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Transverse Momentum Distribution Functions and beyond: setting up the nucleon tomography

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ABSTRACT

Transverse Momentum Distribution and beyond: setting up the nucleon tomography

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In these lecture notes I describe a diagrammatic approach in high energy scattering processeses. Using in particular production processes initiated by a lepton-hadron or a hadron-hadron intitial state we identify the correlators that describe the {\em partons in the hadrons}. In this way one can generalize more rigorous approaches such as the operator product expansion techniques. Generalizations include the treatment of transverse momenta of partons. The latter allows a general treatment that includes all possible correlations between momenta and spins of partons and parent hadrons both in polarized and unpolarized cases. The effects of transverse momenta show up as azimuthal asymmetries in the inclusive production of jets or specific hadrons. Although correlators describe in general squared amplitudes, links can be made to amplitudes in other processes. Examples are form factors and generalized parton distributions. One can also look at extensions to multi-parton scattering phenomena. The parametrization in terms of *universal functions*, such as distribution and fragmentation functions are useful to optimally profit from the kinematic and spin-related degrees of freedom in high-energy processes but the correlators actually also encode interesting hadronic structure that can be studied in lattice approaches or specific models for hadron structure.

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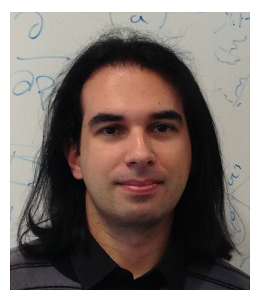

Cristian Pisano

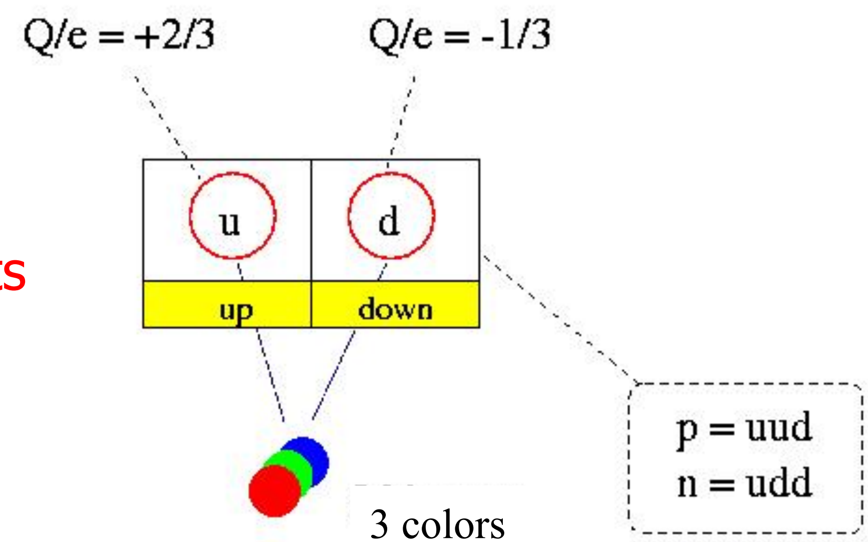

Jian Zhou

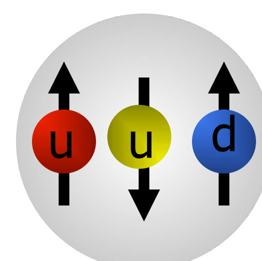
Valence structure of hadrons: global properties

- mass
- charge
- spin
- magnetic moment
- isospin, strangeness
- baryon number

- $M_p \approx M_n \approx 940$ MeV
- $Q_p = 1$, $Q_n = 0$
- $s = ½$
- $g_p \approx 5.59$, $g_n \approx -3.83$
- $I = ½$: (p,n) $S = 0$
- $B = 1$

Quarks as constituents

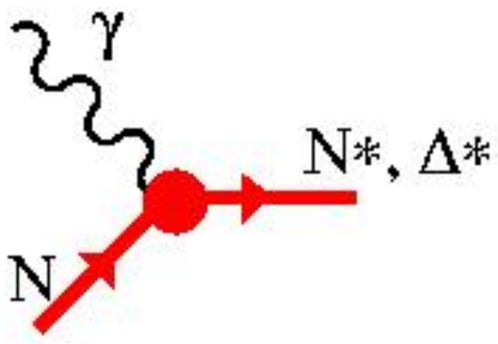


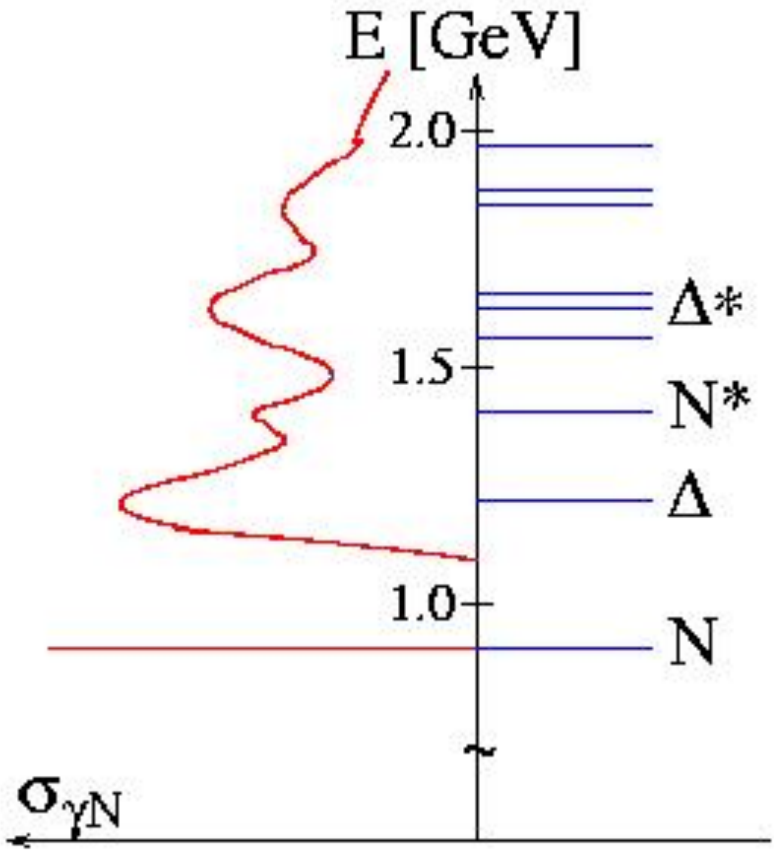


proton

A real look at the proton

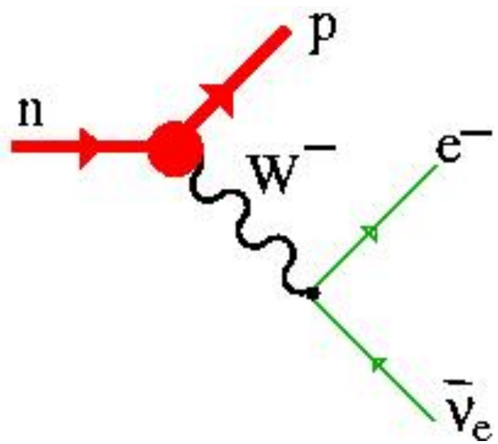
$\gamma + N \rightarrow ....$





Nucleon excitation spectrum
E ~ 1/R ~ 200 MeV
R ~ 1 fm

A (weak) look at the nucleon



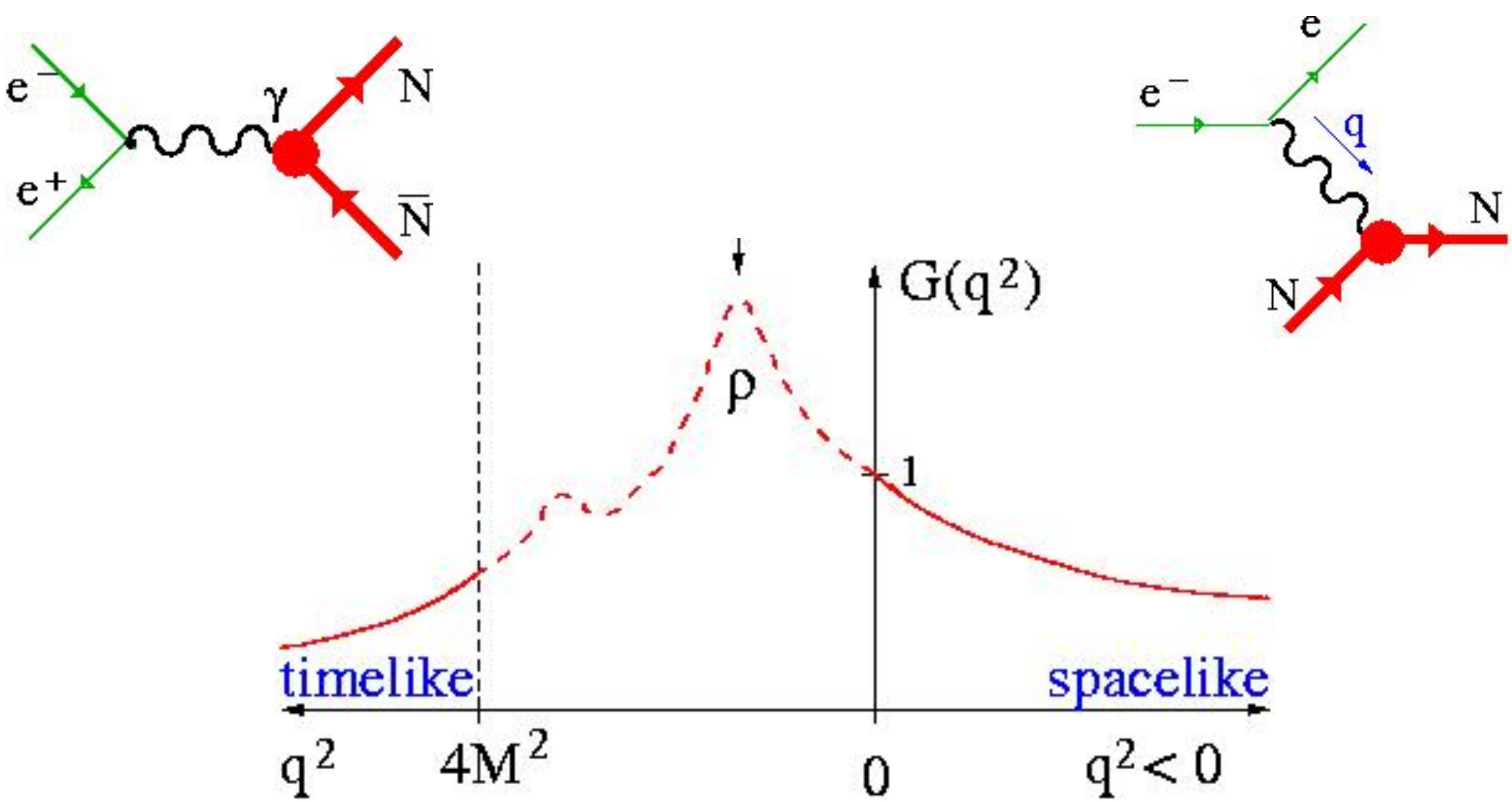
$n \rightarrow p + e^- + \nu$

τ = 900 s
→ Axial charge
$G_A(0) = 1.26$

A virtual look at the proton

$\gamma^* \rightarrow N\bar{N}$

$\gamma^* + N \rightarrow N$



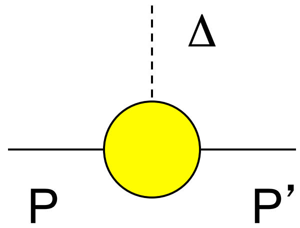
Local – forward and off-forward m.e.

Local operators (coordinate space densities):

$$< P'|O(x)|P> = e^{i\Delta . x}\left[G_1(t) - i\Delta_\mu G_2^\mu(t)\right]$$

$$t = \Delta^2$$

Form factors



Static properties:

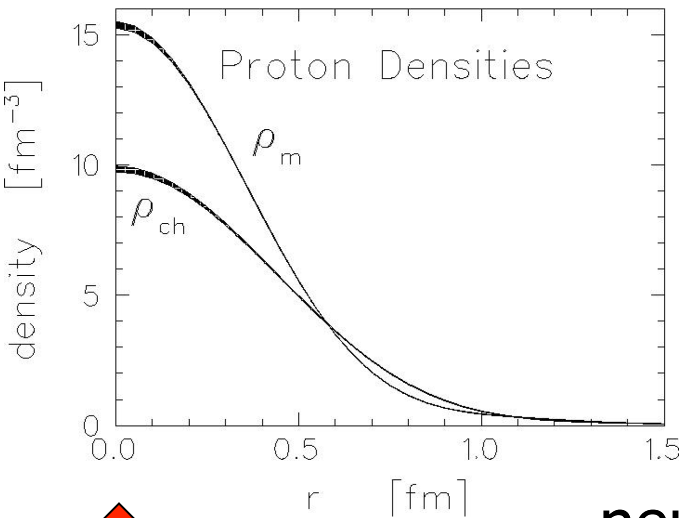
$$G_1(0) = < P|O(x)|P>$$

$$G_2^\mu(0) = < P|x^\mu O(x)|P>$$

Examples:
(axial) charge
mass
spin
magnetic moment
angular momentum

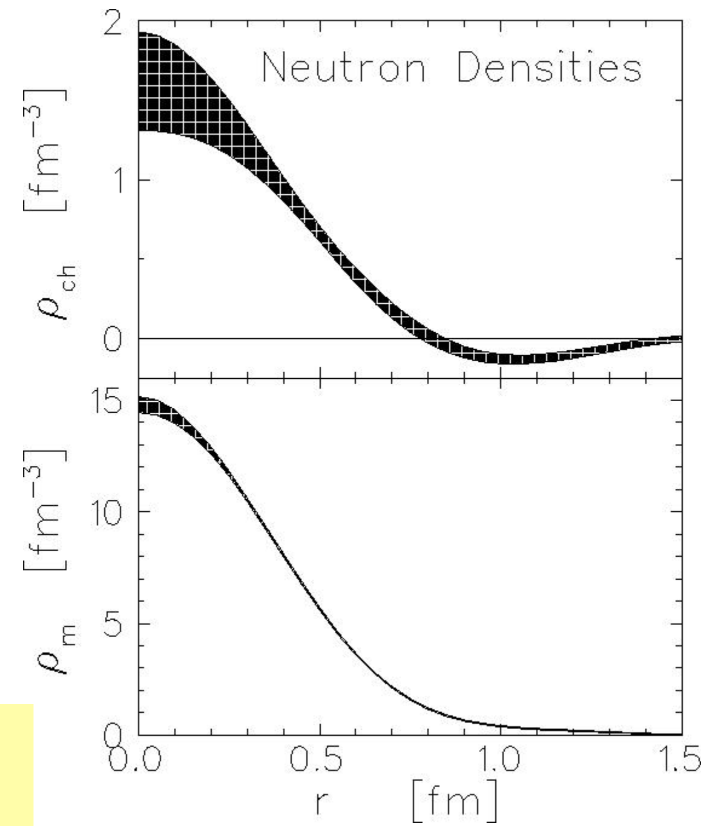
Nucleon densities from virtual look

$$G_i(t) \rightarrow \rho_i(x)$$



proton

neutron



- charge density ≠ 0
- u more central than d?
- role of antiquarks?
- $n = n_0 + p\pi^- + \ldots$?

Quark and gluon operators

Given the QCD framework, the operators are known quarkic or gluonic currents such as

(axial) vector currents

$$V_\mu^q(x) = \bar{q}(x)\gamma_\mu q(x)$$

$$A_\mu^{q'q}(x) = \bar{q}(x)\gamma_\mu\gamma_5 q'(x)$$

probed in specific combinations by photons, Z- or W-bosons

$$J_\mu^{(\gamma)} = \tfrac{2}{3}V_\mu^u - \tfrac{1}{3}V_\mu^d - \tfrac{1}{3}V_\mu^s + \ldots$$

$$J_\mu^{(Z)} = \tfrac{1}{2}\left(V_\mu^u - A_\mu^u\right) - \tfrac{4}{3}\sin^2\theta_W\, V_\mu^u + \ldots$$

$$J_\mu^{(W)} = V_\mu^{ud'} - A_\mu^{ud'} + \ldots$$

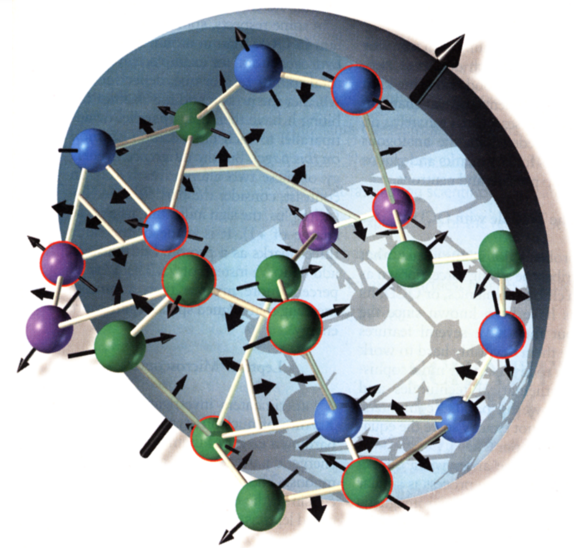
energy-momentum currents

$$T_{\mu\nu}^q(x) \sim \bar{q}(x)\gamma_{\{\mu}D_{\nu\}}q(x)$$

$$T_{\mu\nu}^G(x) \sim G_{\mu\alpha}(x)G^\alpha{}_\nu(x)$$

probed by gravitons

Towards the quarks themselves



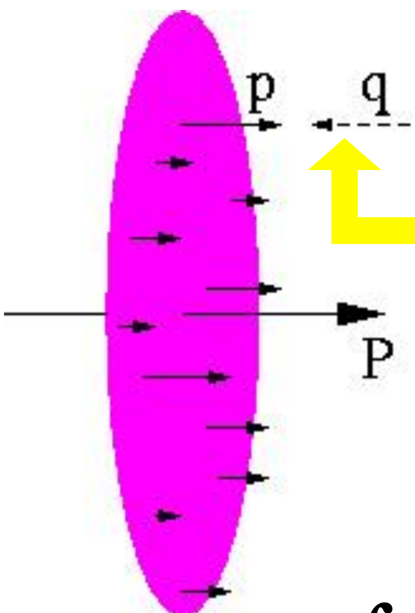
- The current provides the densities but only in specific combinations, e.g. *quarks minus antiquarks* and only flavor weighted
- No information about their correlations, (effectively) pions, or …
- Can we go beyond these global observables (which correspond to local operators)?
- Yes, in high energy (semi-)inclusive measurements we will have access to non-local operators!

- L_{QCD} (quarks, gluons) known!

Non-local probing

Nonlocal forward operators (correlators):

$$< P \,|\, O\left(x - \tfrac{y}{2}, x + \tfrac{y}{2}\right) |\, P > \;=< P \,|\, O\left(-\tfrac{y}{2}, +\tfrac{y}{2}\right) |\, P >$$



Selectivity at high energies: q = p

Specifically useful: ‘squares’

$$O\left(x - \tfrac{y}{2}, x + \tfrac{y}{2}\right) = \Psi^\dagger\left(x - \tfrac{y}{2}\right) \ldots \Psi\left(x + \tfrac{y}{2}\right)$$

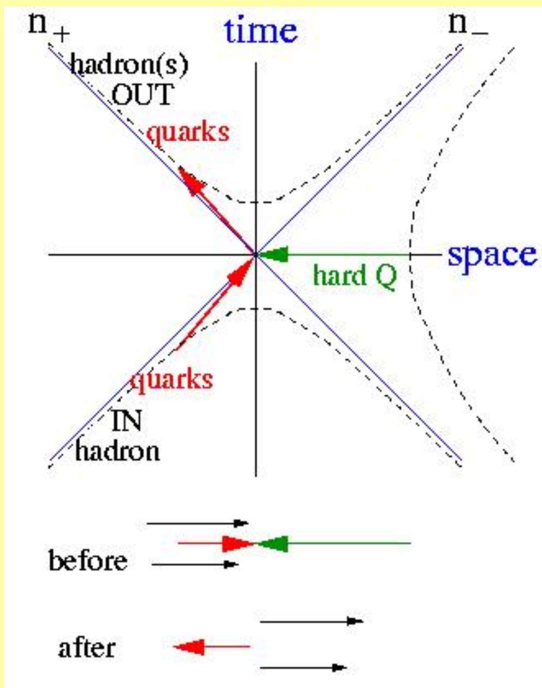
Momentum space densities of Ψ-ons:

$$\int dy\; e^{ip.y} < P \,|\, \Psi^\dagger\left(-\tfrac{y}{2}\right) \Psi\left(+\tfrac{y}{2}\right) |\, P > \; =$$

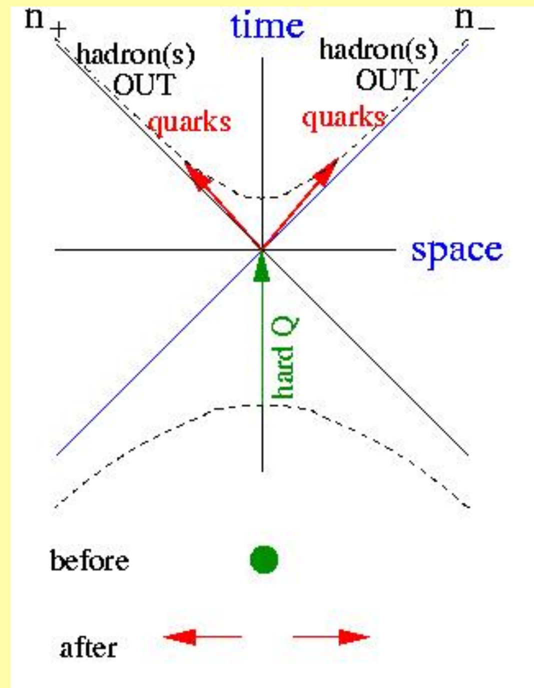
$$= \sum_X \left| < P_X \,|\, \Psi\left(0\right) |\, P > \right|^2 \delta(P_X - P + p) = f(p)$$

A hard look at the proton

- Hard virtual momenta ($\pm q^2 = Q^2 \sim$ many GeV^2) can couple to (two) soft momenta

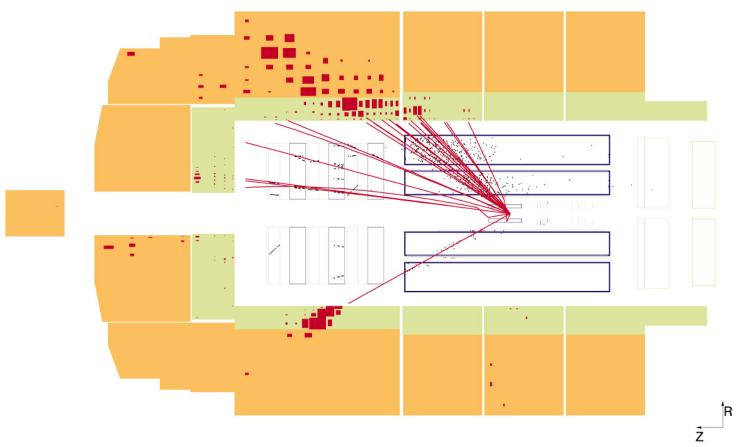


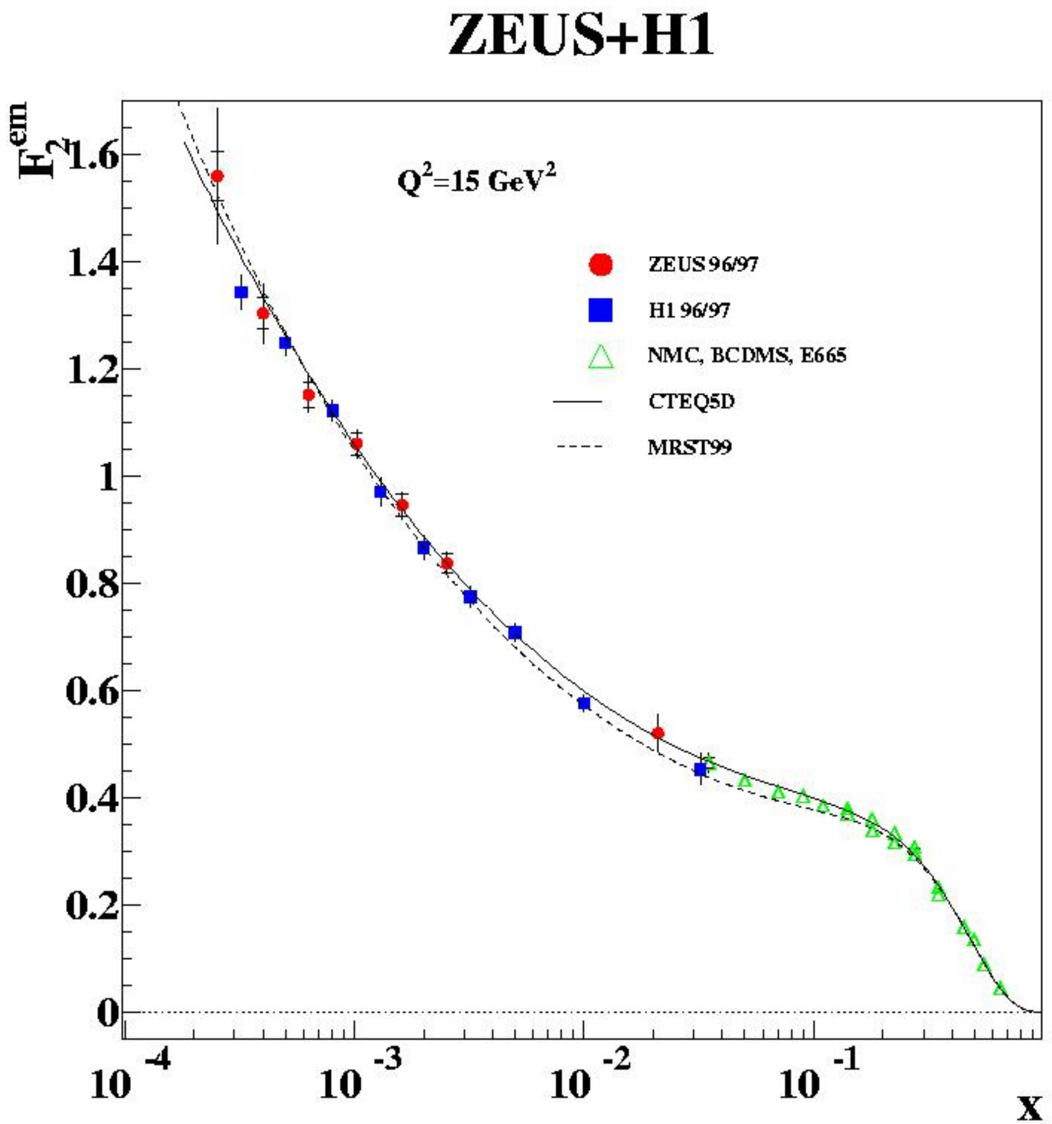
$\gamma^* + N \rightarrow$ jet



$\gamma^* \rightarrow$ jet + jet

Experiments!

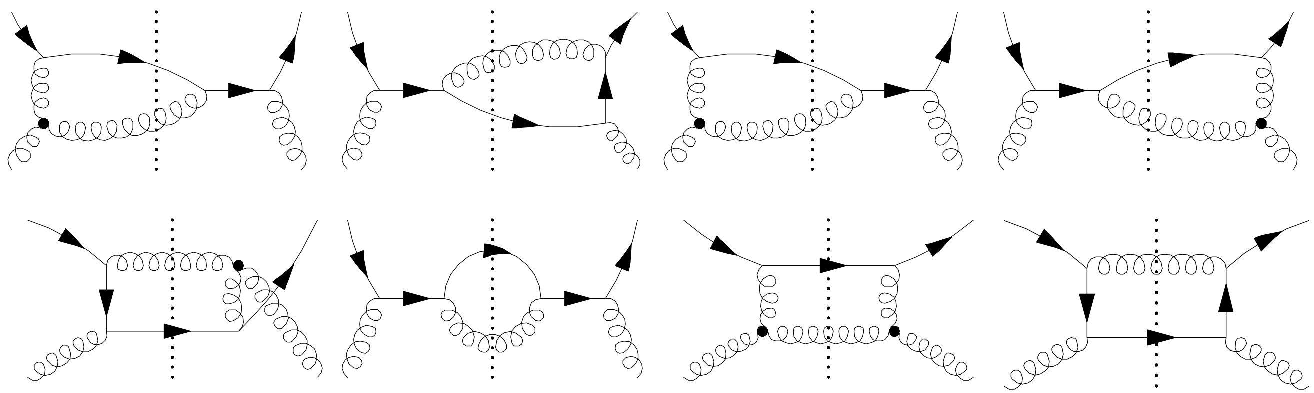




QCD & Standard Model

- QCD framework (including electroweak theory) provides the machinery to calculate cross sections, e.g. $\gamma^* q \rightarrow q$, $q\bar{q} \rightarrow \gamma^*$, $\gamma^* \rightarrow q\bar{q}$, $qq \rightarrow qq$, $qg \rightarrow qg$, etc.
- E.g.

  $qg \rightarrow qg$



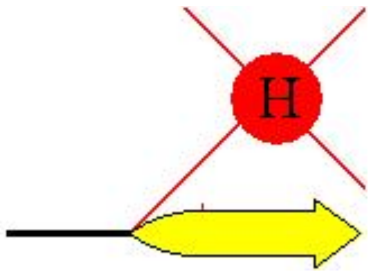
- Calculations work for plane waves

$$\left\langle 0 \left| \psi_i^{(s)}(\xi) \right| p, s \right\rangle = u_i(p,s)\, e^{-ip.\xi}$$

Soft part: hadronic matrix elements

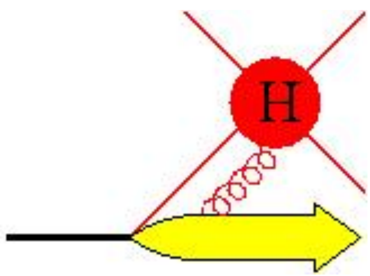
- For hard scattering process involving electrons and photons the link to external particles is, indeed, the ‘one-particle wave function’

$$\left\langle 0 \middle| \psi_i\,(\xi) \middle| p,s \right\rangle = u_i(p,s)\,e^{-ip.\xi}$$



- Confinement, however, implies hadrons as ‘sources’ for quarks

$$\left\langle X \middle| \psi_i\,(\xi) \middle| P \right\rangle e^{+ip.\xi}$$



- … and also as ‘source’ for quarks + gluons

$$\left\langle X \middle| \psi_i\,(\xi)\,A^\mu\,(\eta) \middle| P \right\rangle e^{+i(p-p_1).\xi+ip_1.\eta}$$

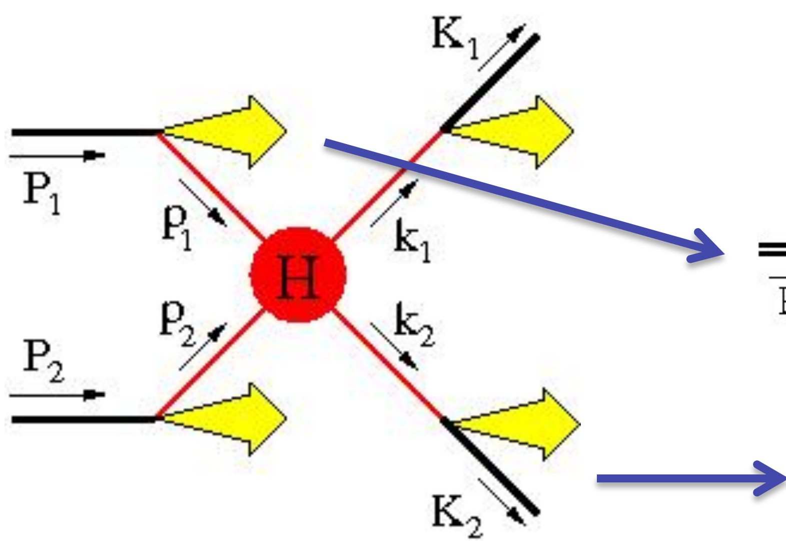
- … and also ….

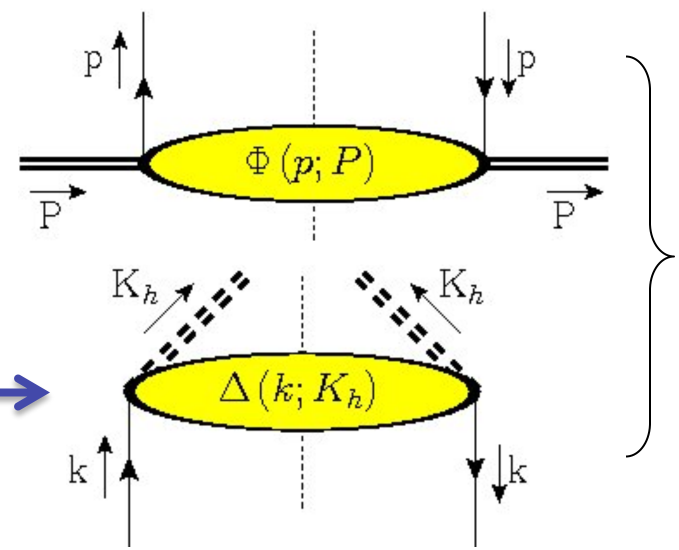
PDFs and PFFs

Basic idea of PDFs and PFFs (also for TMDs) is to obtain a full factorized description of high energy scattering processes

$$\hat{\sigma} = |\, H(p_1, p_2, \ldots)\,|^2$$

calculable





defined (!)
&
portable

$$\sigma(P_1, P_2, \ldots) = \iiint \ldots dp_1 \ldots \Phi_a(p_1, P_1; \mu) \otimes \Phi_b(p_2, P_2; \mu)$$

$$\otimes\, \hat{\sigma}_{ab,c\ldots}(p_1, p_2, \ldots; \mu) \otimes \Delta_c(k_1, K_1; \mu) \ldots.$$

Give a meaning to integration variables!

Hadron correlators

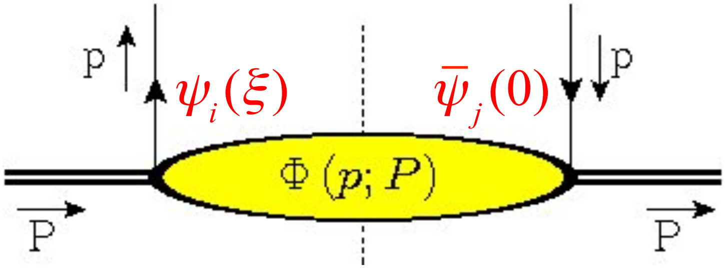
- At high energies no interference and squared amplitudes can be rewritten as correlators of forward matrix elements of parton fields
- Math: $u_i(p,s)\bar{u}_j(p,s) \;\Rightarrow\; \sum_X \langle P|\bar{\psi}_j(0)\,|\,X >< X\,|\,\psi_i(0)|P\rangle \delta(p-P+P_X)$

$$= \sum_X \int \frac{d\xi}{2\pi} \langle P|\bar{\psi}_j(0)\,|\,X >< X\,|\,\psi_i(0)|P\rangle e^{i(p-P+P_X).\xi}$$

$$= \sum_X \int \frac{d\xi}{2\pi} \langle P|\bar{\psi}_j(0)\,|\,X >< X\,|\,\psi_i(\xi)|P\rangle e^{i\,p.\xi}$$

$$= \int \frac{d\xi}{2\pi} e^{i\,p.\xi} \langle P|\bar{\psi}_j(0)\,\psi_i(\xi)|P\rangle$$

- Picture:



Use symmetries (P, T) and hermicity to parametrize these objects!

Hadron correlators

- At high energies no interference and squared amplitudes can be rewritten as correlators of matrix elements of parton fields

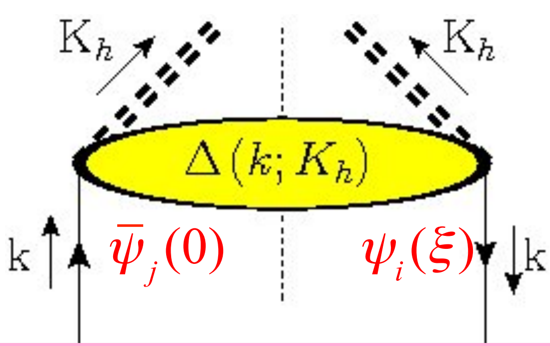
- Math: $u_i(k,s)\bar{u}_j(k,s) \;\Rightarrow\; \sum_X \langle 0|\psi_i(0)|K_h X><K_h X|\bar{\psi}_j(0)|0\rangle \delta(k-K_h-K_X)$

$$= \sum_X \int \frac{d\xi}{2\pi} \langle 0|\psi_i(0)|K_h X><K_h X|\bar{\psi}_j(0)|0\rangle e^{i(k-K_h-K_X).\xi}$$

$$= \sum_X \int \frac{d\xi}{2\pi} \langle 0|\psi_i(\xi)|K_h X><K_h X|\bar{\psi}_j(0)|0\rangle e^{ik.\xi}$$

$$= \int \frac{d\xi}{2\pi} e^{ik.\xi} \langle 0|\psi_i(\xi) a_h^+ a_h \bar{\psi}_j(0)|0\rangle$$

- Picture:



no T-constraint
$T|K_h,X>_{out} = |K_h,X>_{in}$

Collins & Soper, NP B 194 (1982) 445

Role of the hard scale

- In high-energy processes hard momenta are available, such that P.P′ ~ s with a hard scale s >> M^2

- Employ light-like vectors P and n, such that P.n = 1 (e.g. n = P′/P.P′) to make a Sudakov expansion of parton momentum (write s = Q^2)

$$p = \underbrace{x}_{\sim Q} P^\mu + \underbrace{p_T^\mu}_{\sim M} + \underbrace{\sigma}_{\sim M^2/Q} n^\mu$$

$$x = p^+ = p.n \qquad (0 \le x \le 1)$$

$$\sigma = p^- = p.P - xM^2 \sim O(M^2)$$

- Enables expansion in inverse hard scale (twist analysis) for integrated correlators,

$$\Phi(p) = \Phi(x, p_T, p.P) \;\Rightarrow\; \Phi(x, p_T) \;\Rightarrow\; \Phi(x) \;\Rightarrow\; \Phi$$

(Un)integrated correlators

$$\Phi(x,p_T,p.P) = \int \frac{d^4\xi}{(2\pi)^4}\, e^{i\,p.\xi} \left\langle P\middle|\overline{\psi}(0)\,\psi(\xi)\middle|P\right\rangle$$

- unintegrated

$$\Phi(x,p_T;n) = \int \frac{d(\xi.P)d^2\xi_T}{(2\pi)^3}\, e^{i\,p.\xi} \left\langle P\middle|\overline{\psi}(0)\,\psi(\xi)\middle|P\right\rangle_{\boxed{\xi.n=\xi^+=0}}$$

- TMD (light-front)

- σ = p⁻ integration makes time-ordering automatic. The soft part is simply sliced at the light-front

---

$$\Phi(x) = \int \frac{d(\xi.P)}{(2\pi)}\, e^{i\,p.\xi} \left\langle P\middle|\overline{\psi}(0)\,\psi(\xi)\middle|P\right\rangle_{\boxed{\xi.n=\xi_T=0}\ or\ \boxed{\xi^2=0}}$$

- Is already equivalent to a point-like interaction

- collinear (light-cone)

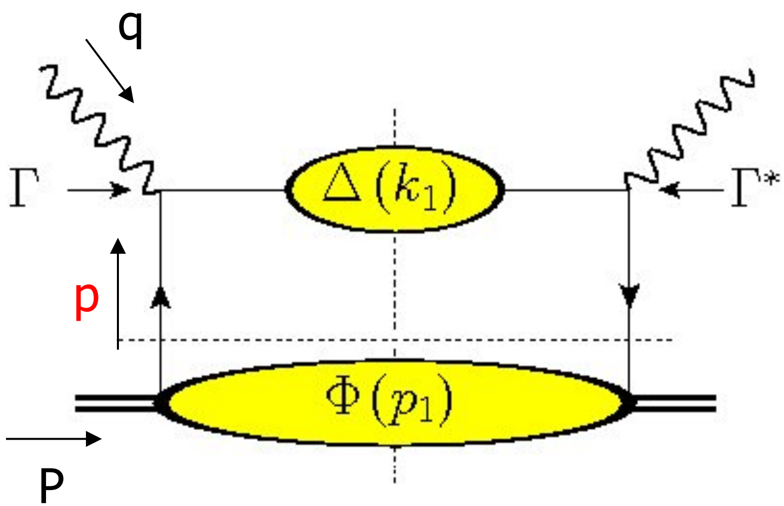
$$\Phi = \left\langle P\middle|\overline{\psi}(0)\,\psi(\xi)\middle|P\right\rangle_{\boxed{\xi=0}}$$

- local

- Local operators with calculable anomalous dimension

Example using correlators (DIS)

Principle for DIS



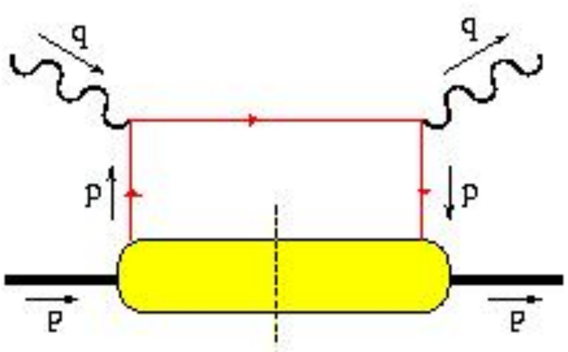
- Instead of partons use correlators

$$\sum_s u(p,s)\,\overline{u}(p,s) \;\Rightarrow\; \Phi(p,P) \qquad\qquad \Delta(k) = \not{k} + m$$

LIGHTCONE DOMINANCE IN DIS

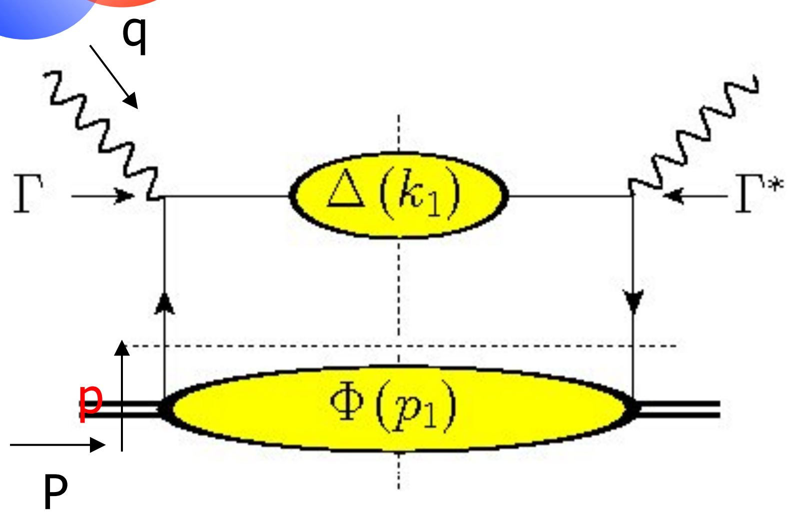
Large scale Q leads in a natural way to the use of lightlike vectors:
$n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ 2\,P\cdot q = \dfrac{Q^2}{x_B} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} P = \dfrac{x_B M^2}{Q\sqrt{2}}\,n_- + \dfrac{Q}{x_B\sqrt{2}}\,n_+ \\ \\ q = \dfrac{Q}{\sqrt{2}}\,n_- - \dfrac{Q}{\sqrt{2}}\,n_+ \end{array}\right.$$



part	'components' $-$	$+$	
HARD	$\sim Q$	$\sim Q$	
$H \to q$	$\sim 1/Q$	$\sim Q$	$\to \int dp^- d^2p_T \ldots$

Principle for DIS



- Instead of partons use correlators

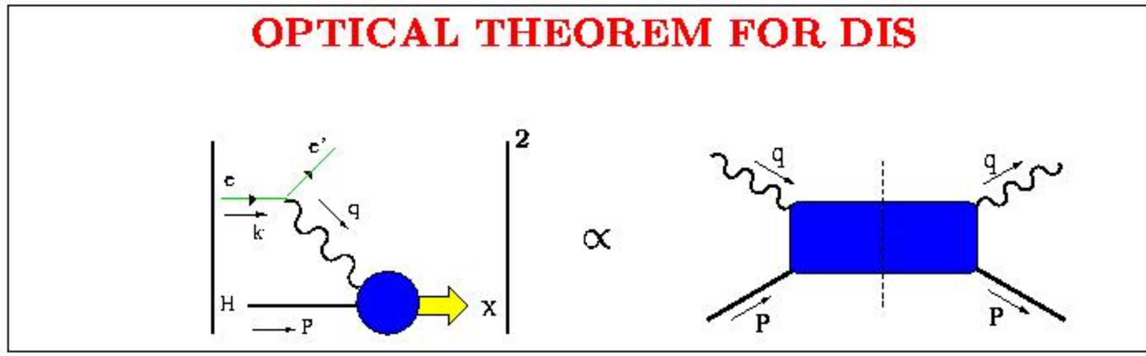
$$\sum_s u(p,s)\,\bar{u}(p,s) \;\Rightarrow\; \Phi(p,P) \qquad\qquad \Delta(k) = \not{k} + m$$

- Expand parton momenta (using P as light-like plus vector)

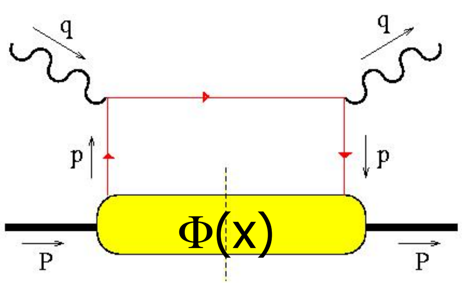
$$p = \underbrace{x P^\mu}_{\sim Q} + \underbrace{p_T^\mu}_{\sim M} + \underbrace{\sigma n^\mu}_{\sim M^2/Q} \qquad\qquad x = p^+ = p.n \sim 1$$

$$\sigma = p.P - xM^2 \sim M^2$$

(calculation of) cross section in DIS



Full calculation



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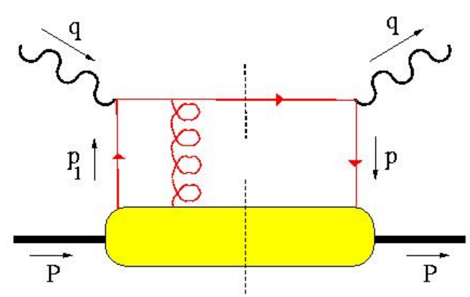

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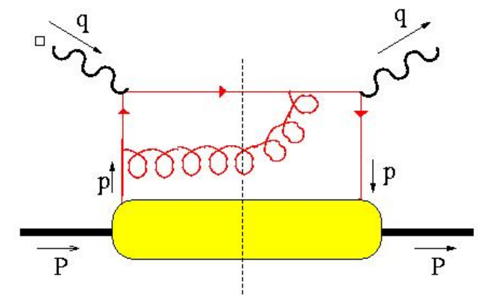
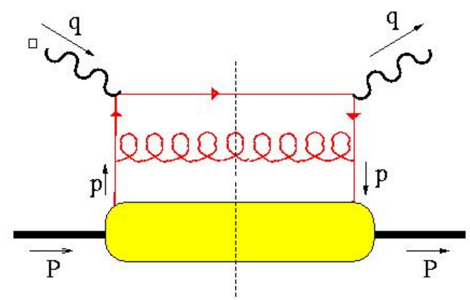


LEADING (in 1/Q)

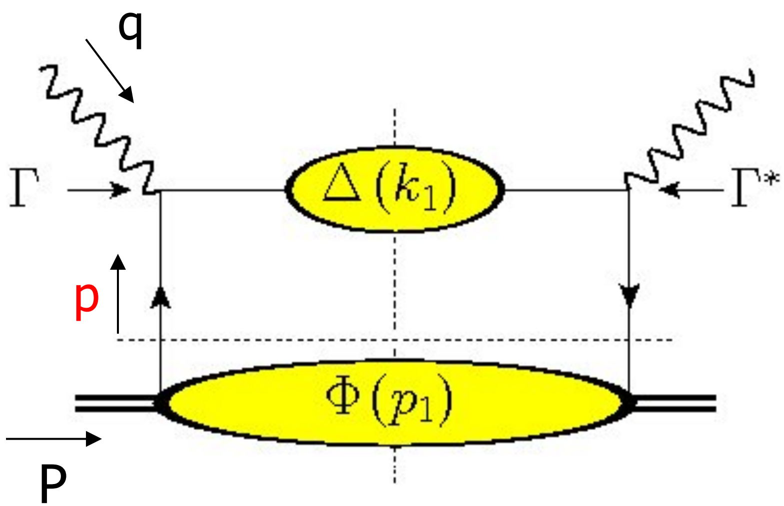
$x = x_B = -q^2/P.q$

+



+ ...

Result for DIS



$$2MW^{\mu\nu}(P,q) = -\tfrac{1}{2}\, g_T^{\mu\nu} \int dx\, dp.P\, d^2 p_T\, Tr[\Phi(p,P)\gamma^+]\delta(x - x_B)$$

$$= -\tfrac{1}{2}\, g_T^{\mu\nu}\, Tr[\Phi(x_B)\gamma^+]$$

Twist analysis (1)

- Dimensional analysis to determine importance in an expansion in inverse hard scale
- Maximize contractions with n

$$\dim[\overline{\psi}(0)\not{n}\,\psi(\xi)] = 2$$

$$\dim[F^{n\alpha}(0)F^{n\beta}(\xi)] = 2$$

$$\dim[\overline{\psi}(0)\not{n}\,A_T^{\alpha}(\eta)\,\psi(\xi)] = 3$$

- ... or maximize # of P's in parametrization of Φ

$$\boxed{\Phi^q(x) = f_1^q(x)\frac{\not{P}}{2} \Leftrightarrow f_1^q(x) = \int \frac{d\lambda}{(2\pi)} e^{ix\lambda}\left\langle P\middle|\overline{\psi}(0)\not{n}\,\psi(\lambda n)\middle|P\right\rangle}$$

- Note that these are densities!

$$\boxed{\overline{\psi}(0)\not{n}\,\psi(\lambda n) = \psi_+^{+}(0)\psi_+(\lambda n)}$$

Parametrization of TMDs

Ingredients in parametrization

- Building blocks: momenta and spins
- Handling of spin in distributions (spin of hadrons can be tuned)
- Handling of spin in fragmentations (spin of produced hadrons cannot be tuned!)
- Color summation in distribution functions
- Color averaging in fragmentation functions

Symmetry constraints

$$\Phi^{T*}(p;P,S) = \gamma_0 \Phi(p;P,S)\gamma_0$$

- Hermiticity

$$\Phi(p;P,S) = \gamma_0 \Phi(\bar{p};\bar{P},-\bar{S})\gamma_0$$

- Parity

$$\Phi^{[U]}(p;P,S) = (-i\gamma_5 C)\Phi^{[-U]}(\bar{p};\bar{P},\bar{S})(-i\gamma_5 C)$$

- Time reversal

$$\Phi^{c}(p;P,S) = C\,\Phi^{T}(-p;P,S)\,C$$

- Charge conjugation (giving antiquark corr)

Parametrization of TMD correlator for unpolarized hadron:

$$\boxed{\Phi^{[\pm]q}(x,p_T) = \left(f_1^q(x,p_T^2) \pm i h_1^{\perp q}(x,p_T^2)\frac{\not{p}_T}{M}\right)\frac{\not{P}}{2}}$$

(unpolarized and transversely polarized quarks)

T-even T-odd

Mulders, Tangerman, Boer; Bacchetta, Diehl, Goeke, Metz, M, Schlegel, JHEP02 (2007) 093

New information in TMD's: f(x,p_T) or D(1/z,k_T)

- Quarks in polarized nucleon: $S = S_L\left(\frac{P}{M} + Mn\right) + S_T$ $\quad S_L^2 + S_T^2 = -1$

$$\Phi^q(p;P,S) \propto x f_1^q(x,p_T^2)\,\not{P} + S_L\,x g_{1L}^q(x,p_T^2)\,\not{P}\gamma_5 + x h_{1T}^q(x,p_T^2)\not{S}_T\not{P}\gamma_5 + \ldots$$

unpolarized quarks

T-polarized quarks in T-polarized N

chiral quarks in L-polarized N

compare

$$u(p,s)\bar{u}(p,s) = \tfrac{1}{2}(\not{p} + m)(1 + \gamma_5\not{s})$$

- … but also

$$\Phi^q(p;P,S) \propto \ldots + \frac{(p_T \cdot S_T)}{M} x g_{1T}^q(x,p_T^2)\not{P}\gamma_5 + \ldots$$

spin ←→ spin

chiral quarks in T-polarized N

New information in TMD's: $f(x,p_T)$ or $D(1/z,k_T)$

- … and T-odd functions

$$\Phi^q(p;P,S) \propto \ldots + i h_1^{\perp q}(x,p_T^2)\frac{\not p_T}{M}\not P + i\frac{(p_T \times S_T)}{M} x f_{1T}^{\perp q}(x,p_T^2)\not P + \ldots$$

T-polarized quarks in unpolarized N (Boer-Mulders)

unpolarized quarks in T-polarized N (Sivers)

spin ←→ orbit

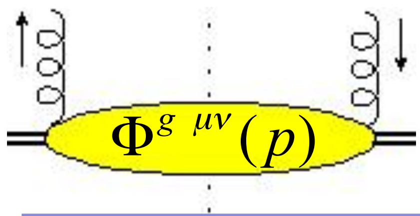
compare

$$u(p,s)\bar{u}(p,s) = \tfrac{1}{2}(\not p + m)(1+\gamma_5 \not s)$$

- Yes, definitely there is new information and even very interesting spin-orbit correlations (single spin!). These are T-odd and because of T-conservation show up in T-odd observables, such as single spin asymmetries, e.g. left-right asymmetry in $p(P_1)p_\uparrow(P_2) \to \pi(K)X$

New information in gluon TMD's: $f(x,p_T)$ or $D(1/z,k_T)$

- Also for gluons there are new features in TMD's



circularly polarized gluons in L-pol. N

spin ←→ spin

$$\Phi^{g\ \mu\nu}(p;P,S) \propto -g_T^{\mu\nu}\, x f_1^{\,g}(x,p_T^2) + iS_L \varepsilon_T^{\mu\nu}\, x g_{1L}^{\,g}(x,p_T^2) + \left(\frac{p_T^\mu p_T^\nu}{M^2} - g_T^{\mu\nu}\frac{p_T^\mu}{2M^2}\right) x h_1^{\perp g}(x,p_T^2) + \ldots$$

unpolarized gluons in unpol. N quarks

linearly polarized gluons in unpol. N (Gluon Boer-Mulders)

spin ←→ orbit

compare

$$\varepsilon^\mu(p,\lambda)\varepsilon^{\nu *}(p,\lambda) = -g_T^{\mu\nu} + \ldots$$

Basis of partons

- 'Good part' of Dirac space is 2-dimensional
- Interpretation of DF's

TWO 'SPIN' STATES FOR (GOOD) QUARK FIELDS

chiral eigenstates:

$$\psi_{R/L} \equiv \frac{1}{2}(1 \pm \gamma_5)\psi : \qquad |R\rangle \quad \text{and} \quad |L\rangle$$

or

transverse spin eigenstates:

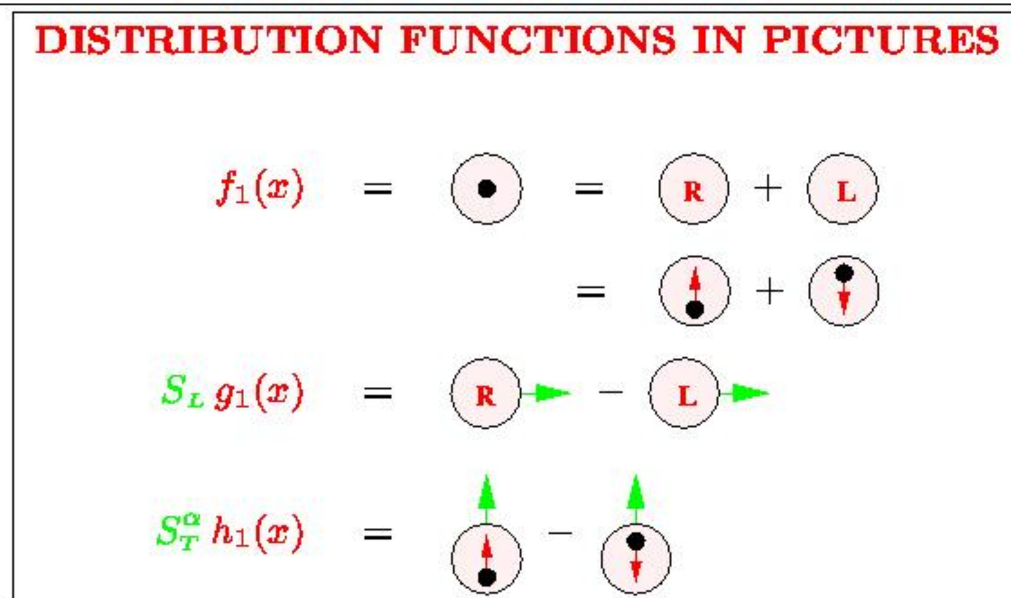
$$\psi_{\uparrow/\downarrow} \equiv \frac{1}{2}(1 \pm \gamma^\alpha \gamma_5)\psi : \qquad |\uparrow\rangle \quad \text{and} \quad |\downarrow\rangle$$

Note: $[\mathcal{P}_{R/L}, \mathcal{P}_+] = [\mathcal{P}_{\uparrow/\downarrow}, \mathcal{P}_+] = 0$

unpolarized quark distribution

helicity or chirality distribution

transverse spin distr. or transversity



$= h_1(x)\, S_T^\alpha$

Matrix representation for M = $[\Phi(x)\gamma^{+}]^{T}$

Bacchetta, Boglione, Henneman & Mulders
PRL 85 (2000) 712

Quark production matrix, directly related to the helicity formalism

Anselmino et al.

MATRIX REPRESENTATION FOR SPIN 1/2

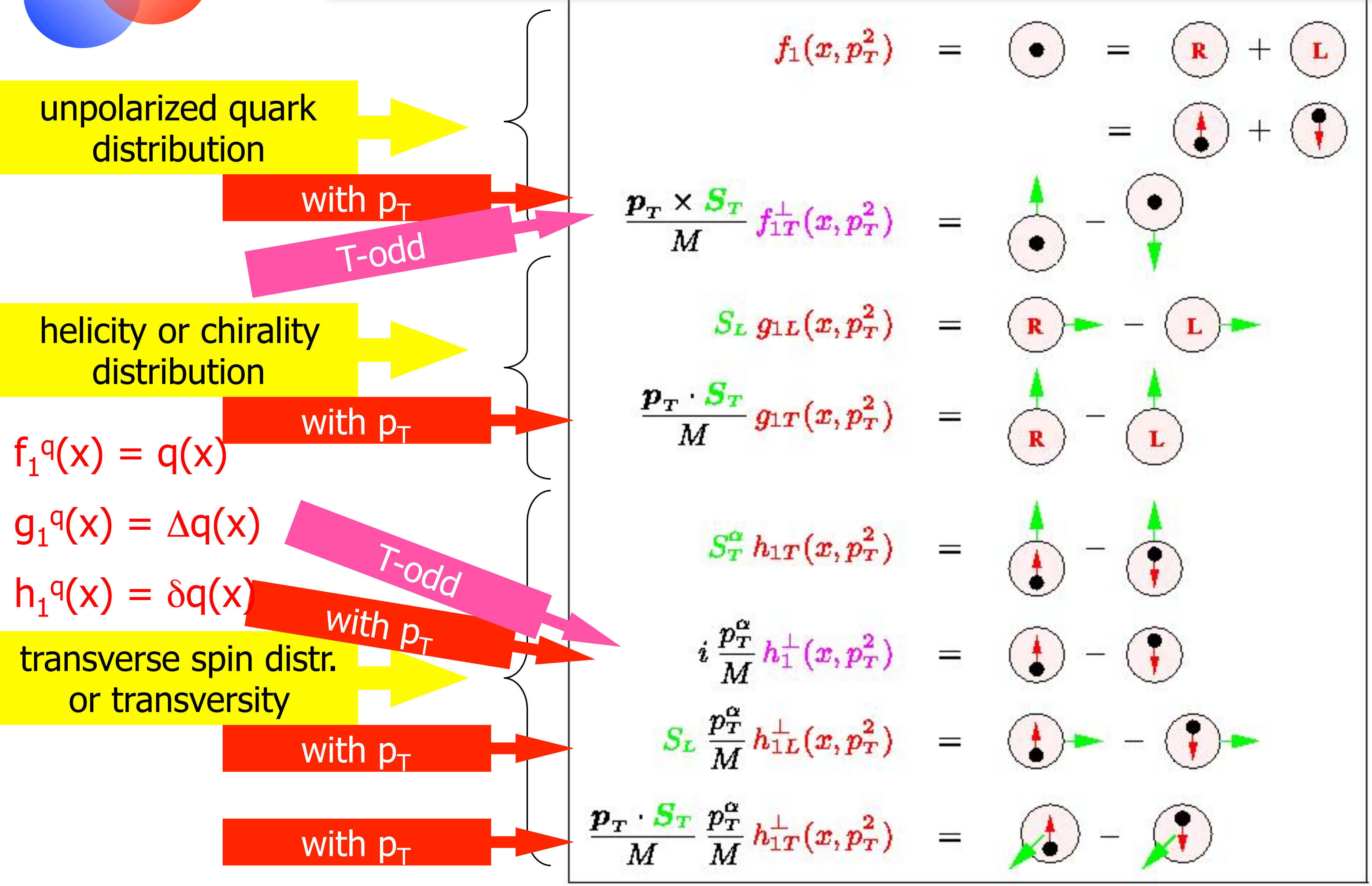
p_T-integrated distribution functions:

For a spin 1/2 hadron (e.g. nucleon) the quark production matrix in quark⊗nucleon spin space is given by

$$M^{(\mathrm{prod})} = \begin{array}{cc} \begin{pmatrix} f_1+g_1 & 0 & 0 & 2\,h_1 \\ 0 & f_1-g_1 & 0 & 0 \\ 0 & 0 & f_1-g_1 & 0 \\ 2\,h_1 & 0 & 0 & f_1+g_1 \end{pmatrix} & \begin{array}{c} \mathrm{R}\rightarrow \\ \leftarrow\mathrm{R} \\ \mathrm{L}\rightarrow \\ \leftarrow\mathrm{L} \end{array} \\ \begin{array}{cccc} \mathrm{R}\rightarrow & \leftarrow\mathrm{R} & \mathrm{L}\rightarrow & \leftarrow\mathrm{L} \end{array} & \end{array}$$

- Off-diagonal elements (RL or LR) are chiral-odd functions
- Chiral-odd soft parts must appear with partner in e.g. SIDIS, DY

Fermionic structure of TMDs



$f_1^q(x) = q(x)$

$g_1^q(x) = \Delta q(x)$

$h_1^q(x) = \delta q(x)$

Matrix representation for M = $[\Phi^{[\pm]}(x,p_T)\gamma^+]^T$

- p_T-dependent functions

MATRIX REPRESENTATION FOR SPIN 1/2

p_T-dependent quark distributions:

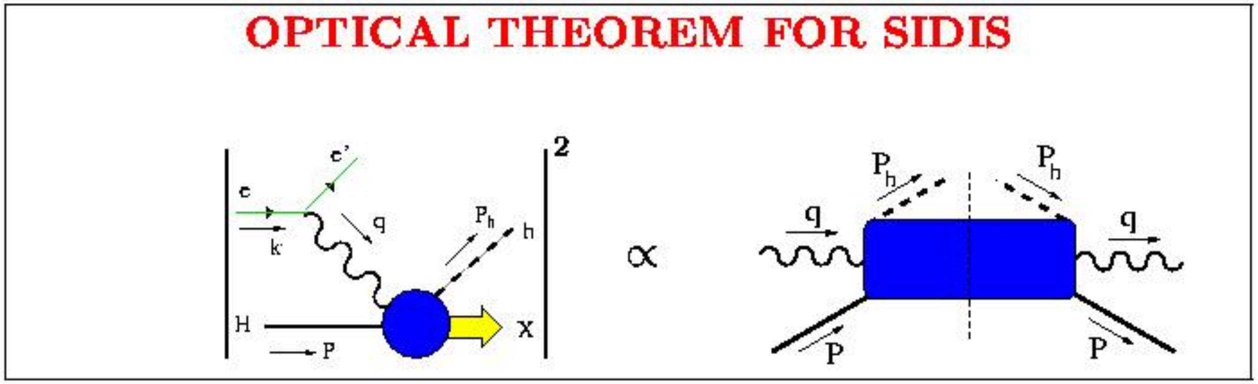
$$\begin{pmatrix} f_1+g_{1L} & \frac{|p_T|}{M}\,e^{i\phi}\,g_{1T} & \frac{|p_T|}{M}\,e^{-i\phi}\,h_{1L}^\perp & 2\,h_1 \\ \frac{|p_T|}{M}\,e^{-i\phi}\,g_{1T} & f_1-g_{1L} & \frac{|p_T|^2}{M^2}\,e^{-2i\phi}\,h_{1T}^\perp & -\frac{|p_T|}{M}\,e^{-i\phi}\,h_{1L}^\perp \\ \frac{|p_T|}{M}\,e^{i\phi}\,h_{1L}^\perp & \frac{|p_T|^2}{M^2}\,e^{2i\phi}\,h_{1T}^\perp & f_1-g_{1L} & -\frac{|p_T|}{M}\,e^{i\phi}\,g_{1T} \\ 2\,h_1 & -\frac{|p_T|}{M}\,e^{i\phi}\,h_{1L}^\perp & -\frac{|p_T|}{M}\,e^{-i\phi}\,g_{1T} & f_1+g_{1L} \end{pmatrix}$$

T-odd: $g_{1T} \to g_{1T} - i\, f_{1T}^\perp$ and $h_{1L}^\perp \to h_{1L}^\perp + i\, h_1^\perp$ (imaginary parts)

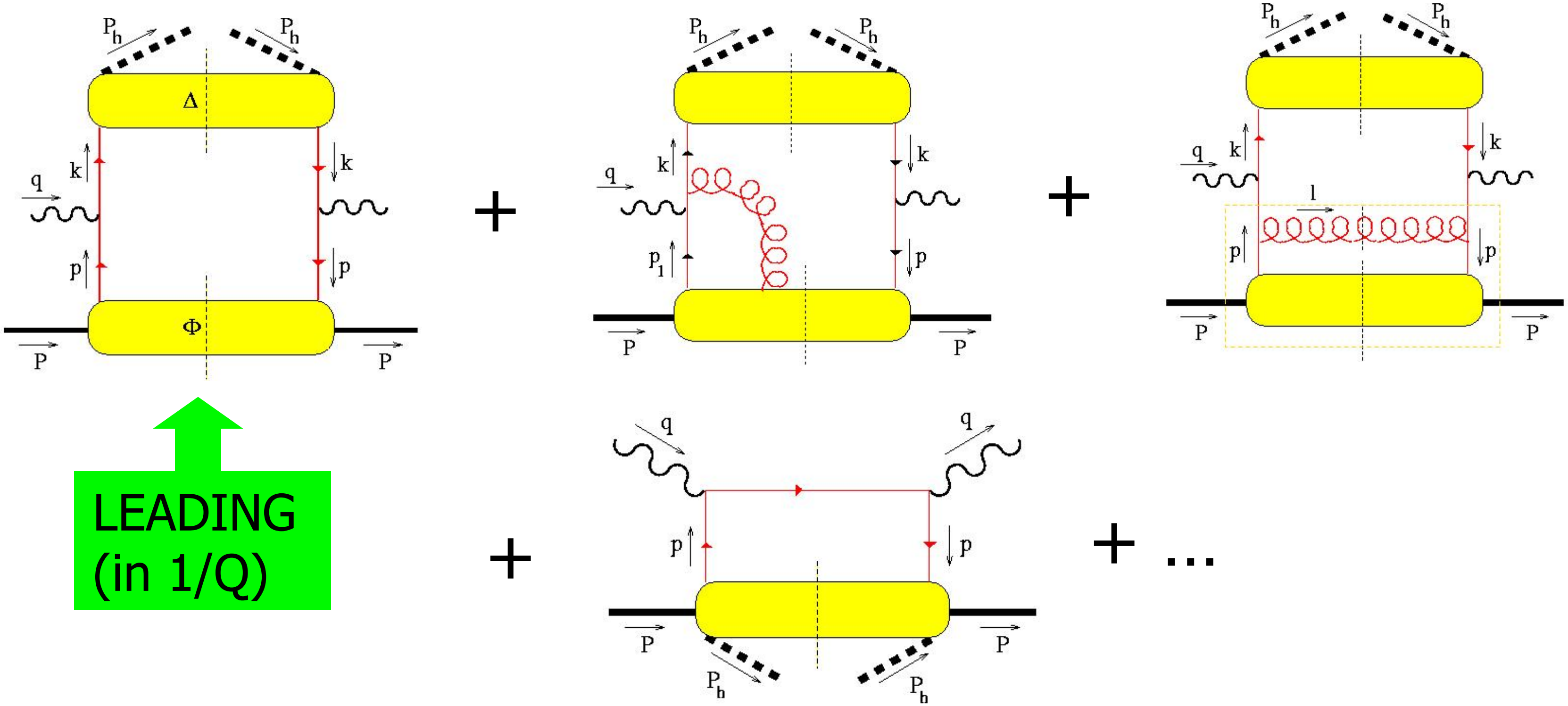
Bacchetta, Boglione, Henneman & Mulders
PRL 85 (2000) 712

Example using TMDs (SIDIS)

(calculation of) cross section in SIDIS



Full calculation

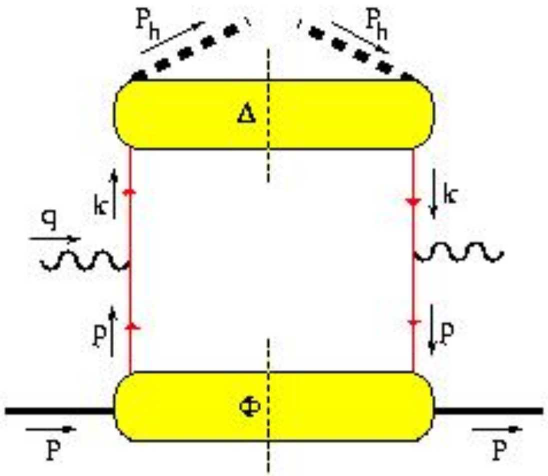


LEADING
(in 1/Q)

LIGHTCONE DOMINANCE IN SIDIS

Large scale Q leads in a natural way to the use of lightlike vectors:
$n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ P_h^2 = M_h^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \\ 2\,P_h\cdot q = -z_h\,Q^2 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} P_h = \frac{z_h\,Q}{\sqrt{2}}\,n_- + \frac{M_h^2}{z_h\,Q\sqrt{2}}\,n_+ \\ q = \frac{Q}{\sqrt{2}}\,n_- - \frac{Q}{\sqrt{2}}\,n_+ + q_T \\ P = \frac{x_B\,M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right.$$



Three external momenta

$P \quad P_h \quad q$

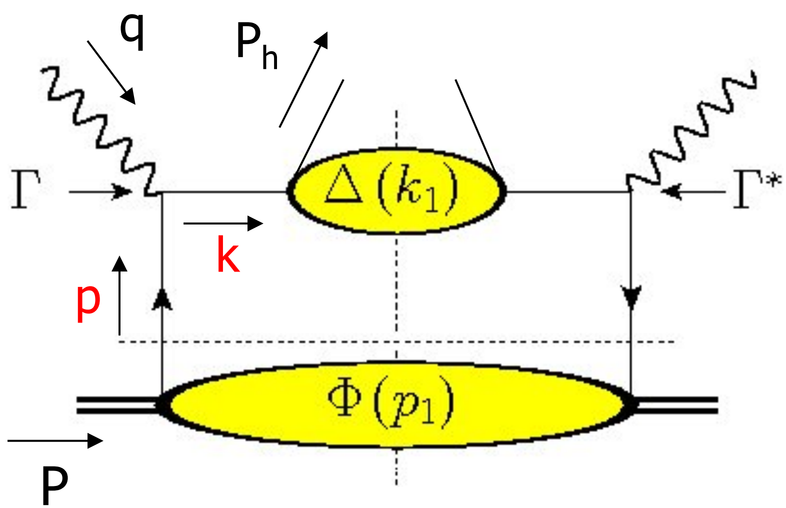
transverse directions relevant

$q_T = q + x_B\, P - P_h/z_h$

or

$q_T = -P_{h\perp}/z_h$

Result for SIDIS



$$q_T = q + x_B P - \frac{P_h}{z_h}$$

$$2MW^{\mu\nu}(P, P_h, q) = \int d^2 p_T \int d^2 k_T$$

$$\times\, Tr[\Phi(x_B, p_T)\gamma^\mu \Delta(z_h, k_T)\gamma^\mu]\,\delta^2(p_T + q_T - k_T)$$

$$= -\tfrac{1}{2} g_T^{\mu\nu} \int d^2 p_T \int d^2 k_T$$

$$\times\, Tr[\Phi(x_B, p_T)\gamma^+]\, Tr[\Delta(z_h, k_T)\gamma^-]\,\delta^2(p_T + q_T - k_T)$$

relevance and measurability of TMDs

Transverse momentum dependence

- Mismatch of hadronic and partonic momenta

$$p - xP = p_T + \ldots = -xP_\perp + \ldots$$

$$k - \tfrac{1}{z}K_h = k_T + \ldots = -\tfrac{1}{z}K_{h\perp} + \ldots$$

- Momentum fractions are linked to scaling variables, e.g. SIDIS (up to $1/Q^2$ corrections):

$$x = p.n / P.n = Q^2 / 2P.q = x_B$$

$$z = K.n / k.n = P.K / P.q = z_h$$

- Transverse momenta are convoluted into a measurable off-collinearity,

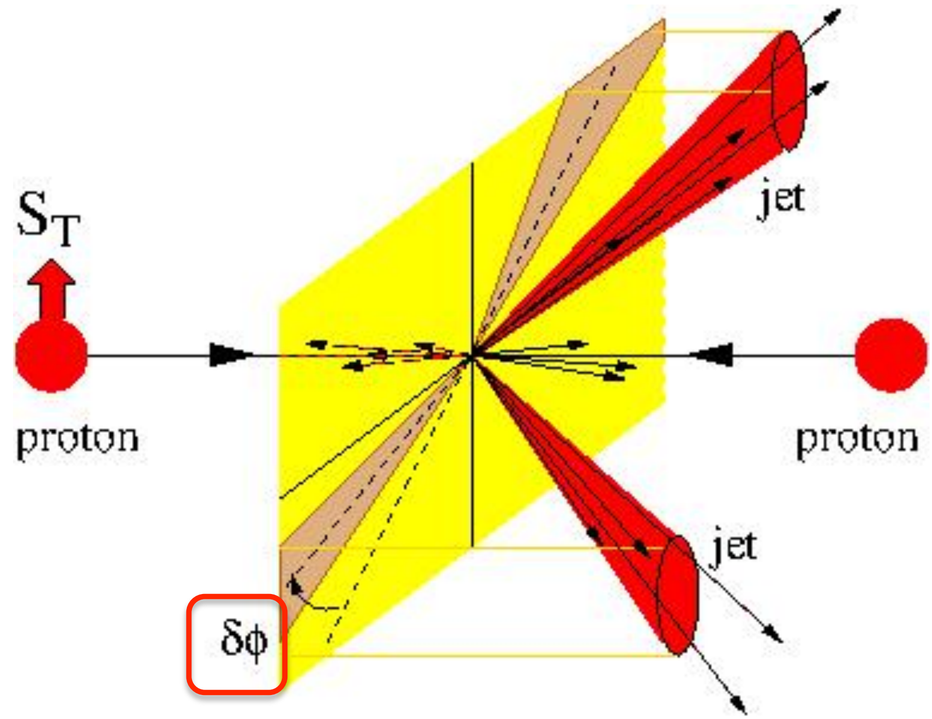
$$\boxed{q_T = q + x_B P - z_h^{-1} K = k_T - p_T}$$

- … or non-alignment of jets in hadron + hadron → jet + jet.

Access to transverse momenta

- Also in more complex situations like hadron-hadron collisions

$$p_1 \approx x_1 P_1 + p_{1T}$$
$$p_2 \approx x_2 P_2 + p_{2T}$$



$$x_1 = p_1.n = \frac{p_1.P_2}{P_1.P_2} = \frac{(k_1 + k_2).P_2}{P_1.P_2}$$

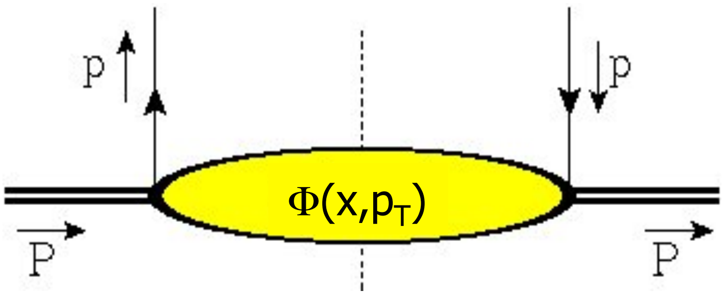
$$q_T = k_{jet,1} + k_{jet,2} - x_1 P_1 - x_2 P_2$$
$$= p_{1T} + p_{2T}$$

Second scale!

Large pT

Large p_T

- p_T-dependence of TMDs



$$\int^{\mu} d^2p_T\, \Phi(x,p_T) = \Phi(x;\mu^2)$$

Fictitious measurement

Large m[2] dependence governed by anomalous dim (i.e. splitting functions)

- $$\Phi(x,p_T) \xrightarrow[p_T^2 > \mu^2]{} \frac{1}{\pi p_T^2} \frac{\alpha_s(p_T^2)}{2\pi} \int_x^1 \frac{dy}{y} P\left(\frac{x}{y}\right) \Phi(y;p_T^2)$$

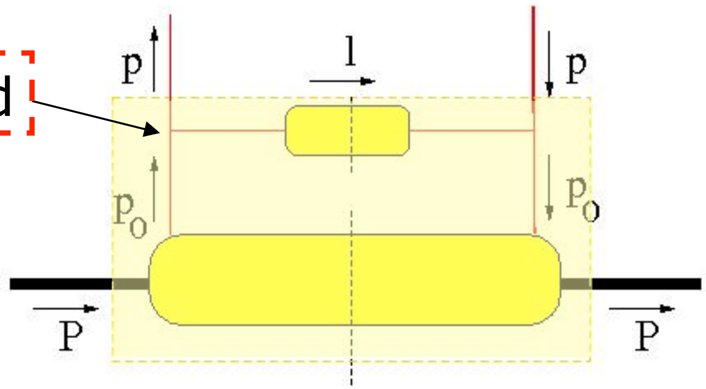
- Consistent matching to collinear situation: CSS formalism

JC Collins, DE Soper and GF Sterman, NP B 250 (1985) 199

Large values of momenta

- Calculable!

hard



$$p^2 \approx \frac{p_T^2 - x_p M_1^2}{1-x_p} < 0$$

$$p.P \approx \frac{x_p (p_T^2 - M_1^2)}{2x(1-x_p)} < 0$$

$$M_R^2 \approx \frac{(x-x_p)p_T^2 + x_p(1-x)M_1^2}{x(1-x_p)} > 0$$

$$p_0 \approx \frac{x}{x_p} P + p_{0T} \qquad (x \le x_p \le 1)$$

$p_{0T} \sim M$

$$l_T \approx -p_T$$

$M << p_T < Q$

$$\Phi(p,P) \rightarrow \frac{\alpha_s}{p_T^2} \ldots \qquad \text{etc.}$$

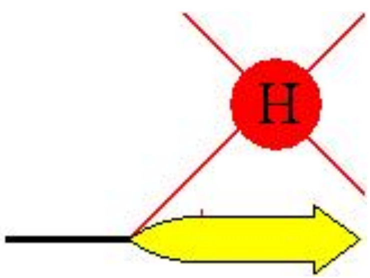
Bacchetta, Boer, Diehl, M
JHEP 0808:023, 2008 (arXiv:0803.0227)

Complications for TMDs

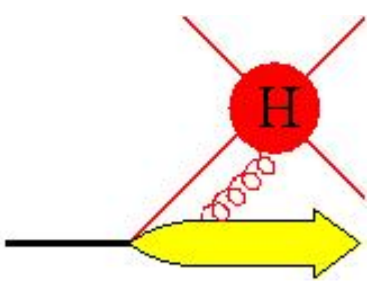
Hadron correlators

- Hadronic correlators establish the diagrammatic link between hadrons and partonic hard scattering amplitude
- Quark, quark + gluon, gluon, …

$$\left\langle 0 \left| \psi_i(\xi) \right| p,s \right\rangle = u_i(p,s)\, e^{-ip.\xi}$$

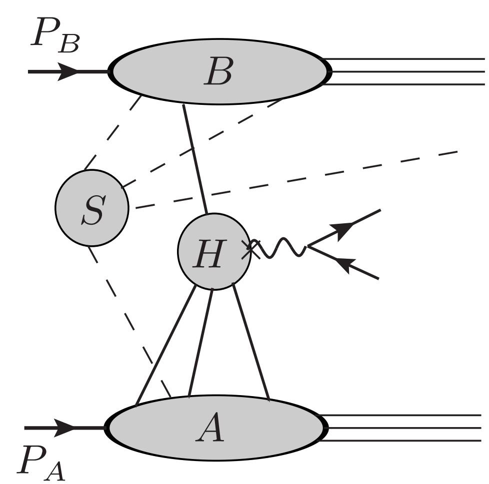


$$\left\langle X \left| \psi_i(\xi) \right| P \right\rangle e^{+ip.\xi}$$



$$\left\langle X \left| \psi_i(\xi) A^\mu(\eta) \right| P \right\rangle e^{+i(p-p_1).\xi + ip_1.\eta}$$

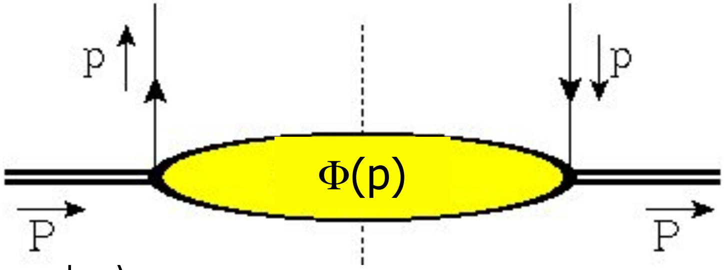
- Disentangling a hard process into collinear parts involving hadrons, hard scattering amplitude and soft factors is non-trivial



J.C. Collins, Foundations of Perturbative QCD, Cambridge Univ. Press 2011

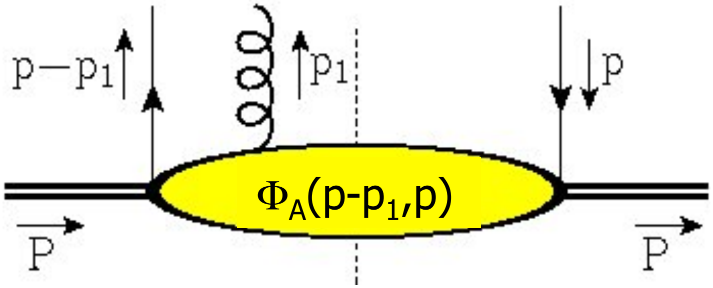
Soft part: hadron correlators

- Forward matrix elements of parton fields describe distribution (and fragmentation) parts



$$\Phi_{ij}(p;P) = \Phi_{ij}(p\,|\,p) = \int \frac{d^4\xi}{(2\pi)^4}\, e^{i\,p.\xi} \left\langle P \middle| \bar{\psi}_j(0)\, \psi_i(\xi) \middle| P \right\rangle$$

- Also needed are multi-parton correlators



$$\Phi^{\alpha}_{A;ij}(p-p_1, p_1\,|\,p) = \int \frac{d^4\xi\, d^4\eta}{(2\pi)^8}\, e^{i(p-p_1).\xi + ip_1.\eta} \left\langle P \middle| \bar{\psi}_j(0)\, A^{\alpha}(\eta)\, \psi_i(\xi) \middle| P \right\rangle$$

Color gauge invariance

- Gauge invariance in a non-local situation requires a gauge link U(0,ξ)

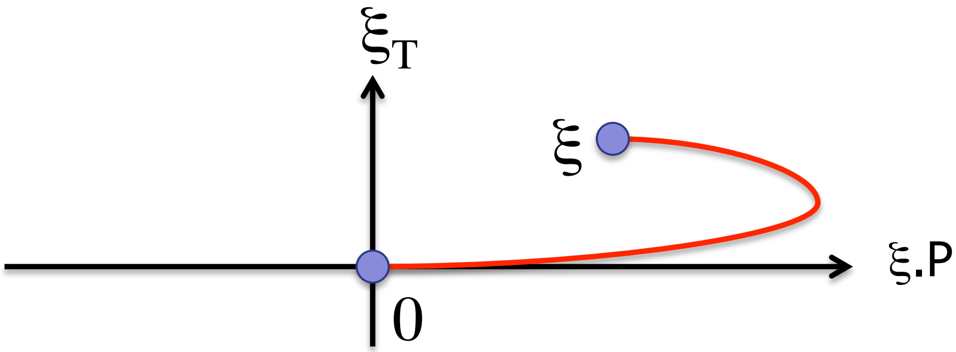
$$\bar{\psi}(0)\psi(\xi) = \sum_n \frac{1}{n!} \xi^{\mu_1} \ldots \xi^{\mu_N} \bar{\psi}(0)\,\partial_{\mu_1} \ldots \partial_{\mu_N} \psi(0)$$

$$U(0,\xi) = \mathcal{P}\,\exp\left(-ig\int_0^\xi ds^\mu A_\mu\right)$$

$$\bar{\psi}(0)U(0,\xi)\psi(\xi) = \sum_n \frac{1}{n!} \xi^{\mu_1} \ldots \xi^{\mu_N} \bar{\psi}(0)\,D_{\mu_1} \ldots D_{\mu_N} \psi(0)$$

- Introduces path dependence for $\Phi(x,p_T)$

$$\Phi^{[U]}(x,p_T) \;\Rightarrow\; \Phi(x)$$



Twist analysis (2)

- Dimensional analysis to determine importance in an expansion in inverse hard scale
- Maximize contractions with n

$$\dim[\overline{\psi}(0)\not{n}\,\psi(\xi)] = 2$$

$$\dim[F^{n\alpha}(0)F^{n\beta}(\xi)] = 2$$

$$\dim[\overline{\psi}(0)\not{n}\,A_T^{\alpha}(\eta)\,\psi(\xi)] = 3$$

- … or maximize # of P's in parametrization of Φ

$$\boxed{\Phi^q(x) = f_1^q(x)\frac{\not{P}}{2} \;\Leftrightarrow\; f_1^q(x) = \int \frac{d\lambda}{(2\pi)} e^{ix\lambda} \left\langle P \middle| \overline{\psi}(0)\not{n}\,\psi(\lambda n) \middle| P \right\rangle}$$

- In addition any number of collinear n.A(ξ) = $A^n(x)$ fields (dimension zero!), but of course in color gauge invariant combinations

dim 0: $i\partial^n \longrightarrow iD^n = i\partial^n + gA^n$

dim 1: $i\partial_T^{\alpha} \longrightarrow iD_T^{\alpha} = i\partial_T^{\alpha} + gA_T^{\alpha}$

Which gauge links?

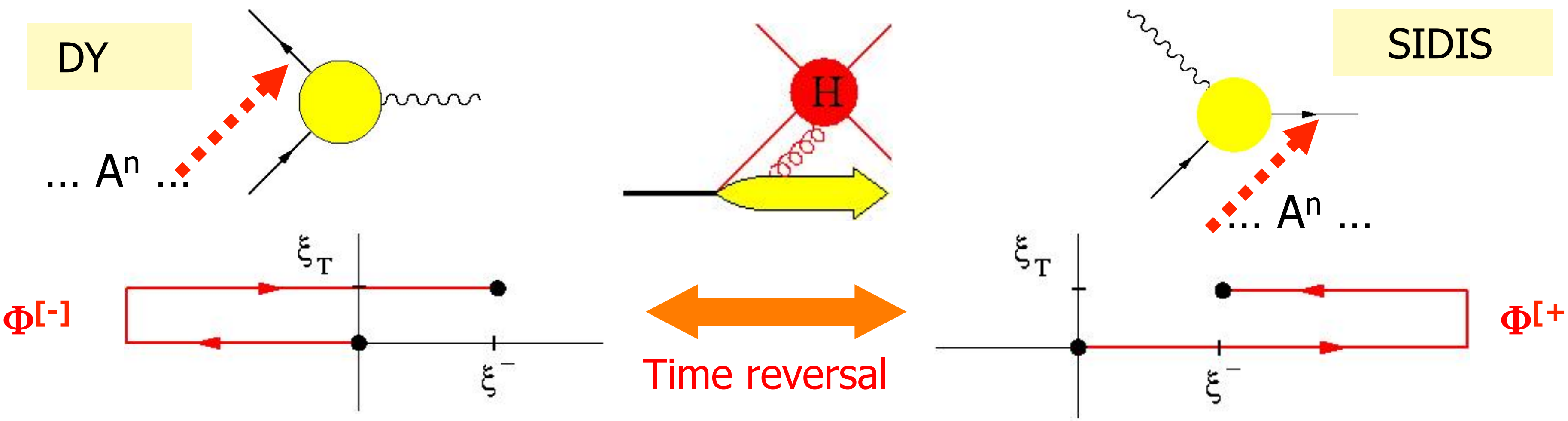
$$\Phi_{ij}^{q[C]}(x, p_T; n) = \int \frac{d(\xi.P)\, d^2\xi_T}{(2\pi)^3}\, e^{i\,p.\xi} \left\langle P \right| \bar{\psi}_j(0)\, U_{[0,\xi]}^{[C]}\, \psi_i(\xi) \left| P \right\rangle_{\xi.n=0}$$

TMD

$$\Phi_{ij}^{q}(x; n) = \int \frac{d(\xi.P)}{(2\pi)}\, e^{i\,p.\xi} \left\langle P \right| \bar{\psi}_j(0)\, U_{[0,\xi]}^{[n]}\, \psi_i(\xi) \left| P \right\rangle_{\xi.n=\xi_T=0}$$

collinear

- Gauge links come from dimension zero (not suppressed!) collinear A.n gluons, but leads for TMD correlators to process-dependence:



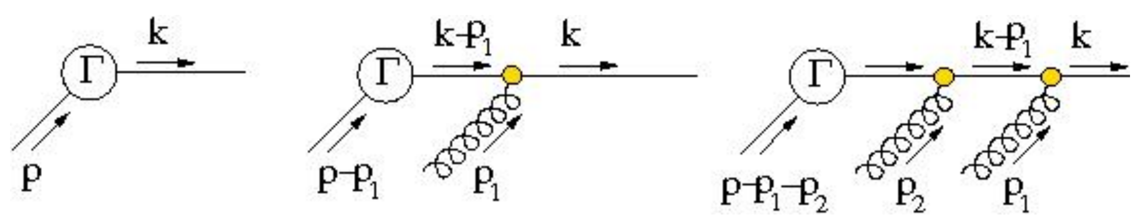
AV Belitsky, X Ji and F Yuan, NP B 656 (2003) 165
D Boer, PJM and F Pijlman, NP B 667 (2003) 201

Some details on the gauge links (1)

- Proper gluon fields (F rather than A, Wilson lines and boundary terms)

$$A^\mu(p_1) = n.A(p_1)\,\frac{P^\mu}{n.P} + i\,A_T^\mu(p_1) + \ldots \;=\; \frac{1}{p_1.n}\left[n.A(p_1)\;p_1^\mu + iG_T^{n\mu}(p_1) + \ldots\right]$$

- Resummation of soft n.A gluons (coupling to outgoing color-line) for one correlator produces a gauge-line (along n)



- Boundary terms give transverse pieces

Which gauge links?

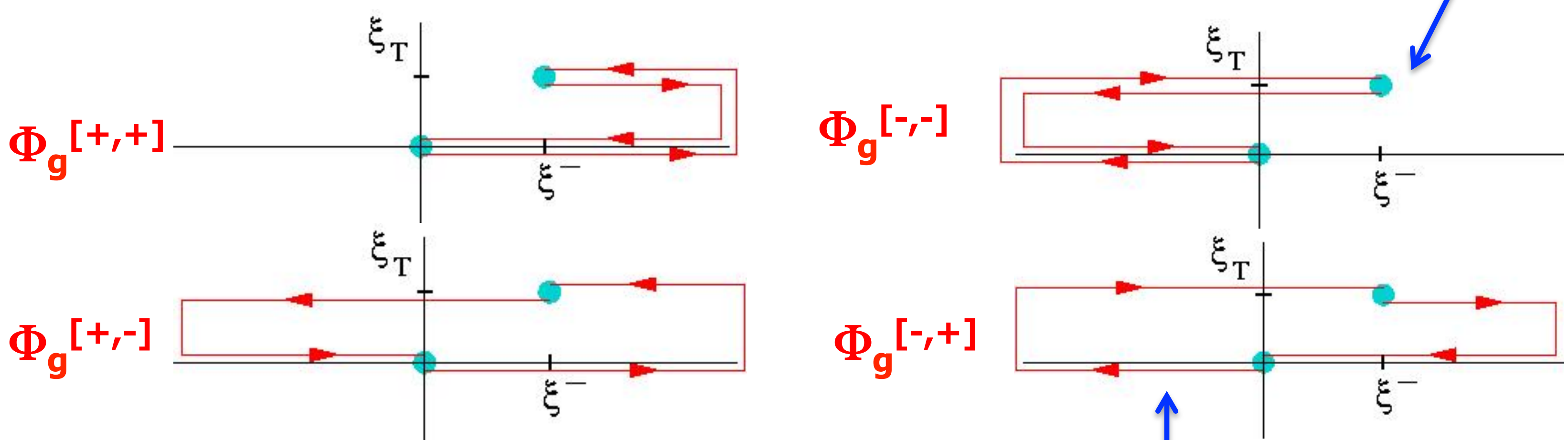
$$\Phi_g^{\alpha\beta[C,C']}(x,p_T;n) = \int \frac{d(\xi.P)\,d^2\xi_T}{(2\pi)^3}\, e^{i\,p.\xi} \left\langle P \left| U^{[C]}_{[\xi,0]} F^{n\alpha}(0)\, U^{[C']}_{[0,\xi]} F^{n\beta}(\xi) \right| P \right\rangle_{\xi.n=0}$$

- The TMD gluon correlators contain two links, which can have different paths. Note that standard field displacement involves C = C'

$$F^{\alpha\beta}(\xi) \rightarrow U^{[C]}_{[\eta,\xi]} F^{\alpha\beta}(\xi)\, U^{[C]}_{[\xi,\eta]}$$

- Basic (simplest) gauge links for gluon TMD correlators:

gg → H



in gg → $Q\bar{Q}$

C Bomhof, PJM, F Pijlman; EPJ C 47 (2006) 147
F Dominguez, B-W Xiao, F Yuan, PRL 106 (2011) 022301

Summarizing: color gauge invariant correlators

- So it looks that at best we have well-defined matrix elements for TMDs but including multiple possiblities for gauge links and each process or even each diagram its own gauge link (depending on flow of color)
- Leading quark TMDs

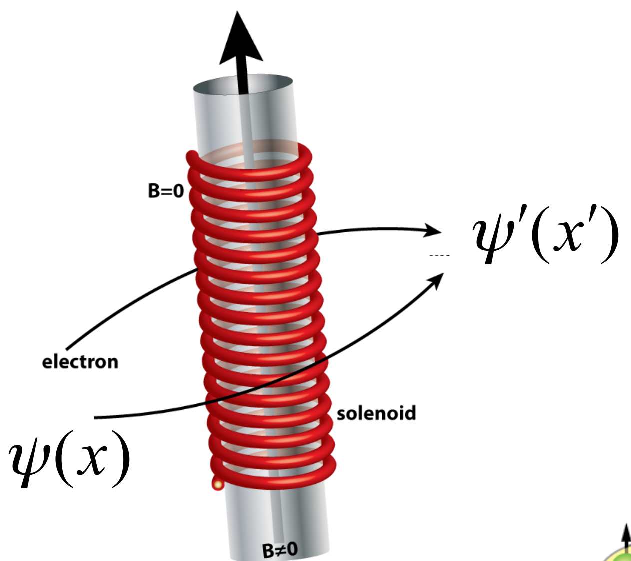
$$\Phi^{[U]}(x,p_T;n) = \Big\{ f_1^{[U]}(x,p_T^2) - f_{1T}^{\perp[U]}(x,p_T^2)\,\frac{\epsilon_T^{p_T S_T}}{M} + g_{1s}^{[U]}(x,p_T)\gamma_5 + h_{1T}^{[U]}(x,p_T^2)\,\gamma_5\,\not{S}_T + h_{1s}^{\perp[U]}(x,p_T)\,\frac{\gamma_5\,\not{p}_T}{M} + i h_1^{\perp[U]}(x,p_T^2)\,\frac{\not{p}_T}{M}\Big\}\frac{\not{P}}{2},$$

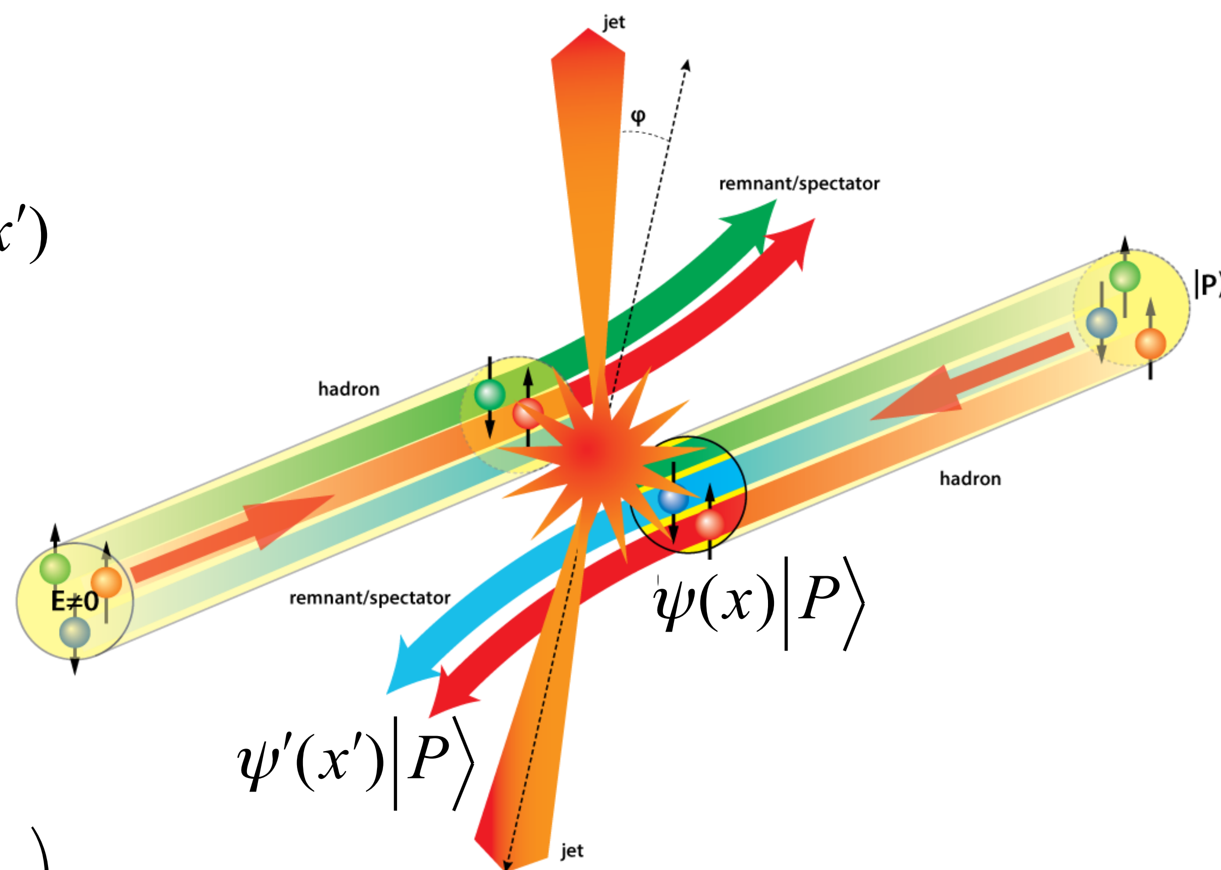
- Leading gluon TMDs:

$$2x\,\Gamma^{\mu\nu[U]}(x,p_T) = -g_T^{\mu\nu}\,f_1^{g[U]}(x,p_T^2) + g_T^{\mu\nu}\frac{\epsilon_T^{p_T S_T}}{M}\,f_{1T}^{\perp g[U]}(x,p_T^2) + i\epsilon_T^{\mu\nu}\;g_{1s}^{g[U]}(x,p_T) + \left(\frac{p_T^\mu p_T^\nu}{M^2} - g_T^{\mu\nu}\frac{p_T^2}{2M^2}\right)h_1^{\perp g[U]}(x,p_T^2) - \frac{\epsilon_T^{p_T\{\mu}\,p_T^{\nu\}}}{2M^2}\,h_{1s}^{\perp g[U]}(x,p_T) - \frac{\epsilon_T^{p_T\{\mu}\,S_T^{\nu\}} + \epsilon_T^{S_T\{\mu}\,p_T^{\nu\}}}{4M}\,h_{1T}^{g[U]}(x,p_T^2).$$

- But wait:
- f_1 is T-even, f_{1T} is T-odd, thus
  - $\Phi^{[+]} + \Phi^{[-]} = f_1$
  - $\Phi^{[+]} - \Phi^{[-]} = f_{1T}$
- This implies
  - $\Phi^{[+]} = f_1 + f_{1T}$
  - $\Phi^{[-]} = f_1 - f_{1T}$
- Example of gluonic pole factors +1 and -1 (to be derived more general).
- These are coupled to processes, since SIDIS needed $\Phi^{[+]}$ and DY needed $\Phi^{[-]}$.

Opportunities to see color-induced phases in QCD





$$\psi(\xi) = \boldsymbol{P}\exp\left(-ig\int_0^{\xi} ds^{\mu} A_{\mu}\right)\psi(0)$$

Figures by Kees Huyser

Next step

Basic strategy: operator product expansion

- Taylor expansion for functions around zero

$$f(z) = \sum_n \frac{f^n}{n!} z^n \qquad f^n = \left.\frac{\partial^n f}{\partial z^n}\right|_{z=0}$$

- Mellin transform for functions on [-1,1] interval

$$f(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dn\, x^{-n} M_n \qquad M_n = \int_0^1 dx\, x^{n-1} f(x)$$

- functions in (transverse) plane

$$f(p_T) = \sum_n \sum_{\alpha_1\ldots\alpha_n} p_T^{\alpha_1}\ldots p_T^{\alpha_n} f_{\alpha_1\ldots\alpha_n} \qquad f_{\alpha_1\ldots\alpha_n} = \left.\partial_{\alpha_1}\ldots\partial_{\alpha_n} f(p_T)\right|_{p_T=0}$$

Operator structure in collinear case (reminder)

- Collinear functions and x-moments

$$\Phi^q(x) = \int \frac{d(\xi.P)}{(2\pi)} e^{i\,p.\xi} \left\langle P \middle| \overline{\psi}(0)\, U^{[n]}_{[0,\xi]}\, \psi(\xi) \middle| P \right\rangle_{\xi.n=\xi_T=0}$$

$$x^{N-1}\,\Phi^q(x) = \int \frac{d(\xi.P)}{(2\pi)} e^{i\,p.\xi} \left\langle P \middle| \overline{\psi}(0)\, (\partial^n_\xi)^{N-1}\, U^{[n]}_{[0,\xi]}\, \psi(\xi) \middle| P \right\rangle_{\xi.n=\xi_T=0}$$

x = p.n

$$= \int \frac{d(\xi.P)}{(2\pi)} e^{i\,p.\xi} \left\langle P \middle| \overline{\psi}(0)\, U^{[n]}_{[0,\xi]}\, (D^n_\xi)^{N-1}\, \psi(\xi) \middle| P \right\rangle_{\xi.n=\xi_T=0}$$

- Moments correspond to local matrix elements of operators that all have the same twist since dim(D^n) = 0

$$\Phi^{(N)} = \left\langle P \middle| \overline{\psi}(0)\, (D^n)^{N-1}\, \psi(0) \middle| P \right\rangle$$

- Moments are particularly useful because their anomalous dimensions can be rigorously calculated and these can be Mellin transformed into the splitting functions that govern the QCD evolution.

Operator structure in TMD case

- For TMD functions one can consider transverse moments

$$\Phi(x,p_T;n)=\int\frac{d(\xi.P)d^2\xi_T}{(2\pi)^3}e^{i\,p.\xi}\left\langle P\middle|\bar{\psi}(0)\,U_{[0,\xi]}^{[\pm]}\,\psi(\xi)\middle|P\right\rangle_{\xi.n=0}$$

$$p_T^{\alpha}\Phi^{[\pm]}(x,p_T;n)=\int\frac{d(\xi.P)d^2\xi_T}{(2\pi)^3}e^{i\,p.\xi}\left\langle P\middle|\bar{\psi}(0)U_{[0,\pm\infty]}D_T^{\alpha}U_{[\pm\infty,\xi]}\psi(\xi)\middle|P\right\rangle_{\xi.n=0}$$

$$p_T^{\alpha_1}p_T^{\alpha_2}\Phi^{[\pm]}(x,p_T;n)=\int\frac{d(\xi.P)d^2\xi_T}{(2\pi)^3}e^{i\,p.\xi}\left\langle P\middle|\bar{\psi}(0)U_{[0,\pm\infty]}D_T^{\alpha_1}D_T^{\alpha_2}U_{[\pm\infty,\xi]}\psi(\xi)\middle|P\right\rangle_{\xi.n=0}$$

- Upon integration, these do involve collinear twist-3 multi-parton correlators

MGA Buffing, A Mukherjee, PJM, PRD 86 (2012) 074030 , Arxiv: 1207.3221 [hep-ph]

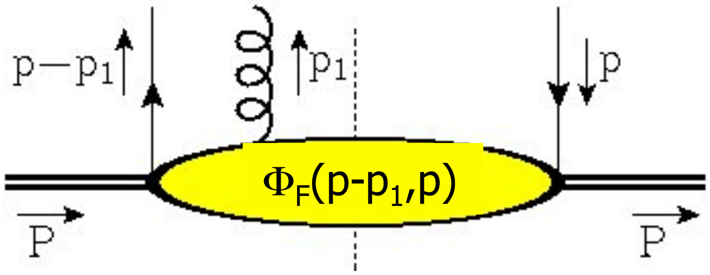
Operator structure in TMD case

- For first transverse moment one needs quark-gluon correlators

$$\Phi_D^\alpha(x-x_1,x_1\,|\,x) = \int \frac{d\xi.P\,d\eta.P}{(2\pi)^2}\, e^{i(p-p_1).\xi + ip_1.\eta} \left\langle P \middle| \bar\psi(0)\, D_T^\alpha(\eta)\, \psi(\xi) \middle| P \right\rangle_{\xi.n=\xi_T=0}$$

$$\Phi_F^\alpha(x-x_1,x_1\,|\,x) = \int \frac{d\xi.P\,d\eta.P}{(2\pi)^2}\, e^{i(p-p_1).\xi + ip_1.\eta} \left\langle P \middle| \bar\psi(0)\, F^{n\alpha}(\eta)\, \psi(\xi) \middle| P \right\rangle_{\xi.n=\xi_T=0}$$

- In principle multi-parton, but we need



$$\Phi_D^\alpha(x) = \int dx_1\, \Phi_D^\alpha(x-x_1,x_1\,|\,x)$$

$$\Phi_A^\alpha(x) = PV \int dx_1\, \frac{1}{x_1}\, \Phi_F^{n\alpha}(x-x_1,x_1\,|\,x)$$

$$\boxed{\tilde\Phi_\partial^\alpha(x) = \Phi_D^\alpha(x) - \Phi_A^\alpha(x)}$$ T-even (gauge-invariant derivative)

$$\boxed{\Phi_G^\alpha(x) = \pi\, \Phi_F^{n\alpha}(x,0\,|\,x)}$$ T-odd (soft-gluon or gluonic pole)

Efremov, Teryaev; Qiu, Sterman; Brodsky, Hwang, Schmidt; Boer, Teryaev, M; Bomhof, Pijlman, M

Operator structure in TMD case

- Transverse moments can be expressed in these particular collinear multi-parton twist-3 correlators (which are not suppressed!)

$$\Phi_{\partial}^{\alpha[U]}(x) = \int d^2p_T\; p_T^{\alpha}\,\Phi^{[U]}(x,p_T;n) = \tilde{\Phi}_{\partial}^{\alpha}(x) + C_G^{[U]}\,\Phi_G^{\alpha}(x)$$

T-even T-odd

T-even T-even T-odd

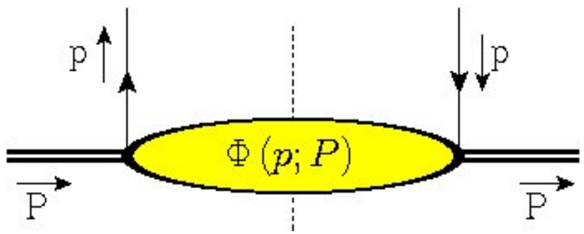
$$\Phi_{\partial\partial}^{\alpha\beta[U]}(x) = \tilde{\Phi}_{\partial\partial}^{\alpha\beta}(x) + C_{GG,c}^{[U]}\,\Phi_{GG,c}^{\alpha\beta}(x) + C_G^{[U]}\left(\tilde{\Phi}_{\partial G}^{\alpha\beta}(x) + \tilde{\Phi}_{G\partial}^{\alpha\beta}(x)\right)$$

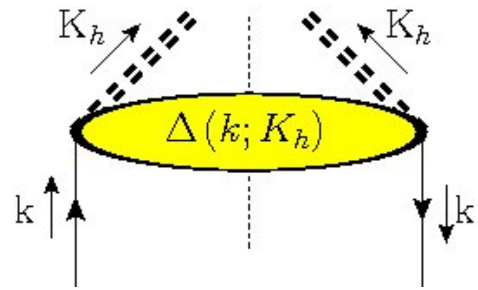
$\mathrm{Tr}_c(GG\,\psi\bar{\psi})$ $\quad$ $\mathrm{Tr}_c(GG)\,\mathrm{Tr}_c(\psi\bar{\psi})$

- $C_G^{[U]}$ calculable gluonic pole factors

U	$U^{[\pm]}$	$U^{[+]}\,U^{[\square]}$	$\frac{1}{N_c}\,\mathrm{Tr}_c(U^{[\square]})\,U^{[+]}$
$\Phi^{[U]}$	$\Phi^{[\pm]}$	$\Phi^{[+\square]}$	$\Phi^{[(\square)+]}$
$C_G^{[U]}$	± 1	3	1
$C_{GG,1}^{[U]}$	1	9	1
$C_{GG,2}^{[U]}$	0	0	4

Distribution versus fragmentation functions





- Operators:

$$\Phi^{[U]}(p\,|\,p) \sim \left\langle P\,|\,\bar{\psi}(0)U_{[0,\xi]}\psi(\xi)\,|\,P\right\rangle$$

$$\Phi_{\partial}^{\alpha[U]}(x) = \tilde{\Phi}_{\partial}^{\alpha}(x) + C_G^{[U]}\,\Phi_G^{\alpha}(x)$$

T-even T-odd (gluonic pole)

$$\boxed{\Phi_G^{\alpha}(x) = \pi\Phi_F^{n\alpha}(x,0\,|\,x) \neq 0}$$

- Operators:

out state

$$\Delta(k\,|\,k) \sim \sum_X \left\langle 0\,|\,\psi(\xi)\,|\,K_h X\right\rangle\left\langle K_h X\,|\,\bar{\psi}(0)\,|\,0\right\rangle$$

$$\boxed{\Delta_G^{\alpha}(x) = \pi\Delta_F^{n\alpha}(\tfrac{1}{z},0\,|\,\tfrac{1}{z}) = 0}$$

$$\Delta_{\partial}^{\alpha[U]}(x) = \tilde{\Delta}_{\partial}^{\alpha}(x)$$

T-even operator combination,
but still T-odd functions!

Collins, Metz; Meissner, Metz; Gamberg, M, Mukherjee, PR D 83 (2011) 071503

Classifying Quark TMDs

- Collecting the right moments gives expansion into full TMD PDFs of definite rank

$$\begin{aligned}
\Phi^{[U]}(x, p_T) \quad = \quad & \Phi(x, p_T^2) + p_{Ti}\tilde{\Phi}_\partial^{i}(x, p_T^2) + p_{Tij}\tilde{\Phi}_{\partial\partial}^{ij}(x, p_T^2) + \ldots \\
& + \sum_c C_{G,c}^{[U]} \left[\qquad p_{Ti}\Phi_{G,c}^{i}(x, p_T^2) + p_{Tij}\tilde{\Phi}_{\{\partial G\},c}^{ij}(x, p_T^2) + \ldots \right] \\
& + \sum_c C_{GG,c}^{[U]} \left[p_T^2 \Phi_{G.G,c}(x, p_T^2) + \quad \ldots \quad + p_{Tij}\Phi_{GG,c}^{ij}(x, p_T^2) + \ldots \right]
\end{aligned}$$

- While for TMD PFFs

$$\Delta^{[U]}(z^{-1}, k_T) = \Delta(z^{-1}, k_T^2) + k_{Ti}\tilde{\Delta}_\partial^{i}(z^{-1}, k_T^2) + k_{Tij}\tilde{\Delta}_{\partial\partial}^{ij}(z^{-1}, k_T^2) + \ldots$$

Classifying Quark TMDs

factor	TMD PDF RANK			
	0	1	2	3
1	$\Phi(x,p_T^2)$	$\tilde{\Phi}_{\partial}(x,p_T^2)$	$\tilde{\Phi}_{\partial\partial}(x,p_T^2)$	$\tilde{\Phi}_{\partial\partial\partial}(x,p_T^2)$
$C_{G,c}^{[U]}$		$\Phi_{G,c}(x,p_T^2)$	$\tilde{\Phi}_{\{G\partial\},c}(x,p_T^2)$	$\tilde{\Phi}_{\{G\partial\partial\},c}(x,p_T^2)$
$C_{GG,c}^{[U]}$			$\Phi_{GG,c}(x,p_T^2)$	$\tilde{\Phi}_{\{GG\partial\},c}(x,p_T^2)$
$C_{GGG,c}^{[U]}$				$\Phi_{GGG,c}(x,p_T^2)$

- Only a finite number needed: rank up to 2(S_{hadron}+s_{parton})
- Rank m shows up as cos(mϕ) and sin(mϕ) azimuthal asymmetries
- No gluonic poles for PFFs

factor	TMD PFF RANK			
	0	1	2	3
1	$\Delta(z^{-1},k_T^2)$	$\tilde{\Delta}_{\partial}(z^{-1},k_T^2)$	$\tilde{\Delta}_{\partial\partial}(z^{-1},k_T^2)$	$\tilde{\Delta}_{\partial\partial\partial}(z^{-1},k_T^2)$

MGA Buffing, A Mukherjee, PJM, PRD2012 , Arxiv: 1207.3221 [hep-ph]

Explicit classification quark TMDs

factor	QUARK TMD PDF RANK UNPOLARIZED HADRON			
	0	1	2	3
1	f_1			
$C_G^{[U]}$		$h_1^\perp$		
$C_{GG,c}^{[U]}$				

- Example: quarks in an unpolarized target are described by just 2 TMD structures

$$\tilde{\Phi}(x,p_T^2) = \left(f_1(x,p_T^2)\right)\frac{\not{P}}{2}$$

T-even

$$\tilde{\Phi}_G^\alpha(x,p_T^2) = \left(i h_1^\perp(x,p_T^2)\frac{\gamma_T^\alpha}{M}\right)\frac{\not{P}}{2}$$

T-odd

[B-M function]

- Gauge link dependence: $h_1^{\perp[U]}(x,p_T^2) = C_G^{[u]} h_1^\perp(x,p_T^2)$

Explicit classification quark TMDs

factor	QUARK TMD PDFs RANK SPIN ½ HADRON			
	0	1	2	3
1	$f_1,\, g_1,\, h_{1T}$	$g_{1T},\, h_{1L}^{\perp}$	$h_{1T}^{\perp(A)}$	
$C_G^{[U]}$		$h_1^{\perp},\, f_{1T}^{\perp}$		
$C_{GG,c}^{[U]}$	$\delta f_1,\, \delta g_1,\, \delta h_{1T}$		$h_{1T}^{\perp(B1)},\, h_{1T}^{\perp(B2)}$	

trace terms

Multiple color possibilities

$$A:\quad \bar{\psi}\partial\partial\psi = Tr_c\left[\partial\partial\psi\bar{\psi}\right]$$

$$B1:\ Tr_c\left[GG\psi\bar{\psi}\right]$$

$$B2:\ Tr_c\left[GG\right]Tr_c\left[\psi\bar{\psi}\right]$$

Explicit classification quark TMDs

factor	QUARK TMD PDFs RANK SPIN ½ HADRON			
	0	1	2	3
1	$f_1,\ g_1,\ h_{1T}$	$g_{1T},\ h_{1L}^{\perp}$	$h_{1T}^{\perp(A)}$	
$C_G^{[U]}$		$h_1^{\perp},\ f_{1T}^{\perp}$		
$C_{GG,c}^{[U]}$	$\delta f_1,\ \delta g_1,\ \delta h_{1T}$		$h_{1T}^{\perp(B1)},\ h_{1T}^{\perp(B2)}$	

Three pretzelocities:

$$A:\ \bar{\psi}\partial\partial\psi = Tr_c\left[\partial\partial\psi\bar{\psi}\right]$$
$$B1:\ Tr_c\left[GG\psi\bar{\psi}\right]$$
$$B2:\ Tr_c\left[GG\right]Tr_c\left[\psi\bar{\psi}\right]$$

Process dependence in f_1, g_1 and h_1
(U-dependent broadening made explicit)

$$f_1^{[U]} = f_1 + C_{GG,c}^{[U]}\delta f_1^{(Bc)}$$
$$h_1^{[U]} = h_{1T} + h_{1T}^{\perp(1)(A)} + C_{GG,c}^{[U]}\left(\delta h_{1T}^{\perp(Bc)} + h_{1T}^{\perp(1)(Bc)}\right)$$

B Boer, MGA Buffing, PJM, work in progress

Explicit classification gluon TMDs

factor	GLUON TMD PDF RANK UNPOLARIZED HADRON			
	0	1	2	3
1	f_1		$h_1^{\perp(A)}$	
$C_{GG,c}^{[U]}$	$\delta f_1^{(Bc)}$		$h_1^{\perp(Bc)}$	

- Note process dependence of unpolarized gluon TMD:

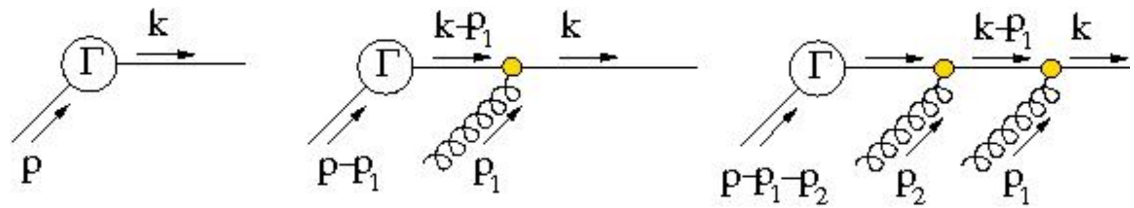
$$f_1^{g[U]} = f_1^g + C_{GG,c}^{[U]} \delta f_1^{g(Bc)}$$

$$h_1^{g\perp[U]} = h_1^{\perp(A)} + C_{GG,c}^{[U]} h_1^{\perp(Bc)}$$

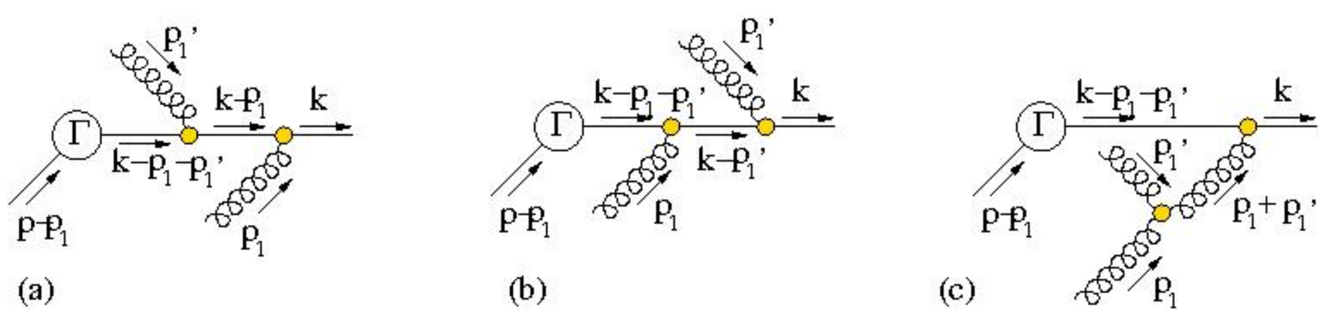
Multiple TMDs in cross sections

Some details on the gauge links (2)

- Resummation of soft n.A gluons (coupling to outgoing color-line) for one correlator produces a gauge-line (along n)

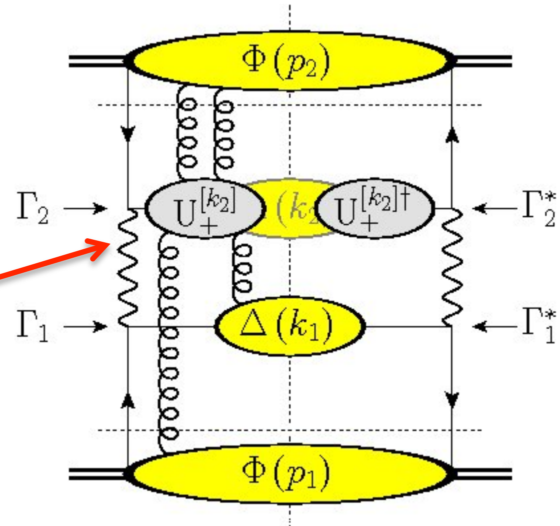


- The lowest order contributions for soft gluons from two different correlators coupling to outgoing color-line resums into gauge-knots: shuffle product of all relevant gauge-lines from that (external initial/final state) line.



Which gauge links?

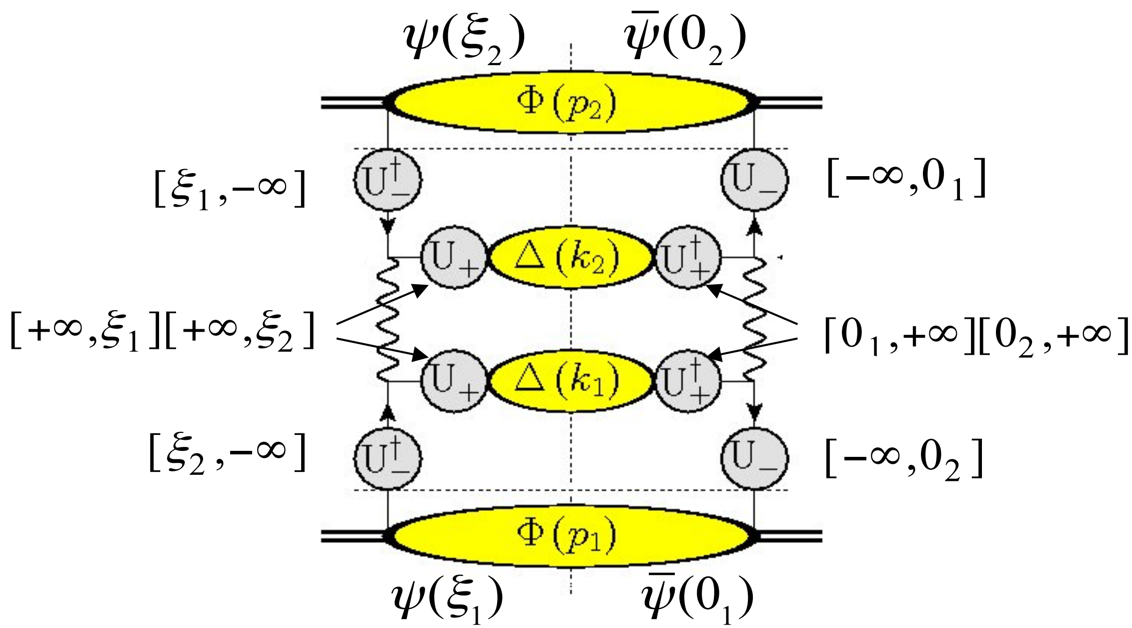
- With more (initial state) hadrons color gets entangled, e.g. in pp
- Gauge knot $U_+[p_1,p_2,\ldots]$



- Outgoing color contributes to a future pointing gauge link in $\Phi(p_2)$ and future pointing part of a gauge loop in the gauge link for $\Phi(p_1)$
- This causes trouble with factorization

T.C. Rogers, PJM, PR D81 (2010) 094006

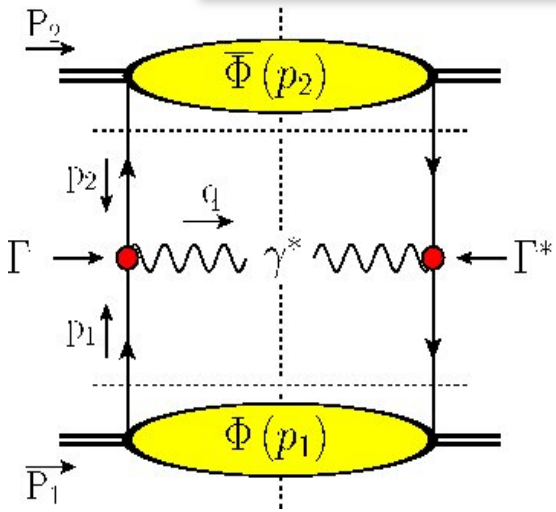
Which gauge links?



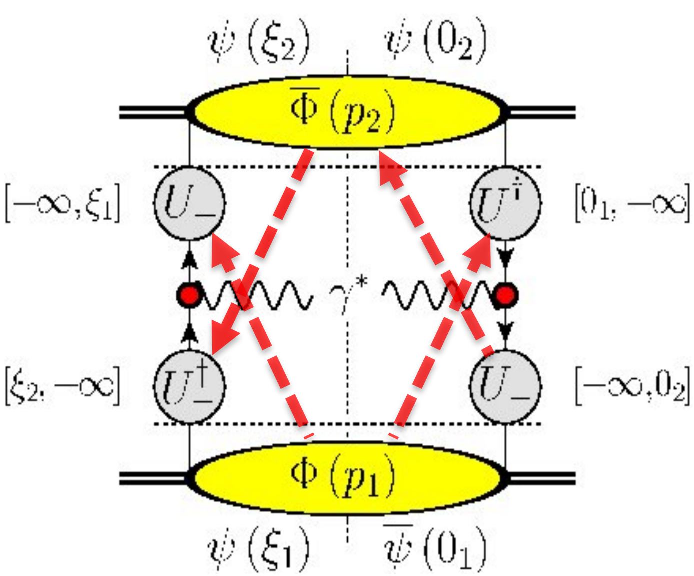
- Can be color-detangled if only p_T of one correlator is relevant (using polarization, …) but must include Wilson loops in final U

MGA Buffing, PJM, JHEP 07 (2011) 065

Correlators in description of hard process (e.g. DY)



$$\xcancel{d\sigma_{\mathrm{DY}} \sim \mathrm{Tr}_c\left[\Phi(x_1,p_{1T})\Gamma^*\overline{\Phi}(x_2,p_{2T})\Gamma\right]}$$
$$\xcancel{= \frac{1}{N_c}\Phi(x_1,p_{1T})\Gamma^*\overline{\Phi}(x_2,p_{2T})\Gamma,}$$



- Complications if the transverse momentum of two initial state hadrons is involved, resulting for DY at measured Q_T in

$$\begin{aligned} d\sigma_{\mathrm{DY}} &= \mathrm{Tr}_c\Big[U_-^\dagger[p_2]\Phi(x_1,p_{1T})U_-[p_2]\Gamma^* \\ &\qquad \times U_-^\dagger[p_1]\overline{\Phi}(x_2,p_{2T})U_-[p_1]\Gamma\Big] \\ &\neq \frac{1}{N_c}\Phi^{[-]}(x_1,p_{1T})\Gamma^*\overline{\Phi}^{[-^\dagger]}(x_2,p_{2T})\Gamma, \end{aligned}$$

Just as for twist-3 squared in collinear DY

Classifying Quark TMDs

factor	TMD RANK			
	0	1	2	3
1	$\Phi(x,p_T^2)$	$\tilde{\Phi}_{\partial}(x,p_T^2)$	$\tilde{\Phi}_{\partial\partial}(x,p_T^2)$	$\tilde{\Phi}_{\partial\partial\partial}(x,p_T^2)$
$C_{G,c}^{[U]}$		$\Phi_{G,c}(x,p_T^2)$	$\tilde{\Phi}_{\{G\partial\},c}(x,p_T^2)$	$\tilde{\Phi}_{\{G\partial\partial\},c}(x,p_T^2)$
$C_{GG,c}^{[U]}$			$\Phi_{GG,c}(x,p_T^2)$	$\tilde{\Phi}_{\{GG\partial\},c}(x,p_T^2)$
$C_{GGG,c}^{[U]}$				$\Phi_{GGG,c}(x,p_T^2)$

$$\sigma(x_1,x_2,q_T) \sim \frac{1}{N_c} f^{[U_1,U_2]}_{R_{G1}R_{G2}}\, \Phi^{[U_1]}(x_1,p_{1T}) \otimes \overline{\Phi}^{[U_2]}(x_2,p_{2T})\, \hat{\sigma}(x_1,x_2),$$

R_G for $\overline{\Phi}^{[-^\dagger]}$	R_G for $\Phi^{[-]}$		
	0	1	2
0	1	1	1
1	1	$-\frac{1}{N_c^2-1}$	$\frac{N_c^2+2}{(N_c^2-2)(N_c^2-1)}$
2	1	$\frac{N_c^2+2}{(N_c^2-2)(N_c^2-1)}$	$\frac{3N_c^4-8N_c^2-4}{(N_c^2-2)^2(N_c^2-1)}$

$$\frac{\mathrm{Tr}_c[T^aT^bT^aT^b]}{\mathrm{Tr}_c[T^aT^a]\,\mathrm{Tr}_c[T^bT^b]} = -\frac{1}{N_c^2-1}\frac{1}{N_c}.$$

MGA Buffing, PJM, PRL (2014), Arxiv: 1309.4681 [hep-ph]

Remember classification of Quark TMDs

factor	QUARK TMD RANK UNPOLARIZED HADRON			
	0	1	2	3
1	f_1			
$C_G^{[U]}$		$h_1^\perp$		
$C_{GG,c}^{[U]}$				

- Example: quarks in an unpolarized target needs only 2 functions
- Resulting in cross section for unpolarized DY at measured Q_T

$$\sigma_{DY}(x_1,x_2,q_T) = \frac{1}{N_c}\Phi(x_1,p_{1T})\otimes\bar{\Phi}(x_2,p_{2T})$$

contains f_1

$$-\frac{1}{N_c}\frac{1}{N_c^2-1}\,q_T^{\alpha\beta}\Phi_G^{\alpha}(x_1,p_{1T})\otimes\bar{\Phi}_G^{\beta}(x_2,p_{2T})$$

contains h_1^{perp}

D. Boer, PRD 60 (1999) 014012; MGA Buffing, PRL (2014) PJM, Arxiv: 1309.4681 [hep-ph]

Definite rank functions and Bessel transforms

- Terms in p_T expansion of TMDs involve

$$\frac{p_{Ti_1\ldots i_m}}{M^m}\,\widetilde{\Phi}^{i_1\ldots i_m}_{\ldots}(x,p_T^2) \qquad \text{or} \qquad \widetilde{\Phi}^{(m/2)}_{\ldots}(x,p_T^2)\,e^{\pm i m\varphi_p}$$

- Use azimuthal integration to get actual p_T^2-dependent TMD PDFs

$$\tilde{\Phi}_{\partial}^{\alpha(1)}(x,p_T^2) = \int \frac{d\varphi}{2\pi}\,p_T^\alpha(\varphi)\Big[\Phi^{[+]}(x,p_T) + \Phi^{[-]}(x,p_T)\Big]$$

$$\Phi_{G}^{\alpha(1)}(x,p_T^2) = \int \frac{d\varphi}{2\pi}\,p_T^\alpha(\varphi)\Big[\Phi^{[+]}(x,p_T) - \Phi^{[-]}(x,p_T)\Big]$$

- This is relevant for lattice calculations as well as experimental analysis
- In general this produces (m/2) moments of the functions

$$\tilde{\Phi}_{\partial}^{\alpha(m/2)}(x,p_T^2) \equiv \left(\frac{-p_T^2}{2M}\right)^{m/2}\tilde{\Phi}_{\partial}^{\alpha}(x,p_T^2)$$

Conclusion with (potential) rewards

- (Generalized) universality studied via operator product expansion, extending the well-known collinear distributions (including polarization 3 for quarks and 2 for gluons) to novel TMD PDF and PFF functions, ordered into functions of definite rank.
- Knowledge of operator structure is important for lattice calculations.
- The rank m is linked to specific cos($m\phi$) and sin($m\phi$) azimuthal asymmetries.
- TMDs encode aspects of hadronic structure, e.g. spin-orbit correlations, such as T-odd transversely polarized quarks or T-even longitudinally polarized gluons in an unpolarized hadron, thus possible applications for precision probing at the LHC, but for sure at a polarized EIC.
- The TMD PDFs appear in cross sections with specific calculable factors that deviate from (or extend on) the naïve parton universality for hadron-hadron scattering.