{(\ell H)}W^{\mu\nu(\ell H)}, \tag{4.23}$$

with the hadron tensor given by

$$\begin{aligned} 2M\,W_{\mu\nu}^{(\ell H)}(q;PS) &= \frac{1}{2\pi}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q+P-P_X)\langle PS|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|PS\rangle \\ &= \frac{1}{2\pi}\int d^4x\; e^{iq\cdot x}\;\langle PS|[J_\mu(x),J_\nu(0)]|PS\rangle. \end{aligned} \tag{4.24}$$

4.2 Scaling variables in lepton-hadron scattering

Before proceeding with the analysis of the hadronic tensor, it is useful to consider the kinematics of this process. In fact, it is convenient to introduce a basis set of vectors that takes into account the essential fact that there is always a spacelike direction defined by the momentum transfer q^μ. Using the target hadron momentum P^μ to construct an orthogonal four vector $\tilde{P}^\mu = P^\mu - (P\cdot q/q^2)\,q^\mu$, which is timelike. Thus we have

$$Z^\mu \equiv -q^\mu, \tag{4.25}$$

$$T^\mu \equiv -\frac{q^2}{P\cdot q}\,\tilde{P}^\mu = q^\mu + 2x_B\,P^\mu. \tag{4.26}$$

For these vectors we have $Z^2 = -Q^2$ and $T^2 = \kappa\,Q^2$ with

$$\kappa = 1 + \frac{M^2\,Q^2}{(P\cdot q)^2} = 1 + \frac{4\,M^2x_B^2}{Q^2} \tag{4.27}$$

taking into account mass corrections $\propto M^2/Q^2$ which will vanish for large Q^2 and $\kappa \to 1$. We will mostly consider normalized vectors $\hat{z}^\mu = Z^\mu/Q$ and $\hat{t}^\mu = T^\mu/(Q\sqrt{\kappa})$. In the space orthogonal to $\hat{z}$ and $\hat{t}$ one has the tensors

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu, \tag{4.28}$$

$$\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho q_\sigma = \frac{1}{(P\cdot q)\sqrt{\kappa}}\,\epsilon^{\mu\nu\rho\sigma}\,P_\rho q_\sigma. \tag{4.29}$$

A possible vector in the perpendical space is $P_{h\perp} = g_\perp^{\mu\nu}\, P_{h\nu}$, defining the orientation of the production plane in semi-inclusive leptoproduction, $\hat{h}^\mu = P_{h\perp}^\mu/|\boldsymbol{P}_{h\perp}|$.

Note that a time-like vector defined from the hadron momentum P_h could be defined as

$$T_h^\mu = -\frac{q^2}{P_h\cdot q}\,\tilde{P}_h^\mu = q^\mu - \frac{2}{z_h}\,P_h^\mu,$$

where $z_h \equiv 2P_h\cdot q/q^2$. In that case $T_h^2 = \kappa_h\, Q^2$ with $\kappa_h = 1 + 4\,M_h^2/(z_h^2 Q^2)$.

It is useful to define a perpendicular vector in the same fashion as done for T^μ. Taking

$$X^\mu = -\frac{P\cdot T}{P\cdot P_h}\,P_{h\perp}^\mu = -\frac{\kappa\, P\cdot q}{P\cdot P_h}\,P_{h\perp}^\mu = \frac{P_h\cdot T}{P\cdot P_h}\,\tilde{P}^\mu - \frac{P\cdot T}{P\cdot P_h}\,\tilde{P}_h^\mu \tag{4.30}$$

Also the lepton momenta l and $l' = l - q$ can be expanded in $\hat{t}$, $\hat{z}$ and a perpendicular component using the scaling variable $y = P\cdot q/P\cdot k$ (in the target restframe reducing to $y = \nu/E$). The result is

$$\begin{aligned} l^\mu = \frac{2-y}{y}\,\frac{1}{\kappa}\,T^\mu - \frac{1}{2}\,Z^\mu + l_\perp^\mu \quad &= \quad \frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)}{2y}\,\frac{Q}{\sqrt{\kappa}}\,\hat{t}^\mu + \frac{\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}}{y}\,\frac{Q}{\sqrt{\kappa}}\,\hat{\ell}^\mu \\ &\overset{Q^2\to\infty}{\longrightarrow} \quad \frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)\,Q}{2y}\,\hat{t}^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}^\mu, \end{aligned} \tag{4.31}$$

where $\hat{\ell}^\mu = l_\perp^\mu/|\boldsymbol{l}_\perp|$, is a spacelike unit-vector in the perpendicular direction lying in the (lepton) scattering plane. The kinematics in the frame where virtual photon and target are collinear (including target rest frame) is illustrated in Fig. 1. With the definition of $\hat{\ell}$, we have for the leptonic tensor[1] neglecting mass corrections ($\kappa = 1$)

$$\begin{aligned} L_{(\ell H)}^{\mu\nu} \quad = \quad \frac{Q^2}{y^2}\Bigg[& -2\left(1-y+\frac{1}{2}\,y^2\right) g_\perp^{\mu\nu} + 4(1-y)\hat{t}^\mu\hat{t}^\nu \\ & +4(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y}\;\hat{t}^{\{\mu}\hat{\ell}^{\nu\}} \\ & -i\lambda_e\, y(2-y)\,\epsilon_\perp^{\mu\nu} - 2i\lambda_e\, y\sqrt{1-y}\;\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\Bigg]. \end{aligned} \tag{4.32}$$

For completeness, we also give the full tensor including mass corrections

$$\begin{aligned} L_{(\ell H)}^{\mu\nu} \quad = \quad \frac{1}{\kappa}\,\frac{Q^2}{y^2}\Bigg[& -2\left(1-y+\frac{1}{4}\,(1+\kappa)\,y^2\right) g_\perp^{\mu\nu} + 4\left(1-y+\frac{1}{4}(1-\kappa)\,y^2\right)\hat{t}^\mu\hat{t}^\nu \\ & +4\left(1-y+\frac{1}{4}(1-\kappa)\,y^2\right)\left(\hat{\ell}^\mu\hat{\ell}^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}\;\hat{t}^{\{\mu}\hat{\ell}^{\nu\}} \\ & -i\lambda_e\,\sqrt{\kappa}\;y(2-y)\,\epsilon_\perp^{\mu\nu} - 2i\lambda_e\,\sqrt{\kappa}\;y\sqrt{1-y+\frac{1}{4}(1-\kappa)\,y^2}\;\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\Bigg]. \end{aligned} \tag{4.33}$$

Often, particularly when Q^2 becomes large, it is useful to employ lightcone components for momenta $p^\mu = [p^-, p^+, \boldsymbol{p}_\perp]$ with $p^\pm \equiv (p^0 \pm p^3)/\sqrt{2}$. In the frame in which the virtual photon and the

[1] A useful relation is

$$\epsilon_{\mu\nu\rho\sigma}\, g_{\alpha\beta} = \epsilon_{\alpha\nu\rho\sigma}\, g_{\mu\beta} + \epsilon_{\mu\alpha\rho\sigma}\, g_{\nu\beta} + \epsilon_{\mu\nu\alpha\sigma}\, g_{\rho\beta} + \epsilon_{\mu\nu\rho\alpha}\, g_{\sigma\beta}$$

or for a vector $a_\perp$ orthogonal to $\hat{t}$ and $\hat{q}$,

$$\epsilon^{\mu\nu\rho\sigma} q_\rho a_{\perp\sigma} = -t^{\,[\mu}\epsilon_\perp^{\nu]\rho} a_{\perp\rho}.$$

initial state hadron have no transverse directions, (referred to as frame I) one obtains the following momenta

lepton-hadron:		
momentum	frame I	relations
$q = k - k'$	$\left[\frac{Q^2}{A\sqrt{2}}, -\frac{A}{\sqrt{2}}, \mathbf{0}_\perp\right]$	
P	$\left[\frac{\overline{x}_B M^2}{A\sqrt{2}}, \frac{A}{\overline{x}_B\sqrt{2}}, \mathbf{0}_\perp\right]$	$\overline{x}_B = -\frac{q^+}{P^+} \approx \frac{Q^2}{2P\cdot q} \approx -\frac{P_h\cdot q}{P_h\cdot P}$
P_h	$\left[\frac{\overline{z}_h Q^2}{A\sqrt{2}}, \frac{AM_{h\perp}^2}{\overline{z}_h Q^2\sqrt{2}}, \boldsymbol{P}_{h\perp}\right]$	$\overline{z}_h = \frac{P_h^-}{q^-} \approx -\frac{2P_h\cdot q}{Q^2} \approx \frac{P\cdot P_h}{P\cdot q}$

The quantity A determines the frame. Frames with different A-values are connected by boosts along the z-direction. The target rest frame corresponds with $A = x_B M$. In the above table the scaling variables $\overline{x}_B$ and $\overline{z}_h$ are ratios of lightcone components. These ratios are only in the limit $Q^2 \to \infty$ equal to the invariants defined before. For the theoretical analysis, however, the lightcone ratios, turn out to be relevant variables. Of course the lightcone ratio can always be reexpressed in invariants, to be specific

$$\overline{x}_B = -\frac{q^+}{P^+} = \frac{2x_B}{1+\sqrt{\kappa}}. \tag{4.34}$$

The ratio on the left-hand side is referred to as the Nachtmann scaling variable.

Also the lepton-momenta can be written down in the frame discussed above

lepton-hadron:		
momentum	frame I	relations
k	$\left[\frac{Q^2}{\overline{y}A\sqrt{2}}, \frac{(1-\overline{y})A}{\overline{y}\sqrt{2}}, \boldsymbol{k}_\perp\right]$	
k'	$\left[\frac{(1-\overline{y})Q^2}{\overline{y}A\sqrt{2}}, \frac{A}{\overline{y}\sqrt{2}}, \boldsymbol{k}_\perp\right]$	$\boldsymbol{k}_\perp^2 = \frac{1-\overline{y}}{\overline{y}^2}Q^2$

The ratio of lightcone momenta $\overline{y}$ is up to target mass corrections ($\propto M^2/Q^2$) equal to the scaling variable y.

$$\overline{y} = \frac{q^-}{k^-} = \frac{y\sqrt{\kappa}}{1-\frac{1}{2}(1-\sqrt{\kappa})\,y}. \tag{4.35}$$

The lepton phase space becomes $d^3k'/E' = E'dE'd\Omega = \pi y s dx_B dy$, and one obtains for Eq. 4.22 (neglecting target mass corrections)

$$\frac{d\sigma^{(\ell H)}}{dx_B\, dz_h\, d^2\boldsymbol{P}_{h\perp}\, dy} = \frac{\pi\alpha^2}{2Q^4}\frac{y}{z_h} L_{\mu\nu}^{(\ell H)}\, 2M\mathcal{W}^{\mu\nu(\ell H)}. \tag{4.36}$$

For the inclusive process one finds (neglecting target mass corrections)

$$\frac{d\sigma^{(\ell H)}}{dx_B\, dy} = \frac{\pi\alpha^2}{Q^4}\, y\, L_{\mu\nu}^{(\ell H)}\, 2MW^{\mu\nu(\ell H)}. \tag{4.37}$$

4.3 Elastic lepton-hadron scattering

We will consider now in more detail electron-hadron scattering interacting electromagnetically. The hadronic tensor can be evaluated for elementary systems, e.g. a charged elementary fermion such as a muon. The cross section for the process $e + \mu \to e + \mu$ can then be calculated exactly (with as only approximation the beforementioned restriction to one photon exchange). For composite particles such as a nucleon or a nucleus the current expectation value is parametrized in terms of *form factors*

and an exact expression of the cross section for $e + N \to e + N$ can be found in terms of these form factors. A similar treatment in terms of form factors can be performed for exclusive processes like $e + N \to e + N^*$, where N^* is an excited state of the nucleon. For inclusive processes or semi-inclusive processes in which case none or not all of the produced particles in the final state are detected one can only consider the most general form for the full hadronic tensor (summed over final states). The hadronic part of the cross section is then parametrized in terms of *structure functions*.

For elastic scattering there is no unknown part X in the final state. The final state phase space is fully known,

$$\frac{M}{\pi} \int \frac{d^3p'}{(2\pi)^3\, 2E'_p} (2\pi)^4 \delta^4(p + q - p') = 2M\, \delta\left((p+q)^2 - M^2\right), \tag{4.38}$$

which in terms of the invariants $M\nu = p \cdot q$ and $Q^2 = -q^2$ becomes

$$\ldots = 2M\, \delta(2p \cdot q + q^2) = \delta\left(\nu - \frac{Q^2}{2M}\right). \tag{4.39}$$

The differential cross section thus becomes (in the target rest frame)

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{Q^4} \frac{E'}{E} \frac{L^{\mu\nu} H_{\mu\nu}}{4M^2}\, \delta\left(\nu - \frac{Q^2}{2M}\right). \tag{4.40}$$

From the expression for the double differential cross section one sees that in the phase space of the scattered electron a restriction arises because the scattering is elastic, reflected in the delta function. Because of this one commonly integrates over the energy of the outgoing particle, i.e. over E' or ν. In the case of integrating over E' the delta function

$$\delta\left(\nu - \frac{Q^2}{2M}\right) = \delta\left(E - E' - \frac{4\, EE'\, \sin^2(\theta/2)}{2M}\right) \tag{4.41}$$

gives a recoil term $E'/E = (1 - Q^2/2ME)$. The result is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{Q^4} \left(1 - \frac{Q^2}{2ME}\right)^2 \frac{L^{\mu\nu} H_{\mu\nu}}{4M^2}. \tag{4.42}$$

Of course the cross section can also be written down in terms of the invariant Q^2 using the explicit relation between Q^2 and θ, $Q^2 = 4E(E - Q^2/2M)\sin^2(\theta/2)$, giving

$$\pi\, dQ^2 = E'^2\, d\Omega = \left(E - \frac{Q^2}{2M}\right)^2 d\Omega, \tag{4.43}$$

and hence

$$\frac{d\sigma}{dQ^2} = \frac{\pi\, \alpha^2}{Q^4\, E^2} \frac{L^{\mu\nu} H_{\mu\nu}}{4M^2}. \tag{4.44}$$

The tensor $H_{\mu\nu}$ contains the unknown part. It is expressed in the expectation values of the current operators between hadron states. The microscopic current operator itself is in principle simple, for electromagnetic interactions

$$\begin{aligned} J_\mu^{(\gamma)} &= \sum_q \bar{q}(x)\, Q\, \gamma_\mu\, q(x) \\ &= \frac{2}{3}\, \bar{u}(x)\, Q\, \gamma_\mu\, u(x) - \frac{1}{3}\, \bar{d}(x)\, Q\, \gamma_\mu\, d(x) - \frac{1}{3}\, \bar{s}(x)\, Q\, \gamma_\mu\, s(x) + \ldots, \end{aligned} \tag{4.45}$$

where $q(x)$, $u(x)$, etc. refer to the fermion field operators for quarks, up quarks, etc. The unknown part is the expectation value of this operator between hadrons that are composed of the quarks.

4.3.1 Elastic scattering off point particles

In the case of elastic scattering off an elementary fermion (one could for instance take the hypothetical example that the target consist of a free quark) the expression for $H_{\mu\nu}$ is the same as that for the lepton current $L_{\mu\nu}$, i.e.

$$\begin{aligned} H_{\mu\nu}(p,p') &= 2p_\mu p'_\nu + 2p_\nu p'_\mu - Q^2\, g_{\mu\nu} \\ &= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) Q^2 + 4\,\tilde{P}_\mu \tilde{P}_\nu, \end{aligned} \tag{4.46}$$

where $\tilde{p}_\mu \equiv p_\mu - (p\cdot q/q^2)\, q_\mu$, which for elastic scattering equals $\tilde{p}_\mu = p_\mu + q_\mu/2 = (p_\mu + p'_\mu)/2$. The tensor structures appearing in the above expression are useful for later purposes and the contraction with $L_{\mu\nu}$ needed to calculate the amplitude squared yields,

$$v_1 = L^{\mu\nu}\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) = 2\,Q^2 \overset{lab}{=} 8\,EE'\,\sin^2(\theta/2), \tag{4.47}$$

$$v_2 = L^{\mu\nu}\,\frac{\tilde{p}_\mu \tilde{p}_\nu}{M^2} = \frac{4(p.k)^2 - 4(p.k)(p.q) - M^2\,Q^2}{M^2} \overset{TRF}{=} 4\,EE'\,\cos^2(\theta/2). \tag{4.48}$$

Using the above contractions, the differential cross section thus becomes (in the target rest frame)

$$\begin{aligned} \frac{d\sigma}{d\Omega\, dE'} &= \frac{4\,\alpha^2\,E'^2}{Q^4}\left(\cos^2(\theta/2) + \frac{Q^2}{2M^2}\,\sin^2(\theta/2)\right)\delta\left(\nu - \frac{Q^2}{2M}\right) \\ &= \left(\frac{d\sigma}{d\Omega}\right)_M \left(1 + \frac{Q^2}{2M^2}\tan^2(\theta/2)\right)\delta\left(\nu - \frac{Q^2}{2M}\right), \end{aligned} \tag{4.49}$$

where the Mott cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_M = \frac{4\,\alpha^2\,E'^2}{Q^4}\,\cos^2(\theta/2). \tag{4.50}$$

From this expression for the double differential cross section one obtains after integration over the energy of the outgoing particle, i.e. over E' the result

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_M \left(1 - \frac{Q^2}{2ME}\right)\left(1 + \frac{Q^2}{2M^2}\tan^2(\theta/2)\right). \tag{4.51}$$

4.3.2 Elastic scattering off composite particles

Given the current expectation value for a spin 0 system,

$$\langle p', s'|J_\mu(0)|p, s\rangle = (p_\mu + p'_\mu)\,F(Q^2), \tag{4.52}$$

it is straightforward to evaluate $H_{\mu\nu}$,

$$H_{\mu\nu} = 4\,\tilde{p}_\mu \tilde{p}_\nu\,F^2(Q^2), \tag{4.53}$$

and the differential cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_M \left(1 - \frac{Q^2}{2ME}\right)\,F^2(Q^2). \tag{4.54}$$

For a spin 1/2 system, one can use

$$\langle p', s'|J_\mu(0)|p, s\rangle = \bar{u}_{s'}(p')\Gamma_\mu(p,p')u_s(p) \tag{4.55}$$

with

$$\Gamma_\mu(p,p') = \gamma_\mu F_1(Q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2M}F_2(Q^2), \tag{4.56}$$

to obtain the expression for the differential cross section after calculating $H_{\mu\nu}$ in analogy with the leptonic tensor,

$$\begin{aligned} H^{(N)}_{\mu\nu} &= \frac{1}{2}\sum_{s,s'}\langle p,s|J_\mu(0)|p',s'\rangle\langle p',s'|J_\nu(0)|p,s\rangle \\ &= \frac{1}{2}Tr\ [\Gamma_\mu(\not{p}'+M)\Gamma_\nu(\not{p}+M)] \\ &= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) Q^2(F_1+F_2)^2 + 4\,\tilde{p}_\mu\tilde{p}_\nu\ \left(F_1^2 + \frac{Q^2}{4M^2}\,F_2^2\right) \\ &= -g_{\perp\,\mu\nu}\,Q^2G_M^2 + \hat{t}_\mu\hat{t}_\nu\,4M^2\,G_E^2 \end{aligned} \tag{4.57}$$

(Recall that the Sachs form factors are given by $G_E = F_1 - (Q^2/4M^2)\,F_2$ and $G_M = F_1 + F_2$). The contraction with the leptonic tensor finally gives the cross section

$$\frac{d\sigma}{d\Omega\,dE'} = \left(\frac{d\sigma}{d\Omega}\right)_M \left(\frac{G_E^2 + Q^2G_M^2/4M^2}{1+Q^2/4M^2} + \frac{Q^2\,G_M^2}{2M^2}\tan^2(\theta/2)\right)\delta\left(\nu - \frac{Q^2}{2M}\right), \tag{4.58}$$

or integrating over E',

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_M \left(1 - \frac{Q^2}{2ME}\right)\left(\frac{G_E^2 + Q^2G_M^2/4M^2}{1+Q^2/4M^2} + \frac{Q^2\,G_M^2}{2M^2}\tan^2(\theta/2)\right). \tag{4.59}$$

This result is the Rosenbluth formula. The essential feature is that the differential cross section divided by the Mott cross section and the recoil term is given by $A(Q^2) + B(Q^2)\,\tan^2(\theta/2)$. Two measurements with different E and θ such that Q^2 is the same are then sufficient to extract the form factors. The only assumption underlying this result is that the interaction is mediated by the exchange of a single photon.

4.4 Inclusive inelastic lepton scattering processes

In the inclusive process $e + H \to e' + X$ the (unobserved) final state X can be the target (elastic scattering) or an excitation thereof. In the case of an elementary fermionic target (e.g. an electron or a hypothetical quark target) the system cannot be broken up. The energy transfer is for given Q^2 fixed, $\nu = Q^2/2m$, where m is the mass of the target. For nucleons the elastic scattering cross section (on top of the $1/Q^4$ of the Mott cross section) is proportional to a form factor squared. At the same time it is possible to excite the system. This gives rise to inelastic contributions in the cross section. When Q^2 and the energy transfer ν are high enough the cross section will reflect elastic scattering off the pointlike constituents of the nucleon and the cross section will become equal to an incoherent sum of the electron-quark cross section. This is known as the deep inelastic scattering region. For electron-nucleus scattering, the cross section shows an even richer structure. First, there is elastic scattering off the nucleus, which vanishes for $Q^2 >> 1/R_A^2$; if the energy transfer is higher than the threshold, $Q^2/2M_A$, nuclear resonances can be excited. In the region around $\nu = Q^2/2M_N$ the cross section is dominated by scattering incoherently off the individual nucleons in the nucleus. The cross section reflects elastic electron-nucleon scattering, hence the name quasi-elastic scattering. The delta-peak in electron-nucleon scattering is smeared out because of the motion of nucleons in the nucleus. This part of the cross section still falls off with increasing Q^2 because of the compositeness of the nucleons. At higher ν values the nucleons in the nucleus also can be excited and finally one also reaches the deep inelastic scattering region, where the scattering process again reflects elastic

scattering off the smallest, pointlike, constituents of the nucleus, the quarks. In this section I will concentrate on the formalism that is used to describe the inclusive scattering process.

For inclusive electron scattering the cross section is given by

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{Q^4}\,\frac{E'}{E}\,L_{\mu\nu}W^{\mu\nu}, \tag{4.60}$$

where $L_{\mu\nu}$ is the leptonic tensor, which we have already discussed, and $W_{\mu\nu}$ is the hadronic tensor, which contains the information on the hadronic part of the scattering process. It can be written as

$$2M\,W_{\mu\nu}(P,q) = \frac{1}{2\pi}\sum_n \int \frac{d^3P_n}{(2\pi)^3\,2E_n}\,\langle P|J^\dagger_\mu(0)|P_n\rangle\,\langle P_n|J_\nu(0)|P\rangle\;(2\pi)^4\,\delta^4(P+q-P_n), \tag{4.61}$$

where $|P\rangle$ represents a target with momentum P. Using $\delta^4(P+q-P_n) = \int d^4x\,\exp(iP\cdot x+iq\cdot x-iP_n\cdot x)$, shifting the argument of the current, $J_\mu(x) = \exp(iP_{op}\cdot x)J_\mu(0)\exp(-iP_{op}\cdot x)$ and using completeness for the intermediate states this can be written as the expectation value of the product of currents $J_\mu(x)J_\nu(0)$. Adding a second term $\propto J_\nu(0)J^\dagger_\mu(0)\delta^4(P-q-P_n)$, which in the physical region ($\nu>0$) is zero because of the spectral conditions of the intermediate states n ($P_n^0 > M$) one can after a similar procedure combine the terms to

$$2M\,W_{\mu\nu} = \frac{1}{2\pi}\int d^4x\; e^{i\,q\cdot x}\,\langle P|[J^\dagger_\mu(x), J_\nu(0)]|P\rangle, \tag{4.62}$$

where summation and averaging over spins is understood.

On the basis of Poincaré invariance, and invariance under parity and time reversal, the most general form allowed for the hadronic tensor in the case of unpolarized scattering is

$$W^{\mu\nu} = \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\,W_1 + \frac{\tilde{P}^\mu\tilde{P}^\nu}{M^2}\,W_2, \tag{4.63}$$

where the structure functions W_1 and W_2 only depend on the invariants in the scattering process, $W_i = W_i(\nu, Q^2)$ where $\nu = P\cdot q/M$, or equivalently $W_i = W_i(x_{_B}, Q^2)$. It turns out to be convenient to define the dimensionless structure functions,

$$F_1(\nu,Q^2) = M\,W_1(\nu,Q^2), \tag{4.64}$$

$$F_2(\nu,Q^2) = \nu\,W_2(\nu,Q^2), \tag{4.65}$$

for which we have

$$M\,W^{\mu\nu} = \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)\,F_1 + \frac{\tilde{P}^\mu\tilde{P}^\nu}{P\cdot q}\,F_2 \tag{4.66}$$

$$= -g^{\mu\nu}_\perp\,F_1 + \hat{t}^\mu\hat{t}^\nu\left(-F_1 + \frac{\kappa}{2x_{_B}}\,F_2\right). \tag{4.67}$$

Working out the contraction between leptonic and hadronic tensor gives the cross section, which in the target rest frame is

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{4\alpha^2E'^2}{Q^4}\,\left[\cos^2(\theta/2)\,W_2(\nu,Q^2) + 2\,\sin^2(\theta/2)\,W_1(\nu,Q^2)\right]. \tag{4.68}$$

The tensor $W_{\mu\nu}$ also appears in the total cross section for $\gamma^*H \to$ everything, where γ^* indicates a virtual photon. For a given virtuality Q^2 of the photon this cross section depends on only one variable, $s = (P+q)^2$, or equivalently on the variable $\nu = P\cdot q/M$,

$$\sigma^{\gamma^*H}(\nu) = \frac{4\pi^2\alpha}{K}\,\epsilon^{\mu *}W_{\mu\nu}\epsilon^\nu, \tag{4.69}$$

where $4M\,K$ is the photon flux factor. This flux factor only is physical for real photons ($Q^2 = 0$). One convention is to take $4M\,K = 4\sqrt{(p\cdot q)^2 - p^2q^2}$, i.e. $K = \sqrt{\nu^2 + Q^2}$. Other possibilities are to take the real photon result $4M\,K = 4\,P\cdot q$ or $K = \nu$ (Hand convention). Another convention that has been used is to equate the final state invariant mass squared $W^2 = (P+q)^2$, i.e. take the result $4M\,K = 2(W^2 - M^2)$ for a massless photon and equate W^2 to the invariant mass in the case of a virtual photon, $W^2 = 2\,P\cdot q + M^2 - Q^2$ or $K = \nu - Q^2/2M$.

Being a (total) cross section for (virtual) photoabsorption, the hadronic tensor is related to the forward (virtual) Compton amplitude through the optical theorem,

$$W_{\mu\nu} = \frac{1}{\pi}\,\mathcal{I}m\;T_{\mu\nu}, \tag{4.70}$$

where

$$2M\,T_{\mu\nu}(P,q) = i\int d^4x\; e^{iq\cdot x}\langle P|\mathcal{T}\,J_\mu(x)J_\nu(0)|P\rangle, \tag{4.71}$$

Using the photon polarization vectors ϵ^μ_α, where α indicates one of the polarization directions (perpendicular to q^μ),

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}\,(0,\mp 1,-i,0) = \mp\frac{1}{\sqrt{2}}(\epsilon_x \pm i\,\epsilon_y), \tag{4.72}$$

$$\epsilon^\mu_L = \frac{1}{\sqrt{Q^2}}\,(q^3,0,0,q^0), \tag{4.73}$$

one gets the longitudinal and transverse structure functions $W_\alpha = \epsilon^{\mu *}_\alpha\,W_{\mu\nu}\epsilon^\nu_\alpha$, which are linear combinations of W_1 and W_2 and which, because of the fact that they are cross sections for (virtual) photons, are positive,

$$W_T = W_+ = W_- = W_1, \tag{4.74}$$

$$W_L = -W_1 + \left(1 + \frac{\nu^2}{Q^2}\right)\,W_2, \tag{4.75}$$

or

$$M\,W_T = F_1, \tag{4.76}$$

$$M\,W_L = -F_1 + \frac{\kappa}{2x_{_B}}\,F_2. \tag{4.77}$$

Note that the expansion of $W^{\mu\nu}$ in terms of the tensors $g^{\mu\nu}_\perp$ and $\hat{t}^\mu\hat{t}^\nu$ given in Eq. 4.67 precisely projects onto the longitudinal and transverse structure functions

In terms of the longitudinal en transverse structure functions,

$$\frac{d\sigma}{d\Omega\,dE'} = \frac{4\alpha^2E'^2}{Q^4}\,\frac{Q^2}{|\boldsymbol{q}|^2}\,\cos^2(\theta/2)\,\left[W_L(\nu,Q^2) + \epsilon^{-1}\,W_T(\nu,Q^2)\right], \tag{4.78}$$

where $|\boldsymbol{q}|^2 = \nu^2 + Q^2$ and

$$\epsilon^{-1} = 1 + 2\,(|\boldsymbol{q}|^2/Q^2)\,\tan^2(\theta/2). \tag{4.79}$$

4.5 Symmetry properties of the structure functions

For the Compton scattering process, $\gamma^*(q) + H(P) \to \gamma^*(q') + H(P')$, the amplitude $T_{\mu\nu}(P,q,q')$ can be expanded, similar as $W_{\mu\nu}$, in terms of amplitudes T_1 and T_2 that depend on the invariants in the scattering process. These are (except the for a given process fixed photon virtuality) two of the (three) Mandelstam variables for the γ^*N process, $s = (P+q)^2$, $t = (P-q)^2$ and $u = (P-q')^2$ (Note that $s + t + u = 2\,M^2 - 2\,Q^2$).

Figure 4.1: *The Compton amplitude for a point fermion.*

First we review the symmetries and analytic properties of the amplitudes T_i and the structure functions W_i. Crossing symmetry relates the amplitudes

$$T^{ab\to cd}(p_a,p_b,p_c,p_d) = T^{\bar{c}b\to\bar{a}d}(-p_c,p_b,-p_a,p_d). \tag{4.80}$$

For instance, for the virtual Compton amplitude $T_1^{\gamma^*\bar{H}\to\gamma^*\bar{H}}(u,s) = T_1^{\gamma^*H\to\gamma^*H}(s,u)$. For the elastic γ^*H and $\gamma^*\bar{H}$ amplitudes themselves the crossing properties for the photon imply $T_1(s,u) = T_1(u,s)$. From the optical theorem the relation between the total cross section for $\gamma^*H \to X$ to the elastic $\gamma^*H \to \gamma^*H$ amplitude, specifically the structure function W_1 is found as the discontinuity over the cut in the physical region, which is the imaginary part of the forward ($t = 0$) amplitude T_1. For the forward amplitude s and u are related and since for $t = 0$ the variable ν equals $\nu = (s-u)/4m$, T_1 is a symmetric function of ν. One has

$$\begin{aligned}
\sigma_T^{\gamma^*H}(\nu) = \frac{4\pi\alpha}{K}\mathcal{I}m\,T_1(s+i\epsilon,u) &= \frac{4\pi\alpha}{K}\mathcal{I}m\,T_1(\nu+i\epsilon), &(4.81)\\
\sigma_T^{\gamma^*\bar{H}}(\nu) = \frac{4\pi\alpha}{K}\mathcal{I}m\,T_1(u+i\epsilon,s) &= \frac{4\pi\alpha}{K}\mathcal{I}m\,T_1(-\nu+i\epsilon)\\
&= -\frac{4\pi\alpha}{K}\mathcal{I}m\,T_1(\nu+i\epsilon), &(4.82)
\end{aligned}$$

where the last equality follows from the symmetry of T_1, $T_1(\nu) = T_1(-\nu)$ and the fact that T_1 is a real analytic function, $T_1(\nu) = T_1^*(\nu^*)$. Note that as $W_{\mu\nu}$ is defined as the commutator of the currents the cross section for the 'crossed' part enters with a negative sign. As an analytic function of ν, however, one precisely has

$$W_1(\nu,Q^2) = \frac{1}{\pi}\mathcal{I}m\;T_1(\nu+i\epsilon) = -W_1(-\nu,Q^2), \tag{4.83}$$

which can also be derived from the translation invariance properties of the commutator defining $W^{\mu\nu}$. The analytic behavior and symmetry in the Bjorken variable $x = Q^2/2M\nu$ is exactly the same as that in ν, i.e. $W_1(x) = -W_1(-x)$ is antisymmetric in x.

A simple example in which all of these properties are illustrated is the forward Compton amplitude for scattering off a point fermion with mass m and charge $Q/e = e_f$ (fig. 4.1). The amplitude T_1 is the coefficient of $g_{\mu\nu}$ in the amplitude $T_{\mu\nu}$,

$$2m\,T_{\mu\nu} = e_f^2\,\frac{\bar{u}(p)\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu u(p)}{(p+q)^2-m^2+i\epsilon} + [\mu\leftrightarrow\nu,\ q\leftrightarrow -q], \tag{4.84}$$

and equals

$$\begin{aligned}
2m\,T_1 &= e_f^2\left(\frac{s-u}{2(s-m^2)} + \frac{u-s}{2(u-m^2)}\right)\\
&= e_f^2\left(\frac{2m\,\nu}{2m\,\nu-Q^2} + \frac{2m\,\nu}{2m\,\nu+Q^2}\right)
\end{aligned}$$

$$= \quad e_f^2 \left(\frac{1}{1-x} + \frac{1}{1+x} \right), \tag{4.85}$$

of which the imaginary part precisely is the structure function for a pointlike fermion,

$$2m\, W_1 = e_f^2 \left[\delta(1-x) - \delta(1+x) \right]. \tag{4.86}$$

4.6 Electroweak lepton - hadron interactions

In electromagnetic interactions the current to which the photon couples, $J_\mu^{(\gamma)}(x)$, is a conserved vector current. Parity and current conservation then restrict the number of structure functions to two. In general, e.g. in the case of weak interactions these restrictions do not apply and a larger number of structure functions is needed in the most general expression of the hadronic tensor.

Starting with the leptonic tensor we consider the electroweak current

$$\langle k', s' | J_\mu(0) | k, s \rangle = e\, \bar{u}(k', s') \Gamma_\mu u(k, s) \tag{4.87}$$

with for an elementary fermion

$$\Gamma_\mu = c_V \gamma_\mu + c_A \gamma_\mu \gamma_5 = \frac{1}{2}\, c_L \gamma_{\mu L} + \frac{1}{2}\, c_R \gamma_{\mu R}, \tag{4.88}$$

where $\gamma_{\mu R/L} = \gamma_\mu (1 \pm \gamma_5)$. For the currents, coupling to the various gauge bosons one has in the Standard Model

$$\Gamma_\mu^{(\gamma)} \quad = \quad Q\, \gamma_\mu \;=\; \frac{1}{2}\, Q\, \gamma_{\mu L} + \frac{1}{2}\, Q\, \gamma_{\mu R}, \tag{4.89}$$

$$\Gamma_\mu^{(Z)} \quad = \quad \frac{(I_W^3 - 2Q\, \sin^2\theta_W)}{2\, \sin\theta_W\, \cos\theta_W}\, \gamma_\mu - \frac{I_W^3}{2\, \sin\theta_W\, \cos\theta_W}\, \gamma_\mu \gamma_5 \tag{4.90}$$

$$= \quad \frac{(I_W^3 - Q\, \sin^2\theta_W)}{2\, \sin\theta_W\, \cos\theta_W}\, \gamma_{\mu L} - \frac{Q\, \sin^2\theta_W}{2\, \sin\theta_W\, \cos\theta_W}\, \gamma_{\mu R}, \tag{4.91}$$

$$\Gamma_\mu^{(W)} \quad = \quad \frac{I_W^\pm}{2\sqrt{2}\, \sin\theta_W}\, \gamma_\mu (1 - \gamma_5) \;=\; \frac{I_W^\pm}{2\sqrt{2}\, \sin\theta_W}\, \gamma_{\mu L}. \tag{4.92}$$

The explicit results for the (anti)leptons are found by inserting the values for the charges (Q) and the weak isospin for the lefthanded leptons (or righthanded antileptons), e.g. $I_W^3 = \pm 1/2$ for the (ν_e, e^-) doublet with $I_W = 1/2$. For antileptons I_W^3 and Q change sign, while also left and right are interchanged, i.e. $c_L(\ell) = c_R(\bar{\ell})$. In the cross sections of electroweak processes and (as we will see below) scattering processes with polarized fermions the following contractions are needed,

$$\frac{1}{2} Tr\, \left[\gamma_\mu (\not{k}' + m) \gamma_\nu (\not{k} + m) \right] \quad = \quad 2\, k_\mu k'_\nu + 2\, k'_\mu k_\nu - Q^2\, g_{\mu\nu} \equiv L_{\mu\nu}^{(S)} \tag{4.93}$$

$$\frac{1}{2} Tr\, \left[\gamma_\mu \gamma_5 (\not{k}' + m) \gamma_\nu \gamma_5 (\not{k} + m) \right] \quad = \quad 2\, k_\mu k'_\nu + 2\, k'_\mu k_\nu - Q^2\, g_{\mu\nu} - 4m^2\, g_{\mu\nu} \approx L_{\mu\nu}^{(S)} \tag{4.94}$$

$$\frac{1}{2} Tr\, \left[\gamma_\mu \gamma_5 (\not{k}' + m) \gamma_\nu (\not{k} + m) \right] \quad = \quad 2i\, \epsilon_{\mu\nu\rho\sigma} q^\rho k^\sigma \equiv L_{\mu\nu}^{(A)} \tag{4.95}$$

$$\frac{1}{2} Tr\, \left[\gamma_\mu (\not{k}' + m) \gamma_\nu \gamma_5 (\not{k} + m) \right] \quad = \quad 2i\, \epsilon_{\mu\nu\rho\sigma} q^\rho k^\sigma = L_{\mu\nu}^{(A)} \tag{4.96}$$

(Note that we use $\epsilon^{0123} = 1$). The approximation that is made in the axial vector - axial vector case is neglecting the fermion mass. Neglecting the mass terms, the leptonic tensor decouples for right/left,

$$\frac{1}{4} Tr\, \left[\gamma_{\mu R/L} (\not{k}' + m) \gamma_{\nu R/L} (\not{k} + m) \right] \quad \equiv \quad L_{\mu\nu}^{(R/L)} \;\approx\; L_{\mu\nu}^{(S)} \pm L_{\mu\nu}^{(A)} \tag{4.97}$$

$$\frac{1}{4} Tr\, \left[\gamma_{\mu R/L} (\not{k}' + m) \gamma_{\nu L/R} (\not{k} + m) \right] \quad \approx \quad 0. \tag{4.98}$$

4.6.1 Charged current cross sections

In analogy with Eq. 4.60 one can write down the cross sections for electroweak processes. Note that there is no averaging over incoming leptons as there is only one lefthanded neutrino (or one righthanded antineutrino). The leptonic tensor (using the current in Eq. 4.88) therefore is

$$(\nu_e, e^-) \text{ scattering}: \qquad Tr\left[\Gamma_\mu(\not{k}' + m)\Gamma_\nu \frac{1-\gamma_5}{2}(\not{k} + m)\right] = c_L^{(W)2}(\nu_e)\, L_{\mu\nu}^{(L)} \tag{4.99}$$

$$(\bar{\nu}_e, e^+) \text{ scattering}: \qquad Tr\left[\Gamma_\mu(\not{k}' + m)\Gamma_\nu \frac{1+\gamma_5}{2}(\not{k} + m)\right] = c_R^{(W)2}(\bar{\nu}_e)\, L_{\mu\nu}^{(R)} \tag{4.100}$$

The cross section therefore becomes

$$\begin{aligned}
&(\nu_e, e^-) \text{ scattering} \\
&\frac{d\sigma}{d\Omega\, dE'} = \frac{\alpha^2}{(Q^2 + M_W^2)^2}\frac{E'}{E}\, c_L^{(W)2}\, L_{\mu\nu}^{(L)} W_{(WW)}^{\mu\nu}, \\
&\qquad = \frac{G_F^2}{8\pi^2}\left(\frac{M_W^2}{Q^2 + M_W^2}\right)^2 \frac{E'}{E}\, L_{\mu\nu}^{(L)} W_{(WW)}^{\mu\nu}, \qquad (4.101) \\
&(\bar{\nu}_e, e^+) \text{ scattering} \\
&\frac{d\sigma}{d\Omega\, dE'} = \frac{G_F^2}{8\pi^2}\left(\frac{M_W^2}{Q^2 + M_W^2}\right)^2 \frac{E'}{E}\, L_{\mu\nu}^{(R)} W_{(WW)}^{\mu\nu}, \qquad (4.102)
\end{aligned}$$

where the Fermi-coupling is given by $G_F/\sqrt{2} = e^2/8M_W^2 \sin^2\theta_W$ and $W_{(WW)}^{\mu\nu}$ is the hadronic tensor defined as in Eq. 4.62 with the current

$$J_\mu^{(W)} = \sum_q \bar{q}(x)\, I_W^{\pm}\, \gamma_{\mu L}\, q(x), \tag{4.103}$$

being the hadronic current coupling to the W-boson, usually referred to as weak charged current. Note that the factor $(e/2\sqrt{2}\sin\theta_W)$ that appears in the hadronic current coupling to the W-boson as given in Eq. 4.92 is already absorbed in the expression for the cross section.

The most general expression allowed for the hadronic tensor (neglecting pieces which turn out to be proportional to the lepton mass) is

$$W_{\mu\nu}(P,q) = W_{\mu\nu}^{(S)}(P,q) + W_{\mu\nu}^{(A)}(P,q) \tag{4.104}$$

with

$$\begin{aligned}
W_{\mu\nu}^{(S)} &= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) W_1 + \frac{\tilde{P}_\mu \tilde{P}_\nu}{M^2} W_2, \\
W_{\mu\nu}^{(A)} &= -i\,\epsilon_{\mu\nu\rho\sigma} q^\rho P^\sigma \frac{W_3}{2\,M^2}.
\end{aligned} \tag{4.105}$$

For the cross section one needs the contractions between leptonic and hadronic tensor, of which the symmetric ones have been given before (Eqs 4.47 and 4.48),

$$\begin{aligned}
v_1 &= L_{\mu\nu}^{(S)}\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) = 2\,Q^2 \overset{lab}{=} 8\,EE'\sin^2(\theta/2), \\
v_2 &= L_{\mu\nu}^{(S)} \frac{\tilde{P}^\mu \tilde{P}^\nu}{M^2} = \frac{4(P.k)^2 - 4(P.k)(P.q) - M^2\,Q^2}{M^2} \overset{lab}{=} 4\,EE'\cos^2(\theta/2), \\
v_3 &= L_{\mu\nu}^{(A)} \frac{-i\,\epsilon^{\mu\nu\rho\sigma} q_\rho P_\sigma}{2\,M^2} = -q^2 \frac{P\cdot(k+k')}{M^2} \overset{lab}{=} 4\,EE'\frac{E+E'}{M}\sin^2(\theta/2).
\end{aligned} \tag{4.106}$$

The cross sections for charged current neutrino and antineutrino scattering then become

$$\frac{d\sigma^{(\nu/\bar{\nu})}}{d\Omega\, dE'} = \frac{G_F^2\, E'^2}{2\,\pi^2}\,\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2\,\left[\cos^2(\theta/2)\,W_2 + 2\,\sin^2(\theta/2)\,W_1 \mp \frac{E+E'}{M}\,W_3\right]. \tag{4.107}$$

The structure functions can also be projected out using photon polarizations,

$$W_L = -W_1 + \frac{|\boldsymbol{q}|^2}{Q^2}\,W_2, \tag{4.108}$$

$$W_+ = W_1 + \frac{q_z}{2M}\,W_3, \tag{4.109}$$

$$W_- = W_1 - \frac{q_z}{2M}\,W_3 \tag{4.110}$$

($q_z = q^3$). Finally, one often encounters the expressions in terms of combinations of structure functions that exhibit scaling in the Bjorken variable x. In terms of the scaling variables

$$x = \frac{Q^2}{2\,P\cdot q} = \frac{Q^2}{2\,M\nu} \qquad \text{and} \qquad y = \frac{P\cdot q}{P\cdot k} = \frac{\nu}{E} \tag{4.111}$$

and the scaling functions

$$F_1 = M\,W_1, \qquad F_2 = \nu\,W_2, \qquad F_3 = \nu\,W_3, \tag{4.112}$$

one obtains using $E'\,dE'd\Omega = \pi\,2ME\,y\,dx\,dy$ (if one prefers $2ME$ rewritten as $(s_{eN}-M^2)$ in order to get a fully covariant expression)

$$\frac{d\sigma^{(\nu/\bar{\nu})}}{dx\,dy} = \frac{G_F^2}{2\pi}\,2ME\,\left(\frac{M_W^2}{Q^2+M_W^2}\right)^2\,\left[\left(1-y-\frac{M^2\,xy}{2ME}\right)\,F_2 + xy^2\,F_1 \mp xy\left(1-\frac{y}{2}\right)\,F_3\right]. \tag{4.113}$$

4.6.2 Neutral current cross sections

For deep inelastic electron scattering where the electron energy becomes of the order of the mass of the heavy vector bosons, both photon and Z^0 exchange will contribute. Also at low energies Z^0 exchange may be relevant if one considers left-right asymmetries, e.g. in polarized scattering experiments (see next section). In these cases the amplitude consists of two terms and the cross sections contains three contributions,

(e^-,e^-) scattering

$$\frac{d\sigma^{(\gamma\gamma)}}{d\Omega\, dE'} = \frac{\alpha^2}{Q^4}\,\frac{E'}{E}\,\frac{1}{2}\left(c_L^{(\gamma)2}\,L_{\mu\nu}^{(L)} + c_R^{(\gamma)2}\,L_{\mu\nu}^{(R)}\right)W_{(\gamma\gamma)}^{\mu\nu} = \frac{\alpha^2}{Q^4}\,\frac{E'}{E}\,L_{\mu\nu}^{(S)}W_{(\gamma\gamma)}^{\mu\nu}, \tag{4.114}$$

$$\begin{aligned}\frac{d\sigma^{(\gamma Z)}}{d\Omega\, dE'} &= 2\,\frac{\alpha^2}{Q^2(Q^2+M_Z^2)}\,\frac{E'}{E}\,\frac{1}{2}\left(c_L^{(\gamma)}c_L^{(Z)}\,L_{\mu\nu}^{(L)} + c_R^{(\gamma)}c_R^{(Z)}\,L_{\mu\nu}^{(R)}\right)W_{(\gamma Z)}^{\mu\nu},\\ &= \frac{G_F\,\alpha}{2\pi\sqrt{2}\,Q^2}\,\frac{M_Z^2}{Q^2+M_Z^2}\,\frac{E'}{E}\,\left[(1-4\sin^2\theta_W)L_{\mu\nu}^{(S)} - L_{\mu\nu}^{(A)}\right]\,W_{(\gamma Z)}^{\mu\nu},\end{aligned} \tag{4.115}$$

$$\begin{aligned}\frac{d\sigma^{(ZZ)}}{d\Omega\, dE'} &= \frac{\alpha^2}{(Q^2+M_Z^2)^2}\,\frac{E'}{E}\,\frac{1}{2}\left(c_L^{(Z)2}\,L_{\mu\nu}^{(L)} + c_R^{(Z)2}\,L_{\mu\nu}^{(R)}\right)W_{(ZZ)}^{\mu\nu},\\ &= \frac{G_F^2}{16\,\pi^2}\,\left(\frac{M_Z^2}{Q^2+M_Z^2}\right)^2\,\frac{E'}{E}\,\Big[(1-4\sin^2\theta_W+8\sin^4\theta_W)\,L_{\mu\nu}^{(S)}\\ &\qquad - (1-4\sin^2\theta_W)\,L_{\mu\nu}^{(A)}\Big]\,W_{(ZZ)}^{\mu\nu}.\end{aligned} \tag{4.116}$$

In this expression a factor $2\sin\theta_W\cos\theta_W$ for each hadronic current coupling to the Z^0 has been included in the denominator and hence the hadronic tensors are defined as in Eq. 4.62 with the currents

$$\begin{aligned} J_\mu^{(\gamma)} &= \sum_q \bar{q}(x)\, Q\, \gamma_\mu\, q(x), & (4.117)\\ J_\mu^{(Z)} &= \sum_q \bar{q}(x)\,(I_W^3 - Q\sin^2\theta_W)\,\gamma_{\mu L}\, q(x) - \sum_q \bar{q}(x)\, Q\sin^2\theta_W\,\gamma_{\mu R}\, q(x) \\ &= \sum_q \bar{q}(x)\,(I_W^3 - 2\,Q\sin^2\theta_W)\,\gamma_\mu\, q(x) - \sum_q \bar{q}(x)\, I_W^3\,\gamma_\mu\gamma_5\, q(x), & (4.118) \end{aligned}$$

being the hadronic current coupling to the neutral bosons, usually referred to as weak neutral current.

4.7 Polarized inclusive lepton - hadron scattering

Note that for spin-polarized leptons in the initial state we have

$$\begin{aligned} L_{\mu\nu}^{(s)} &= Tr\,\left[\gamma_\mu(\not{k}' + m)\gamma_\nu(\not{k} + m)\,\frac{1 \pm \gamma_5\,\not{s})}{2}\right] \\ &= 2\,k_\mu k'_\nu + 2\,k'_\mu k_\nu - Q^2\,g_{\mu\nu} \pm 2i\,m\,\epsilon_{\mu\nu\rho\sigma}q^\rho s^\sigma. & (4.119) \end{aligned}$$

Note that for light particles or particles at high energies helicity states ($\hat{s} = \hat{\boldsymbol{k}}$) become chirality eigenstates. For $L_{\mu\nu}$ the equivalence is easily seen because for $s^\mu = (|\boldsymbol{k}|/m, E\hat{\boldsymbol{k}}/m)$ ($s^2 = -1$ and $s\cdot p = 0$) one obtains in the limit $E \approx |\boldsymbol{k}|$ the result $m\,s^\mu \approx k^\mu$. Therefore the leptonic tensor for helicity states is

$$L_{\mu\nu}^{(\lambda=\pm 1/2)} \approx L_{\mu\nu}^{(R/L)} = L_{\mu\nu}^{(S)} \pm L_{\mu\nu}^{(A)}. \quad (4.120)$$

4.7.1 Asymmetry in $\vec{e}$ - hadron scattering

For deep inelastic polarized electron scattering the difference of cross sections for electrons polarized parallel and antiparallel to the beam direction is denoted $d\sigma_{R-L} \equiv d\sigma(e_R) - d\sigma(e_L)$. For scattering off an unpolarized hadron such an asymmetry only arises from the weak (parity violating) interactions. For energies much lower than the mass of the Z^0 boson (i.e. $Q^2 \ll M_Z^2$) one has for the electromagnetic and the interference term,

polarized (e^-, e^-) scattering

$$\begin{aligned} \frac{d\sigma_{R-L}^{(\gamma\gamma)}}{d\Omega\, dE'} &= \frac{\alpha^2}{Q^4}\,\frac{E'}{E}\,\left(c_R^{(\gamma)2}\,L_{\mu\nu}^{(R)} - c_L^{(\gamma)2}\,L_{\mu\nu}^{(L)}\right) W_{(\gamma\gamma)}^{\mu\nu} = 0, & (4.121)\\ \frac{d\sigma_{R-L}^{(\gamma Z)}}{d\Omega\, dE'} &= 2\,\frac{\alpha^2}{Q^2(Q^2+M_Z^2)}\,\frac{E'}{E}\,\left(c_R^{(\gamma)}c_R^{(Z)}\,L_{\mu\nu}^{(R)} - c_L^{(\gamma)}c_L^{(Z)}\,L_{\mu\nu}^{(L)}\right) W_{(\gamma Z)}^{\mu\nu}, \\ &= -\frac{G_F\,\alpha}{\pi\sqrt{2}\,Q^2}\,\frac{M_Z^2}{Q^2+M_Z^2}\,\frac{E'}{E}\,\left[L_{\mu\nu}^{(S)} - (1-4\sin^2\theta_W)L_{\mu\nu}^{(A)}\right] W_{(\gamma Z)}^{\mu\nu}, & (4.122) \end{aligned}$$

or an asymmetry given by

$$\begin{aligned} A &= \frac{d\sigma(e_R) - d\sigma(e_L)}{d\sigma(e_R) + d\sigma(e_L)} & (4.123)\\ &= -\frac{G_F\,Q^2}{\pi\sqrt{2}\,\alpha}\,\frac{\left[L_{\mu\nu}^{(S)} - (1-4\sin^2\theta_W)L_{\mu\nu}^{(A)}\right] W_{(\gamma Z)}^{\mu\nu}}{L_{\mu\nu}^{(S)}\,W_{(\gamma\gamma)}^{\mu\nu}} \end{aligned}$$

$$= -\frac{G_F\,Q^2}{\pi\sqrt{2}\,\alpha}\,\frac{\left(1-y-\frac{M^2\,xy}{2ME}\right)\,F_2^{(\gamma Z)}+xy^2\,F_1^{(\gamma Z)}-xy\left(1-\frac{y}{2}\right)\,(1-4\sin^2\theta_W)F_3^{(\gamma Z)}}{\left(1-y-\frac{M^2\,xy}{2ME}\right)\,F_2^{(\gamma\gamma)}+xy^2\,F_1^{(\gamma\gamma)}} \quad (4.124)$$

4.7.2 Inclusive $\vec{e}$ - $\vec{p}$ scattering

In the situation where the target is polarized, one can measure a number of new structure functions compared to the case of an unpolarized target. For a spin 1/2 target one can employ the spin vector n^μ ($n^2 = -1$ and $n \cdot P = 0$). The most general expression for the hadronic tensor is

$$W_{\mu\nu}^{(pol)} = W_{\mu\nu}^{(S)} + i\epsilon_{\mu\nu\rho\sigma}q^\rho n^\sigma\,(MG_1+\nu G_2) - i\epsilon_{\mu\nu\rho\sigma}q^\rho P^\sigma\,(q\cdot n)\,\frac{G_2}{M}. \quad (4.125)$$

The resulting cross sections can be disentangled from the experimental cross sections as each of the terms have their characteristic behavior,

$$v_4 = L_{\mu\nu}^{(A)}\cdot\, i\,\epsilon^{\mu\nu\rho\sigma}q_\rho n_\sigma = 2\,q^2\,n\cdot(k+k') \overset{lab}{=} 8\,EE'\,\sin^2(\theta/2)\,\hat{\boldsymbol{n}}\cdot(\boldsymbol{k}+\boldsymbol{k'}) \quad (4.126)$$

$$v_5 = L_{\mu\nu}^{(A)}\,\frac{-i\,\epsilon^{\mu\nu\rho\sigma}q_\rho P_\sigma\,(q\cdot n)}{M} = -2\,q^2\,P\cdot(k+k')\,(q\cdot n)$$
$$\overset{lab}{=} -8\,EE'\,(E+E')\sin^2(\theta/2)\,\hat{\boldsymbol{n}}\cdot\boldsymbol{q}. \quad (4.127)$$

Choosing the polarization vector under an angle α with the incoming lepton momentum $\boldsymbol{k}$ in a plane that makes an angle ϕ with the scattering plane (formed by $\boldsymbol{k}$ and $\boldsymbol{k'}$, see table 4.1) one obtains for the cross section difference for leptons polarized parallel and antiparallel to the momentum $\boldsymbol{k}$ respectively,

$$\frac{d\Delta\sigma}{d\Omega\,dE'} = \frac{4\,\alpha^2}{Q^2}\,\frac{E'}{E}\left\{\cos\alpha\left[(E+E'\cos\theta)\,M\,G_1 - Q^2\,G_2\right]\right.$$
$$\left. + E'\sin\theta\,\sin\alpha\cos\phi(MG_1+2E\,G_2)\right\}. \quad (4.128)$$

The resulting cross section difference for electrons and protons polarized parallel and antiparallel to the beam is then given by

$$\frac{d\sigma^{(\uparrow\uparrow-\uparrow\downarrow)}}{d\Omega\,dE'} = \frac{4\,\alpha^2}{Q^2}\,\frac{E'}{E}\,\left[(E+E'\cos\theta)\,M\,G_1 - Q^2\,G_2\right], \quad (4.129)$$

while for protons polarized perpendicular to the beam (in the scattering plane),

$$\frac{d\sigma^{(\uparrow\rightarrow-\uparrow\leftarrow)}}{d\Omega\,dE'} = \frac{4\,\alpha^2}{Q^2}\,\frac{E'}{E}\,E'\sin\theta\,(MG_1+2E\,G_2). \quad (4.130)$$

Table 4.1: *The momenta and polarization vector in the target rest frame for inclusive polarized lepton-hadron scattering ($\hat{k} = \hat{z}$)*

k	$= (E, 0, 0, E)$
k'	$= (E', E'\sin\theta, 0, E'\cos\theta)$
n	$= (0, \sin\alpha\cos\phi, \sin\alpha\sin\phi, \cos\alpha)$

4.8 Generalization of the expansion of structure functions

For the analysis of the hadronic tensor it is useful to consider vectors that are orthogonal to q. We consider vectors $\tilde{a}$ obtained by subtracting from a the projection along q are useful,

$$\tilde{a}^\mu \equiv \tilde{g}^{\mu\nu} a_\nu = a^\mu - \frac{a\cdot q}{q^2}\, q^\mu, \tag{4.131}$$

where

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}. \tag{4.132}$$

For lepton-hadron it is convenient to use a Cartesian basis defined by the spacelike vector q^μ and the momentum of the target hadron. The other hadron then fixes the azimuthal direction.

$$Z^\mu \equiv -q^\mu, \tag{4.133}$$

$$T^\mu \equiv -\frac{q^2}{P\cdot q}\,\tilde{P}^\mu = -\frac{q^2}{P\cdot q}\,P^\mu + q^\mu, \tag{4.134}$$

$$\begin{aligned} X^\mu &\equiv \frac{P_h\cdot T}{P_h\cdot P}\,\tilde{P}^\mu - \frac{P\cdot T}{P\cdot P_h}\,\tilde{P}_h^\mu \approx -\frac{P_h\cdot q}{P_h\cdot P}\,\tilde{P}^\mu - \frac{P\cdot q}{P\cdot P_h}\,\tilde{P}_h^\mu \\ &\approx q^\mu - \frac{P_h\cdot q}{P_h\cdot P}\,P^\mu - \frac{P\cdot q}{P\cdot P_h}\,P_h^\mu \end{aligned} \tag{4.135}$$

$$Y^\mu \equiv \frac{1}{P\cdot P_h}\,\epsilon^{\mu\nu\rho\sigma} P_\nu P_{h\rho} q_\sigma, \tag{4.136}$$

These vectors satisfy $T^2 = Q^2$, $Z^2 = -Q^2$, and $X^2 \approx Y^2 \approx -Q_T^2$. We will used normalized vectors $\hat{t} = T/Q$, $\hat{z} = Z/Q$, etc. The perpendicular tensors are given by

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} - \frac{T^\mu T^\nu}{T^2} \qquad (\ell H) \tag{4.137}$$

$$\epsilon_\perp^{\mu\nu} \equiv \frac{1}{Q^2}\epsilon^{\mu\nu\rho\sigma} T_\rho q_\sigma = -\frac{1}{Q^2}\epsilon^{\mu\nu\rho\sigma} T_\rho Z_\sigma = \frac{1}{P\cdot q}\,\epsilon^{\mu\nu\rho\sigma} P_\rho q_\sigma, \qquad (\ell H) \tag{4.138}$$

Note that the azimuthal vector $X^\mu = -P_{h\perp}^\mu/z = -g_\perp^{\mu\nu} P_{h\,\nu}/z$.

In terms of these vectors and tensors, the expansion of the hadronic tensor for unpolarized inclusive lepton-hadron scattering becomes

$$M\,W^{\mu\nu}(P,q) = -g_\perp^{\mu\nu}\,F_1 + \hat{t}^\mu \hat{t}^\nu\,F_L, \tag{4.139}$$

where

$$F_1 = MW_T = MW_1, \tag{4.140}$$

$$F_L = MW_L \approx -F_1 + \frac{F_2}{2x_{\scriptscriptstyle B}}. \tag{4.141}$$

Also the lepton momenta can be expressed in the cartesian vectors. For ℓH we have

$$k^\mu = \frac{2-y}{y}\,T^\mu - \frac{1}{2}\,Z^\mu + k_\perp^\mu = \frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)Q}{2y}\,\hat{t}^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}^\mu, \tag{4.142}$$

$$k^\mu = \frac{2-y}{y}\,T^\mu + \frac{1}{2}\,Z^\mu + k_\perp^\mu = -\frac{Q}{2}\,\hat{q}^\mu + \frac{(2-y)Q}{2y}\,\hat{t}^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}^\mu. \tag{4.143}$$

This leads to the leptonic tensor

$$L_{(\ell H)}^{\mu\nu} = \frac{Q^2}{y^2}\left[-2\left(1-y+\frac{1}{2}\,y^2\right) g_\perp^{\mu\nu} + 4(1-y)\hat{t}^\mu\hat{t}^\nu\right.$$

$$+4(1-y)\left(\hat{\ell}^\mu\hat{\ell}^\nu+\frac{1}{2}\,g_\perp^{\mu\nu}\right)+2(2-y)\sqrt{1-y}\;\hat{t}^{\{\mu}\hat{\ell}^{\nu\}}$$
$$-i\lambda\,y(2-y)\,\epsilon_\perp^{\mu\nu}-2i\lambda\,y\sqrt{1-y}\;\hat{\ell}_\rho\epsilon_\perp^{\rho[\mu}\hat{t}^{\nu]}\Bigg]. \qquad (4.144)$$

The relevant contractions can be expressed in the scaling variables, e.g. for unpolarized inclusive scattering

$w^{\mu\nu}$	$L_{\mu\nu}w^{\mu\nu}$
$-g_\perp^{\mu\nu}$	$\frac{4Q^2}{y^2}\,(1-y+\frac{1}{2}y^2)$
$\hat{t}^\mu\hat{t}^\nu$	$\frac{4Q^2}{y^2}(1-y)$

4.9 The meaning of spin vectors

As a final topic in this section, we will consider here the general types of squared matrix elements that have been encountered and the role of the spin vector in them. This includes processes that can be 'factorized', e.g. a process followed by a decay of an unstable particle (e.g. a Λ decaying in πN). The formalism for hard processes is similar; here (part of) the final state $|f\rangle$ can be traced back to the initial state $|i\rangle$ through production and decay of a quark q.

Starting with the quantities

$$\begin{aligned}
\tilde{\mathcal{W}}^{\mu\nu}_{ij,kl}(q,P,P_h) &= \langle h;k|T|H;i\rangle\,\langle H;j|T^\dagger|h;l\rangle \\
&- \text{ generalized tensor in } 2\times 2 \text{ H-spin-space (ij) and in } 2\times 2 \text{ h-spin-space (kl)}, \\
\rho_{ij}(P,S) &= \langle H;i|\sum_\alpha|\alpha\rangle p(\alpha)\langle\alpha|H;j\rangle \overset{\text{H rest frame}}{\equiv} \frac{1}{2}\Big(\mathbf{1}_{ij}+\sigma_{ij}\cdot\boldsymbol{S}\Big) \\
&- \text{ initial state density matrix , defining } S, \\
R^*_{kl}(P_h,f) &= \langle h;k|f\rangle\,\langle f|h;l\rangle \overset{\text{h rest frame}}{\equiv} w(f)\,\Big(\mathbf{1}_{kl}+\sigma^*_{kl}\cdot\boldsymbol{S}_h\Big) \\
&- \text{ decay matrix into final states f in } 2\times 2 \text{ h-spin-space (kl), defining } S_h,
\end{aligned}$$

where the latter is normalized to $\sum_f R^*_{kl}(f)=\delta_{kl}$. Note that the summation over f is just symbolic for all kinematic variables appearing in the decay. An example of $w(f)$ is the center-of-mass distribution $w(\theta_{\pi N})$ in the decay of the Λ baryon. Starting with some (general) initial state, one has the following result for the semi-inclusive hadronic tensor in which the decay products of hadron h are detected,

$$\mathcal{W}^{\mu\nu}(i\to f)=\rho_{ji}(P,S)\,\tilde{\mathcal{W}}^{\mu\nu}_{ij,kl}(q,P,P_h)\,R_{lk}(P_h,f)\equiv\tilde{\mathcal{W}}^{\mu\nu}_{kl}(q;P,S;P_h)\,R_{lk}(P_h,f). \qquad (4.145)$$

This shows for instance why the spin vector, originating from $\rho(P,S)$ can only appear linearly in the parametrization of the tensor. Summing (or integrating) over the decay products of h one is not able to measure the polarization of h,

$$\sum_f\mathcal{W}^{\mu\nu}(i\to h\to f)=\tilde{\mathcal{W}}^{\mu\nu}_{kk}(q;P,S;P_h)\equiv\mathcal{W}^{\mu\nu}(q;P,S;P_h,S_h=0). \qquad (4.146)$$

The general result in the h-spin-space can (in the rest-frame of hadron h) be written as the unpolarized result times the production density matrix parametrized in terms of the polarization $\mathcal{P}_h$,

$$\tilde{\mathcal{W}}^{\mu\nu}_{kl}(q;P,S;P_h)\equiv\mathcal{W}^{\mu\nu}(q;P,S;P_h,0)\,\frac{1}{2}\Big(\mathbf{1}_{kl}+\sigma_{kl}\cdot\mathcal{P}_h(q;P,S;P_h)\Big). \qquad (4.147)$$

Thus one can write in the h-rest frame,

$$\begin{aligned}\mathcal{W}^{\mu\nu}(i\to f) &= \mathcal{W}^{\mu\nu}(q;P,S;P_h,0)\,w(f)\,\Big(1+\mathcal{P}(q;P,S;P_h)\cdot\boldsymbol{S}_h(P_h;f)\Big)\\ &\equiv \mathcal{W}^{\mu\nu}(q;P,S;P_h,S_h)\,w(f).\end{aligned}\tag{4.148}$$

This shows that also the spin vector S_h appears only linearly, while it multiplies the polarization vector as appearing in the production matrix. Parametrizing the polarization vector $\mathcal{P}_h$ similar as S_h in a longitudinal polarization Λ_h and a transverse polarization $\mathcal{P}_{hT}$, which because the polarization vector parametrizes a density matrix satisfy $\Lambda_h^2+\mathcal{P}_{hT}^2\leq 1$, one can in a general frame write for $\mathcal{P}\cdot\boldsymbol{S}_h$ $=\Lambda_h\,\lambda_h+\mathcal{P}_{h\perp}\cdot\boldsymbol{S}_{h\perp}$.

4.10 Scaling variables in lepton-hadron scattering

We will use lightcone components for momenta $p^\mu=[p^-,p^+,\boldsymbol{p}_T]$ with $p^\pm\equiv(p^0\pm p^3)/\sqrt{2}$ and consider two different frames I and II. In frame I the virtual photon has no transverse directions, in frame II (only useful if a final state hadrons is considered) the hadrons do not have transverse components.

lepton-hadron:			
momentum	frame I	frame II	relations
$q=k-k'$	$\left[\frac{Q^2}{A\sqrt{2}},-\frac{A}{\sqrt{2}},\mathbf{0}_\perp\right]$	$\left[\frac{Q^2}{A\sqrt{2}},-\frac{A}{\sqrt{2}},\boldsymbol{q}_T\right]$	$\boldsymbol{q}_T=-\frac{1}{z}\,\boldsymbol{P}_{h\perp}$
P	$\left[\frac{x_B\,M^2}{A\sqrt{2}},\frac{A}{x_B\sqrt{2}},\mathbf{0}_\perp\right]$	$\left[\frac{x_B\,M^2}{A\sqrt{2}},\frac{A}{x_B\sqrt{2}},\mathbf{0}_T\right]$	$x_B=-\frac{q^+}{P^+}\approx\frac{Q^2}{2P\cdot q}\approx-\frac{P_h\cdot q}{P_h\cdot P}$
P_h	$\left[\frac{zQ^2}{A\sqrt{2}},\frac{AM_{h\perp}^2}{zQ^2\sqrt{2}},\boldsymbol{P}_{h\perp}\right]$	$\left[\frac{zQ^2}{A\sqrt{2}},\frac{AM_h^2}{zQ^2\sqrt{2}},\mathbf{0}_T\right]$	$z=\frac{P_h^-}{q^-}\approx-\frac{2P_h\cdot q}{Q^2}\approx\frac{P\cdot P_h}{P\cdot q}$

As indicated in the above table the scaling variables x_B and z are ratios of lightcone components. In fact it is appropriate to take the lightcone ratios in frame II as the defining relations and treat the differences with the invariants $P\cdot q$ etcetera as mass corrections which are of order $1/Q^2$, to be specific if we want to keep the definition $x_B=Q^2/2P\cdot q$ one has

$$-\frac{q^+}{P^+}=\frac{2x_B}{1+\sqrt{1+\frac{4M^2x_B^2}{Q^2}}}.\tag{4.149}$$

This is referred to as the Nachtmann scaling variable. Also the lepton-momenta can be written down in the frames discussed above using

$$y=\frac{q^-}{k^-}\approx\frac{P\cdot q}{P\cdot k}=\frac{Q^2}{x_B\,s},\tag{4.150}$$

which in the target rest frame ($A=x_BM$) equals $y=\nu/E$ ($\nu=P\cdot q/M$).

lepton-hadron:			
momentum	frame I	frame II	relations
k	$\left[\frac{Q^2}{yA\sqrt{2}},\frac{(1-y)A}{y\sqrt{2}},\boldsymbol{k}_\perp\right]$	$\left[\frac{yQ^2}{A\sqrt{2}},\frac{\boldsymbol{k}_T^2A}{yQ^2\sqrt{2}},\boldsymbol{k}_T\right]$	$\boldsymbol{k}_T=\boldsymbol{k}_\perp+\frac{\boldsymbol{q}_T}{y}$
k'	$\left[\frac{(1-y)Q^2}{yA\sqrt{2}},\frac{A}{y\sqrt{2}},\boldsymbol{k}_\perp\right]$	$\left[\frac{(1-y)Q^2}{yA\sqrt{2}},\frac{y(\boldsymbol{k}_T^2-2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)A}{(1-y)Q^2\sqrt{2}},\boldsymbol{k}_T-\boldsymbol{q}_T\right]$	$\boldsymbol{k}_\perp^2=\frac{1-y}{y^2}Q^2$

The lepton phase space becomes $d^3k'/E'=E'dE'd\Omega=\pi ysdx_Bdy$, and one obtains for Eq. 4.22

$$\frac{d\sigma^{(\ell H)}}{dx_B\,dz\,d^2\boldsymbol{q}_T\,dy}=\frac{\pi\,\alpha^2}{2Q^4}\,yz\,L_{\mu\nu}^{(\ell H)}\,2M\mathcal{W}^{\mu\nu(\ell H)}.\tag{4.151}$$

For the inclusive process one finds

$$\frac{d\sigma^{(\ell H)}}{dx_B\, dy} = \frac{\pi\,\alpha^2}{Q^4}\, y\, L_{\mu\nu}^{(\ell H)}\, 2M W^{\mu\nu(\ell H)}. \tag{4.152}$$

4.11 Electron-positron annihilation

4.11.1 The hadronic tensor

Consider the process $e^- + e^+ \to h_1 + h_2 + X$, with two hadrons belonging to opposite jets emerging with momenta P_1 and P_2. The annihilating incoming leptons with momenta k^μ and k'^μ produce a high mass photon with momentum $q = k + k'$ with $q^2 \equiv Q^2 \geq 0$. The unobserved outstate will be denoted by $|P_X\rangle$. We will consider the general case of polarized leptons with helicities λ and λ' and production of hadrons of which the spin states are characterized by spin vectors S_1 and S_2, respectively (see Appendix to this section for the meaning of spin vectors in production processes). We have the following relations

$$P_1^2 = M_1^2, \quad P_2^2 = M_2^2, \tag{4.153}$$
$$k^2 = k'^2 = m^2 \approx 0, \tag{4.154}$$
$$S_1^2 = S_2^2 = -1, \tag{4.155}$$
$$P_1 \cdot S_1 = P_2 \cdot S_2 = 0. \tag{4.156}$$

We will work mostly in the limit where Q^2, $P_1 \cdot q$ and $P_2 \cdot q$ are large, keeping the ratios $2P_1 \cdot q/Q^2$ and $2P_2 \cdot q/Q^2$ finite. The invariant ampitude for the process is given by

$$\mathcal{M} = \bar{v}(k', \lambda')\gamma^\mu u(k, \lambda)\frac{e^2}{Q^2}\langle P_X; P_1 S_1; P_2 S_2|J_\mu(0)|0\rangle. \tag{4.157}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathcal{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(e^+e^-)} H^{\mu\nu(e^+e^-)}, \tag{4.158}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(e^+e^-)}(k\lambda; k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}k^\rho k'^\sigma\right). \tag{4.159}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(e^+e^-)}(P_X; P_1 S_1; P_2 S_2) = \langle 0|J_\mu(0)|P_X; P_1 S_1; P_2 S_2\rangle\langle P_X; P_1 S_1; P_2 S_2|J_\nu(0)|0\rangle, \tag{4.160}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\mathcal{R}, \tag{4.161}$$

with the flux factor

$$F = 4\sqrt{(k\cdot k')^2 - k^2 k'^2} \;\approx\; 2Q^2 \tag{4.162}$$

and the Lorentz invariant phase space

$$d\mathcal{R} = (2\pi)^4\delta^4(k + k' - P_X - P_1 - P_2)\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3P_1}{(2\pi)^3 2P_1^0}\frac{d^3P_2}{(2\pi)^3 2P_2^0}. \tag{4.163}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$\mathcal{W}_{\mu\nu}^{(e^+e^-)}(q;P_1S_1;P_2S_2) = \frac{1}{(2\pi)^4}\int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q-P_X-P_1-P_2)H_{\mu\nu}^{(e^+e^-)}(P_X;P_1S_1;P_2S_2). \tag{4.164}$$

One obtains the cross section (including a factor 1/2 from averaging over incoming polarizations)

$$\frac{P_1^0\,P_2^0\;d\sigma^{(e^+e^-)}}{d^3P_1\,d^3P_2} = \frac{\alpha^2}{4\,Q^6}L_{\mu\nu}^{(e^+e^-)}\mathcal{W}^{\mu\nu(e^+e^-)}, \tag{4.165}$$

where the lepton tensor is given by the expression between brackets in Eq. (4.159) and the hadron tensor by Eq. (4.164).

Note that for a single produced hadron one finds

$$E_h\,\frac{d\sigma}{d^3P_h} = \frac{\alpha^2}{2\,Q^6}\,L_{\mu\nu}^{(e^+e^-)}W^{\mu\nu(e^+e^-)}, \tag{4.166}$$

where the hadron tensor is given by

$$W_{\mu\nu}^{(e^+e^-)}(q;P_hS_h) = \frac{1}{(2\pi)}\int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q-P_X-P_h)\langle 0|J_\mu(0)|P_X;P_hS_h\rangle\langle P_X;P_hS_h|J_\nu(0)|0\rangle. \tag{4.167}$$

For the annihilation cross section one finds

$$\sigma(e^+e^-\to hadrons) = \frac{4\pi^2\,\alpha^2}{Q^6}\,L_{\mu\nu}^{(e^+e^-)}R^{\mu\nu(e^+e^-)}, \tag{4.168}$$

where the tensor $R_{\mu\nu}$ is given by

$$\begin{aligned} R_{\mu\nu}^{(e^+e^-)}(q) &= \int \frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(q-P_X)\langle 0|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|0\rangle \\ &= \int d^4x\; e^{iq\cdot x}\,\langle 0|[J_\mu(0),J_\nu(0)]|0\rangle. \end{aligned} \tag{4.169}$$

4.11.2 Scaling variables for e^+e^- annihilation

It often will turn out to be convenient to use lightcone directions with for momenta the notation $p^\mu = [p^-,p^+,\boldsymbol{p}_T]$ with $p^\pm \equiv (p^0 \pm p^3)/\sqrt{2}$. We will consider two different frames referred to as I and II. In frame I the virtual photon has no transverse directions, in frame II (only useful if two or more hadrons are considered) the hadrons do not have transverse components.

electron-positron:			
momentum	frame I	frame II	relations
$q=k+k'$	$\left[\frac{Q^2}{A\sqrt{2}},\frac{A}{\sqrt{2}},\mathbf{0}_\perp\right]$	$\left[\frac{Q^2}{A\sqrt{2}},\frac{A}{\sqrt{2}},\boldsymbol{q}_T\right]$	$\boldsymbol{q}_T = -\frac{1}{z_1}\boldsymbol{P}_{1\perp}$
P_2	$\left[\frac{M_2^2}{z_2A\sqrt{2}},\frac{z_2A}{\sqrt{2}},\mathbf{0}_\perp\right]$	$\left[\frac{M_2^2}{z_2A\sqrt{2}},\frac{z_2A}{\sqrt{2}},\mathbf{0}_T\right]$	$z_2 = \frac{P_2^+}{q^+} \approx \frac{2P_2\cdot q}{Q^2} \approx \frac{P_1\cdot P_2}{P_1\cdot q}$
P_1	$\left[\frac{z_1Q^2}{A\sqrt{2}},\frac{AM_{1\perp}^2}{z_1Q^2\sqrt{2}},\boldsymbol{P}_{1\perp}\right]$	$\left[\frac{z_1Q^2}{A\sqrt{2}},\frac{AM_{1\perp}^2}{z_1Q^2\sqrt{2}},\mathbf{0}_T\right]$	$z_1 = \frac{P_1^-}{q^-} \approx \frac{2P_1\cdot q}{Q^2} \approx \frac{P_2\cdot P_1}{P_2\cdot q}$

The connection between frames I and II is made by a Lorentz transformation that leaves either the plus or minus component unchanged. For instance one that leaves the minus component unchanged and involves a parameter b^- and a two-component vector $\boldsymbol{b}$

$$[a^-,a^+,\boldsymbol{a}] \;\longrightarrow\; \left[a^-,a^+-\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{b^-}+\frac{\boldsymbol{b}^2\,a^-}{2(b^-)^2},\boldsymbol{a}-\frac{a^-}{b^-}\,\boldsymbol{b}\right]. \tag{4.170}$$

As indicated in the above table the scaling variables z_1 and z_2 are ratios of lightcone components. In fact it is appropriate to take the lightcone ratios in frame II as the defining relations and treat the differences with the invariants $P_1 \cdot q$ etcetera as mass corrections which are of order $1/Q^2$.

Also the lepton-momenta can be written down in the frames discussed above using

$$y = \frac{k^-}{q^-} \approx \frac{P_2 \cdot k}{P_2 \cdot q}, \tag{4.171}$$

which in the lepton rest frame equal $y = (1 \pm \cos\theta_2)/2$ with θ_2 the angle of hadron (or jet) with respect to the momentum of the incoming leptons.

Electron-positron:			
momentum	frame I	frame II	relations
k	$\left[\frac{yQ^2}{A\sqrt{2}}, \frac{(1-y)A}{\sqrt{2}}, \boldsymbol{k}_\perp\right]$	$\left[\frac{yQ^2}{A\sqrt{2}}, \frac{\boldsymbol{k}_T^2 A}{yQ^2\sqrt{2}}, \boldsymbol{k}_T\right]$	$\boldsymbol{k}_T = \boldsymbol{k}_\perp + y\boldsymbol{q}_T$
k'	$\left[\frac{(1-y)Q^2}{A\sqrt{2}}, \frac{yA}{\sqrt{2}}, -\boldsymbol{k}_\perp\right]$	$\left[\frac{(1-y)Q^2}{A\sqrt{2}}, \frac{(\boldsymbol{k}_T^2 - 2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)A}{(1-y)Q^2\sqrt{2}}, \boldsymbol{q}_T - \boldsymbol{k}_T\right]$	$\boldsymbol{k}_\perp^2 = y(1-y)Q^2$

In the e^-e^+ rest frame $d^3P_1 d^3P_2/P_1^0P_2^0 = (dz_1/z_1)(z_2Q^2\,dz_2/4)d\boldsymbol{P}_{1\perp}d\Omega_2 = \pi Q^2\, z_1 z_2\, dz_1\, dz_2\, dy\, d^2\boldsymbol{q}_T$, so Eq. 4.165 becomes

$$\frac{d\sigma^{(e^+e^-)}}{dz_1\, dz_2\, d^2\boldsymbol{q}_T\, dy} = \frac{\pi\,\alpha^2}{2Q^4}\, z_1 z_2\, L_{\mu\nu}^{(e^+e^-)} \mathcal{W}^{\mu\nu(e^+e^-)}. \tag{4.172}$$

For the production of a single hadron

$$\frac{d\sigma^{(e^+e^-)}}{dz\, dy} = \frac{\pi\,\alpha^2}{Q^4}\, z\, L_{\mu\nu}^{(e^+e^-)} W^{\mu\nu(e^+e^-)}. \tag{4.173}$$

4.12 The Drell-Yan process

4.12.1 The hadronic tensor

Consider the process $A + B \to \ell + \bar{\ell} + X$, where two spin-1/2 hadrons with momenta P_A and P_B interact and two outgoing leptons are measured with momenta k and k'. The leptons are assumed to originate from a high mass photon with momentum $q = k + k'$ with $q^2 \equiv Q^2 \geq 0$. The unobserved outstate will be denoted by $|P_X\rangle$, having a total momentum P_X. We will consider the case of pure incoming spin states, characterized by the spin vectors S_A and S_B, respectively, and observed lepton helicities λ and λ'.

We have the following relations

$$P_A^2 = M_A^2, \quad P_B^2 = M_B^2, \tag{4.174}$$
$$(P_A + P_B)^2 = s, \tag{4.175}$$
$$k^2 = k'^2 = m^2 \approx 0, \tag{4.176}$$
$$S_A^2 = S_B^2 = -1, \tag{4.177}$$
$$P_A \cdot S_A = P_B \cdot S_B = 0. \tag{4.178}$$

In the deep inelastic limit $Q^2, s \to \infty$, with the ratio $\tau = Q^2/s$ fixed. The invariant amplitude for the process is given by

$$\mathcal{M} = \bar{u}(k,\lambda)\gamma^\mu v(k',\lambda')\frac{e^2}{Q^2}\langle P_X|J_\mu(0)|P_A S_A; P_B S_B\rangle. \tag{4.179}$$

The square of this amplitude can be split into a purely leptonic and a purely hadronic part, according to

$$|\mathcal{M}|^2 = \frac{e^4}{Q^4} L_{\mu\nu}^{(DY)} H^{\mu\nu(DY)}, \tag{4.180}$$

with the lepton tensor (neglecting the lepton masses) being

$$L_{\mu\nu}^{(DY)}(k\lambda;k'\lambda') = \delta_{\lambda\lambda'}\left(2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} + 2i\lambda\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma\right). \tag{4.181}$$

The product of hadronic current matrix elements is written as

$$H_{\mu\nu}^{(DY)}(P_X;P_AS_A;P_BS_B) = \langle P_AS_A;P_BS_B|J_\mu(0)|P_X\rangle\langle P_X|J_\nu(0)|P_AS_A;P_BS_B\rangle, \tag{4.182}$$

where a summation over spins of the unobserved out state is understood. The total cross section is given by

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2 d\mathcal{R}, \tag{4.183}$$

with the flux factor

$$F = 4\sqrt{(P_A\cdot P_B)^2 - M_A^2M_B^2} \;\approx\; 2s \tag{4.184}$$

and the Lorentz invariant phase space

$$d\mathcal{R} = (2\pi)^4\delta^4(P_A+P_B-P_X-k-k')\frac{d^3P_X}{(2\pi)^3 2P_X^0}\frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3k'}{(2\pi)^3 2k'^0}. \tag{4.185}$$

Integrating $H_{\mu\nu}$ over P_X, gives the usual hadron tensor

$$\begin{aligned}\mathcal{W}_{\mu\nu}^{(DY)}(q;P_AS_A;P_BS_B) &= \frac{1}{(2\pi)^4}\int\frac{d^3P_X}{(2\pi)^3 2P_X^0}(2\pi)^4\delta^4(P_A+P_B-P_X-q)\\ &\qquad\times H_{\mu\nu}^{(DY)}(P_X;P_AS_A;P_BS_B)\\ &= \frac{1}{(2\pi)^4}\int d^4x\;\; e^{iq\cdot x}\langle P_AS_A;P_BS_B|[J_\mu(0),J_\nu(x)]|P_AS_A;P_BS_B\rangle\end{aligned} \tag{4.186}$$

For the last equality completeness of the out states and causality has been used. The remaining phase space is conveniently written as

$$\frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3k'}{(2\pi)^3 2k'^0} = \frac{d^4q}{(2\pi)^4}\frac{d\Omega}{32\pi^2}, \tag{4.187}$$

where the angles are those of the lepton axis in the rest frame of the two leptons. In terms of the fine-structure constant $\alpha \equiv e^2/4\pi$, we then obtain the Drell-Yan cross section (including a factor 2 from the summation over the lepton polarizations)

$$\frac{d\sigma^{(DY)}}{d^4q\,d\Omega} = \frac{\alpha^2}{2s\,Q^4}L_{\mu\nu}^{(DY)}\mathcal{W}^{\mu\nu(DY)}, \tag{4.188}$$

where the lepton tensor is given by the symmetric part in Eq. (4.181) and the hadron tensor by Eq. (4.186).

4.12.2 Scaling variables for Drell-Yan scattering

We will use lightcone components for momenta $p^\mu = [p^-,p^+,\boldsymbol{p}_T]$ with $p^\pm \equiv (p^0\pm p^3)/\sqrt{2}$ and consider two different frames I and II. In frame I the virtual photon has no transverse directions, in frame II the hadrons do not have transverse components.

Drell-Yan:			
momentum	frame I	frame II	relations
$q = k + k'$	$\left[\frac{Q^2}{A\sqrt{2}}, \frac{A}{\sqrt{2}}, \mathbf{0}_\perp\right]$	$\left[\frac{Q^2}{A\sqrt{2}}, \frac{A}{\sqrt{2}}, \boldsymbol{q}_T\right]$	
P_A	$\left[\frac{x_A M_A^2}{A\sqrt{2}}, \frac{A}{x_A\sqrt{2}}, -\frac{\boldsymbol{q}_T}{2x_A}\right]$	$\left[\frac{x_A M_A^2}{A\sqrt{2}}, \frac{A}{x_A\sqrt{2}}, \mathbf{0}_T\right]$	$x_A = \frac{q^+}{P_A^+} \approx \frac{Q^2}{2P_A\cdot q} \approx \frac{P_B\cdot q}{P_B\cdot P_A}$
P_B	$\left[\frac{Q^2}{x_B A\sqrt{2}}, \frac{x_B A M_B^2}{Q^2\sqrt{2}}, -\frac{\boldsymbol{q}_T}{2x_B}\right]$	$\left[\frac{Q^2}{x_B A\sqrt{2}}, \frac{x_B A M_B^2}{Q^2\sqrt{2}}, \mathbf{0}_T\right]$	$x_B = \frac{q^-}{P_B^-} \approx \frac{Q^2}{2P_B\cdot q} \approx \frac{P_A\cdot q}{P_A\cdot P_B}$

The usual scaling variables x_A and x_B in the Drell-Yan process can be written in terms of the ratios of scalar products of the momenta. These variables run between zero and unity (at least when $Q^2 \to \infty$) For the lepton kinematics one can use

$$y = \frac{k^-}{q^-} \approx \frac{k'^+}{q^+}, \tag{4.189}$$

which in the lepton rest frame equal $y = (1 \pm \cos\theta)/2$ with θ the angle of the leptons with respect to an axis that is approximately parallel to the momentum of hadrons A and B (in fact it is the z-direction to be specified below).

Drell-Yan:			
momentum	frame I	frame II	relations
k	$\left[\frac{yQ^2}{A\sqrt{2}}, \frac{(1-y)A}{\sqrt{2}}, \boldsymbol{k}_\perp\right]$	$\left[\frac{yQ^2}{A\sqrt{2}}, \frac{\boldsymbol{k}_T^2 A}{yQ^2\sqrt{2}}, \boldsymbol{k}_T\right]$	$\boldsymbol{k}_T = \boldsymbol{k}_\perp + \frac{1}{2}\boldsymbol{q}_T$
k'	$\left[\frac{(1-y)Q^2}{A\sqrt{2}}, \frac{yA}{\sqrt{2}}, -\boldsymbol{k}_\perp\right]$	$\left[\frac{(1-y)Q^2}{A\sqrt{2}}, \frac{(\boldsymbol{k}_T^2 - 2\boldsymbol{k}_T\cdot\boldsymbol{q}_T)A}{(1-y)Q^2\sqrt{2}}, \boldsymbol{q}_T - \boldsymbol{k}_T\right]$	$\boldsymbol{k}_\perp^2 = y(1-y)Q^2$

Eq. 4.188 becomes

$$\frac{d\sigma^{(DY)}}{dx_A\, dx_B\, d^2\boldsymbol{q}_T\, dy} = \frac{\pi\,\alpha^2}{Q^4}\, L_{\mu\nu}^{(DY)} \mathcal{W}^{\mu\nu(DY)}. \tag{4.190}$$

Chapter 5

THEORETICAL ANALYSIS OF LEPTON-HADRON SCATTERING

5.1 Introduction

In this chapter we will analyse lepton-hadron scattering in terms of the underlying hadron structure, the quarks and gluons. The phenomenological analysis is for deep inelastic lepton-hadron scattering known as the quark-parton model. In general the theoretical anslysis in this field involves quite some formalism as one tries to extract information on the hadronic system combining kinematical flexibility and the known electromagnetic and weak currents of quarks and leptons. A great help is the fact that for high momentum transfer one can make a short distance expansion of the product of currents that appears in the hadronic tensor. This expansion in terms of local operators is known as the operator product expansion (OPE).

5.2 The principle of scaling

I will give two examples of scaling to show that it is a quite general phenomenon, which arises if the cross section for scattering off a composite target can be considered as an incoherent sum of scattering of the constituents.

The first example is electron-nucleus scattering, a process that at sufficiently high energies (in this case a few hundred MeV, which is above typical nuclear excitations but lower than typical excitation energies for baryon resonances) can be described as an incoherent sum of scattering from the individual nucleons. Correspondingly one writes for the cross section - or equivalently the structure functions - schematically

$$\begin{aligned} \sigma_{eA}(P,q) \propto W(\nu,Q^2) \quad &\propto \quad \int dk\ n(k)\ \sigma_{eN}(k,q) \\ &\propto \quad \int dk\ n(k)\ F_N^2(Q^2)\ \delta(2\,k\cdot q + q^2), \end{aligned} \tag{5.1}$$

where dk indicates the integration over all relevant momentum components and $n(k)$ is the momentum distribution of nucleons in the nucleus. The integral can be evaluated by studying the argument of the delta function. For nucleons $E = \sqrt{\boldsymbol{k}^2 + M_N^2} \approx M_N(1 + \boldsymbol{k}^2/2M_N^2 + \ldots)$ and the delta function is approximated by $\delta(2E\nu - 2k_{\|}|\boldsymbol{q}| - Q^2) \approx \delta(2M_N\nu - 2k_{\|}|\boldsymbol{q}| - Q^2)$. This allows us to perform the

integral in Eq. 5.1,

$$W(\nu,Q^2) \propto \int dk_\parallel \; F(k_\parallel) \; F_N^2(Q^2) \; \delta\left(k_\parallel - \frac{2M_N\nu - Q^2}{2|\boldsymbol{q}|}\right), \tag{5.2}$$

where we have introduced the longitudinal momentum distribution $F(k_\parallel) = \int d^2\boldsymbol{k}_\perp \, n(k)$. Introducing the scaling variable

$$y = \frac{2M_N\nu - Q^2}{2|\boldsymbol{q}|}, \tag{5.3}$$

and taking properly into account all factors one derives

$$W(\nu,Q^2) = \frac{M_A}{|\boldsymbol{q}|} \, F_N^2(Q^2) \, F(y). \tag{5.4}$$

For the longitudinal and transverse structure functions W_L and W_T the relevant combinations of nucleon form factors are $F_L^2 = G_E^2$ and $F_T^2 = Q^2 \, G_M^2/4M^2$. From the relation one sees that the scaling function $F(y)$ can be obtained from the experimental cross sections by dividing by the appropriate form factors or from the total cross section dividing by the appropriate combination of form factors.

The next example I want to discuss is deep inelastic scattering, following the same approach. For this one considers the Bjorken limit where $Q^2 \to \infty$ and $\nu \to \infty$ with their ratio $x_{_B}$ fixed. In that case $|\boldsymbol{q}| = \sqrt{Q^2+\nu^2} \approx \nu + Q^2/2\nu + \ldots$, and it is convenient to use lightcone momenta $p^\pm \equiv (p^0 \pm p^3)/\sqrt{2}$. With $q^3 = -|\boldsymbol{q}|$ one obtains for the lightcone momenta, $q^- \to \infty$ and $q^+ \approx -Q^2/2\nu\sqrt{2} = -M_N x_{_B}/\sqrt{2}$. One can again evaluate the impulse approximation expression for the structure function using $\delta(2k\cdot q + q^2) = \delta(2\,k^+q^- + 2\,k^-q^+ - Q^2) \approx \delta(2\sqrt{2}\,\nu k^+ - Q^2)$,

$$W(\nu,Q^2) \;\propto\; \int dk \; n(k) \; \delta(2k\cdot q + q^2) \tag{5.5}$$

$$\;\propto\; \int dk^+ \; f\left(\frac{k^+}{P^+}\right) \delta\left(\frac{k^+}{P^+} - x_{_B}\right) \tag{5.6}$$

or

$$W(\nu,Q^2) \propto f(x_{_B}), \tag{5.7}$$

where the function f is the lightcone momentum (k^+/P^+) distribution (P^+ refers to the target, i.e. in the target rest frame $P^+ = M/\sqrt{2}$).

5.3 The parton model

5.3.1 Photon-parton cross sections

The first way to derive the parton model results is to consider $\gamma^* q \to q$ as an elastic subprocess, for which the current-current product is given by

$$\hat{H}_{\mu\nu}(p,q) = \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) Q^2 + 4\,\tilde{p}_\mu\tilde{p}_\nu, \tag{5.8}$$

and

$$\hat{W}_{\mu\nu}(p,q) = \frac{\hat{H}_{\mu\nu}(p,q)}{2m} \, \delta(2p\cdot q + q^2). \tag{5.9}$$

The quark-parton cross section is given by

$$\hat{\sigma}(\gamma^* q) = \frac{4\pi^2\alpha}{2p\cdot q} \, \epsilon^*_\mu \hat{W}^{\mu\nu} \epsilon_\nu \tag{5.10}$$

(Note that there are in principle ambiguities here because of the flux factor for virtual photons). Explicitly,

$$\hat{\sigma}_T(\gamma^* q) = 4\pi^2\alpha\, e_q^2\, \delta(2p\cdot q - Q^2), \tag{5.11}$$

$$\hat{\sigma}_L(\gamma^* q) = 4\pi^2\alpha\, e_q^2\, \frac{4m^2}{Q^2}\, \delta(2p\cdot q - Q^2) \ll \hat{\sigma}_T. \tag{5.12}$$

In the next step this partonic cross section is folded with a probability function for finding partons in the target. For this purpose it is convenient to give the explicit momenta as done before using lightcone components, $p = [p^-, p^+, \boldsymbol{p}_\perp]$ where $p^\pm = (p^0 \pm p^3)/\sqrt{2}$, including the quark momentum p (now neglecting systematically M^2/Q^2 contributions),

lepton-hadron:		
momentum	frame I	relations
q	$\left[\frac{Q^2}{A\sqrt{2}}, -\frac{A}{\sqrt{2}}, \mathbf{0}_\perp\right]$	
p	$\left[p^-, \frac{A}{\xi\sqrt{2}}, \boldsymbol{p}_\perp\right]$	$\xi = -\frac{q^+}{p^+} = \frac{Q^2}{2p\cdot q}$
P	$\left[\frac{x_B M^2}{A\sqrt{2}}, \frac{A}{x_B\sqrt{2}}, \mathbf{0}_\perp\right]$	$x_B = -\frac{q^+}{P^+} = \frac{Q^2}{2P\cdot q}$

On-shell we would have

$$p^- = \frac{m^2 + \boldsymbol{p}_\perp^2}{2p^+} = \frac{m_\perp^2}{2p^+}. \tag{5.13}$$

However, the quark momentum need not to be on-mass-shell, but to derive the parton model results it is sufficient that $p^- \sim P^- \ll q^-$. The cross section for $\gamma^* q$ (elastic scattering) is given by

$$\hat{\sigma}_T = \frac{4\pi^2\alpha}{Q^2}\, \delta(1-\xi). \tag{5.14}$$

Introducing probability functions $f_i(x)$ for finding partons carrying momentum fraction $x = k^+/P^+ = x_B/\xi$ of the target lightcone momentum, leads to

$$\begin{aligned} \sigma_T &= \sum_i e_i^2 \int dx\, f_i(x)\, \frac{4\pi^2\alpha}{Q^2}\, \delta\left(1 - \frac{x_B}{x}\right) \\ &= \frac{4\pi^2\alpha}{Q^2} \sum_i e_i^2\, x_B\, f_i(x_B), \end{aligned} \tag{5.15}$$

i.e. since we know from the previous chapter

$$\sigma_T = \frac{4\pi^2\alpha}{P\cdot q}\, MW_T = \frac{8\pi^2\alpha}{Q^2}\, x_B\, F_1, \tag{5.16}$$

we get

$$F_1(x_B) = \frac{1}{2} \sum_i e_i^2\, f_i(x_B). \tag{5.17}$$

As $\hat{\sigma}_L \propto 1/Q^2 \to 0$ one obtains $F_L = 0$ or the Callan-Gross relation,

$$F_2(x_B) = 2x_B\, F_1(x_B). \tag{5.18}$$

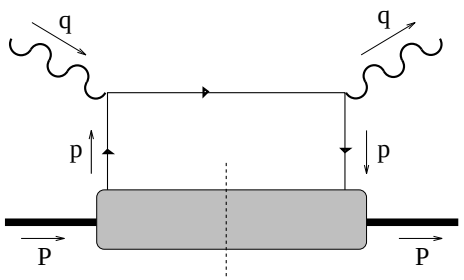


Figure 5.1: *The handbag diagram for inclusive deep inelastic scattering off a hadronic target.*

5.3.2 Diagrammatic approach

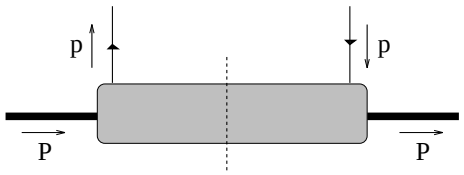
Another way in which the parton model is obtained is by just considering one diagram (which later will turn out to be the leading one), the socalled quark handbag diagram (see Fig. 5.1) and its antiquark equivalent. The basic expression corresponding to the handbag diagram restricting us to the quark part is

$$\begin{aligned} 2M\,W^{\mu\nu}(P,q) &= \sum_q e_q^2 \int dp^-\,dp^+\,d^2\boldsymbol{p}_\perp\; Tr\left(\Phi(p)\,\gamma^\mu(\not{p}+\not{q}+m)\gamma^\nu\right)\,\delta((p+q)^2-m^2) \\ &\approx \sum_q e_q^2 \int dp^-\,dp^+\,d^2\boldsymbol{p}_\perp\; Tr\left(\Phi(p)\,\gamma^\mu\,\frac{\not{q}}{2q^-}\,\gamma^\nu\right)\,\delta(p^+ + q^+) \\ &\approx -g_\perp^{\mu\nu}\,\frac{1}{2}\int dp^-\,d^2\boldsymbol{p}_\perp\; Tr\left(\gamma^+\,\Phi(p)\right)\Big|_{p^+=x_B\,P^+} + \ldots, \end{aligned} \tag{5.19}$$

where $\Phi(p)$ is the forward antiquark-target scattering amplitude,

$$\Phi_{ij}(p,P,S) = \frac{1}{(2\pi)^4}\int d^4x\; e^{i\,p\cdot x}\,\langle P,S|\overline{\psi}_j(0)\psi_i(x)|P,S\rangle, \tag{5.20}$$

diagrammatically represented by



Comparing with the general form of the hadronic tensor, we read off (including now also the antiquark part)

$$2\,F_1(x) = 2M\,W_1(x,Q^2) = \sum_q e_q^2\,\left[q(x)+\overline{\psi^q}(x)\right], \tag{5.21}$$

with

$$q(x) = \frac{1}{4\pi}\int dx^-\; e^{-ixP^+x^-}\,\langle P|\overline{\psi}(x)\gamma^+\psi(0)|P\rangle\Big|_{x^+=\boldsymbol{x}_\perp=0}, \tag{5.22}$$

$$\overline{q}(x) = -\frac{1}{4\pi}\int dx^-\; e^{ixP^+x^-}\,\langle P|\overline{\psi}(0)\gamma^+\psi(x)|P\rangle\Big|_{x^+=\boldsymbol{x}_\perp=0}, \tag{5.23}$$

satisfying $\overline{q}(x) = -q(-x)$. This result for the distribution functions can alternatively be derived by inserting free currents in the hadronic tensor for the current commutator and using the expression for the free field commutator.

To convince oneself that the above expressions for $q(x)$ and $\overline{q}(x)$ actually can be interpreted as quark momentum distributions one needs to realize that $\overline{\psi}(x)\gamma^+\psi(0) = \psi_+^\dagger(x)\psi_+(0)$ where $\psi_\pm = P_\pm\,\psi$ are projections obtained with projection operators

$$P_\pm = \frac{1}{2}\gamma^\mp\gamma^\pm = \frac{1}{2}(1 \pm \gamma^0\gamma^3). \tag{5.24}$$

One then can insert a complete set of states and obtain

$$\begin{aligned} q(x) &= \frac{1}{2\pi\sqrt{2}}\int dx^-\; e^{ixP^+x^-}\,\langle P|\psi_+^\dagger(x)|n\rangle\langle n|\psi_+(0)|P\rangle\Big|_{x^+=\boldsymbol{x}_\perp=0} \\ &= \sum_n \frac{1}{\sqrt{2}}\,\delta(P^+ + q^+ - p_n^+)\,|\langle n|\psi_+(0)|P\rangle|^2\,, \end{aligned} \tag{5.25}$$

which represents the probability that a quark is annihilated from $|P\rangle$ giving a state $|n\rangle$ with $p_n^+ = (1-x)P^+$.

Analogously to the unpolarized structure functions one can obtain for the polarized structure functions

$$2\,g_1(x_{\scriptscriptstyle B}) = \sum_q e_q^2\,[\Delta q(x_{\scriptscriptstyle B}) + \Delta\overline{q}(x_{\scriptscriptstyle B})] \tag{5.26}$$

where

$$\Delta q(x) = \frac{1}{4\pi}\int dx^-\; e^{-ixP^+x^-}\,\langle P|\bar{\psi}(x)\gamma^+\gamma_5\psi(0)|P\rangle\Big|_{x^+=\boldsymbol{x}_\perp=0}\,, \tag{5.27}$$

$$\Delta\overline{q}(x) = -\frac{1}{4\pi}\int dx^-\; e^{ixP^+x^-}\,\langle P|\bar{\psi}(0)\gamma^+\gamma_5\psi(x)|P\rangle\Big|_{x^+=\boldsymbol{x}_\perp=0}\,, \tag{5.28}$$

This represents the difference of chiral even and odd quarks (in infinite momentum frame quarks parallel or antiparallel to proton spin). The corresponding quark fields are projected out by

$$P_{R/L} = \frac{1}{2}(1 \pm \gamma_5), \tag{5.29}$$

which commute with the projectors $P_\pm$. In this way one obtains distributions $q_R(x)$ and $q_L(x)$ for which $q(x) = q_R(x) + q_L(x)$ and $\Delta q(x) = q_R(x) - q_L(x)$.

The expressions derived above are lightcone correlation functions. Already from the kinematics of deep inelastic scattering one can see that the process probes the lightcone. We have already seen that both q^0 and q^3 go to infinity but that taking the sum and the difference only one of them goes to infinity. Choosing $\boldsymbol{q}$ along the negative z-axis one has $q^- = (\nu + |\boldsymbol{q}|)/\sqrt{2} \to \infty$ and $q^+ = (\nu - |\boldsymbol{q}|)/\sqrt{2}$ $\approx -M_N x_B/\sqrt{2}$. This corresponds in the hadronic tensor which involves a Fourier transform of the product of currents to $|x^+| \approx 1/q^- \to 0$ and $|x^-| \approx 1/|q^+| \to 1/Mx_B$ or $|\boldsymbol{x}| \approx |t| \approx 1/Mx_B$. Thus, depending on the value of x_B the distances and times not necessarily are small, but one has $x^2 = x^+x^- - \boldsymbol{x}_\perp^2 \approx -\boldsymbol{x}_\perp^2 \leq 0$, while on the other hand causality requires that $x^2 \geq 0$. Therefore, one sees that deep inelastic scattering probes the lightcone, $x^2 \approx 0$.

5.4 Parton distributions in electroweak processes

The explicit flavor and spin dependence of the structure functions in electroweak processes depend on the probe being a γ, Z^0 or $W^\pm$ boson, for which we have seen that the relevant currents to be used

in the hadronic part were

$$J_\mu^{(\gamma)} = \sum_q \overline{\psi^q}(x)\, Q\, \gamma_\mu\, \psi^q(x), \tag{5.30}$$

$$\begin{aligned} J_\mu^{(Z)} &= \sum_q \overline{\psi^q}(x)\, (I_W^3 - Q \sin^2\theta_W)\, \gamma_{\mu L}\, \psi^q(x) - \sum_q \overline{\psi^q}(x)\, Q \sin^2\theta_W\, \gamma_{\mu R}\, \psi^q(x) \\ &= \sum_q \overline{\psi^q}(x)\, (I_W^3 - 2\, Q \sin^2\theta_W)\, \gamma_\mu\, \psi^q(x) - \sum_q \overline{\psi^q}(x)\, I_W^3\, \gamma_\mu\gamma_5\, \psi^q(x), \end{aligned} \tag{5.31}$$

$$\begin{aligned} J_\mu^{(W)} &= \sum_q \overline{\psi^q}(x)\, I_W^{\pm}\, \gamma_{\mu L}\, \psi^q(x) \\ &= \sum_q \overline{\psi^q}(x)\, I_W^{\pm}\, \gamma_\mu\, \psi^q(x) - \sum_q \overline{\psi^q}(x)\, I_W^{\pm}\, \gamma_\mu\gamma_5\, \psi^q(x). \end{aligned} \tag{5.32}$$

In this chapter we have so far only used the $W_{\mu\nu}^{\gamma\gamma}$ in the examples. Including the flavors explicitly,

$$J_\mu^{(\gamma)} = \frac{2}{3}\,\overline{\psi^u}\gamma_\mu\psi^u - \frac{1}{3}\,\overline{\psi^d}\gamma_\mu\psi^d - \frac{1}{3}\,\overline{\psi^s}\gamma_\mu\psi^s + \ldots, \tag{5.33}$$

one obtains for ep scattering the following expression in terms of the distribution functions $u_p(x)$, $d_p(x)$, etc.,

$$\begin{aligned} \frac{F_2^{(\gamma\gamma)}(ep)}{x} = 2\, F_1^{(\gamma\gamma)}(ep) &= \frac{4}{9}\,(u_p + \overline{u}_p) + \frac{1}{9}\,(d_p + \overline{d}_p) + \frac{1}{9}\,(s_p + \overline{s}_p) + \ldots \\ &\equiv \frac{4}{9}\,(u + \overline{u}) + \frac{1}{9}\,(d + \overline{d}) + \frac{1}{9}\,(s + \overline{s}) + \ldots\ . \end{aligned} \tag{5.34}$$

Using isospin summetry, $u_p = d_n$, $d_p = u_n$, $s_p = s_n$ one has for en scattering

$$\frac{F_2^{(\gamma\gamma)}(en)}{x} = 2\, F_1^{(\gamma\gamma)}(en) = \frac{1}{9}\,(u + \overline{u}) + \frac{4}{9}\,(d + \overline{d}) + \frac{1}{9}\,(s + \overline{s}) + \ldots. \tag{5.35}$$

As the difference between quarks and antiquarks contributes to the quantum numbers, it is convenient to divide the quark distribution in a valence part and a sea part

$$q(x) = q_v(x) + q_s(x), \tag{5.36}$$

where

$$q_v(x) \equiv q(x) - \overline{q}(x). \tag{5.37}$$

The quark distributions are positive definite, so for instance

$$\frac{1}{4} \le \frac{F_2(en)}{F_2(ep)} = \frac{(u + \overline{u}) + 4(d + \overline{d}) + (s + \overline{s}) + \ldots}{4(u + \overline{u}) + (d + \overline{d}) + (s + \overline{s}) + \ldots} \le 4. \tag{5.38}$$

If one looks at the experimental result one sees near $x \approx 0$ a ratio that is about 1, indicating dominance of sea quarks with $u\overline{u}$ and $d\overline{d}$ pairs in equal amounts. Near $x \approx 1$ the valence quarks dominate. Naively one might expect $u = 2\,d$ and all others zero, i.e. a ratio of $2/3$. The experimentally observed limit for $x \to 1$ tends to $1/4$, the lower limit, which is reached for $d \ll u$, i.e. dominance of the u-quark in the proton (and therefore the d-quark in the neutron).

Including the weak interactions one can in general write for the most important structure functions in deep inelastic scattering

$$2\, F_1^{(V_1V_2)} = \frac{F_2^{(V_1V_2)}}{x} = \sum_q A_q^{(V_1V_2)}\, [q(x) + \overline{q}(x)]\,, \tag{5.39}$$

$$F_3^{(V_1V_2)} = \sum_q B_q^{(V_1V_2)}\, [q(x) - \overline{q}(x)]\,, \tag{5.40}$$

$$g_1^{(V_1V_2)} = \sum_q A_q^{(V_1V_2)}\, [\Delta q(x) + \Delta\overline{q}(x)]\,, \tag{5.41}$$

with

$$A_q^{(V_1V_2)} = \frac{1}{2}\left(g_{Lq}^{(V_1)}\, g_{Lq}^{(V_2)} + g_{Rq}^{(V_1)}\, g_{Rq}^{(V_2)}\right), \tag{5.42}$$

$$B_q^{(V_1V_2)} = \frac{1}{2}\left(g_{Lq}^{(V_1)}\, g_{Lq}^{(V_2)} - g_{Rq}^{(V_1)}\, g_{Rq}^{(V_2)}\right), \tag{5.43}$$

From the currents one finds the coefficients from the charges (Q) or weak isospin ($\boldsymbol{I}_W$) of the quarks,

$$g_{Lq}^{(\gamma)} = g_{Rq}^{(\gamma)} = Q, \tag{5.44}$$

$$g_{Lq}^{(W)} = 2\, I_W^{\pm}, \quad g_{Rq}^{(W)} = 0, \tag{5.45}$$

$$g_{Lq}^{(Z)} = 2\, I_W^3 - 2\, Q\, \sin^2\theta_W, \quad g_{Rq}^{(Z)} = -2\, Q\, \sin^2\theta_W. \tag{5.46}$$

Some explicit examples for weak interactions through the charged current,

$$J_\mu^{(W)} = \overline{u}\gamma_\mu(1-\gamma_5)[d\,\cos\theta_c + s\,\sin\theta_c] + \overline{c}\gamma_\mu(1-\gamma_5)[s\,\cos\theta_c - d\,\sin\theta_c], \tag{5.47}$$

are for $\nu p \to \mu^- X$,

$$\frac{F_2^{(WW)}(x)}{x} = 2\, F_1^{(WW)}(x) = 2\,[d(x) + s(x) + \overline{u}(x) + \overline{c}(x)], \tag{5.48}$$

$$F_3^{(WW)}(x) = 2\,[d(x) + s(x) - \overline{u}(x) - \overline{c}(x)], \tag{5.49}$$

while for $\overline{\nu} p \to \mu^+ X$,

$$\frac{F_2^{(WW)}(x)}{x} = 2\, F_1^{(WW)}(x) = 2\left[u(x) + c(x) + \overline{d}(x) + \overline{s}(x)\right], \tag{5.50}$$

$$F_3^{(WW)}(x) = 2\left[u(x) + c(x) - \overline{d}(x) - \overline{s}(x)\right]. \tag{5.51}$$

Note that the quarks that are eigenstates of I_W^3 are linear combinations of the strong interaction eigenstates with mixing angle θ_c. Including all quarks of both (lowest) families no dependence on θ_c appears in the inclusive result. Below the charm threshold factors $\cos^2\theta_c$ and $\sin^2\theta_c$ will appear. These same structure functions can also be measured in $e^+ p \to \overline{\nu} X$ and $e^- p \to \nu X$ respectively if the kinematics is fixed from the direction of the produced jet (HERA). Note that the cross sections multiplying the structure functions for eN processes contain a factor $1/2$ compared with the expressions in section 4.6.1 because of the spin-averaging for the incoming electrons.

For the interference structure functions one has explicitly

$$A_u^{(\gamma Z)} = A_c^{(\gamma Z)} = \frac{1}{3}\left(1 - \frac{8}{3}\sin^2\theta_W\right), \tag{5.52}$$

$$B_u^{(\gamma Z)} = B_c^{(\gamma Z)} = \frac{1}{3}, \tag{5.53}$$

$$A_d^{(\gamma Z)} = A_s^{(\gamma Z)} = \frac{1}{6}\left(1 - \frac{4}{3}\sin^2\theta_W\right), \tag{5.54}$$

$$B_d^{(\gamma Z)} = B_s^{(\gamma Z)} = \frac{1}{6}. \tag{5.55}$$

For the pure Z-exchange contribution in neutral current neutrino scattering one can get similar explicit expressions.

The above explicit partonic expressions for the structure functions allow one to disentangle the quark distributions from electroweak scattering data.

5.5 Sum rules

As probability distributions the quark distribution functions satisfy a number of obvious sum rules, such as

$$\int_0^1 dx\ [u(x) - \overline{u}(x)] = n_u = 2, \tag{5.56}$$

$$\int_0^1 dx\ \left[d(x) - \overline{d}(x)\right] = n_d = 1, \tag{5.57}$$

$$\int_0^1 dx\ [s(x) - \overline{s}(x)] = n_s = 2, \tag{5.58}$$

corresponding to the (net) number of each of these quark species in the proton. On the basis of this one finds a number of sum rules for the structure functions, such as the *Adler sum rule*,

$$\int_0^1 \frac{dx}{x}\ \left[F_2^{\overline{\nu}p}(x) - F_2^{\nu p}(x)\right] = 2\left[n_u - n_d + n_c - n_s\right] = 2, \tag{5.59}$$

the *Bjorken sum rule*,

$$\int_0^1 dx\ \left[F_1^{\overline{\nu}p}(x) - F_1^{\nu p}(x)\right] = n_u - n_d + n_c - n_s = 1, \tag{5.60}$$

the *Gross-Llewellyn Smith sum rule*,

$$\int_0^1 dx\ \left[F_3^{\overline{\nu}p}(x) + F_3^{\nu p}(x)\right] = 2\left[n_u + n_d + n_s + n_c\right] = 6, \tag{5.61}$$

the *Gottfried sum rule* which is based on a symmetric sea distribution, $\overline{u}(x) = \overline{d}(x)$,

$$\int_0^1 \frac{dx}{x}\ [F_2^{ep}(x) - F_2^{en}(x)] = \frac{1}{3}\int_0^1 dx[u(x) - d(x) + \overline{u}(x) - \overline{d}(x)] = \frac{1}{3}. \tag{5.62}$$

The experimental result (NMC), giving 0.240 ± 0.016 indicates possibly that $\overline{U}(x) \neq \overline{d}(x)$. Most of these sum rules will acquire a QCD correction. So also, the quantity

$$\int_0^1 dx\, x[u(x) + \overline{u}(x) + d(x) + \overline{d}(x) + s(x) + \overline{s}(x) + \ldots] \equiv \int_0^1 dx\, x\, \Sigma(x) = \epsilon_q \leq 1, \tag{5.63}$$

represents the total momentum fraction of the proton carried by quarks. It must obviously be smaller than one.

For the polarized structure functions one can obtain similar estimates using the naive flavor-spin structure of the proton based on $SU(6)$ symmetry,

$$|p\uparrow\rangle = \frac{1}{\sqrt{18}}\Bigg(2\ u_\uparrow u_\uparrow d_\downarrow - u_\uparrow u_\downarrow d_\uparrow - u_\downarrow u_\uparrow d_\uparrow + \ [d \text{ at places 1 and 2}]\ \Bigg).\Bigg), \tag{5.64}$$

From this wave function one finds in terms of a normalized one-quark distribution $q(x)$ the 'naive' results

$$\begin{aligned} u_\uparrow(x) &= \frac{5}{3}\, q(x), & u_\downarrow(x) &= \frac{1}{3}\, q(x), \\ d_\uparrow(x) &= \frac{1}{3}\, q(x), & d_\downarrow(x) &= \frac{2}{3}\, q(x), \end{aligned} \tag{5.65}$$

or

$$u(x) = 2\,q(x), \qquad \Delta u(x) = \frac{4}{3}\,q(x),$$
$$d(x) \;=\; q(x), \qquad \Delta d(x) = -\frac{1}{3}\,q(x), \tag{5.66}$$

and all other distributions (strange quarks or antiquarks) are zero. In this case one obtains naive sum rule results for the polarized distributions,

$$\int_0^1 dx\,[\Delta u(x) + \Delta\overline{u}(x)] = \Delta u = \frac{4}{3}, \tag{5.67}$$
$$\int_0^1 dx\,[\Delta d(x) + \Delta\overline{d}(x)] = \Delta d = -\frac{1}{3}. \tag{5.68}$$

Note that the sum $\Delta\Sigma = \Delta u + \Delta d + \Delta s + \ldots$ represents (as probability distributions) the total number of quarks parallel to the proton spin. If the proton spin comes from the quark spins, as is the case for the above $SU(6)$ wave function multiplying a spherically symmetric spatial wave function, one expects this to be one. It leads to

$$\Gamma_1^p = \int_0^1 dx\,g_1^p(x) = \frac{1}{2}\left[\frac{4}{9}\,\Delta u + \frac{1}{9}\,\Delta d + \frac{1}{9}\,\Delta s\right] = \frac{5}{18} \approx 0.28, \tag{5.69}$$
$$\Gamma_1^n = \int_0^1 dx\,g_1^n(x) = \frac{1}{2}\left[\frac{4}{9}\,\Delta d + \frac{1}{9}\,\Delta u + \frac{1}{9}\,\Delta s\right] = 0, \tag{5.70}$$

which is in disagreement with the experimental result with the experimental result, $\Gamma_1^p = 0.13 \pm 0.02$ and the result $\Gamma_1^n = -0.05 \pm 0.03$, obtained from the deuteron sum rule $\Gamma_1^d = (\Gamma_1^p + \Gamma_1^n)\,\left(1 + \frac{3}{2}\omega_D\right)$ (with $\omega_D \approx 0.05$) the D-wave probability in the deuteron.

A different view of these sum rules comes from the expressions in terms of correlation functions. From the expression 5.23, it is easy to obtain

$$\begin{aligned}\int_0^1 dx\,[q(x) - \overline{q}(x)] &= \int_{-1}^1 dx\,q(x) \\ &= \frac{\langle P|\bar\psi(x)\gamma^+\psi(0)|P\rangle}{2P^+} = n_q,\end{aligned} \tag{5.71}$$

where n_q is the coefficient in the expectation value $\langle P|\bar\psi(x)\gamma^\mu\psi(0)|P\rangle = 2\,n_q\,P^\mu$. Using the vector currents to obtain the quantum numbers for flavor (upness, downness, strangeness, etc.) n_q is precisely the quark number. In general one obtains

$$\begin{aligned}\int_{-1}^1 dx\;x^{n-1}\;q(x) &= \int_0^1 dx\;x^{n-1}\,[q(x) + (-)^n\,\overline{q}(x)] \\ &= \frac{1}{2P^+}\langle P|\bar\psi(0)\,\gamma^+\left(\frac{i\partial^+}{P^+}\right)^{n-1}\psi(0)|P\rangle.\end{aligned} \tag{5.72}$$

The moments of the structure functions are related to expectation values of particular quark operators. One problem that occurs here is that these expectation values are in fact precisely the Green's functions with operator insertions, discussed in chapter 1. In general these depend on the renormalization scale μ^2 and thus a similar renormalization scale dependence must be present for the structure functions. In the next sections these QCD corrections will be discussed in more detail. Note that the sum rule in Eq. 5.71 is scale independent, as it concerns a conserved current. The situation is different for the second moment,

$$\begin{aligned}\int_{-1}^1 dx\;x\,q(x,\ln\mu^2) &= \int_0^1 dx\;x\,[q(x,\ln\mu^2) + \overline{q}(x,\ln\mu^2)] \\ &= \frac{\langle P|\bar\psi(0)\,i\gamma^+\partial^+\psi(0)|P\rangle_{(\mu^2)}}{2\,(P^+)^2} = \epsilon_q(\mu^2),\end{aligned} \tag{5.73}$$

where ϵ_q is defined in $\langle P|\bar{\psi}(0)\, i\gamma_\mu \partial_\nu \psi(0)|P\rangle = 2\,\epsilon_q\, P_\mu P_\nu + \ldots$ and is the relative contribution of quarks to the energy momentum tensor of the proton (the dots indicate trace terms $\propto M^2\, g_{\mu\nu}$).

For polarized structure functions the lowest moment is given by

$$\begin{aligned}\int_{-1}^{1} dx\, \Delta q(x, \ln \mu^2) &= \int_0^1 dx\, [\Delta q(x, \ln \mu^2) + \Delta \overline{q}(x, \ln \mu^2)] \\ &= \frac{\langle P|\bar{\psi}(0)\gamma^+\gamma_5\psi(0)|P\rangle_{(\mu^2)}}{2\,P^+} = \Delta_q(\mu^2) \end{aligned} \tag{5.74}$$

where Δq appears in the expectation value for the axial current, $\langle P|\bar{\psi}(x)\gamma^+\gamma_5\psi(0)|P\rangle = 2M\,\Delta q\, S^\mu$ and S^μ is the spin vector for the nucleon (S^2 = -1; $S \cdot p = 0$). Also this is in general not independent of the renormalization scale as the flavor singlet axial current is not conserved (because of the ABJ-anomaly). For the axial currents that are important in the weak decays of baryons, e.g. for the neutron $\propto \tau_+\, \gamma^\mu\gamma_5$, the current is conserved (up to mass effects) and the corresponding matrix elements are scale independent. From the neutron decay one deduces the (scale independent) flavor nonsinglet combinations

$$\Delta q^3 = \Delta u(\mu^2) - \Delta d(\mu^2) = G_A = 1.259, \tag{5.75}$$

while from hyperon decays one finds (using $SU(3)$ symmetry),

$$\Delta q^8 = \Delta u(\mu^2) + \Delta d(\mu^2) - 2\,\Delta s(\mu^2) \approx 0.6, \tag{5.76}$$

A sum rule involving only flavor nonsinglet combinations is for example the *polarized Bjorken sum rule*,

$$\int_0^1 dx\, [g_1^p(x) - g_1^n(x)] = \frac{1}{6}\Delta q^3 = \frac{G_A}{6}. \tag{5.77}$$

Assuming that in the proton $\Delta s = 0$ one can solve $\Delta u \approx 0.93$ and $\Delta d \approx -0.33$. Although these values deviate from the naive values of $\Delta u = 4/3$ and $\Delta d = -1/3$, one can understand this for instance from the consideration of spinor densities $\overline{\psi}\gamma_5\gamma_3\psi = \psi^\dagger\sigma_z\psi$ in a model, such as the bag model. The lower component of the solution gives a reduction of this density compared to the vector density $\overline{\psi}\gamma^0\psi = \psi^\dagger\psi$. Using the phenomenological reduction factor $G_A/(5/3) \approx 0.75$, corrections one finds

$$\int dx\; g_1^p(x) = \frac{1}{2}\left(\frac{4}{9}\Delta u + \frac{1}{9}\Delta d + \frac{1}{9}\Delta s\right) = \frac{1}{9}\,\Delta\Sigma(\mu^2) + \frac{1}{12}\,\Delta q^3 + \frac{1}{36}\,\Delta q^8 \;\approx\; 0.21\,. \tag{5.78}$$

We will come back to this sum rule in a later section.

5.6 The operator product expansion

The basic idea of the operator product expansion (OPE) is that the product of two operators simplifies in the limit that their arguments coincide,

$$\lim_{x\to 0} A(x)B(0) \propto \sum_\beta C^{(\beta)}(x^2; g) O^{(\beta)}(0), \tag{5.79}$$

where $C^{(\beta)}$ are c-number functions that also depend on the parameters of the theory, and $O^{(\beta)}$ are local operators. After taking care of all Lorentz indices the functions are invariant functions depending on x^2, which in the (well-defined) Euclidean formulation corresponds to a short-distance expansion.

An explicit example is the $g_{\mu\nu}$ contribution in the current-current commutator for free quark currents,

$$\begin{aligned}[J_\mu(x), J_\nu(0)] &= [:\overline{\psi}(x)\gamma_\mu\psi(x):, :\overline{\psi}(0)\gamma_\nu\psi(0):] \\ &= \frac{-g_{\mu\nu}}{2\pi}\,[\partial_\rho\,\delta(x^2)\,\epsilon(x^0)]\,:\overline{\psi}(x)\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\psi(x):\end{aligned} \tag{5.80}$$

$$= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(x^2)\,\epsilon(x^0)\right]\sum_{n=0}^{\infty}\frac{x^{\mu_1}\ldots x^{\mu_n}}{n!} \times :[\partial_{\mu_1}\ldots\partial_{\mu_n}\overline{\psi}(0)]\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\,\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(0):. \quad (5.81)$$

In calculating matrix elements one encounters on the right hand side the matrix elements of local operators. The important features of the OPE are:

- The OPE yields an expansion in which the target dependence is in the matrix elements. The coefficient functions are independent of the target (factorization!). Their singularity structure plays a crucial role.
- The OPE gives a systematic expansion of the operators in terms of twist and spin, that corresponds to an expansion in powers of $1/Q^2$ for the matrix elements.
- In the operator expectation values (Green's functions) the renormalization scale μ enters. It is precisely cancelled by the scale dependence in $C^{(\beta)}(x^2;g,\mu)$. The latter scale dependence can be calculated using perturbative QCD.

First, consider the expansion for the Compton amplitude, in terms of a set of operators (suppressing the indices on the currents),

$$\mathcal{T}\,J(x)J(0) \propto \sum_{\beta,n} C_n^{(\beta)}(x^2;g)\,x^{\mu_1}\ldots x^{\mu_n}\,O^{(\beta)}_{\mu_1\ldots\mu_n}(0), \quad (5.82)$$

where $O^{(\beta)}_{\mu_1\ldots\mu_n}$ represents a set of (irreducible) symmetric and traceless operators. Their expectation value is given as

$$\langle P|O^{(\beta)}_{\mu_1\ldots\mu_n}(0)|P\rangle_{(\mu^2)} = \Theta^{(\beta)}(\mu^2)[P_{\mu_1}\ldots P_{\mu_n}+\ldots], \quad (5.83)$$

where the dots are dictated by the tracelessness of O. After Fourier transforming one has for the forward Compton amplitude or the hadronic tensor W, after Fourier transforming $C(x^2)$,

$$\begin{aligned} 2M\,T(q^2,\nu) &\propto \sum_{\beta,n}\tilde{C}_n^{(\beta)}(q^2;g,\mu^2)\,(-)^n\,\frac{q^{\mu_1}\ldots q^{\mu_n}}{(q^2)^n}\,\langle P|O^{(\beta)}_{\mu_1\ldots\mu_n}(0)|P\rangle_{(\mu^2)} \\ &= \sum_{\beta,n}\tilde{C}_n^{(\beta)}(q^2;g,\mu^2)\,\Theta_n^{(\beta)}(\mu^2)\left[\left(\frac{1}{2x}\right)^n+\mathcal{O}\left(\frac{1}{Q^2}\right)\right]. \end{aligned} \quad (5.84)$$

Dimensional analysis gives

$$d\left[C_n^{(\beta)}(x^2)\right] = 6 - d\left[O_n^{(\beta)}\right] + n, \quad (5.85)$$

$$d\left[\tilde{C}_n^{(\beta)}(q^2)\right] = 2 - d\left[O_n^{(\beta)}\right] + n \equiv 2 - t, \quad (5.86)$$

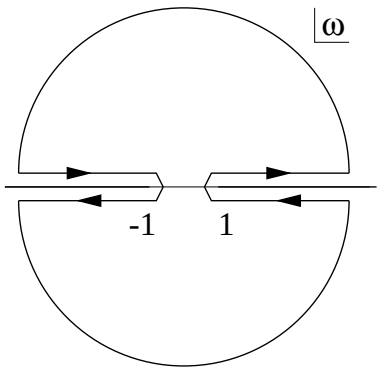
where

$$t = d\left[O_n^{(\beta)}\right] - n \quad (5.87)$$

is the *twist* of the operator O. The dimension of $C(q^2;g,\mu^2)$ must be provided by q^2, with logarithmic behavior arising from the anomalous dimensions of the operator (as we will see below),

$$\tilde{C}_n^{(\beta)}(q^2;g,\mu^2) = \frac{c_n^{(\beta)}\left(\ln(q^2/\mu^2)\right)}{(q^2)^{t-2}}. \quad (5.88)$$

In order to make contact with the deep inelastic structure functions we want to connect the expansion coefficients in $\omega \equiv 1/x$ with observable cross sections. For this we use the analytic properties of the forward Compton amplitude (considered in the previous chapter), $T_1(\omega) = T_1(-\omega)$ and the real-analyticity $T_1^*(\omega^*) = T_1(\omega)$. Considering the following contour in the complex ω plane with branch cuts in the physical regions (where $\mathcal{I}m\ T_1 \neq 0$)



one obtains

$$\begin{aligned} T_1(\omega) &= \frac{1}{2\pi\, i}\left[\int_{-\infty}^{-1} d\omega'\, \frac{T(\omega'+i\epsilon)}{\omega'-\omega} + \int_{-1}^{-\infty} d\omega'\, \frac{T(\omega'-i\epsilon)}{\omega'-\omega}\right. \\ &\qquad \left. + \int_{\infty}^{1} d\omega'\, \frac{T(\omega'-i\epsilon)}{\omega'-\omega} + \int_{1}^{\infty} d\omega'\, \frac{T(\omega'+i\epsilon)}{\omega'-\omega}\right] \\ &= \frac{1}{\pi}\int_1^\infty d\omega' \left[\frac{\mathcal{I}m\; T(\omega'+i\epsilon)}{\omega'+\omega} + \frac{\mathcal{I}m\; T(\omega'+i\epsilon)}{\omega'-\omega}\right] \\ &= \frac{2}{\pi}\int_1^\infty d\omega'\, \frac{\omega'\,\mathcal{I}m\; T(\omega')}{\omega'^2-\omega^2}. \end{aligned} \tag{5.89}$$

In case of bad convergence one can use a subtracted relation,

$$T_1(\omega) - T_1(0) = \frac{2\omega^2}{\pi}\int_1^\infty d\omega'\, \frac{\mathcal{I}m\, T(\omega')}{\omega'\,(\omega'^2-\omega^2)}, \tag{5.90}$$

etc. Expanding around zero one finds

$$T_1(\omega) = \frac{2}{\pi} \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \int_1^\infty \frac{d\omega'}{\omega'} \left(\frac{\omega}{\omega'}\right)^n \mathcal{I}m\; T_1(\omega'). \tag{5.91}$$

Note that this expansion around $\omega = 1/x = 0$ is in fact a short distance expansion since in deep inelastic processes, the relevant contributions in the current-current matrix elements came from $|\boldsymbol{x}| \approx t \approx 1/Mx$. Recalling that the structure functions $W_i(q^2,\nu)$ or $F_i(x,q^2)$ are found as the imaginary part of the forward Compton amplitude, to be precise $(1/\pi)\mathcal{I}m\; T(q^2,\nu)$. Thus

$$2M\, T(q^2,\nu) \propto 2i \sum_{n \text{ even}} M_n(q^2) \left(\frac{1}{x}\right)^n, \tag{5.92}$$

where

$$M_n(q^2) = \frac{1}{\pi}\int_0^1 dx\, x^{n-1}\, \mathcal{I}m\; T(q^2,\nu). \tag{5.93}$$

Thus one sees that the moments of the structure functions $F(x,Q^2) = \mathcal{I}m\, T(x,Q^2)/\pi$ are given by

$$M_n(Q^2) = \int_0^1 dx\, x^{n-1}\, F(x,Q^2) \;\propto\; \frac{c_n^{(\beta)}\left(\ln(q^2/\mu^2)\right)\;\Theta_n^{(\beta)}(\mu^2)}{(q^2)^{t-2}}. \tag{5.94}$$

The dominant contributions in the moments and thus also the structure functions come from twist two operators. They come in two classes, quark operators

$$O^F_{\mu_1\ldots\mu_n} = i^{n-1}\mathcal{S}\,\overline{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_n}\,\psi + \ldots, \tag{5.95}$$

where $\mathcal{S}$ indicates symmetrization in the indices, D_μ are covariant derivatives; furthermore, there are also gluon operators

$$O^G_{\mu_1\ldots\mu_n} = (2\,i)^{n-2}\mathcal{S}\,F_{\mu_1\lambda}D_{\mu_2}\ldots D_{\mu_{n-1}}F_{\mu_n}{}^{\lambda} + \ldots\ . \tag{5.96}$$

The quark operators can of course be supplemented with flavor operators. Note that the sum rules derived for the parton distributions written as correlation functions and the expansion of the field operators in the correlation functions are precisely the (free) quark operators from the above set.

Twist 4 operators contributing to the unpolarized structure functions are for instance of the form $\mathcal{S}\,\overline{\psi}\gamma_{\mu_1}\psi\,\overline{\psi}\gamma_{\mu_2}\psi$ or $\overline{\psi}[D^\rho, F_{\rho\mu_1}]\gamma_{\mu_2}\psi$. They specifically contribute to the lowest moments of the contributions in deep inelastic scattering arising from diagrams in which more than two partons emerge from the soft scattering part.

5.7 QCD corrections in deep inelastic scattering

From the operator product expansion, it is easy to obtain the $\ln Q^2$ dependence of the moments. Consider for the operator expansion $\mathcal{T}J_A(x)J_B(0) = \sum_i C_i(x^2;g,\mu)\,O_i(0)$ the insertions in any Green's function,

$$G^{(n)}_{JJ} \overset{x\to 0}{\approx} \sum_i C_i(x^2;g)\,G^{(n)}_{O_i}, \tag{5.97}$$

where

$$G^{(n)}_{JJ}(x,x_1,\ldots,x_n;g,\mu) = \langle 0|\mathcal{T}J_A(x)J_B(0)\,\phi(x_1)\ldots\phi(x_n)|0\rangle, \tag{5.98}$$

$$G^{(n)}_{O_i}(x,x_1,\ldots,x_n;g,\mu) = \langle 0|\mathcal{T}O_i(0)\,\phi(x_1)\ldots\phi(x_n)|0\rangle. \tag{5.99}$$

The renormalization group equations read

$$[D - n\,\gamma(g) + \gamma_A(g) + \gamma_B(g)]\,G^{(n)}_{JJ} = 0, \tag{5.100}$$

$$[D - n\,\gamma(g) + \gamma_i(g)]\,G^{(n)}_{O_i} = 0, \tag{5.101}$$

where $D = \mu(\partial/\partial\mu) + \beta(g)\,(\partial/\partial g)$. Inserting in the expansion in Eq. 5.97 the RGE's for G_{JJ} and G_{O_i} one obtains an RGE for the coefficient function,

$$[D + \gamma_A(g) + \gamma_B(g) - \gamma_i(g)]\,C_i(x^2;g,\mu) = 0, \tag{5.102}$$

or for the Fourier-transform

$$[D + \gamma_A(g) + \gamma_B(g) - \gamma_i(g)]\,\tilde{C}_i(q^2;g,\mu) = 0. \tag{5.103}$$

The equation is independent of the specific Green's function in which the operators have been inserted, i.e. the $\ln Q^2$ behavior is independent of the target. If the currents in $J(x)J(0)$ are conserved currents the corresponding gamma functions are zero. We now can use the results in chapter 1 and obtain the QCD-corrections in leading order. After extraction of the $(q^2)^{2-t}$ factor one finds for the coefficients c_i

$$c_i(Q^2;g,\mu) = c_i(\mu^2;\overline{g}(\tau),\mu)\ \exp\left(-\int_0^\tau d\tau'\gamma_i(\overline{g}(\tau'))\right) \tag{5.104}$$

$$= c_i(\mu^2;\overline{g}(\tau),\mu)\ \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{\gamma_{0i}/2b_0}, \tag{5.105}$$

$$= c_i(1;\overline{g}(\tau))\ \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{d_i}, \tag{5.106}$$

where $\tau = \ln(Q^2/\mu^2)$ and γ_{0i} is the coefficient in the leading term in $\gamma_i(g) \approx (\gamma_{0i}/4\pi)\alpha_s$, while b_0 multiplies the leading term in the β function, $b_0 = (33-2f)/3$, and $d_i \equiv \gamma_{0i}/2b_0$.

The next to leading order (NLO) is a matter of calculating the γ function for the appropriate operators and calculating the coefficient functions,

$$\gamma_i(g) = \gamma_{0i}\,\left(\frac{\alpha_s}{4\pi}\right) + \gamma_{1i}\,\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots, \tag{5.107}$$

$$c_i(1;g) = 1 + B_i\,\left(\frac{\alpha_s}{4\pi}\right) + \ldots\ . \tag{5.108}$$

Many of these coefficients are now know to this or even higher orders, which has been an enormous task.

For sum rules connected themselves to a conserved current the anomalous dimension $\gamma_i(g) = 0$ and the results for the coefficients B_i immediately give the NLO corrections to sum rules. There is no correction to the Adler sum rule, but one has for the Bjorken sum rule,

$$\int_0^1 dx\ \left[F_1^{\overline{\nu}p}(x) - F_1^{\nu p}(x)\right] = 1 - \frac{2}{3}\,\frac{\alpha_s}{\pi}, \tag{5.109}$$

and for the Gross-Llewellyn Smith sum rule,

$$\int_0^1 dx\ \left[F_3^{\overline{\nu}p}(x) + F_3^{\nu p}(x)\right] = 6\,\left(1 - \frac{\alpha_s}{\pi}\right). \tag{5.110}$$

The Gottfried sum rule gets a very small correction. The polarized Bjorken sum rule gets a correction $(1-\alpha_s/\pi)$, or explicitly (including also the next order),

$$\int_0^1 dx\ \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right] = \frac{G_A}{6}\,\left(1 - \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2}\,(-4.5833 + f/3)\right), \tag{5.111}$$

where f is the number of (active) flavors.

The g_1^p sum rule by itself involves the singlet axial current, which is not conserved. Its anomalous dimension is given by

$$\gamma_1^S(\alpha_s) = 16f\,\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots, \tag{5.112}$$

with as leading order solution

$$\Delta\Sigma(Q^2) = \exp\left(\frac{16f}{4\pi\,\beta_0}(\alpha_s(Q^2) - \alpha_s(Q_0^2))\right)\,\Delta\Sigma(Q_0^2). \tag{5.113}$$

Togeter with the coefficient B^S this gives the corrections for the singlet contribution in the spin sum rules

$$\Gamma_1^S(Q^2) = \left(1 - \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2}\,(-4.5833 + 1.16248f)\right)\,\Delta\Sigma(Q^2). \tag{5.114}$$

Including this the deviation of the proton sumrule from the naive result can be fairly well understood.

5.8 Evolution equations for the structure functions

It is possible to extend the intuitive folding picture that we have used in section 5.3.1 to obtain the QCD correction in a different approach, which provides also a practical way to calculate the coefficient functions $c_i(g)$ and gamma functions $\gamma_i(g)$.

Extending the result for the transverse structure function written as a delta function contribution,

$$\sigma_T = \sum_i \int_0^1 dz\, f_i(z)\,\left[e_i^2\,\frac{4\pi^2\alpha}{Q^2}\,x\,\delta(x-z) + \delta\hat{\sigma}_i(z,Q^2) + \ldots\right], \tag{5.115}$$

one obtains contributions from the process $\gamma^* q \to Gq$ process, which is no longer proportional to $\delta(x-z)$ because of the fact that there are two particles in the final state. The the amplitude is

$$|\mathcal{M}|^2 = 32\pi^2\, e_i^2\, \alpha\alpha_s\, \frac{4}{3}\, \left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} + \frac{2\hat{u}Q^2}{\hat{s}\hat{t}}\right) \tag{5.116}$$

and contributes

$$\delta\sigma(\hat{s}) = e_i^2\, \frac{2\pi\alpha\alpha_s}{(\hat{s}+Q^2)^2}\, \frac{4}{3}\left(-\frac{\hat{t}}{\hat{s}} - \frac{\hat{s}}{\hat{t}} - \frac{2\hat{t}Q^2 + 2\hat{s}Q^2 + 2Q^4}{\hat{s}\hat{t}}\right) d\hat{t} \tag{5.117}$$

We now express the momenta (using lightcone components) for $\gamma^*(q)$ + quark $(k) \to$ gluon (p_G) + quark (p_q) as

$$q \equiv \left[\frac{Q^2}{xA\sqrt{2}}, -\frac{xA}{\sqrt{2}}, \mathbf{0}_\perp\right] \qquad \text{(photon)}, \tag{5.118}$$

$$k \equiv \left[\frac{m^2}{zA\sqrt{2}}, \frac{zA}{\sqrt{2}}, \mathbf{0}_\perp\right] \approx \left[0, \frac{zA}{\sqrt{2}}, \mathbf{0}_\perp\right] \qquad \text{(parton)}, \tag{5.119}$$

$$p_G = \left[\zeta q^-, \frac{\boldsymbol{p}_\perp^2}{2\zeta q^-}, \boldsymbol{p}_\perp\right] \approx \left[\frac{\zeta Q^2}{xA\sqrt{2}}, 0, \boldsymbol{p}_\perp\right], \tag{5.120}$$

$$p_q = \left[(1-\zeta)q^-, \frac{\boldsymbol{p}_\perp^2}{2(1-\zeta)q^-}, -\boldsymbol{p}_\perp\right] \approx \left[\frac{(1-\zeta)Q^2}{xA\sqrt{2}}, 0, -\boldsymbol{p}_\perp\right]. \tag{5.121}$$

Note that z can be written as x/ξ, where $\xi \equiv -q^+/k^+ = x/z$ is the Bjorken scaling variable for the subprocess. We will neglect all particles masses, in which case $p_q^2 = 0$ gives

$$\boldsymbol{p}_\perp^2 = \frac{\zeta(1-\zeta)(1-\xi)}{\xi}\, Q^2. \tag{5.122}$$

The invariants for the subprocess become

$$\hat{s} = (k+q)^2 = \frac{1-\xi}{\xi}\, Q^2 = \frac{1}{\zeta(1-\zeta)}\, \boldsymbol{p}_\perp^2, \tag{5.123}$$

$$\hat{t} = (p_G - q)^2 = -\frac{1-\zeta}{\xi}\, Q^2 = -\frac{1}{\zeta(1-\xi)}\, \boldsymbol{p}_\perp^2, \tag{5.124}$$

The kinematic range of the process $\hat{s} \geq 0$, $-(\hat{s}+Q^2) \leq t \leq 0$ in principle restricts the ranges of (ξ, ζ) to $0 \leq \xi \leq 1$ and $0 \leq \zeta \leq 1$. The cross section becomes

$$\delta\hat{\sigma}_i(\xi, Q^2) = e_i^2\, \frac{2\pi\alpha\alpha_s}{Q^2}\, \frac{4}{3}\, \xi \left[\frac{1-\zeta-2\,\xi}{1-\xi} + \frac{1+\xi^2}{(1-\xi)(1-\zeta)}\right] d\zeta. \tag{5.125}$$

Integrating for inclusive scattering over the final state, i.e. integrating over ζ one sees that there are singular points, specifically for $\zeta = 1$, which corresponds to a gluon radiated with $\boldsymbol{p}_\perp = 0$. These divergences are therefore referred to as *collinear divergences*. There are again several ways of regularizing, either by giving quarks and gluons masses or by dimensional regularization. In this case a $\boldsymbol{p}_\perp$ cutoff also provides a regularization. The restriction $\boldsymbol{p}_\perp^2 \geq \mu^2$ modifies the allowed region in the (ξ, ζ) plane. For a given ξ (not to close to unity) the integration is limited to

$$\frac{\xi}{1-\xi}\, \frac{\mu^2}{Q^2} \leq \zeta \leq 1 - \frac{\xi}{1-\xi}\, \frac{\mu^2}{Q^2}. \tag{5.126}$$

With this regularization the result becomes

$$\delta\hat{\sigma}_i(\xi, Q^2) = e_i^2\, \frac{2\pi\alpha\alpha_s}{Q^2}\, \frac{4}{3}\, \xi \left[\frac{1-4\,\xi}{2(1-\xi)} + \frac{1+\xi^2}{(1-\xi)}\, \ln\left(\frac{1-\xi}{\xi}\right) + \frac{1+\xi^2}{(1-\xi)}\, \ln\left(\frac{Q^2}{\mu^2}\right)\right] \tag{5.127}$$

$$= e_i^2\, \frac{2\pi\alpha\alpha_s}{Q^2}\, \xi \left[B(\xi) + P_{qq}(\xi)\, \ln\left(\frac{Q^2}{\mu^2}\right)\right]. \tag{5.128}$$

where

$$B(\xi) = \frac{4}{3}\left[\frac{1-4\,\xi}{2(1-\xi)} + \frac{1+\xi^2}{(1-\xi)}\,\ln\left(\frac{1-\xi}{\xi}\right)\right], \tag{5.129}$$

$$P_{qq}(\xi) = \frac{4}{3}\,\frac{1+\xi^2}{(1-\xi)}. \tag{5.130}$$

The functions P_{qq} are called *splitting functions*. Combining the results of the (leading) contribution (omitting for now the term $B(\xi)$) one has

$$\sigma_T = \frac{4\pi^2\alpha}{Q^2}\,x\,\sum_i e_i^2 \int_x^1 \frac{dz}{z}\,f_i(z)\left[\delta\left(1-\frac{x}{z}\right) + \frac{\alpha_s}{2\pi}\,P_{qq}\left(\frac{x}{z}\right)\,\ln\left(\frac{Q^2}{\mu^2}\right)\right]. \tag{5.131}$$

This can be rewritten in parton form

$$\sigma_T = \frac{4\pi^2\alpha}{Q^2}\,x\,\sum_i e_i^2\,f_i(x,\ln Q^2), \tag{5.132}$$

where the functions $f_i(x,\ln Q^2)$ satisfy

$$\frac{\partial f_i(x,\ln Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dz}{z}\,f_i(z)\,P_{qq}\left(\frac{x}{z}\right). \tag{5.133}$$

As in the case of integrating over ζ above one still encounters divergences, in this case for $\xi = x/z \to 1$. This also gives rise to $\ln(Q^2/\mu^2)$ corrections. They are considered separately by considering the $1/(1-\xi)$ appearing in the splitting functions as functionals,

$$\begin{aligned}
\int_0^{1-\delta} dx\,\frac{f(x)}{1-x} &= -\int f(x)\,d\ln(1-x) \\
&= \int_0^{1-\delta} dx\,\frac{f(x)-f(1)}{1-x} \;-\; f(1)\,\ln\delta \\
&= \int_0^{1} dx\,\frac{f(x)-f(1)}{1-x} \;-\; \ln\delta\int dx\,f(x)\,\delta(1-x) \\
&\equiv \int_0^{1} dx\,\frac{f(x)}{(1-x)_+} \;-\; \ln\delta\int dx\,f(x)\,\delta(1-x)
\end{aligned}$$

or

$$\frac{1}{1-x} = \frac{1}{(1-x)_+} - \delta(1-x)\,\ln\delta. \tag{5.134}$$

Including these and other singular contributions (vertex corrections to the process $\gamma^* + q \to q$) one then has

$$f_i(x,Q^2) = \int_x^1 \frac{dz}{z}\,f_i(z)\left\{\delta\left(1-\frac{x}{z}\right) + \frac{\alpha_s}{2\pi}\,\ln\left(\frac{Q^2}{\mu^2}\right)\left[a\,\delta\left(1-\frac{x}{z}\right) + P_{qq}\left(\frac{x}{z}\right)\right]\right\}, \tag{5.135}$$

The piece between curly brackets can be seen as the probability density $\mathcal{P}_{qq}$ of finding a quark inside a quark with fraction $\xi = x/z$ of the parent quark to first order in α_s. Instead of calculating all contributions and checking all cancellations, it is easier to see what the final result must be from

$$\int d\xi\,\mathcal{P}_{qq}(\xi) = 1. \tag{5.136}$$

Keeping the form in Eq. 5.131 one can write the full piece between square brackets in Eq. 5.135 as

$$P_{qq}(\xi) = \frac{4}{3}\,\frac{1+\xi^2}{(1-\xi)_+} + 2\,\delta(1-\xi). \tag{5.137}$$

In a similar way other splitting functions can be calculated, e.g. P_{qG} from the proces $\gamma^* G \to q\overline{q}$. Summarizing one has

$$\frac{dq(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right)\, q(z,\tau) + P_{qG}\left(\frac{x}{z}\right)\, G(z,\tau)\right], \tag{5.138}$$

$$\frac{d\overline{q}(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right)\, \overline{q}(z,\tau) + P_{qG}\left(\frac{x}{z}\right)\, G(z,\tau)\right], \tag{5.139}$$

$$\frac{dG(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{Gq}\left(\frac{x}{z}\right)\, \sum_f \left(q_f(z,\tau) + \overline{q}_f(z,\tau)\right) + P_{GG}\left(\frac{x}{z}\right)\, G(z,\tau)\right]. \tag{5.140}$$

It is convenient to distinguish singlet and nonsinglet distribution functions at this point. The singlet distribution is the sum over all (relevant) flavors,

$$\Sigma(x,\tau) = \sum_f \left(q_f(x,\tau) + \overline{q}_f(x,\tau)\right), \tag{5.141}$$

everything else is 'nonsinglet', e.g. for three flavors

$$\begin{aligned} q_v &= q - \overline{q}, \qquad \text{for} \quad q = u, d, s \\ q^3 &= (u + \overline{u}) - (d + \overline{d}), \\ q^8 &= (u + \overline{u}) + (d + \overline{d}) - 2\,(s + \overline{s}). \end{aligned}$$

For the nonsinglet structure function we have

$$\frac{dq(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\, P_{qq}\left(\frac{x}{z}\right)\, q(z,\tau), \tag{5.142}$$

for the singlet

$$\frac{d\Sigma(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right)\, \Sigma(z,\tau) + 2f\, P_{qG}\left(\frac{x}{z}\right)\, G(z,\tau)\right], \tag{5.143}$$

$$\frac{dG(x,\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{Gq}\left(\frac{x}{z}\right)\, \Sigma(z,\tau) + P_{GG}\left(\frac{x}{z}\right)\, G(z,\tau)\right], \tag{5.144}$$

i.e. the nonsinglet has a simple evolution equation, while the singlet distribution mixes with the gluon distribution.

The splitting functions are given by

$$P_{qq}(x) = \frac{4}{3}\,\frac{1+x^2}{(1-x)_+} + 2\,\delta(1-x), \tag{5.145}$$

$$P_{Gq}(x) = \frac{4}{3}\,\frac{1+(1-x)^2}{x}, \tag{5.146}$$

$$P_{qG}(x) = \frac{1}{2}\,\left(x^2 + (1-x)^2\right), \tag{5.147}$$

$$P_{GG}(x) = 6\,\left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \frac{33-2f}{6}\,\delta(1-x). \tag{5.148}$$

(Note that the coefficient for the δ-function in P_{GG} is determined from momentum conservation.

For the polarized distribution functions a similar set of splitting functions can be derived,

$$\Delta P_{qq}(x) = P_{qq}(x), \tag{5.149}$$

$$\Delta P_{Gq}(x) = \frac{4}{3}\frac{1-(1-x)^2}{x}, \tag{5.150}$$

$$\Delta P_{qG}(x) = \frac{1}{2}\left(x^2-(1-x)^2\right), \tag{5.151}$$

$$\Delta P_{GG}(x) = 6\left[(1+x^4)\left(\frac{1}{x}+\frac{1}{(1-x)_+}\right)-\frac{(1-x)^3}{x}\right]+\frac{33-2f}{6}\delta(1-x). \tag{5.152}$$

5.9 Solutions of the evolution equations

The evolution equations can be solved most easily by considering moments. Starting from

$$\begin{aligned}\frac{dq(x,\tau)}{d\tau} &= \frac{\alpha_s(\tau)}{2\pi}\int_x^1 \frac{dz}{z} P_{qq}\left(\frac{x}{z}\right) q(z,\tau), \\ &= \frac{\alpha_s(\tau)}{2\pi}\int_0^1 dz \int_0^1 dy \delta(x-yz)\, P_{qq}(y)\, q(z,\tau),\end{aligned} \tag{5.153}$$

and defining

$$M_n(\tau) \equiv \int_0^1 dx\, x^{n-1}\, q(x,\tau), \tag{5.154}$$

$$A_n \equiv \int_0^1 dy\, y^{n-1}\, P_{qq}(y,\tau) = \frac{4}{3}\left[-\frac{1}{2}+\frac{1}{n(n+1)}-2\sum_{j=2}^{n}\frac{1}{j}\right], \tag{5.155}$$

one has

$$\frac{dM_n(\tau)}{d\tau} = \frac{\alpha_s(\tau)}{2\pi} A_n\, M_n(\tau). \tag{5.156}$$

Using the leading order QCD result for $\alpha_s(\tau)$ (which reads $d\alpha_x/d\tau = -b_0(\alpha_s/4\pi)$) this is easily solved, giving

$$\frac{M_n(Q^2)}{M_n(\mu^2)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{d_n}, \tag{5.157}$$

where $d_n = -2\,A_n/b_0$. Comparison with Eqs 5.105 and 5.106 show that the moments of the splitting functions are up to a factor equal to the coefficients γ_0,

$$A_n = -\frac{\gamma_0}{4}. \tag{5.158}$$

ΛAs an example, consider the second moment, for which $A_2 = -16/9$. The result for d_2 is

$$d_2 = \frac{32}{9\,b_0} = \frac{32}{99-6\,f)} \stackrel{f=3}{=} \frac{32}{81},$$

and the second moment of the nonsinglet distributions, e.g. the fraction of momentum carried by the valence quarks, vanishes for $Q^2 \to \infty$, e.g.

$$M_2(Q^2) = \int dx\, x[q(x,\ln Q^2) - \overline{q}(x,\ln Q^2)] \;\propto\; (\alpha_s(Q^2))^{d_2}. \tag{5.159}$$

For sea quarks and gluons mixing occurs and one has

$$\frac{d}{d\tau}\begin{pmatrix}\Sigma_n(\tau)\\ G_n(\tau)\end{pmatrix} = \frac{\alpha_s(\tau)}{2\pi}\begin{pmatrix}A_n(qq) & 2f\,A_n(qG)\\ A_n(Gq) & A_n(GG)\end{pmatrix}\begin{pmatrix}\Sigma_n(\tau)\\ G_n(\tau)\end{pmatrix}. \tag{5.160}$$

For the second moment the anomalous dimensions can be calculated from the splitting functions or through the relation with the anomalous dimensions for the relevant operators. As can be seen from

Eq. 5.6 or 5.96, the relevant operators are those for which the anomalous dimensions are given in Eq. 2.191. One has

$$\begin{pmatrix} A_2(qq) & 2f\,A_2(qG) \\ A_2(Gq) & A_2(GG) \end{pmatrix} = \begin{pmatrix} -\frac{16}{9} & \frac{f}{3} \\ \frac{16}{9} & -\frac{f}{3} \end{pmatrix}. \tag{5.161}$$

The eigenvalues of the A_2 matrix are *zero* and $-(16+3f)/9$. Thus one linear combination of Σ_2 and G_2 (the sum!) does not vanish. This corresponds to momentum conservation. One can write

$$\begin{pmatrix} \Sigma_2(Q^2) \\ G_2(Q^2) \end{pmatrix} = \begin{pmatrix} \frac{3f}{16+3f} \\ \frac{16}{16+3f} \end{pmatrix} + \begin{pmatrix} S(Q^2) \\ -S(Q^2) \end{pmatrix}, \tag{5.162}$$

where

$$\frac{S(Q^2)}{S(\mu^2)} = \left(\frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)}\right)^{(32+6f)/9b_0}. \tag{5.163}$$

5.9.1 Inverse Mellin transforms of moments

In principle the evolution of the structure functions can be done using the evolution equations. Using the moments is actually quite convenient, although one needs all of them. A very useful method in practice is to parametrize the distributions at one Q^2 with a function for which the moments can be easily calculated, evolve the moments and apply an inverse Mellin transform. One has

$$M_n(Q^2) = \int_0^1 dx\, x^{n-1}\, q(x,Q^2), \tag{5.164}$$

$$q(x,Q^2) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, M_n(Q^2), \tag{5.165}$$

where c must be such that M_n has no singularities in the complex n-plane right of the line $\mathcal{R}e\, n = c$. A similar relation exists for the moments of the splitting functions, or if we use the relation between A_n and γ_{0n},

$$\gamma_{0n} = -4\int_0^1 dx\, x^{n-1}\, P_{qq}(x), \tag{5.166}$$

$$P_{qq}(x) = \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, \gamma_{0n}, \tag{5.167}$$

We have discussed the extensions to higher orders in Eqs 5.107 and 5.108 involving γ_1 and B. The γ_1 coefficients are precisely moments of the next orders in the splitting functions. If

$$P_{qq}(x) = P_{qq}^{(0)}(x) + \frac{\alpha_s(Q^2)}{2\pi}\, P_{qq}^{(1)}(x), \tag{5.168}$$

then

$$\gamma_{1n} = -8\int_0^1 dx\, x^{n-1}\, P_{qq}^{(1)}(x), \tag{5.169}$$

$$P_{qq}^{(1)}(x) = \frac{1}{16\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, \gamma_{1n}. \tag{5.170}$$

The coefficients B correspond to the moments of the additional pieces which appear in the expressions for the structure functions, such as the $B(x)$ appearing in the calculation of F_1 in Eq. 5.128. If one extends the delta-function contribution,

$$C(x) = \delta(1-x) + \frac{\alpha_s(Q^2)}{4\pi}B(x), \tag{5.171}$$

then

$$B_n = \int_0^1 dx\, x^{n-1}\, B(x), \tag{5.172}$$

$$B(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n}\, B_n. \tag{5.173}$$

While the splitting functions are universal (process independent), the contributions $C(x)$ are in general process dependent. For instance the longitudinal structure function (for which the dominant parton model result is zero) picks up a contribution of which the result in Eq. 5.128 is a part that for electroproduction is given by

$$F_L(x,Q^2) = \frac{\alpha_s(Q^2)}{\pi}\left[\frac{4}{3}\int_x^1 \frac{dz}{z}\left(\frac{x}{z}\right)^2 F_2(z,Q^2) + \left(2\sum_q e_q^2\right)\int_x^1 \frac{dz}{z}\left(\frac{x}{z}\right)^2\left(1-\frac{x}{z}\right) z\, G(z,Q^2)\right]. \tag{5.174}$$

Bibliography

[1] E.S. Abers and B.W. Lee, Phys. Rep. 9C (1973) 1.

[2] J. Ashman et al, Nucl. Phys. B328 (1989) 1.

[3] F.E. Close, *An Introduction to Quarks and Partons*, Academic Press 1979.

[4] P. Cvitanović, Phys. Rev. 14 (1976) 1536.

[5] D.Z. Freedman, K. Johnson and J.I. Latorre, Nucl. Phys. B 1992, MIT preprint CTP#1972 (1991).

[6] F. Halzen and A.D. Martin, *Quarks and Leptons*, Wiley 1984.

[7] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw-Hill 1985.

[8] R.L. Jaffe, in *Relativistic Dynamics and Quark Nuclear Physics*, eds M.B. Johnson and A. Picklesheimer, Wiley, 1986

[9] H. Kendall, Proc. V. Intern. Symp. on Electron and Photon Interactions at High Energies (Cornell University, 1971); J.I. Friedman and H.W. Kendall, Ann. Rev. Nucl. Part. Sci. 22 (1972) 203.

[10] W.J. Marciano and H. Pagels, Phys. Rep. 36 (1978) 137

[11] Yu.L. Dokshitzer, V.A. Khoze, A.H. Mueller and S.I. Troyan, *Basics of perturbative QCD*, Editions Frontières 1991.

[12] P.J. Mulders, Physics report 185 (1990) 83.

[13] T. Muta, *Foundations of Quantum Chromodynamics*, World Scientific 1987.

[14] R.G. Roberts, *The structure of the proton, deep inelastic scattering*, Cambridge University Press 1990.

[15] L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 1985.

[16] M. Veltman, *Diagrammatica*, Cambridge University Press, 1994.

[17] S. Weinberg, *The quantum theory of fields (Vol I)*, Cambridge University Press, 1995.

[18] B. de Wit and J. Smith *Field Theory in Particle Physics*, North-Holland, 1986.

[19] For a review see W.J. Marciano and H. Pagels, Phys. Rep. 36 (1978) 137.

[20] G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977) 298; G. Altarelli, Phys. Rep. 81 (1982) 1.

[21] N. Isgur and M.B. Wise, Phys. Lett. B232 (1989) 113.

[22] C. Itzykson and M. Nauenberg, Rev. Mod. Phys. 38 (1966) 95.

[23] N. Isgur, in: *New aspects of subnuclear Physics*, Proc. 1978 Erice Summer School "Ettore Majorana", ed. A. Zichichi, Plenum 1980.

[24] R.L. Jaffe, in: *Pointlike structures inside and outside hadrons*, Proc. 1979 Erice Summer School "Ettore Majorana", ed. A. Zichichi, Plenum 1981.

[25] T. DeGrand, R.L. Jaffe, K. Johnson and J. Kiskis, Phys. Rev. D12 (1975) 2060.

[26] A.W. Thomas, Adv. Nucl. Phys. 13 (1983) 1.

[27] I. Zahed and G.E. Brown, Phys. Rep. 142 (1986) 1; U.-G. Meissner, Phys. Rep. 161 (1988) 213.

[28] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn and V.F. Weisskopf, Phys. Rev. D9 (1974) 3471.

[29] J.F. Donoghue and K. Johnson, Phys. Rev. D21 (1980) 1975; C.E. DeTar, Phys. Rev. D24 (1981) 752, 762.

[30] J.J. de Swart, Rev. Mod. Phys. 35 (1963) 916.

[31] Particle Data Group, Phys. Lett. B239 (1990) 1.

[32] S. Adler, Phys. Rev. 117 (1969) 2426; J. Bell and R. Jackiw, Nuovo Cimento A60 (1969) 47.