Transverse Momentum Distribution Functions and beyond: setting up the nucleon tomography

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In these lecture notes I describe a diagrammatic approach in high energy scattering processeses. Using in particular production processes initiated by a lepton-hadron or a hadron-hadron intitial state we identify the correlators that describe the em partons in the hadrons. In this way one can generalize more rigorous approaches such as the operator product expansion techniques. Generalizations include the treatment of transverse momenta of partons. The latter allows a general treatment that includes all possible correlations between momenta and spins of partons and parent hadrons both in polarized and unpolarized cases. The effects of transverse momenta show up as azimuthal asymmetries in the inclusive production of jets or specific hadrons. Although correlators describe in general squared amplitudes, links can be made to amplitudes in other processes. Examples are form factors and generalized parton distributions. One can also look at extensions to multi-parton scattering phenomena. The parametrization in terms of universal functions, such as distribution and fragmentation functions are useful to optimally profit from the kinematic and spin-related degrees of freedom in high-energy processes but the correlators actually also encode interesting hadronic structure that can be studied in lattice approaches or specific models for hadron structure.
Collaborators (a.o.)

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Valence structure of hadrons: global properties

- mass
- charge
- spin
- magnetic moment
- isospin, strangeness
- baryon number

- \( M_p \approx M_n \approx 940 \text{ MeV} \)
- \( Q_p = 1, \ Q_n = 0 \)
- \( s = \frac{1}{2} \)
- \( g_p \approx 5.59, \ g_n \approx -3.83 \)
- \( I = \frac{1}{2}: \ (p,n) \quad S = 0 \)
- \( B = 1 \)

Quarks as constituents

\[
\begin{array}{c|c}
\text{u} & \text{d} \\
\hline
\text{up} & \text{down} \\
\end{array}
\]

\( Q/e = +2/3 \)  \( Q/e = -1/3 \)

p = uud
\( n = udd \)

3 colors

proton
A real look at the proton

\[ \gamma + N \rightarrow \ldots \]

Nucleon excitation spectrum
\[ E \sim 1/R \sim 200 \text{ MeV} \]
\[ R \sim 1 \text{ fm} \]
A (weak) look at the nucleon

\[ n \rightarrow p + e^- + \nu \]

\[ \tau = 900 \text{ s} \]
\[ \rightarrow \text{Axial charge} \]
\[ G_A(0) = 1.26 \]
A virtual look at the proton

\[ \gamma^* \rightarrow N \bar{N} \]

\[ \gamma^* + N \rightarrow N \]

\[ G(q^2) \]

\[ q^2 \]

\[ 4M^2 \]

\[ 0 \]

\[ q^2 < 0 \]

timelike

spacelike
Local operators (coordinate space densities):

\[
< P' \mid O(x) \mid P > = e^{i \Delta \cdot x} \left[ G_1(t) - i \Delta_{\mu} G_2^{\mu}(t) \right]
\]

\[
t = \Delta^2
\]

Form factors

Static properties:

\[
G_1(0) = < P \mid O(x) \mid P >
\]

\[
G_2^{\mu}(0) = < P \mid x^{\mu} O(x) \mid P >
\]

Examples:
- (axial) charge
- mass
- spin
- magnetic moment
- angular momentum
Nucleon densities from virtual look

\[ G_i(t) \rightarrow \rho_i(x) \]

- charge density \( \neq 0 \)
- \( u \) more central than \( d \)?
- role of antiquarks?
- \( n = n_0 + p\pi^- + \ldots ? \)
Given the QCD framework, the operators are known quarkic or gluonic currents such as

**Quark and gluon operators**

(axial) vector currents

\[ V^q_\mu (x) = \bar{q}(x) \gamma_\mu q(x) \]

\[ A^{q'q}_\mu (x) = \bar{q}(x) \gamma_\mu \gamma_5 q'(x) \]

energy-momentum currents

\[ T^q_{\mu \nu} (x) \sim \bar{q}(x) \gamma_{\{\mu} D_{\nu\}} q(x) \]

\[ T^G_{\mu \nu} (x) \sim G_{\mu \alpha}(x) G^\alpha_{\nu}(x) \]

probed in specific combinations by photons, Z- or W-bosons

\[ J^{(\gamma)}_\mu = \frac{2}{3} V^u_\mu - \frac{1}{3} V^d_\mu - \frac{1}{3} V^s_\mu + ... \]

\[ J^{(Z)}_\mu = \frac{1}{2} \left( V^u_\mu - A^u_\mu \right) - \frac{4}{3} \sin^2 \theta_W V^u_\mu + ... \]

\[ J^{(W)}_\mu = V^{ud'}_\mu - A^{ud'}_\mu + ... \]

probed by gravitons
Towards the quarks themselves

- The current provides the densities but only in specific combinations, e.g. quarks minus antiquarks and only flavor weighted
- No information about their correlations, (effectively) pions, or ...
- Can we go beyond these global observables (which correspond to local operators)?
- Yes, in high energy (semi-)inclusive measurements we will have access to non-local operators!
- $L_{QCD}$ (quarks, gluons) known!
Non-local probing

Nonlocal forward operators (correlators):

\[ < P \left| O \left( x - \frac{y}{2}, x + \frac{y}{2} \right) \right| P > = < P \left| O \left( -\frac{y}{2}, +\frac{y}{2} \right) \right| P > \]

Specifically useful: ‘squares’

\[ O \left( x - \frac{y}{2}, x + \frac{y}{2} \right) = \Psi^\dagger \left( x - \frac{y}{2} \right) \cdots \Psi \left( x + \frac{y}{2} \right) \]

Selectivity at high energies: \( q = p \)

Momentum space densities of \( \Psi \)-ons:

\[
\int d\gamma \ e^{ip\cdot\gamma} < P \left| \Psi^\dagger \left( -\frac{y}{2} \right) \Psi \left( +\frac{y}{2} \right) \right| P > = \\
= \sum_X \left| < P_X \left| \Psi \left( 0 \right) \right| P > \right|^2 \delta(P_X - P + p) = f(p)
\]
A hard look at the proton

- Hard virtual momenta \((\pm q^2 = Q^2 \sim \text{many GeV}^2)\) can couple to (two) soft momenta

\[ \gamma^* + N \rightarrow \text{jet} \]

\[ \gamma^* \rightarrow \text{jet + jet} \]
Experiments!
QCD framework (including electroweak theory) provides the machinery to calculate cross sections, e.g. $\gamma^* q \rightarrow q$, $qq \rightarrow \gamma^*$, $\gamma^* \rightarrow q\bar{q}$, $qq \rightarrow qq$, $qg \rightarrow qg$, etc.

E.g.

$qg \rightarrow qg$

Calculations work for plane waves

$$\langle 0 | \psi_i^{(s)} (\xi) | p, s \rangle = u_i (p, s) e^{-ip \cdot \xi}$$
For hard scattering process involving electrons and photons the link to external particles is, indeed, the ‘one-particle wave function’

\[
\langle 0 \left| \psi_i (\xi) \right| p, s \rangle = u_i (p, s) e^{-ip.\xi}
\]

Confinement, however, implies hadrons as ‘sources’ for quarks

\[
\langle X \left| \psi_i (\xi) \right| P \rangle e^{+ip.\xi}
\]

... and also as ‘source’ for quarks + gluons

\[
\langle X \left| \psi_i (\xi) A^\mu (\eta) \right| P \rangle e^{+i(p-p_1).\xi + ip_1.\eta}
\]

... and also ....
PDFs and PFFs

Basic idea of PDFs and PFFs (also for TMDs) is to obtain a full factorized description of high energy scattering processes

\[
\sigma = |H(p_1, p_2, \ldots)|^2
\]

calculable

defined (!) & portable

\[
\sigma(P_1, P_2, \ldots) = \iiint \ldots dp_1 \ldots \Phi_a(p_1, P_1; \mu) \otimes \Phi_b(p_2, P_2; \mu) \otimes \delta_{ab, \ldots} (p_1, p_2, \ldots; \mu) \otimes \Delta_c(k_1, K_1; \mu) \ldots
\]

Give a meaning to integration variables!
At high energies no interference and squared amplitudes can be rewritten as correlators of forward matrix elements of parton fields

Math:

$$u_i(p,s)\bar{u}_j(p,s) \Rightarrow \sum_X \left\langle P \left| \bar{\psi}_j(0) \right| X \right\rangle \left\langle X \left| \psi_i(0) \right| P \right\rangle \delta(p - P + P_X)$$

$$= \sum_X \int \frac{d\xi}{2\pi} \left\langle P \left| \bar{\psi}_j(0) \right| X \right\rangle \left\langle X \left| \psi_i(0) \right| P \right\rangle e^{i(p-P+P_X)\cdot\xi}$$

$$= \sum_X \int \frac{d\xi}{2\pi} \left\langle P \left| \bar{\psi}_j(0) \right| X \right\rangle \left\langle X \left| \psi_i(\xi) \right| P \right\rangle e^{ip\cdot\xi}$$

$$= \int \frac{d\xi}{2\pi} e^{ip\cdot\xi} \left\langle P \left| \bar{\psi}_j(0) \psi_i(\xi) \right| P \right\rangle$$

Use symmetries (P, T) and hermicity to parametrize these objects!
Hadron correlators

At high energies no interference and squared amplitudes can be rewritten as correlators of matrix elements of parton fields

Math: \( u_i(k, s) \bar{u}_j(k, s) \Rightarrow \sum_X \langle 0 | \psi_i(0) | K_h X \rangle \langle K_h X | \bar{\psi}_j(0) | 0 \rangle \delta(k - K_h - K_X) \)

\[ = \sum_X \int \frac{d\xi}{2\pi} \langle 0 | \psi_i(0) | K_h X \rangle \langle K_h X | \bar{\psi}_j(0) | 0 \rangle e^{i(k - K_h - K_X)\cdot \xi} \]

\[ = \sum_X \int \frac{d\xi}{2\pi} \langle 0 | \psi_i(\xi) | K_h X \rangle \langle K_h X | \bar{\psi}_j(0) | 0 \rangle e^{i k \cdot \xi} \]

\[ = \int \frac{d\xi}{2\pi} e^{i k \cdot \xi} \langle 0 | \psi_i(\xi) a_h^+ a_h \bar{\psi}_j(0) | 0 \rangle \]

no T-constraint
\( T|K_h X \rangle_{\text{out}} = |K_h X \rangle_{\text{in}} \)
Role of the hard scale

- In high-energy processes hard momenta are available, such that $P.P' \sim s$ with a hard scale $s >> M^2$

- Employ light-like vectors $P$ and $n$, such that $P.n = 1$ (e.g. $n = P'/P.P'$) to make a Sudakov expansion of parton momentum (write $s = Q^2$)

\[
p = xP^\mu + p_T^\mu + \sigma n^\mu
\]
\[
x = p^+ = p.n \quad (0 \leq x \leq 1)
\]
\[
\sim Q \quad \sim M \quad \sim M^2/Q
\]
\[
\sigma = p^- = p.P - xM^2 \sim O(M^2)
\]

- Enables expansion in inverse hard scale (twist analysis) for integrated correlators,

\[
\Phi(p) = \Phi(x, p_T, p.P) \Rightarrow \Phi(x, p_T) \Rightarrow \Phi(x) \Rightarrow \Phi
\]
(Un)integrated correlators

\[ \Phi(x, p_T, p_P) = \int \frac{d^4 \xi}{(2\pi)^4} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) \psi(\xi) \right| P \right\rangle \]

- Unintegrated

\[ \Phi(x, p_T ; n) = \int \frac{d(\xi \cdot P)d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) \psi(\xi) \right| P \right\rangle \bigg|_{\xi \cdot n = \xi^+ = 0} \]

- TMD (light-front)

\[ \sigma = p^- \text{ integration makes time-ordering automatic.} \]
\[ \text{The soft part is simply sliced at the light-front} \]

\[ \Phi(x) = \int \frac{d(\xi \cdot P)}{(2\pi)} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) \psi(\xi) \right| P \right\rangle \bigg|_{\xi \cdot n = \xi_T = 0 \text{ or } \xi^2 = 0} \]

- Collinear (light-cone)

\[ \Phi = \left\langle P \left| \bar{\psi}(0) \psi(\xi) \right| P \right\rangle \bigg|_{\xi = 0} \]

- Local

\[ \text{Local operators with calculable anomalous dimension} \]
Example using correlators (DIS)
• Instead of partons use correlators

\[ \sum_s u(p,s) \bar{u}(p,s) \Rightarrow \Phi(p,P) \]

\[ \Delta(k) = k + m \]
LIGHTCONE DOMINANCE IN DIS

Large scale $Q$ leads in a natural way to the use of lightlike vectors:
$n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$

\[
\begin{align*}
q^2 &= -Q^2 \\
P^2 &= M^2 \\
2P \cdot q &= \frac{Q^2}{x_B}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
P = \frac{x_B M^2}{Q \sqrt{2}} n_- + \frac{Q}{x_B \sqrt{2}} n_+ \\
q = \frac{Q}{\sqrt{2}} n_- - \frac{Q}{\sqrt{2}} n_+
\end{cases}
\end{align*}
\]

\[
\begin{array}{c|cc}
\text{part} & \text{'components'} & \\
& - & + \\
\hline
\text{HARD} & \sim Q & \sim Q \\
H \rightarrow q & \sim \frac{1}{Q} & \sim Q \\
\rightarrow \int dp^- d^2 p_T & & \ldots
\end{array}
\]
Principle for DIS

- Instead of partons use correlators

\[ \sum_s u(p,s) \bar{u}(p,s) \Rightarrow \Phi(p,P) \quad \Delta(k) = k + m \]

- Expand parton momenta (using P as light-like plus vector)

\[ p = xP^\mu + p_T^\mu + \sigma n^\mu \]

\[ x = p^+ = p.n \sim 1 \]

\[ \sim Q \quad \sim M \quad \sim M^2/Q \]

\[ \sigma = p.P - xM^2 \sim M^2 \]
(calculation of) cross section in DIS

Full calculation

\[ \Phi(x) \]

LEADING (in $1/Q$)

\[ x = x_B = -q^2/P \cdot q \]
Result for DIS

\[ 2MW_{\mu\nu}(P,q) = -\frac{1}{2} g_T^{\mu\nu} \int dx dp.P \ d^2 p_T \ Tr[\Phi(p,P)\gamma^+] \delta(x - x_B) \]

\[ = -\frac{1}{2} g_T^{\mu\nu} \ Tr[\Phi(x_B)\gamma^+] \]
Twist analysis (1)

- Dimensional analysis to determine importance in an expansion in inverse hard scale

- Maximize contractions with $n$

\[
\dim[\bar{\psi}(0)\eta \psi(\xi)] = 2
\]
\[
\dim[F^{\alpha}(0)F^{\beta}(\xi)] = 2
\]
\[
\dim[\bar{\psi}(0)\eta A_{T}^{\alpha}(\eta)\psi(\xi)] = 3
\]

- ... or maximize # of $P$’s in parametrization of $\Phi$

\[
\Phi^{q}(x) = f_{1}^{q}(x)\frac{P}{2} \iff f_{1}^{q}(x) = \int \frac{d\lambda}{(2\pi)} e^{ix\lambda} \left\langle P \left| \bar{\psi}(0)\eta \psi(\lambda n) \right| P \right\rangle
\]

- Note that these are densities!

\[
\bar{\psi}(0)\eta \psi(\lambda n) = \psi^{+}(0)\psi^{+}(\lambda n)
\]
Parametrization of TMDs
Ingredients in parametrization

- Building blocks: momenta and spins
- Handling of spin in distributions (spin of hadrons can be tuned)
- Handling of spin in fragmentations (spin of produced hadrons cannot be tuned!)
- Color summation in distribution functions
- Color averaging in fragmentation functions
Symmetry constraints

\[ \Phi^{T^*}(p; P, S) = \gamma_0 \Phi(p; P, S) \gamma_0 \]

\[ \Phi(p; P, S) = \gamma_0 \Phi(\bar{p}; \bar{P}, -\bar{S}) \gamma_0 \]

\[ \Phi^{[U]}(p; P, S) = (-i \gamma_5 C) \Phi^{[-U]}(\bar{p}; \bar{P}, \bar{S})(-i \gamma_5 C) \]

\[ \Phi^c(p; P, S) = C \Phi^T(-p; P, S) C \]

Parametrization of TMD correlator for unpolarized hadron:

\[ \Phi^{[\pm]q}(x, p_T) = \left( f_1^q(x, p_T^2) \pm i h_1^{\perp q}(x, p_T^2) \frac{p_T}{M} \right) \frac{P}{2} \]

(unpolarized and transversely polarized quarks)

Mulders, Tangerman, Boer; Bacchetta, Diehl, Goeke, Metz, M, Schlegel, JHEP02 (2007) 093
New information in TMD’s: $f(x,p_T)$ or $D(1/z,k_T)$

- Quarks in polarized nucleon:
  \[ S = S_L \left( \frac{P}{M} + Mn \right) + S_T \]
  \[ S_L^2 + S_T^2 = -1 \]

\[ \Phi^q(p; P, S) \propto x f_1^q(x, p^2_T) P + S_L x g_{1L}^q(x, p^2_T) P \gamma_5 \]
\[ + x h_{1T}^q(x, p^2_T) S_T P \gamma_5 + \ldots \]

unpolarized quarks

T-polarized quarks in T-polarized N

chiral quarks in L-polarized N

compare
\[ u(p, s)\bar{u}(p, s) = \frac{1}{2} (p + m)(1 + \gamma_5 \gamma) \]

... but also

\[ \Phi^q(p; P, S) \propto \ldots + \frac{(p_T \cdot S_T)}{M} x g_{1T}^q(x, p^2_T) P \gamma_5 + \ldots \]

spin $\leftrightarrow$ spin

chiral quarks in T-polarized N
New information in TMD’s: $f(x,p_T)$ or $D(1/z,k_T)$

- ... and T-odd functions

$$\Phi^q(p;P,S) \propto ... + i h_{1q}^{1q}(x, p_T^2) \frac{p_T}{M} \mathcal{P} + i \frac{(p_T \times S_T)}{M} x f_{1T}^{1q}(x, p_T^2) \mathcal{P} + ...$$

T-polarized quarks in unpolarized N (Boer-Mulders)

unpolarized quarks in T-polarized N (Sivers)

Yes, definitely there is new information and even very interesting spin-orbit correlations (single spin!). These are T-odd and because of T-conservation show up in T-odd observables, such as single spin asymmetries, e.g. left-right asymmetry in $p(P_1)p_\uparrow(P_2) \rightarrow \pi(K)X$
New information in gluon TMD’s: $f(x,p_T)$ or $D(1/z,k_T)$

Also for gluons there are new features in TMD’s

$\Phi_g^{\mu\nu}(p; P, S) \propto -g_T^{\mu\nu} x f_1^g(x, p_T^2) + i S_L \varepsilon_T^{\mu\nu} x g_{1L}^g(x, p_T^2)$

$$+ \left( \frac{p_T^\mu p_T^\nu}{M^2} - g_T^{\mu\nu} \frac{p_T^\mu}{2M^2} \right) x h_{1g}^{\perp}(x, p_T^2) + \ldots$$

compare

$\varepsilon^\mu(p, \lambda) \varepsilon^{\nu*}(p, \lambda) = -g_T^{\mu\nu} + \ldots$

- Unpolarized gluons in unpol. N quarks
- Linearly polarized gluons in L-pol. N (Gluon Boer-Mulders)
- Circularly polarized gluons in L-pol. N
- Spin $\leftrightarrow$ orbit
Basis of partons

- ‘Good part’ of Dirac space is 2-dimensional
- Interpretation of DF’s

**TWO ‘SPIN’ STATES FOR (GOOD) QUARK FIELDS**

Chiral eigenstates:

\[ \psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi : \left| \begin{array}{c} \uparrow \end{array} \right| \text{ and } \left| \begin{array}{c} \downarrow \end{array} \right| \]

or

Transverse spin eigenstates:

\[ \psi_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^\alpha \gamma_5)\psi : \left| \begin{array}{c} \uparrow \gamma_\alpha \end{array} \right| \text{ and } \left| \begin{array}{c} \downarrow \gamma_\alpha \end{array} \right| \]

Note: \([P_{R/L}, P_+] = [P_{\uparrow/\downarrow}, P_+] = 0\)

**DISTRIBUTION FUNCTIONS IN PICTURES**

\[ f_1(x) = \bullet = \left| \begin{array}{c} \bullet \end{array} \right| = \left| \begin{array}{c} \bullet \end{array} \right| + \left| \begin{array}{c} \bullet \end{array} \right| \]

\[ S_L g_1(x) = \left| \begin{array}{c} \bullet \end{array} \right| - \left| \begin{array}{c} \bullet \end{array} \right| \]

\[ S_T^\alpha h_1(x) = \left| \begin{array}{c} \bullet \end{array} \right| - \left| \begin{array}{c} \bullet \end{array} \right| \]

\[ \frac{1}{2} \left[ \frac{1}{2} (1 \pm \gamma_5) \psi \right] = \int (2\pi)^4 \left| \begin{array}{c} \psi(0) \end{array} \right| \left[ (1 \pm \gamma_5) \psi(0) \right] \right|_{\epsilon^+ = \epsilon_T = 0} = h_1(x) S_T^\alpha \]
Quark production matrix, directly related to the helicity formalism

- Off-diagonal elements (RL or LR) are chiral-odd functions
- Chiral-odd soft parts must appear with partner in e.g. SIDIS, DY

\[ M^{(\text{prod})} = \begin{pmatrix}
  f_1 + g_1 & 0 & 0 & 2h_1 \\
  0 & f_1 - g_1 & 0 & 0 \\
  0 & 0 & f_1 - g_1 & 0 \\
  2h_1 & 0 & 0 & f_1 + g_1 \\
\end{pmatrix} \]

**Matrix representation for spin 1/2**

\( p_T \)-integrated distribution functions:
For a spin 1/2 hadron (e.g. nucleon) the quark production matrix in quark\( \otimes \)nucleon spin space is given by

\[ M = [\Phi(x)\gamma^+]^T \]

Bacchetta, Boglione, Henneman & Mulders
PRL 85 (2000) 712
unpolarized quark distribution

helicity or chirality distribution

transverse spin distribution or transversity

\[ f_1^q(x) = q(x) \]

\[ g_1^q(x) = \Delta q(x) \]

\[ h_1^q(x) = \delta q(x) \]
Matrix representation for \( M = [\Phi^{[\pm]}(x,p_T)\gamma^+]^T \)

- \( p_T \)-dependent functions

**Matrix representation for spin 1/2**

\( p_T \)-dependent quark distributions:

\[
\begin{pmatrix}
  f_1 + g_{1L} & \frac{|p_T|}{M} e^{i\phi} g_{1T} & \frac{|p_T|}{M} e^{-i\phi} h_{1L} & 2h_1 \\
  \frac{|p_T|}{M} e^{-i\phi} g_{1T} & f_1 - g_{1L} & \frac{|p_T|^2}{M^2} e^{-2i\phi} h_{1T} & -\frac{|p_T|}{M} e^{-i\phi} h_{1L} \\
  \frac{|p_T|}{M} e^{i\phi} h_{1L} & \frac{|p_T|^2}{M^2} e^{2i\phi} h_{1T} & f_1 - g_{1L} & -\frac{|p_T|}{M} e^{i\phi} g_{1T} \\
  2h_1 & -\frac{|p_T|}{M} e^{i\phi} h_{1L} & -\frac{|p_T|}{M} e^{-i\phi} g_{1T} & f_1 + g_{1L}
\end{pmatrix}
\]

T-odd: \( g_{1T} \rightarrow g_{1T} - i f_{1T} \) and \( h_{1L} \rightarrow h_{1L} + i h_{1L} \) (imaginary parts)

Bacchetta, Boglione, Henneman & Mulders
PRL 85 (2000) 712
Example using TMDs (SIDIS)
(calculation of) cross section in SIDIS

Full calculation

LEADING (in 1/Q)

OPTICAL THEOREM FOR SIDIS
LIGHTCONE DOMINANCE IN SIDIS

Large scale $Q$ leads in a natural way to the use of lightlike vectors: $n_+^2 = n_-^2 = 0$ and $n_+ \cdot n_- = 1$

\[
\begin{align*}
q^2 &= -Q^2 \\
P^2 &= M^2 \\
P_h^2 &= M_h^2 \\
2 P \cdot q &= \frac{Q^2}{x_B} \\
2 P_h \cdot q &= -z_h Q^2
\end{align*}
\]

\[
\begin{align*}
P_h &= \frac{z_h Q}{\sqrt{2}} n_- + \frac{M_h^2}{z_h Q \sqrt{2}} n_+ \\
q &= \frac{Q}{\sqrt{2}} n_- - \frac{Q}{\sqrt{2}} n_+ + q_T \\
P &= \frac{x_B M^2}{Q \sqrt{2}} n_- + \frac{Q}{x_B \sqrt{2}} n_+ 
\end{align*}
\]

Three external momenta $P \quad P_h \quad q$
transverse directions relevant $q_T = q + \frac{x_B P - P_h}{z_h}$
or $q_T = -\frac{P_{h\perp}}{z_h}$
Result for SIDIS

\[ q_T = q + x_B P - \frac{P_h}{z_h} \]

\[ 2MW^{\mu\nu}(P, P_h, q) = \int d^2 p_T \int d^2 k_T \]

\[ \times \text{Tr} [\Phi(x_B, p_T)\gamma^\mu \Delta(z_h, k_T)\gamma^\nu] \delta^2 (p_T + q_T - k_T) \]

\[ = -\frac{1}{2} g_T^{\mu\nu} \int d^2 p_T \int d^2 k_T \]

\[ \times \text{Tr} [\Phi(x_B, p_T)\gamma^+] \text{Tr} [\Delta(z_h, k_T)\gamma^-] \delta^2 (p_T + q_T - k_T) \]
relevance and measurability of TMDs
Transverse momentum dependence

- Mismatch of hadronic and partonic momenta
  \[ p - xP = p_T + ... = -xP_\perp + ... \]
  \[ k - \frac{1}{z} K_h = k_T + ... = -\frac{1}{z} K_{h\perp} + ... \]

- Momentum fractions are linked to scaling variables, e.g. SIDIS (up to \(1/Q^2\) corrections):
  \[ x = p.n / P.n = Q^2 / 2P.q = x_B \]
  \[ z = K.n / k.n = P.K / P.q = z_h \]

- Transverse momenta are convoluted into a measurable off-collinearity,
  \[ q_T = q + x_B P - z_h^{-1} K = k_T - p_T \]

- ... or non-alignment of jets in hadron + hadron → jet + jet.
Access to transverse momenta

Also in more complex situations like hadron-hadron collisions

\[ p_1 \approx x_1 P_1 + p_{1T} \]
\[ p_2 \approx x_2 P_2 + p_{2T} \]

\[ x_1 = p_1.n = \frac{p_1.P_2}{P_1.P_2} = \frac{(k_1 + k_2).P_2}{P_1.P_2} \]

\[ q_T = k_{jet,1} + k_{jet,2} - x_1 P_1 - x_2 P_2 \]
\[ = p_{1T} + p_{2T} \]

Boer & Vogelsang
Large pT
**Large \( p_T \)**

- \( p_T \)-dependence of TMDs

\[
\int d^2 p_T \, \Phi(x, p_T) = \Phi(x; \mu^2)
\]

Fictitious measurement

Large \( m^2 \) dependence governed by anomalous dim (i.e. splitting functions)

- \( \Phi(x, p_T) \rightarrow \frac{1}{\pi p_T^2} \frac{\beta_s(p_T^2)}{2\pi} \int dy \frac{1}{y} P\left(\frac{x}{y}\right) \Phi(y; p_T^2) \)

- Consistent matching to collinear situation: CSS formalism

JC Collins, DE Soper and GF Sterman, NP B 250 (1985) 199
Large values of momenta

\[ p^2 \approx \frac{p_T^2 - x_p M_1^2}{1 - x_p} < 0 \]

\[ p.P = \frac{x_p (p_T^2 - M_1^2)}{2x(1 - x_p)} < 0 \]

\[ M_R^2 \approx \frac{(x - x_p) p_T^2 + x_p (1 - x) M_1^2}{x(1 - x_p)} > 0 \]

\[ p_0 \approx \frac{x}{x_p} P + p_{0T} \quad (x \leq x_p \leq 1) \]

\[ l_T \approx -p_T \quad p_{0T} \sim M \]

\[ \Phi(p, P) \xrightarrow{\rightarrow} \frac{\alpha_s}{p_T^2} \ldots \quad \text{etc.} \]

Bacchetta, Boer, Diehl, M
Complications for TMDs
Hadron correlators

- Hadronic correlators establish the diagrammatic link between hadrons and partonic hard scattering amplitude
- Quark, quark + gluon, gluon, ...

\[
\left\langle 0 \left| \psi_i (\xi) \right| p, s \right\rangle = u_i (p, s) e^{-ip \cdot \xi}
\]

\[
\left\langle X \left| \psi_i (\xi) \right| P \right\rangle e^{ip \cdot \xi}
\]

\[
\left\langle X \left| \psi_i (\xi) A^\mu (\eta) \right| P \right\rangle e^{i(p-p_1) \cdot \xi + ip_1 \cdot \eta}
\]

Disentangling a hard process into collinear parts involving hadrons, hard scattering amplitude and soft factors is non-trivial.

Fig. 5.17. (a) An important reduced graph for the amplitude for the Drell-Yan process. (b) Space-time diagram for collinear subgraphs.

In light-front coordinates, we write the momenta as

\[
P_A = \left( P_A^+, \frac{m_A^2}{2P_A^+}, 0, T \right)
\]

\[
P_B = \left( \frac{m_B^2}{2P_B^+}, P_B^-, 0, T \right)
\]

\[
q = \left( x_p^A P_A^+ + \sqrt{1+q^2T/Q_s^2}, x_p^B P_B^- - \sqrt{1+q^2T/Q_s^2}, q^T \right)
\]

Here the scaling variables are defined by

\[
x^A = \frac{Q^y}{\sqrt{s}}, x^B = \frac{Q^{y-2}}{\sqrt{s}}
\]

where

\[y = \frac{1}{2} \ln \frac{q^2 + (P_B^−−P_B^+)^2 + (q^T)^2}{Q_s^4/4} + \frac{P_B^−−P_B^+}{q^T}
\]

\[Q_s = \sqrt{q^2} \]

In the center-of-mass, the large components of the hadron momenta are \(P_A^+\) and \(P_B^-,\) both equal to \(\sqrt{s/2}p_{opt}.\)

Frequently, the cross section is integrated over \(q^T\), and is presented as \(d^2\sigma/(dQ^2 dy)\).

We first discuss the DY amplitude. Its reduced graphs are constructed by an elementary generalization of the construction for DIS.

When we have two collinear subgraphs, \(A\) and \(B\), associated with each incoming particle.

As in DIS, we classify the reduced graphs by the number of outgoing directions of lines from the hard scattering \(H\).

Now \(H\) has incoming lines from each of the \(A\) and \(B\) subgraphs, and has the virtual photon taking out momentum. This allows the minimal situation, illustrated in Fig. 5.17, with no extra collinear groups at all going out from \(H\).

The soft subgraph can create particles in the final state that fill in the rapidity gap between the beam remnants.

This is illustrated by the microscopic view of a collision shown in Fig. 5.18 (which corresponds to Fig. 2.2 for DIS). Here we have shown the

J.C. Collins, Foundations of Perturbative QCD, Cambridge Univ. Press 2011
Soft part: hadron correlators

- Forward matrix elements of parton fields describe distribution (and fragmentation) parts

\[ \Phi_{ij}(p; P) = \Phi_{ij}(p \mid p) = \int \frac{d^4\xi}{(2\pi)^4} e^{ip\cdot\xi} \langle P \bar{\psi}_j(0) \psi_i(\xi) \mid P \rangle \]

- Also needed are multi-parton correlators

\[ \Phi_{A;ij}^{\alpha}(p - p_1, p_1 \mid p) = \int \frac{d^4\xi d^4\eta}{(2\pi)^8} e^{i(p-p_1)\cdot\xi + ip_1\cdot\eta} \langle P \bar{\psi}_j(0) A^{\alpha}(\eta) \psi_i(\xi) \mid P \rangle \]
Gauge invariance in a non-local situation requires a gauge link \( U(0, \xi) \)

\[
\overline{\psi}(0) \psi(\xi) = \sum_n \frac{1}{n!} \xi^{\mu_1} ... \xi^{\mu_N} \overline{\psi}(0) \partial_{\mu_1} ... \partial_{\mu_N} \psi(0)
\]

\[
U(0, \xi) = \mathcal{P} \exp \left( -ig \int_0^\xi ds^\mu A_\mu \right)
\]

\[
\overline{\psi}(0) U(0, \xi) \psi(\xi) = \sum_n \frac{1}{n!} \xi^{\mu_1} ... \xi^{\mu_N} \overline{\psi}(0) D_{\mu_1} ... D_{\mu_N} \psi(0)
\]

Introduces path dependence for \( \Phi(x, p_T) \)

\[
\Phi^{[U]}(x, p_T) \Rightarrow \Phi(x)
\]
Twist analysis (2)

- Dimensional analysis to determine importance in an expansion in inverse hard scale
- Maximize contractions with $n$

\[
\text{dim}[\bar{\psi}(0) \gamma \psi(\xi)] = 2 \\
\text{dim}[F^{\alpha}(0)F^{\beta}(\xi)] = 2 \\
\text{dim}[\bar{\psi}(0) \gamma A_\alpha^T(\eta) \psi(\xi)] = 3
\]

... or maximize # of $P$’s in parametrization of $\Phi$

\[
\Phi^q(x) = f_1^q(x) \frac{P}{2} \iff f_1^q(x) = \int \frac{d\lambda}{(2\pi)} e^{ix\lambda} \left\langle P \left| \bar{\psi}(0) \gamma \psi(\lambda n) \right| P \right\rangle
\]

- In addition any number of collinear $n.A(\xi) = A^n(x)$ fields (dimension zero!), but of course in color gauge invariant combinations

\[
\text{dim } 0: \quad i\partial^n \rightarrow iD^n = i\partial^n + gA^n \\
\text{dim } 1: \quad i\partial_T^\alpha \rightarrow iD_T^\alpha = i\partial_T^\alpha + gA_T^\alpha
\]
Which gauge links?

\[ \Phi_{ij}^{\mu[C]}(x, p_T; n) = \int \frac{d(\xi.P)d^2\xi}{(2\pi)^3} e^{i p.x} \langle P | \bar{\psi}_j(0) U^{[C]}_{[0,\xi]} \psi_i(\xi) | P \rangle_{\xi,n=0} \]

\[ \Phi_{ij}^{\mu}(x; n) = \int \frac{d(\xi.P)}{(2\pi)} e^{i p.x} \langle P | \bar{\psi}_j(0) U^{[n]}_{[0,\xi]} \psi_i(\xi) | P \rangle_{\xi,n=\xi_T=0} \]

◆ Gauge links come from dimension zero (not suppressed!) collinear A.n gluons, but leads for TMD correlators to process-dependence:

DY

... A\textsuperscript{n} ...

SIDIS

... A\textsuperscript{n} ...

Φ\textsuperscript{[-]}

Time reversal

Φ\textsuperscript{[+]}

AV Belitsky, X Ji and F Yuan, NP B 656 (2003) 165
D Boer, PJM and F Pijlman, NP B 667 (2003) 201
Some details on the gauge links (1)

■ Proper gluon fields (F rather than A, Wilson lines and boundary terms)

\[ A^\mu(p_1) = n.A(p_1) \frac{P^\mu}{n.P} + i A^\mu_T(p_1) + \ldots = \frac{1}{p_1.n} \left[ n.A(p_1) p_1^\mu + i G^{n^\mu}_T(p_1) + \ldots \right] \]

■ Resummation of soft n.A gluons (coupling to outgoing color-line) for one correlator produces a gauge-line (along n)

■ Boundary terms give transverse pieces
Which gauge links?

\[
\Phi^{\alpha\beta[C,C']}_g(x, p_T; n) = \int \frac{d(\xi.P)d^2\xi_T}{(2\pi)^3} e^{ip.\xi} \left\langle P \left| U^{[C]}_{[\xi,0]} F^{n\alpha}(0) U^{[C]}_{[0,\xi]} F^{n\beta}(\xi) \right| P \right\rangle_{\xi,n=0}
\]

◆ The TMD gluon correlators contain two links, which can have different paths. Note that standard field displacement involves \( C = C' \)

\[
F^{\alpha\beta}(\xi) \rightarrow U^{[C]}_{[\eta,\xi]} F^{\alpha\beta}(\xi) U^{[C]}_{[\xi,\eta]}
\]

◆ Basic (simplest) gauge links for gluon TMD correlators:

C Bomhof, PJM, F Pijlman; EPJ C 47 (2006) 147
F Dominguez, B-W Xiao, F Yuan, PRL 106 (2011) 022301
Summarizing: color gauge invariant correlators

- So it looks that at best we have well-defined matrix elements for TMDs but including multiple possibilities for gauge links and each process or even each diagram its own gauge link (depending on flow of color)
- Leading quark TMDs:

\[
\Phi^\nu[U](x, p_T; n) = \left\{ f_1^U(x, p_T^2) - f_{1T}^\perp[U](x, p_T^2) \frac{\epsilon_p T S_T}{M} + g_{1s}^T(x, p_T) \gamma_5 \\
+ h_{1T}^U(x, p_T^2) \gamma_5 \frac{\rho^T}{M} + i h_{1s}^T(x, p_T^2) \frac{\rho_T}{M} \right\} \frac{p_T}{2},
\]

- Leading gluon TMDs:

\[
2x \Gamma^{\mu\nu}[U](x, p_T) = -g_T^{\mu\nu} f_1^g[U](x, p_T^2) + g_T^{\mu\nu} \frac{\epsilon_p T S_T}{M} f_{1T}^\perp(g[U](x, p_T^2) \\
+ \frac{\epsilon_T^{\mu\nu}}{2M^2} h_{1s}^g[U](x, p_T^2) - \frac{\epsilon_T^{\mu\nu}}{2M^2} h_{1s}^g[U](x, p_T^2) + \epsilon_T^{\mu\nu} \frac{S_T}{M} h_{1T}^g[U](x, p_T^2) \frac{p_T}{4M}.
\]
But wait:

- $f_1$ is T-even, $f_{1T}$ is T-odd, thus
  - $\Phi^+ + \Phi^- = f_1$
  - $\Phi^+ - \Phi^- = f_{1T}$

This implies

- $\Phi^+ = f_1 + f_{1T}$
- $\Phi^- = f_1 - f_{1T}$

Example of gluonic pole factors +1 and -1 (to be derived more general).

These are coupled to processes, since SIDIS needed $\Phi^+$ and DY needed $\Phi^-$. 
Opportunities to see color-induced phases in QCD

\[ \psi(\xi) = P \exp \left( -ig \int_0^\xi ds^\mu A_\mu \right) \psi(0) \]
Next step
Basic strategy: operator product expansion

- Taylor expansion for functions around zero

\[ f(z) = \sum_n \frac{f^n}{n!} z^n \]

\[ f^n = \frac{\partial^n f}{\partial z^n} \bigg|_{z=0} \]

- Mellin transform for functions on [-1,1] interval

\[ f(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dn x^{-n} M_n \]

\[ M_n = \int_0^1 dx x^{n-1} f(x) \]

- Functions in (transverse) plane

\[ f(p_T) = \sum_n \sum_{\alpha_1...\alpha_n} p_T^\alpha_1 ... p_T^\alpha_n f_{\alpha_1...\alpha_n} \]

\[ f_{\alpha_1...\alpha_n} = \partial_{\alpha_1} ... \partial_{\alpha_n} f(p_T) \bigg|_{p_T=0} \]
Collinear functions and x-moments

\[ \Phi^q(x) = \int \frac{d(\xi \cdot P)}{(2\pi)} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) U^{[n]}_{[0,\xi]} \psi(\xi) \right| P \right\rangle_{\xi, n = \xi_T = 0} \]

\[ x^{N-1} \Phi^q(x) = \int \frac{d(\xi \cdot P)}{(2\pi)} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) \left( \frac{\partial}{\partial \xi} \right)^{N-1} U^{[n]}_{[0,\xi]} \psi(\xi) \right| P \right\rangle_{\xi, n = \xi_T = 0} \]

\[ x = p.n \]

- Moments correspond to local matrix elements of operators that all have the same twist since \( \text{dim}(D^n) = 0 \)

\[ \Phi^{(N)} = \left\langle P \left| \bar{\psi}(0) (D^n)^{N-1} \psi(0) \right| P \right\rangle \]

- Moments are particularly useful because their anomalous dimensions can be rigorously calculated and these can be Mellin transformed into the splitting functions that govern the QCD evolution.
Operator structure in TMD case

- For TMD functions one can consider transverse moments

\[
\Phi(x, p_T; n) = \int \frac{d(\xi.P) d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) U^{[\pm]}_{[0, \xi]} \psi(\xi) \right| P \right\rangle_{\xi, n=0}
\]

\[
p_T^\alpha \Phi^{[\pm]}(x, p_T; n) = \int \frac{d(\xi.P) d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) U^{[0, \pm \infty]} D_T^\alpha U^{[\pm \infty, \xi]} \psi(\xi) \right| P \right\rangle_{\xi, n=0}
\]

\[
p_T^\alpha_1 p_T^\alpha_2 \Phi^{[\pm]}(x, p_T; n) = \int \frac{d(\xi.P) d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \left\langle P \left| \bar{\psi}(0) U^{[0, \pm \infty]} D_T^\alpha_1 D_T^\alpha_2 U^{[\pm \infty, \xi]} \psi(\xi) \right| P \right\rangle_{\xi, n=0}
\]

- Upon integration, these do involve collinear twist-3 multi-parton correlators
Operator structure in TMD case

- For first transverse moment one needs quark-gluon correlators
  
  \[ \Phi_D^\alpha(x - x_1, x_1 | x) = \int \frac{d\xi \cdot P \, d\eta \cdot P}{(2\pi)^2} e^{i(p-p_1) \cdot \xi + i \eta \cdot \eta} \left\langle P \left| \bar{\psi}(0) D_T^\alpha(\eta) \psi(\xi) \right| P \right\rangle_{\xi \cdot n = \xi_T = 0} \]

  \[ \Phi_F^\alpha(x - x_1, x_1 | x) = \int \frac{d\xi \cdot P \, d\eta \cdot P}{(2\pi)^2} e^{i(p-p_1) \cdot \xi + i \eta \cdot \eta} \left\langle P \left| \bar{\psi}(0) F^\alpha(n) \psi(\xi) \right| P \right\rangle_{\xi \cdot n = \xi_T = 0} \]

- In principle multi-parton, but we need

  \[ \Phi_D^\alpha(x) = \int dx_1 \Phi_D^\alpha(x - x_1, x_1 | x) \]

  \[ \Phi_A^\alpha(x) = PV \int dx_1 \frac{1}{x_1} \Phi_F^{n\alpha}(x - x_1, x_1 | x) \]

  \[ \Phi_D^\alpha(x) - \Phi_A^\alpha(x) = T\text{-even (gauge-invariant derivative)} \]

  \[ \Phi_G^\alpha(x) = \pi \Phi_F^{n\alpha}(x, 0 | x) = T\text{-odd (soft-gluon or gluonic pole)} \]
Transverse moments can be expressed in these particular collinear multi-parton twist-3 correlators (which are not suppressed!)

\[
\Phi^{\alpha[U]}_{\partial}(x) = \int d^2 p_T \left( p_T^{\alpha} \Phi^{[U]}(x, p_T; n) = \tilde{\Phi}^{\alpha}_{\partial}(x) + C^{[U]}_G \Phi^{\alpha}_{G}(x) \right)
\]

\[
\Phi^{\alpha\beta[U]}_{\partial\partial}(x) = \tilde{\Phi}^{\alpha\beta}_{\partial\partial}(x) + C^{[U]}_{G \cdot G, c} \Phi^{\alpha\beta}_{G \cdot G, c}(x) + C^{[U]}_G \left( \tilde{\Phi}^{\alpha\beta}_{\partial G}(x) + \tilde{\Phi}^{\alpha\beta}_{G\partial}(x) \right)
\]

- **C\textsubscript{G}[U] calculable gluonic pole factors**

- **Tr\textsubscript{c}(GG \bar{\psi} \psi)**

- **Tr\textsubscript{c}(GG) Tr\textsubscript{c}(\psi \bar{\psi})**

<table>
<thead>
<tr>
<th>( U )</th>
<th>( U^{[\pm]} )</th>
<th>( U^{[+] \cdot U^{[\Box]} \cdot U^{[+]} \cdot \frac{1}{N_c} \cdot \text{Tr}_c(U^{[\Box]} \cdot U^{[+]}) )</th>
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</thead>
<tbody>
<tr>
<td>( \Phi^{[U]} )</td>
<td>( \Phi^{[\pm]} )</td>
<td>( \Phi^{[+,\Box]} )</td>
</tr>
<tr>
<td>( C^{[U]}_G )</td>
<td>±1</td>
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</tr>
<tr>
<td>( C^{[U]}_{G \cdot G, 1} )</td>
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</tr>
<tr>
<td>( C^{[U]}_{G \cdot G, 2} )</td>
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<td>0</td>
</tr>
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</table>
Operators:

\[ \Phi^{[U]}(p | p) \sim \langle P | \bar{\psi}(0) U_{[0, \xi]} \psi(\xi) | P \rangle \]

\[ \Phi^{[U]}_\alpha(x) = \tilde{\Phi}^{[U]}_\alpha(x) + C^G_\alpha \Phi^G_\alpha(x) \]

\[ \Phi^G_\alpha(x) = \pi \Phi^{nG}_\alpha(x, 0 | x) \neq 0 \]

T-even

T-odd (gluonic pole)

\[ \Delta^G_\alpha(x) = \pi \Delta^{nG}_F \left( \frac{1}{Z}, 0 | \frac{1}{Z} \right) = 0 \]

\[ \Delta^{[U]}_\alpha(x) = \tilde{\Delta}^{[U]}_\alpha(x) \]

T-even operator combination, but still T-odd functions!
Classifying Quark TMDs

- Collecting the right moments gives expansion into full TMD PDFs of definite rank

\[
\Phi^{[U]}(x, p_T) = \Phi(x, p_T^2) + p_{Ti} \tilde{\Phi}^i_\partial (x, p_T^2) + p_{Tij} \tilde{\Phi}^{ij}_\partial\partial (x, p_T^2) + \ldots \\
+ \sum_c C^{[U]}_{G,c} \left[ p_{Ti} \Phi^i_{G,c} (x, p_T^2) + p_{Tij} \tilde{\Phi}^{ij}_{\{\partial G\},c} (x, p_T^2) + \ldots \right] \\
+ \sum_c C^{[U]}_{GG,c} \left[ p_T^2 \Phi^{GG,c} (x, p_T^2) + \ldots + p_{Tij} \Phi^{ij}_{GG,c} (x, p_T^2) + \ldots \right]
\]

- While for TMD PFFs

\[
\Delta^{[U]}(z^{-1}, k_T^2) = \Delta(z^{-1}, k_T^2) + k_{Ti} \tilde{\Delta}^i_\partial (z^{-1}, k_T^2) + k_{Tij} \tilde{\Delta}^{ij}_\partial\partial (z^{-1}, k_T^2) + \ldots
\]
Classifying Quark TMDs

Only a finite number needed: rank up to $2(S_{\text{hadron}}+s_{\text{parton}})$

Rank $m$ shows up as $\cos(m\phi)$ and $\sin(m\phi)$ azimuthal asymmetries

No gluonic poles for PFFs

<table>
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<tr>
<th>factor</th>
<th>TMD PDF RANK</th>
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<tbody>
<tr>
<td>0</td>
<td>$\Phi(x, p_T^2)$</td>
</tr>
<tr>
<td>1</td>
<td>$C_{G,c}^{[U]}(x, p_T^2)$</td>
</tr>
<tr>
<td>2</td>
<td>$C_{GG,c}^{[U]}(x, p_T^2)$</td>
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<table>
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<th>TMD PFF RANK</th>
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<td>0</td>
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</tr>
<tr>
<td>1</td>
<td>$C_{GGG,c}^{[U]}(x, p_T^2)$</td>
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Explicit classification quark TMDs

<table>
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<tr>
<th>factor</th>
<th>QUARK TMD PDF RANK UNPOLARIZED HADRON</th>
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<tr>
<td>1</td>
<td>$f_1$</td>
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<tr>
<td>$C_G^{[U]}$</td>
<td></td>
</tr>
<tr>
<td>$C_{GG,c}^{[U]}$</td>
<td></td>
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</tbody>
</table>

- Example: quarks in an unpolarized target are described by just 2 TMD structures
  
  $\tilde{\Phi}(x, p_T^2) = \left( f_1(x, p_T^2) \right) \frac{P}{2}$

  $\tilde{\Phi}_G^\alpha(x, p_T^2) = \left( i h_1^\perp(x, p_T^2) \frac{\gamma_\alpha}{M} \right) \frac{P}{2}$

  [B-M function]

- Gauge link dependence: $h_1^\perp^{[U]}(x, p_T^2) = C_G^{[u]} h_1^\perp(x, p_T^2)$
Explicit classification quark TMDs

<table>
<thead>
<tr>
<th>factor</th>
<th>QUARK TMD PDFs</th>
<th>RANK</th>
<th>SPIN</th>
<th>½</th>
<th>HADRON</th>
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<td></td>
<td>$g_{1T}, h_{1L}^\perp$</td>
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<td></td>
<td>$h_{1T}^{(A)}$</td>
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<td>3</td>
<td></td>
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<tr>
<td>$C_G^{[U]}$</td>
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<tr>
<td></td>
<td>$h_1^\perp, f_{1T}^\perp$</td>
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<tr>
<td>$C_{GG,c}^{[U]}$</td>
<td>$\delta f_1, \delta g_1, \delta h_{1T}$</td>
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<td>$h_{1T}^{(B1)}, h_{1T}^{(B2)}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Multiple color possibilities

$A : \bar{\psi} \partial \partial \psi = Tr_c \left[ \partial \partial \psi \bar{\psi} \right]$

$B1 : Tr_c \left[ GG \psi \bar{\psi} \right]$

$B2 : Tr_c \left[ GG \right] Tr_c \left[ \psi \bar{\psi} \right]$
Explicit classification quark TMDs

<table>
<thead>
<tr>
<th>factor</th>
<th>QUARK TMD PDFs</th>
<th>RANK</th>
<th>SPIN ½</th>
<th>HADRON</th>
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<tr>
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<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<tr>
<td>1</td>
<td>$f_1, g_1, h_{1T}$</td>
<td>$g_{1T}, h_{1L}^{\perp}$</td>
<td>$h_{1T}^{\perp(A)}$</td>
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</tr>
<tr>
<td>$C_{G}^{[U]}$</td>
<td>$h_{1}^{\perp}, f_{1T}^{\perp}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$C_{GG,c}^{[U]}$</td>
<td>$\delta f_1, \delta g_1, \delta h_{1T}$</td>
<td>$h_{1T}^{\perp(B1)}, h_{1T}^{\perp(B2)}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Three pretzelocities:

Process dependence in $f_1, g_1$ and $h_1$
(U-dependent broadening made explicit)

$$f_1^{[U]} = f_1 + C_{GG,c}^{[U]} \delta f_1^{(Bc)}$$

$$h_1^{[U]} = h_{1T}^{\perp} + h_{1T}^{\perp(1)(A)} + C_{GG,c}^{[U]} \left( \delta h_{1T}^{\perp(Bc)} + h_{1T}^{\perp(1)(Bc)} \right)$$

$$A: \bar{\psi} \partial \partial \psi = Tr_c \left[ \partial \partial \psi \bar{\psi} \right]$$

$$B1: Tr_c \left[ GG \psi \bar{\psi} \right]$$

$$B2: Tr_c \left[ GG \right] Tr_c \left[ \psi \bar{\psi} \right]$$

B Boer, MGA Buffing, PJM, work in progress
### Explicit classification gluon TMDs

<table>
<thead>
<tr>
<th>factor</th>
<th>GLUON TMD PDF RANK UNPOLARIZED HADRON</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$f_1$</td>
</tr>
<tr>
<td>$C_{GG,c}^{[U]}$</td>
<td>$\delta f_{1}^{(Bc)}$</td>
</tr>
</tbody>
</table>

- Note process dependence of unpolarized gluon TMD:

$$f_{1}^{g[U]} = f_{1}^{g} + C_{GG,c}^{[U]} \delta f_{1}^{g(Bc)}$$

$$h_{1}^{g \perp [U]} = h_{1 \perp (A)} + C_{GG,c}^{[U]} h_{1 \perp (Bc)}$$
Multiple TMDs in cross sections
Resummation of soft n.A gluons (coupling to outgoing color-line) for one correlator produces a gauge-line (along n).

The lowest order contributions for soft gluons from two different correlators coupling to outgoing color-line resums into gauge-knots: shuffle product of all relevant gauge-lines from that (external initial/final state) line.
With more (initial state) hadrons color gets entangled, e.g. in pp

Gauge knot $U_+[p_1, p_2, ...]$

Outgoing color contributes to a future pointing gauge link in $\Phi(p_2)$ and future pointing part of a gauge loop in the gauge link for $\Phi(p_1)$

This causes trouble with factorization

T.C. Rogers, PJM, PR D81 (2010) 094006
Which gauge links?

- Can be color-detangled if only $p_T$ of one correlator is relevant (using polarization, ...) but must include Wilson loops in final $U$
Correlators in description of hard process (e.g. DY)

\[ d\sigma_{\text{DY}} \sim \text{Tr}_c \left[ \Phi(x_1, p_{1T}) \Gamma^* \overline{\Phi}(x_2, p_{2T}) \Gamma \right] \]

\[ = \frac{1}{N_c} \Phi(x_1, p_{1T}) \Gamma^* \overline{\Phi}(x_2, p_{2T}) \Gamma, \]

- Complications if the transverse momentum of two initial state hadrons is involved, resulting for DY at measured \( Q_T \) in

\[ d\sigma_{\text{DY}} = \text{Tr}_c \left[ U^\dagger_{-}[p_2] \Phi(x_1, p_{1T}) U_-[p_2] \Gamma^* \times U^\dagger_{-}[p_1] \overline{\Phi}(x_2, p_{2T}) U_-[p_1] \Gamma \right] \]

\[ \neq \frac{1}{N_c} \Phi[-](x_1, p_{1T}) \Gamma^* \overline{\Phi}[{-}^\dagger](x_2, p_{2T}) \Gamma, \]

Just as for twist-3 squared in collinear DY
Classifying Quark TMDs

<table>
<thead>
<tr>
<th>factor</th>
<th>TMD RANK</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\Phi(x, p_T^2)$</td>
</tr>
<tr>
<td>$C_{G,c}^{[U]}$</td>
<td>$\Phi_{G,c}(x, p_T^2)$</td>
</tr>
<tr>
<td>$C_{GG,c}^{[U]}$</td>
<td>$\Phi_{GG,c}(x, p_T^2)$</td>
</tr>
<tr>
<td>$C_{GGG,c}^{[U]}$</td>
<td>$\Phi_{GGG,c}(x, p_T^2)$</td>
</tr>
</tbody>
</table>

$$\sigma(x_1, x_2, q_T) \sim \frac{1}{N_c} f_{R_{G1}R_{G2}} [U_1, U_2] \Phi[U_1](x_1, p_{1T})$$

$$\otimes \Phi^{[U_2]}(x_2, p_{2T}) \hat{\sigma}(x_1, x_2),$$

$$\frac{\text{Tr}_c[T^a T^b T^a T^b]}{\text{Tr}_c[T^a T^a] \text{Tr}_c[T^b T^b]} = -\frac{1}{N_c^2 - 1} \frac{1}{N_c}.$$
Remember classification of Quark TMDs

<table>
<thead>
<tr>
<th>factor</th>
<th>QUARK TMD RANK UNPOLARIZED HADRON</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$C^{[U]}_G$</td>
<td></td>
</tr>
<tr>
<td>$C^{[U]}_{GG,c}$</td>
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</tr>
</tbody>
</table>

- Example: quarks in an unpolarized target needs only 2 functions
- Resulting in cross section for unpolarized DY at measured $Q_T$

$$\sigma_{DY}(x_1,x_2,q_T) = \frac{1}{N_c} \Phi(x_1,p_{1T}) \otimes \Phi(x_2,p_{2T})$$

contains $f_1$

$$- \frac{1}{N_c} \frac{1}{N_c^2 - 1} q^{\alpha\beta} \Phi^\alpha_G(x_1,p_{1T}) \otimes \Phi^\beta_G(x_2,p_{2T})$$

contains $h_1^\perp$
Definite rank functions and Bessel transforms

- Terms in $p_T$ expansion of TMDs involve
  \[ \frac{p_T i_1 \cdots i_m}{M^m} \tilde{\Phi}^{i_1 \cdots i_m}(x, p^2_T) \quad \text{or} \quad \tilde{\Phi}^{(m/2)}(x, p^2_T) e^{\pm im\varphi_p} \]

- Use azimuthal integration to get actual $p_T^2$-dependent TMD PDFs
  \[
  \tilde{\Phi}^{(1)}_{\alpha}(x, p^2_T) = \int \frac{d\varphi}{2\pi} \, p_T^\alpha(\varphi) \left[ \Phi^+[x, p_T] + \Phi^-[x, p_T] \right] \\
  \Phi^{(1)}_G(x, p^2_T) = \int \frac{d\varphi}{2\pi} \, p_T^\alpha(\varphi) \left[ \Phi^+[x, p_T] - \Phi^-[x, p_T] \right]
  \]

- This is relevant for lattice calculations as well as experimental analysis

- In general this produces $(m/2)$ moments of the functions
  \[ \tilde{\Phi}^{(m/2)}_{\alpha}(x, p^2_T) \equiv \left( \frac{-p_T^2}{2M} \right)^{m/2} \tilde{\Phi}^{\alpha}(x, p^2_T) \]
(Generalized) universality studied via operator product expansion, extending the well-known collinear distributions (including polarization 3 for quarks and 2 for gluons) to novel TMD PDF and PFF functions, ordered into functions of definite rank.

Knowledge of operator structure is important for lattice calculations.

The rank m is linked to specific $\cos(m\phi)$ and $\sin(m\phi)$ azimuthal asymmetries.

TMDs encode aspects of hadronic structure, e.g. spin-orbit correlations, such as T-odd transversely polarized quarks or T-even longitudinally polarized gluons in an unpolarized hadron, thus possible applications for precision probing at the LHC, but for sure at a polarized EIC.

The TMD PDFs appear in cross sections with specific calculable factors that deviate from (or extend on) the naïve parton universality for hadron-hadron scattering.