NEUTRINO MASSES AND MIXINGS*

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I. INTRODUCTION

The neutrino was proposed in 1930 by Pauli and solved the problem of the "missing energy" in beta decay, $n \to p + e^- + \bar{\nu}$. It had to be a fermion with the same spin as the electron, but its mass should be extremely small, because the threshold being in essence the sum of proton and electron masses. The neutrino was only experimentally established in 1956 by Reines and Cowan via the inverse beta decay process, $\bar{\nu} + p \to n + e^+$.

In the last decade, we have obtained quite detailed information on neutrinos, both mass differences and couplings, with the absolute scale of the mass spectrum and the precise nature of the mixings still being open. We know that there are two quite different differences in the squared masses of which the smaller one plays a role for solar neutrinos and the larger difference is relevant for atmospheric neutrinos. The precise hierarchy of mass states is still open. Also the fermionic nature is still open. Are neutrinos Majorana or Dirac fermions which has consequences for the CP violating phases.

In these lectures we will concentrate on basic theoretical aspects needed to understand the oscillation mechanisms. In the Review of Particle Properties[1] a complete list of references and up to date listings of experimental results can be found. In the first lecture, we will look at 'simple' quantum mechanical aspects of oscillations in vacuum and matter. In the second and third lecture, we will look at the difference between Dirac and Majorana neutrinos and the structure of masses and couplings depending on the type and the embedding in and beyond the Standard Model.

II. NEUTRINO VACUUM OSCILLATIONS

The phenomenon of neutrino oscillations is a first year quantum mechanics problem. The quantum mechanical states at production (say a muon neutrino arising from pion decay) is not an eigenstate of the hamiltonian. Assuming two relevant neutrino species, there are two orthogonal eigenstates of the Hamiltonian which can be used as our basis

$$\nu_1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \nu_2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1}$$

They are eigenstates with definite energies corresponding to slightly different masses m_1 and m_2, i.e. $E_1 = \sqrt{p^2 + m_1^2}$ and $E_2 = \sqrt{p^2 + m_2^2}$. Thus we have in general

$$\psi(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\,E_1 t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\,E_2 t} = \begin{pmatrix} c_1\, e^{-i\,E_1 t} \\ c_2\, e^{-i\,E_2 t} \end{pmatrix}. \tag{2}$$

Assuming that a muon-neutrino produced in the atmosphere is a linear superposition of the two mass eigenstates ν_1 and ν_2, one has at time $t = 0$ and time t

$$\nu(0) = \nu_\mu \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \nu(t) = \begin{pmatrix} \cos\theta\ e^{-i\,E_1 t} \\ \sin\theta\ e^{-i\,E_2 t} \end{pmatrix},$$

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and the probability to find at time t again a muon-neutrino is

$$|\langle\nu_\mu|\nu(t)\rangle|^2 = \left| \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta\, e^{-i\,E_1 t} \\ \sin\theta\, e^{-i\,E_2 t} \end{pmatrix} \right|^2 = 1 - \sin^2 2\theta\, \sin^2\left(\frac{E_1 - E_2}{2}\, t\right). \tag{3}$$

For the situation that m_1, $m_2 \ll p$ one has $E_1 \approx E_2$, both roughly equal to $E = p$ with the difference being

$$E_1 - E_2 \approx \frac{m_1^2 - m_2^2}{2E} \equiv \frac{\Delta m^2}{2E},$$

After travelling a distance L with approximately the speed of light c one has a survival probability

$$P(\nu_\mu \to \nu_\mu) = 1 - \sin^2 2\theta\, \sin^2\left(\frac{\Delta m^2\, L}{4E}\right) \equiv 1 - \sin^2 2\theta\, \sin^2\left(\pi\, \frac{L}{\lambda_V}\right) \tag{4}$$

with

$$\lambda_V = 4\pi\, \frac{E}{\Delta m^2}. \tag{5}$$

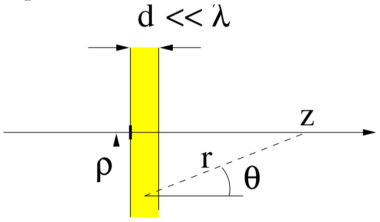
or for numerical purposes

$$\lambda_V\ [\text{km}] = 2.5\, \frac{E\ [\text{GeV}]}{\Delta m^2\ [\text{eV}^2]}. \tag{6}$$

Numerically a wavelength $\lambda_V = 1000$ km corresponds for a neutrino with a typical energy of 1 GeV to a mass squared difference of $\Delta m^2 = (0.05\ \text{eV})^2$. The present-day values for the atmospheric mixing parameters are $\Delta m^2 = \Delta m_A^2 \approx 2.4 \times 10^{-3}\ \text{eV}^2$ and $\sin^2\theta = \sin^2\theta_A \approx 0.42$.

III. PROPAGATION OF PLANE WAVES IN MATTER

We consider here a straightforward treatment of propagation in matter, which will be applied to neutrinos. Suppose a plane wave crosses a slab of matter (width $d \ll \lambda$), as shown below.



$$z^2 + \rho^2 = r^2 \implies \begin{array}{l}\text{At fixed } z: \\ \rho\, d\rho = r\, dr\end{array}$$

$$z = r\,\cos\theta \implies \begin{array}{l}\text{At fixed } z: \\ \frac{\partial \cos\theta}{\partial r} = \frac{\partial (z/r)}{\partial r} = -\frac{z}{r^2} = -\frac{\cos\theta}{r}\end{array}$$

The wave function (continuous) is given by

$$\psi(z) = e^{i\,kz} \qquad \text{for } z \le 0, \tag{7}$$

$$\psi(z) = e^{i\,n\,kz} \qquad \text{for } 0 \le z \le d(\text{ in slab}), \tag{8}$$

$$\psi(z) = e^{i\,kz + i\,kd(n-1)} \qquad \text{for } z \ge d. \tag{9}$$

In order to calculate the wave function in the point indicated in the figure above, we add the contributions from all scattering centers in the slab. Assuming the density to be N, we obtain

$$\begin{aligned}\psi(z) &= e^{i\,kz} + \int_0^\infty 2\pi\,\rho\, d\rho\, N\, d\, \frac{e^{i\,kr}}{r}\, f(k,\theta) \\ &= e^{i\,kz} + 2\pi\, N\, d \int_z^\infty dr\ e^{i\,kr}\, f(k,\theta)\end{aligned}$$

We estimate the integral by successive partial integrations, omitting always the contributions from $r = \infty$ (this can be done neatly by considering wave packets),

$$\begin{aligned}\int_z^\infty dr\ e^{i\,kr}\, f(k,\theta) &= \left.\frac{e^{i\,kr}}{i\,k}\, f(k,\theta)\right|_z^\infty + \frac{i}{k} \int_z^\infty dr\ e^{i\,kr}\, \frac{\partial f}{\partial r} \\ &= \frac{i}{k}\, e^{i\,kz} \left[f(k,\theta) + \frac{i}{k}\frac{\partial f}{\partial r} + \left(\frac{i}{k}\right)^2 \frac{\partial^2 f}{\partial r^2} + \ldots \right]_{\theta=0}.\end{aligned}$$

Since $\partial f/\partial r \propto 1/r$ (see figure above) we find for large r

$$\psi(z) = e^{i\,kz}\left[1 + i\,\frac{2\pi\,N\,d}{k}\,f(k,0)\right], \tag{10}$$

to be compared with the above result in terms of the refraction index n, which for $n \approx 1$ becomes

$$\psi(z) = e^{i\,kz}\left[1 + i\,kd(n-1)\right] \tag{11}$$

We obtain $n = n' + i\,n''$ with

$$n' = 1 + \frac{2\pi\,N}{k^2}\,\mathcal{R}e\,f(k,0), \tag{12}$$

$$n'' = \frac{2\pi\,N}{k^2}\,\mathcal{I}m\,f(k,0) = \frac{N\,\sigma_T}{2\,k}. \tag{13}$$

For the latter we have used the optical theorem, $\mathcal{I}m\,f(k,0) = (k/4\pi)\,\sigma_T$. From the wave function in the slab ($e^{i\,nkz}$) we see that the imaginary part corresponds to the damping of the wave,

$$|\psi(z)| \;\propto\; \exp(-n''kz) = \exp(-N\sigma_T z/2) = \exp(-z/\lambda_c), \tag{14}$$

with the collision length

$$\lambda_c = \frac{2}{N\sigma_T}. \tag{15}$$

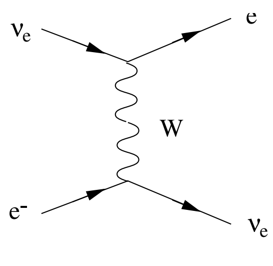
As an example, for neutrinos the total cross sections are of the order of 10^{-42} m^2 and we have

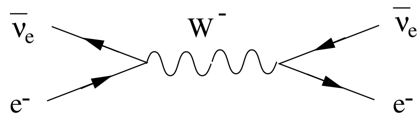
$$\lambda_c = \frac{[2\times 10^{12}\ \mathrm{m}]}{(\rho/\rho_{\mathrm{water}})\times((\sigma_T/E)/[10^{-42}\ \mathrm{m}^2/\mathrm{GeV}])\times(E/[1\ \mathrm{GeV}])}. \tag{16}$$

Note that 2×10^{12} m is about 13 times the distance Earth-Sun.

IV. THE MSW EFFECT FOR SOLAR NEUTRINOS

The results of the previous section can be used to study propagating neutrinos in the sun. In matter neutrinos undergo phase rotations because the elastic scattering in matter, which proceed via t-channel exchange of Z^0-bosons (Mikheyev, Smirnov and Wolfenstein effect). This happens for all neutrino types, so all phase rotations are the same. For electron neutrinos and antineutrinos we have an additional contribution in the scattering amplitude via t-channel or s-channel W-exchange respectively,





Elastic $\overline{\nu}_e e^-$ scattering via W-exchange

Elastic $\nu_e e^-$ scattering via W-exchange

These elastic processes are in matter only possible for electron neutrinos or antineutrinos since they require scattering off the corresponding lepton, in this case the electron. The amplitude for $\overline{\nu}_e(k) + e^-(p) \to \overline{\nu}_e(k')e^-(p')$ becomes (including a factor 1/2 for averaging over electron spins)

$$\begin{aligned} -i\,\mathcal{M}(s,\theta) &= -\frac{g^2}{2}\,\frac{1}{2}\,(\overline{v}_{\nu L}(k)\gamma^\rho u_{eL}(p))\,\frac{-i\,g_{\rho\sigma}+\ldots}{s+M_W^2}\,(\overline{u}_{eL}(p')\gamma^\sigma v_{\nu L}(k')) \\ &= i\,G_F\,\sqrt{2}\,(\overline{v}_{\nu L}(k)\gamma^\rho u_{eL}(p))\,(\overline{u}_{eL}(p')\gamma_\rho v_{\nu L}(k')) \end{aligned} \tag{17}$$

For the forward scattering amplitude ($p = p'$ and $k = k'$) we obtain

$$\begin{aligned} -i\,\mathcal{M}(s,0) &= i\,G_F\,\sqrt{2}\,\mathrm{Tr}\left(\frac{1-\gamma_5}{2}\,\not{k}\gamma^\rho\not{p}\gamma_\rho\right) \\ &= -i\,G_F\,2\sqrt{2}\,\mathrm{Tr}\left(\frac{1-\gamma_5}{2}\,\not{k}\not{p}\right) \\ &= -i\,2\sqrt{2}\,G_F\,s \end{aligned} \tag{18}$$

The charged current part of the forward quantummechanical scattering amplitude becomes (with $\sqrt{s} = 2E = 2k$)

$$f_{cc}(k,0) = -\frac{\mathcal{M}(s,0)}{8\pi\,\sqrt{s}} = -\frac{G_F}{\pi\sqrt{2}}\,k. \tag{19}$$

The additional phase in matter for electron-type neutrinos is

$$\exp\left(i\,\frac{2\pi\,N_e\,f_{cc}}{k}\,x\right) = \exp\left(i\,N_e\,G_F\sqrt{2}\,x\right) = \exp\left(i\,N_e\,G_F\sqrt{2}\,t\right),$$

corresponding to a wavelength

$$\lambda_e = \frac{2\pi}{N_e\,G_F\sqrt{2}}, \tag{20}$$

numerically of the order of

$$\lambda_e \approx \frac{2\times 10^7\ \mathrm{m}}{(\rho/\rho_{\mathrm{water}})}.$$

Again let us consider a two-neutrino mixing scenario with $E_1 \approx E_2 \approx E$ and $E_1 - E_2 \approx \Delta m^2/2E \ll E$, and assume the (in this case relevant) electron-neutrino to be a linear combination of vacuum neutrino eigenstates ν_1 and ν_2. We assume that $\langle\nu_1|\nu_e\rangle = \cos\theta$ and $\langle\nu_2|\nu_e\rangle = \sin\theta$. In this case we will, unlike the case of oscillations considered in section 2, work on the basis ν_e and another flavor, say ν_ℓ. This implies that in matter the effective hamiltonian becomes

$$\begin{aligned} H &= H_{\mathrm{vacuum}} + \frac{N_e\,G_F}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= E\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{N_e\,G_F}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\quad + \frac{\Delta m^2}{4E}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &\approx E\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta m^2}{4E}\begin{pmatrix} D-\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -(D-\cos 2\theta) \end{pmatrix} \end{aligned} \tag{21}$$

where

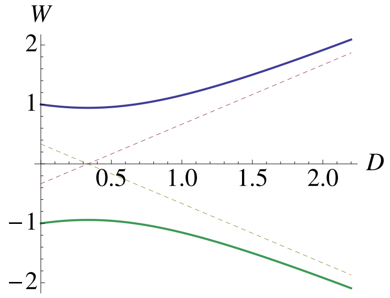
$$D = \sqrt{2}\,G_F\,N_e\,\frac{2E}{\Delta m^2} = \frac{\lambda_V}{\lambda_e}. \tag{22}$$

The eigenvalues of H in matter then are $E \pm W\,\Delta m^2/4E$, where

$$W^2 = (D-\cos 2\theta)^2 + \sin^2 2\theta. \tag{23}$$

Defining the angle θ_m via $W\cos 2\theta_m \equiv \cos 2\theta - D$ and $W\sin 2\theta_m \equiv \sin 2\theta$, we recover at $D = 0$ the usual mixing with $\theta_m = \theta$. For finite D the angle $\theta_m = \theta_m(D)$ and we can write the Hamiltonian as

$$H \approx E\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta m^2}{4E}\begin{pmatrix} -W\,\cos 2\theta_m & W\,\sin 2\theta_m \\ W\,\sin 2\theta_m & +W\,\cos 2\theta_m \end{pmatrix}. \tag{24}$$



Let us investigate this for a typical two-state solar neutrino situation with $\cos\theta = c_{12} \approx \sqrt{2/3}$ (and $\sin\theta = \sqrt{1/3}$). Up to the overall factor $\Delta m^2/4E$, the diagonal elements of the Hamiltonian $\cos 2\theta \pm D$ are given as the dashed lines in the Figure. The eigenvalues $\pm W$ are given as the solid lines. The results have been plotted as a function of D, proportional to the matter density.

The eigenstates of the matrix in the above Hamiltonian are at $\pm W$. At $D \neq 0$, they are just $W = \pm 1$. At the level crossing for the diagonal elements, at $D - \cos 2\theta = 0$ (in our case $D = 1/3$), the level distance is minimal and $\theta_m = \pi/4$. At large values of D, the eigenvalues approach $\pm W \approx \pm D$ and $\theta_m \to \pi/2$.

The corresponding eigenstates in matter are

$$\nu_1 \propto \begin{pmatrix} \cos\theta_m \\ \sin\theta_m \end{pmatrix} \quad \text{but also} \quad \nu_2 \propto \begin{pmatrix} -\sin\theta_m \\ \cos\theta_m \end{pmatrix}, \tag{25}$$

where $\theta_m(D)$ is the mixing angle in matter. If a ν_e is produced at finite density D below the critical density $D = \cos 2\theta$, one is in the combinations of the two levels at low D and the situation is like vacuum oscillations. The ν_e is divided over ν_1 and ν_2 with probabilities $P_1 = \cos^2\theta_m \approx \cos^2\theta \approx 2/3$ and $P_2 = \sin^2\theta_m \approx \sin^2\theta \approx 1/3$. If a ν_e is produced at very large density D far above the critical density, the neutrinos are mostly (probability $sin^2\theta_m \to 1$) in the upper ν_2 level and, moving adiabatically through the Sun, end up at the upper level ν_2 when reaching the surface.

Numerically one has

$$D = \frac{\lambda_V}{\lambda_e} = \frac{E[\text{MeV}] \times (\rho/\rho_{\text{water}})}{10^7 \times \Delta m^2[\text{eV}^2]},$$

i.e. for $E \sim 1$ MeV and $(\rho/\rho_{\text{water}}) \sim 100$ one finds $D \sim 1$ for $\Delta m^2 \sim 10^{-5}$ eV^2. The present-day values for the solar neutrino oscillations are $\Delta m^2 = \Delta m^2_\odot \approx 7.6 \times 10^{-5}$ eV^2 and $\sin^2\theta = \sin^2\theta_\odot \approx 0.31$.

V. THE ADIABATIC APPROXIMATION IN QUANTUM MECHANICS

The adiabatic behavior is contrasted with the *sudden approximation* that is used in the case of atmospheric neutrino oscillations. In that case a wave function $\psi(0)$ at a given instant is expressed in terms of eigenstates of a new Hamiltonian, H_+, which determines the time evolution for $t > 0$. Its validity can be investigated and requires the timescale of the change to be much faster than the typical timescales related to the initial motion. In the adiabatic approximation, the change in the Hamiltonian is slow. Although the treatment is similar on some parts, the difference with time-dependent perturbation theory is that the whole system (full spectrum) changes, usually due to the environment, expressed as $H(t) = H(N(t))$, where $N(t)$ in the case of neutrinos is the matter density in the Sun.

We simply start with writing down instantaneous sets of (normalized) eigenfunctions,

$$H(t)\phi_n(t) = E_n(t)\,\phi_n(t), \tag{26}$$

and solve for

$$i\hbar\,\frac{\partial}{\partial t}\psi = H(t)\,\psi(t). \tag{27}$$

For the solution we use the ansatz

$$\psi(t) = \sum_n c_n(t)\,\phi_n(t)\,e^{i\theta_n(t)}, \tag{28}$$

with

$$i\theta_n(t) = \frac{1}{i\hbar}\int_0^t dt'\; E_n(t'), \tag{29}$$

satisfying $i\dot{\theta}_n(t) = E_n(t)/i\hbar = -i\,\omega_n(t)$. Singling out this phase is just convenient. For a time-independent Hamiltonian with time-independent eigenfunctions and eigenvalues we actually get $i\theta_n(t) = -i\omega_n t$.

We find from the Schrödinger equation that

$$\dot{c}_p = -\sum_n \langle\phi_p|\dot{\phi}_n\rangle\, c_n\, e^{i(\theta_n-\theta_p)}. \tag{30}$$

Note all quantities in this expression are time-dependent. The matrix element can be related to the matrix element of $\dot{H}$, starting from Eq. 26, $\dot{H}\phi_n = -H\dot{\phi}_n + E_n\dot{\phi}_n + \dot{E}_n\phi_n$, giving

$$\langle\phi_p|\dot{H}|\phi_n\rangle = (E_n - E_p)\,\langle\phi_p|\dot{\phi}_n\rangle. \tag{31}$$

This gives the result

$$\dot{c}_p = -\langle\phi_p|\dot{\phi}_p\rangle\, c_p - \sum_{n\neq p} \frac{\langle\phi_p|\dot{H}|\phi_n\rangle}{E_n - E_p}\, c_n\, e^{i(\theta_n-\theta_p)}. \tag{32}$$

In the *adiabatic limit* the change of the Hamiltonian is assumed to be small compared to the intrinsic time-dependence, which is of the order of $\hbar/\Delta E$, where ΔE are typical energies or energy differences in the spectrum. Therefore, omitting the second term, and starting with $\psi(0) = \phi_n(0)$ in the nth eigenstate, one gets in the adiabatic limit

$$\psi(t) = e^{i\theta_n(t)}\, e^{i\gamma_n(t)}\, \phi_n(t), \tag{33}$$

where the phase γ_n, defined as

$$\gamma_n(t) \equiv i\int_0^t dt'\ \langle\phi_n|\dot{\phi}_n\rangle, \tag{34}$$

incorparates the effect of the (first) term on the righthandside in Eq. 32. As defined, the phase γ_n is real, because it's imaginary part (the real part of $\langle\phi_n|\dot{\phi}_n\rangle$) is zero for normalized wave functions. The phase γ_n is known as *Berry's phase*. It is not relevant for solar neutrinos. In that case, only the fact that at the surface of the Sun the neutrinos travelling to the Earth are mostly in level ν_2 if produced originally at high density. The condition on the adiabaticity can be checked, knowing the minimal level distance to be $E_2 - E_1 = \Delta m^2\,\sin 2\theta/2E$.

VI. SPIN IN QUANTUM MECHANICS

In the next sections we repeat a few topics needed to look at the Dirac equation and the treatment of Majorana neutrinos. The first topic is that of symmetry groups, starting with the rotations.

Looking in 3-dimensional space, rotations around the z-axis are given by

$$\begin{pmatrix} V'^1 \\ V'^2 \\ V'^3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix}. \tag{35}$$

From the infinitesimal transformation $\mathcal{R}(\theta,\hat{z}) = 1 + i\theta\, J^3$, and the corresponding rotations around x and y axes, one immediately can obtain the (defining) generators and their commutation relations,

$$[J^i, J^j] = i\hbar\,\epsilon^{ijk}\, J^k \tag{36}$$

(where $\epsilon^{123} = 1$). Including spin the rotations in quantummechanics are represented by unitary operators

$$U(\boldsymbol{\theta}) = \exp(i\,\boldsymbol{\theta}\cdot\boldsymbol{J}), \tag{37}$$

with the above commutation relations for the generators $\boldsymbol{J}$, but allowing different representations besides the defining three-dimensional one.

Starting with the commutation relations one can study spin states. First we note that an operator that commutes with all three spin operators (a socalled Casimir operator) is $\boldsymbol{J}^2$,

$$[\boldsymbol{J}^2, J^i] = 0. \tag{38}$$

Only one of the three spin operators can be used to label states, for which we without loss of generality can take J^3. In addition we can use $\boldsymbol{J}^2$, which commutes with J^3. We write states $\chi_m^{(s)} = |s,m\rangle$ satisfying

$$\boldsymbol{J}^2 |J,M\rangle = \hbar^2\, J(J+1)|J,M\rangle, \tag{39}$$

$$J^3|J,M\rangle = M\hbar\,|J,M\rangle, \tag{40}$$

where we have used the fact that the eigenvalue of $\boldsymbol{J}^2$ must be positive. We recombine the operators J^1 and J^2 into

$$J_\pm \equiv J^1 \pm i\, J^2. \tag{41}$$

The commutation relations for these operators are,

$$[\boldsymbol{J}^2, J_\pm] = 0, \tag{42}$$

$$[J^3, J_\pm] = \pm\hbar\, J_\pm, \tag{43}$$

$$[J_+, J_-] = 2\hbar\, J^3. \tag{44}$$

Up to an overall sign convention one can show that

$$\begin{aligned} J_\pm|J,M\rangle &= \hbar\sqrt{J(J+1) - M(M\pm 1)}\,|J,M+1\rangle, \\ &= \hbar\sqrt{(J\mp M)(J\pm M+1)}\,|J,M+1\rangle \end{aligned} \tag{45}$$

This shows that given a state $|J,M\rangle$, we have a whole series of states

$$\ldots |J,M-1\rangle,\ |J,M\rangle,\ |J,M+1\rangle, \ldots$$

But, we can also easily see that since $\boldsymbol{J}^2 - (J^3)^2 = (J^1)^2 + (J^2)^2$ must be an operator with positive definite eigenstates that $J(J+1) - M^2 \geq 0$, i.e. $|M| \leq \sqrt{J(J+1)}$ or strictly $|M| < J+1$. From the actions for the raising and lowering operators one sees that this inequality requires $M_{max} = J$ as one necessary state to achieve a cutoff of the series of states on the upper side, while $M_{min} = -J$ is required as a necessary state to achieve a cutoff of the series of states on the lower side. Moreover to have both cutoffs the step operators require that the difference $M_{max} - M_{min} = 2\,J$ must be an integer, i.e. the only allowed values of spin quantum numbers are

$$\begin{aligned} &J = 0,\ 1/2,\ 1,\ 3/2, \ldots, \\ &M = J, J-1, \ldots, -J. \end{aligned}$$

Thus for spin states with a given quantum number J, there exist $2J+1$ states.

In the space of spin states with a given quantum number J, we can write the spin operators as $(2J+1)\times(2J+1)$ matrices. The most well-known and simplest nontrivial example is the case spin $J = 1/2$, for which a basis is given by

$$\chi_\uparrow = |1/2, +1/2\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_\downarrow = |1/2, -1/2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using the definition of the quantum numbers in Eq. 40 one finds that

$$J^3 = \hbar \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad J_+ = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_- = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

For spin 1/2 we find the familiar spin matrices, $\boldsymbol{J} = \hbar\boldsymbol{\sigma}/2$,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With the representation matrices we can consider rotated states using the unitary matrices U ($U^\dagger = U^{-1}$),

$$\begin{aligned} &\xi \to U\xi, \\ &U(\boldsymbol{\theta}) = \exp(i\,\boldsymbol{\theta}\cdot\sigma/2). \end{aligned} \tag{46}$$

The useful relation

$$U(\boldsymbol{\theta})\,(\boldsymbol{\sigma}\cdot\boldsymbol{a})\,U^{-1}(\boldsymbol{\theta}) = \boldsymbol{\sigma}\cdot\mathcal{R}(\boldsymbol{\theta})\boldsymbol{a}, \tag{47}$$

shows that when ξ is an eigenstates of $\boldsymbol{\sigma}\cdot\boldsymbol{a}$, then $U\xi$ is an eigenstate of $\boldsymbol{\sigma}\cdot\mathcal{R}\boldsymbol{a}$. For a rotation around the y-axis over an angle θ, one finds the following two rotated states

$$\chi_\uparrow(\hat{\boldsymbol{z}}') = \left(\begin{array}{c}\cos(\theta/2)\\ \sin(\theta/2)\end{array}\right), \qquad \chi_\downarrow(\hat{\boldsymbol{z}}') = \left(\begin{array}{c}-\sin(\theta/2)\\ \cos(\theta/2)\end{array}\right),$$

eigenstates of $\boldsymbol{J}\cdot\hat{\boldsymbol{z}}'$ with eigenvalues $\pm\hbar/2$. In general the rotated eigenstates are written as

$$\chi_M^{(J)}(\hat{\boldsymbol{n}}) = \left(\begin{array}{c} d^{(J)}_{JM}(\theta)\\ \vdots \\ d^{(J)}_{M'M}(\theta)\\ \vdots \\ d^{(J)}_{-JM}(\theta)\end{array}\right). \tag{48}$$

where $d_{M'M}(\theta)$ are the d-functions. These are in fact just matrix elements of the spin rotation matrix $\exp(-i\,\theta\,J^2)$ between states quantized along the z-direction. Extended to include azimuthal dependence it is customary to use the rotation matrix $e^{-i\,\phi\,J^3}\,e^{-i\,\theta\,J^2}\,e^{-i\,\chi\,J^3}$ and the functions are called $D_{M'M}(\phi,\theta,\chi)$.

Finally a remark on the conjugate representation, for which the transformation matrices are U^*. Using $\epsilon = i\sigma^2$ (satisfying $\epsilon^* = \epsilon$, $\epsilon^{-1} = \epsilon^\dagger = \epsilon^T = -\epsilon$), which relates

$$-\boldsymbol{\sigma}^* = \epsilon\,\boldsymbol{\sigma}\,\epsilon^{-1} = -\epsilon\,\boldsymbol{\sigma}\,\epsilon, \tag{49}$$

shows that $\epsilon\xi^*$ transforms as ξ and hence the equivalence of spin representations with their conjugate representations, something that cannot be done in for instance $SU(3)$.

VII. SPIN IN RELATIVISTIC THEORIES

In the previous section spin has been introduced as a representation of the rotation group ($SU(2)$) without worrying much about the rest of the symmetries of the world. We have considered the rotation group, more specifically its generators $\boldsymbol{J}$ and then looked for representations among them the two-dimensional (defining) one, $\boldsymbol{J} = \boldsymbol{\sigma}/2$.

The starting point in relativistic theories is the Poincaré group, consisting of the Lorentz group and the translations. The Lorentz transformations are defined as transformations in Minkowski space,

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu, \tag{50}$$

containing the well-known rotations and the boosts. For instance rotations $V' = \mathcal{R}V$ around the z-axis are given by

$$\begin{pmatrix} V'^0\\ V'^1\\ V'^2\\ V'^3\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} V^0\\ V^1\\ V^2\\ V^3\end{pmatrix}, \tag{51}$$

and the boosts $V' = \mathcal{M}V$ along the z-direction by

$$\begin{pmatrix} V'^0\\ V'^1\\ V'^2\\ V'^3\end{pmatrix} = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sinh\phi & 0 & 0 & \cosh\phi\end{pmatrix}\begin{pmatrix} V^0\\ V^1\\ V^2\\ V^3\end{pmatrix} \tag{52}$$

with $-\infty < \phi < \infty$. Note that the velocity $\beta = v = v/c$ (taking $c = 1$ as usual). and the Lorentz contraction factor $\gamma = (1-\beta^2)^{-1}$ corresponding to the boost are related to ϕ as $\gamma = \cosh\phi$, $\beta\gamma = \sinh\phi$ Applying the boost to the momentum vector of a particle at rest one obtains $E = M\,\cosh\phi$ and $|\boldsymbol{p}| = M\,\sinh\phi$. The rotations and boosts together constitute in fact only the socalled proper orthochrone Lorentz transformations. The full Lorentz group also includes space and time inversion.

The generators $\boldsymbol{J}$ of rotations $U(\boldsymbol{\theta}) = \exp(i\boldsymbol{\theta}\cdot\boldsymbol{J})$ and the generators $\boldsymbol{K}$ of the boosts $U(\boldsymbol{\theta}) = \exp(i\boldsymbol{\phi}\cdot\boldsymbol{K})$ are given by are given by

$$J^3 \equiv M^{12} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad K^3 \equiv M^{03} = \begin{pmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -i & 0 & 0 & 0\end{pmatrix}$$

etc. They satisfy the commutation relations

$$[J^i, J^j] = i\,\epsilon^{ijk}\,J^k,$$
$$[J^i, K^j] = i\,\epsilon^{ijk}\,K^k,$$
$$[K^i, K^j] = -i\,\epsilon^{ijk}\,J^k.$$

VIII. SPIN 1/2 AND THE LORENTZ GROUP

It is straightforward to see that both the following two-dimensional representations satisfy the commutation relations of the Lorentz group,

$$\text{Type I:} \qquad \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \boldsymbol{K} = -i\,\frac{\boldsymbol{\sigma}}{2}, \tag{53}$$
$$\text{Type II:} \qquad \boldsymbol{J} = \frac{\boldsymbol{\sigma}}{2}, \quad \boldsymbol{K} = i\,\frac{\boldsymbol{\sigma}}{2}, \tag{54}$$

Correspondingly one has spinors ξ and η transforming similarly under unitary rotations ($U^\dagger = U^{-1}$, $\overline{U} \equiv (U^\dagger)^{-1} = U$)

$$\xi \to U\xi, \qquad \eta \to U\eta,$$
$$\overline{U}(\boldsymbol{\theta}) = U(\boldsymbol{\theta}) = \exp(i\,\boldsymbol{\theta}\cdot\sigma/2) \tag{55}$$

but differently under hermitean boosts ($H^\dagger = H$, $\overline{H} \equiv (H^\dagger)^{-1} = H^{-1}$), namely

$$\xi \longrightarrow H\xi, \qquad \eta \to \overline{H}\eta,$$
$$H(\boldsymbol{\phi}) = \exp(\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2) \quad \text{and} \quad \overline{H}(\boldsymbol{\phi}) = \exp(-\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2). \tag{56}$$

A practical boost for the spinors is the one taking a spinor from rest momentum $\boldsymbol{p}$,

$$H(\boldsymbol{p}) = \frac{M + E + \boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2M(E+M)}}. \tag{57}$$

Introducing the four matrices

$$\sigma^\mu \equiv (\boldsymbol{I}, \boldsymbol{\sigma}), \qquad \bar{\sigma}^\mu \equiv (\boldsymbol{I}, -\boldsymbol{\sigma}), \tag{58}$$

(note that these are fake four-vectors) one has

$$U\,\sigma^\mu a_\mu\,U^{-1} = \sigma^\mu \mathcal{R} a_\mu, \qquad H\,\sigma^\mu a_\mu\,\bar{H} = \sigma^\mu \mathcal{M} a_\mu$$
$$U\,\bar{\sigma}^\mu a_\mu\,U^{-1} = \bar{\sigma}^\mu \mathcal{R} a_\mu, \qquad \bar{H}\,\bar{\sigma}^\mu a_\mu\,H = \bar{\sigma}^\mu \mathcal{M} a_\mu$$

while the matrix $\epsilon = i\sigma^2$ (satisfying $\epsilon^* = \epsilon$, $\epsilon^{-1} = \epsilon^\dagger = \epsilon^T = -\epsilon$), can be used to relate $\bar{\sigma}^{\mu *} = \epsilon\,\sigma^\mu\,\epsilon^{-1} = -\epsilon\,\sigma^\mu\,\epsilon$, implying

$$U^* = \epsilon\,U\,\epsilon^{-1} = -\epsilon\,U\,\epsilon \quad \text{and} \quad H^* = \epsilon\,\bar{H}\,\epsilon^{-1} = -\epsilon\,\bar{H}\,\epsilon, \tag{59}$$

Furthermore one has $H^2(\boldsymbol{p}) = \bar{\sigma}^\mu p_\mu/M = (E + \boldsymbol{\sigma}\cdot\boldsymbol{p})/M$.

The most general lagrangian density for spin 1/2 representations invariant under boosts and rotations is

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}\,\xi^\dagger(i\,\sigma^\mu \overset{\leftrightarrow}{\partial}_\mu)\xi + \frac{1}{2}\,\eta^\dagger(i\,\bar{\sigma}^\mu \overset{\leftrightarrow}{\partial}_\mu)\eta - \frac{1}{2}\left(M_D\,\eta^\dagger\xi - M_D^*\,\eta^T\xi^*\right) - \frac{1}{2}\left(M_D^*\,\xi^\dagger\eta - M_D\,\xi^T\eta^*\right) \\ & - \frac{1}{2}\left(M_M^*\,\xi^\dagger\epsilon\,\xi^* - M_M\,\xi^T\epsilon\,\xi\right) - \frac{1}{2}\left(M_M\,\eta^\dagger\epsilon\,\eta^* - M_M^*\,\eta^T\epsilon\,\eta\right),\end{aligned} \tag{60}$$

where M_M and M_D are referred to as Majorana and Dirac masses. Note that this lagrangian is a real number if ξ and η are Grassmann numbers for which $\alpha\beta = -\beta\alpha$ and $(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^*$.

Considering discrete transformations, one knows that parity leaves $\boldsymbol{J}$ invariant, but changes the sign of $\boldsymbol{K}$. Thus for the spin 1/2 representations:

$$\xi \xrightarrow{\mathcal{P}} \eta, \qquad \eta \xrightarrow{\mathcal{P}} \xi. \tag{61}$$

Conjugate spin 1/2 representations transform under U^* and H^*. As we have seen $\epsilon\,\xi^*$ and $\epsilon\,\eta^*$ are equivalent to ξ and η as far as rotations is concerned. The spinors ξ^* and η^*, however, are not appropriate type I or type II spinors. Type I spinors are ξ and $\epsilon\,\eta^*$, while type II spinors are η and $\epsilon\,\xi^*$. Therefore charge conjugation can be defined as

$$\xi \xrightarrow{\mathcal{C}} -\epsilon\,\eta^*, \qquad \eta \xrightarrow{\mathcal{C}} \epsilon\,\xi^*. \tag{62}$$

The equations of motion can be obtained easily using the Euler-Lagrange equations, e.g.

$$i\,\bar\sigma^\mu\partial_\mu\,\eta - M_M\,\epsilon\,\eta^* - M_D^*\,\xi = 0, \tag{63}$$
$$i\,\sigma^\mu\partial_\mu\,\xi - M_M\,\epsilon\,\xi^* - M_D\eta = 0, \tag{64}$$
$$i\,\bar\sigma^\mu\partial_\mu\,\epsilon\,\xi^* - M_M^*\,\xi + M_D\epsilon\,\eta^* = 0, \tag{65}$$

implying also $(\partial^2 + |M_M|^2 + |M_D|^2)\,\xi = 0$ and similarly for η. Thus the solutions are plane waves, but the (linear) equations impose conditions. Writing

$$\xi = \xi(p)\,e^{-i\,p\cdot x} \qquad \text{with} \quad \xi(p) = H(p)\xi_0, \tag{66}$$
$$\eta = \eta(p)\,e^{-i\,p\cdot x} \qquad \text{with} \quad \eta(p) = \bar H(p)\eta_0, \tag{67}$$

one obtains with $M = \sqrt{|M_M|^2 + |M_D|^2}$

$$\begin{aligned} &M\,\xi_0 - M_M\,\epsilon\,\xi_0^* - M_D\,\eta_0 = 0,\\ &M\,\eta_0 - M_M\,\epsilon\,\eta_0^* - M_D^*\,\xi_0 = 0. \end{aligned} \tag{68}$$

Let us first discuss a general feature for massless fermions, $M_D = M_M = 0$. In this case one has definite helicity states

$$\sigma^\mu p_\mu\,\xi(p) = 0 \longrightarrow (E - \boldsymbol{\sigma}\cdot\boldsymbol{p})\,\xi(p) = 0 \longrightarrow \xi(p) = \xi_+(p), \tag{69}$$
$$\bar\sigma^\mu p_\mu\,\eta(p) = 0 \longrightarrow (E + \boldsymbol{\sigma}\cdot\boldsymbol{p})\,\eta(p) = 0 \longrightarrow \eta(p) = \eta_-(p). \tag{70}$$

We will consider the massive examples below for some special cases.

IX. DIRAC FERMION

For a *Dirac fermion* one considers the case $M_M = 0$. Writing M_D as $|M_D|\,e^{i\phi}$, one sees from the constraint in Eq. 68 that it requires $\eta_0 = e^{-i\phi}\,\xi_0$. In order to incorporate parity, the fields ξ and η in the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\,\xi^\dagger(i\,\sigma^\mu\,\overleftrightarrow{\partial}_\mu)\xi + \frac{1}{2}\,\eta^\dagger(i\,\bar\sigma^\mu\,\overleftrightarrow{\partial}_\mu)\eta - M_D\,\eta^\dagger\xi - M_D^*\,\xi^\dagger\eta \\ &= \begin{pmatrix}\xi^\dagger & \eta^\dagger\end{pmatrix}\begin{pmatrix}\frac{1}{2}\,i\sigma^\mu\,\overleftrightarrow{\partial}_\mu & -M_D^* \\ -M_D & \frac{1}{2}\,i\bar\sigma^\mu\,\overleftrightarrow{\partial}_\mu\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\eta^\dagger & \xi^\dagger\end{pmatrix}\begin{pmatrix}-M_D & \frac{1}{2}\,i\bar\sigma^\mu\,\overleftrightarrow{\partial}_\mu \\ \frac{1}{2}\,i\sigma^\mu\,\overleftrightarrow{\partial}_\mu & -M_D^*\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \end{aligned} \tag{71}$$

are written in terms of a four-component spinor

$$\psi \equiv \begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{72}$$

giving

$$\mathcal{L} = \overline{\psi}\left(\frac{1}{2}\,i\,\overleftrightarrow{\not\partial}_\mu - M - iM'\gamma_5\right)\psi \tag{73}$$

with $\not\partial = \gamma^\mu\partial_\mu$, $\overline{\psi} \equiv \psi^\dagger\,\gamma^0$, $M = \text{Re}\ M_D$, $M' = \text{Im}\ M_D$ and γ matrices defined as

$$\gamma^0 = \begin{pmatrix}0 & \boldsymbol{I}\\ \boldsymbol{I} & 0\end{pmatrix}, \qquad \gamma^i = \begin{pmatrix}0 & -\sigma^i\\ \sigma^i & 0\end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix}\boldsymbol{I} & 0\\ 0 & -\boldsymbol{I}\end{pmatrix} = i\,\gamma^0\gamma^1\gamma^2\gamma^3. \tag{74}$$

The mass-term can actually be choosen real by a chiral rotation of the spinor, $\psi \to \exp(i\alpha\gamma_5)\,\psi$. Under parity and charge conjugation one has (omitting the space-time arguments which also change)

$$\psi = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \xrightarrow{\mathcal{P}} \begin{pmatrix}\eta\\ \xi\end{pmatrix} = \gamma^0\,\psi \tag{75}$$

$$\psi = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \xrightarrow{\mathcal{C}} \begin{pmatrix}-\epsilon\,\eta^*\\ \epsilon\,\xi^*\end{pmatrix} = C\,\overline{\psi}^T \qquad \text{with } C = i\gamma^2\gamma^0 = \begin{pmatrix}-i\sigma^2 & 0\\ 0 & i\sigma^2\end{pmatrix}. \tag{76}$$

The components of the Dirac spinor, ξ and η, are referred to as *chiral right* (righthanded, R) and *chiral left* (lefthanded, L) components. They are projected with the help of γ_5, $\psi_{R/L} \equiv \frac{1}{2}\,(1 \pm \gamma_5)\,\psi$. The Dirac Lagrangian split up into lefthanded and righthanded fields thus in essence is of the form of the $\xi - \eta$ lagrangian,

$$\mathcal{L} = \overline{\psi}_R \frac{1}{2}\, i \overset{\leftrightarrow}{\not{\partial}_\mu}\; \psi_R + \overline{\psi}_L \frac{1}{2}\, i \overset{\leftrightarrow}{\not{\partial}_\mu}\; \psi_L - M\,(\overline{\psi}_R\,\psi_L + \overline{\psi}_L\,\psi_R), \tag{77}$$

i.e. if $M = 0$ the lagrangian decouples into two independent lefthanded and righthanded fields, which are solutions of the free equations. In that case chiral states coincide with helicity states, as can be immediately seen in the explicit expressions above.

The γ-matrices can be defined independent of any representation. They form a Clifford algebra and the γ_5 anti-commutes with all other gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \{\gamma^\mu, \gamma_5\} = 0. \tag{78}$$

Under hermitian conjugation the gamma matrices behave as

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \qquad \left(\gamma^5\right)^\dagger = \gamma^5. \tag{79}$$

The charge conjugation matrix C has the following important properties

$$C^{-1}\gamma^\mu C = -\left(\gamma^\mu\right)^T, \qquad C^{-1}\gamma^5 C = \left(\gamma^5\right)^T, \tag{80}$$

and

$$C^{-1} = C^\dagger = C^T = -C. \tag{81}$$

Finally note that the spinors obtained after parity and charge conjugation can be chosen independent of the representation as

$$\psi^P = \gamma^0\,\psi \tag{82}$$

$$\psi^C = C\overline{\psi}^T = C\gamma^0\psi^* \text{ (if } \gamma^0 \text{ real).} \tag{83}$$

X. MAJORANA FERMION

To investigate this case in general we start with a lagrangian containing both type of fermions, ξ and η and in first instance allow different Majorana masses (thus breaking parity). It is convenient to rewrite the fields as $\eta = \chi_1$ and $\xi = \epsilon\,\chi_2^*$. This leads to the lagrangian

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}\,\chi_1^\dagger(i\,\bar{\sigma}^\mu \overset{\leftrightarrow}{\partial}_\mu)\chi_1 + \frac{1}{2}\,\chi_2^\dagger(i\,\bar{\sigma}^\mu \overset{\leftrightarrow}{\partial}_\mu)\chi_2 \\ & - \frac{1}{2}\left(M_D\,\chi_1^\dagger\,\epsilon\,\chi_2^* - M_D^*\,\chi_1^T\,\epsilon\,\chi_2\right) - \frac{1}{2}\left(M_D\,\chi_2^\dagger\,\epsilon\,\chi_1^* - M_D^*\,\chi_2^T\,\epsilon\,\chi_1\right) \\ & - \frac{1}{2}\left(M_1\,\chi_1^\dagger\,\epsilon\,\chi_1^* - M_1^*\,\chi_1^T\,\epsilon\,\chi_1\right) - \frac{1}{2}\left(M_2\,\chi_2^\dagger\,\epsilon\,\chi_2^* - M_2^*\,\chi_2^T\,\epsilon\,\chi_2\right).\end{aligned} \tag{84}$$

It is the lagrangian for two (lefthanded) fermions with for $\chi = (\chi_1, \chi_2)$ a mass matrix $\chi^\dagger M \chi^*$ + h.c.,

$$M = \begin{pmatrix} M_1 & M_D \\ M_D & M_2 \end{pmatrix} = \begin{pmatrix} M_1 & |M_D|\,e^{i\phi} \\ |M_D|\,e^{i\phi} & M_2 \end{pmatrix}, \tag{85}$$

assuming in the last expression that M_1 and M_2 are real and non-negative. This choice is possible without loss of generality because the phases can be absorbed into χ_1 and χ_2.

This is a mixing problem (with an symmetric complex mass matrix) leading to two (real) mass eigenstates. Assuming the mass matrix to be diagonal for $\phi = U\chi$ one sees that $U\,M\,U^T = M_0$. This implies $U^*\,M^\dagger\,U^\dagger = M_0$ and thus a 'normal' diagonalization of the (hermitean) matrix $MM^\dagger$,

$$U\,(MM^\dagger)\,U^\dagger = M_0^2, \tag{86}$$

Thus one obtains from

$$MM^\dagger = \begin{pmatrix} M_1^2 + |M_D|^2 & |M_D|\,\left(M_1\,e^{-i\phi} + M_2\,e^{+i\phi}\right) \\ |M_D|\,\left(M_1\,e^{+i\phi} + M_2\,e^{-i\phi}\right) & M_2^2 + |M_D|^2 \end{pmatrix}, \tag{87}$$

the eigenvalues

$$m_{1/2}^2 = \frac{1}{2}\left[M_1^2 + M_2^2 + 2|M_D|^2 \pm \sqrt{(M_1^2 - M_2^2)^2 + 4|M_D|^2\,\left(|M_D|^2 + M_1^2 + M_2^2 + 2M_1M_2\,\cos(2\phi)\right)}\right], \tag{88}$$

and we are left with the case $M_D = 0$ of two decoupled (lefthanded) Majorana fields, $\eta_1 = U\chi_1$ and $\eta_2 = U\chi_2$, each of them of the form

$$\mathcal{L} = \frac{1}{2}\,\eta_i^\dagger(i\,\bar\sigma^\mu \overleftrightarrow{\partial}_\mu)\eta_i - \frac{1}{2}\,\left(m_i\,\eta_i^\dagger\,\epsilon\,\eta_i^* - m_i^*\,\eta_i^T\,\epsilon\,\eta_i\right), \tag{89}$$

with a real mass. The mass-terms can then actually be combined into a single term

$$-m_\eta\,\eta^\dagger\,\epsilon\,\eta^* = -\frac{m_\eta}{2}\,\left(\eta^\dagger\,\epsilon\,\eta^* - \eta^T\,\epsilon\,\eta\right).$$

One can also decide to split up the 'kinetic' term in the same fashion as the mass term,

$$\frac{1}{2}\,\eta^\dagger(i\,\bar\sigma^\mu \overleftrightarrow{\partial}_\mu)\eta = \frac{1}{4}\,\eta^\dagger(i\,\bar\sigma^\mu \overleftrightarrow{\partial}_\mu)\eta - \frac{1}{4}\,\eta^T\,\epsilon\,(i\,\sigma^\mu \overleftrightarrow{\partial}_\mu)\,\epsilon\,\eta^*.$$

By combining $-\epsilon\,\eta^*$ and η into a self-conjugate ($\psi = \psi^C$) four-component spinor,

$$\psi \equiv \begin{pmatrix} -\epsilon\,\eta^* \\ \eta \end{pmatrix}, \tag{90}$$

one finds the lagrangian,

$$\mathcal{L} = \frac{1}{4}\,\overline{\psi}\,i\,\overleftrightarrow{\not\partial}\,\psi - \frac{1}{2}\,M\,\overline{\psi}\,\psi. \tag{91}$$

with the same γ-matrices as before. This differs by a factor 1/2 from the usual Dirac lagrangian, which comes because we for a self-conjugate field are essentially doubling a lagrangian in this case, without adding degrees of freedom. Separating into left- and right-handed components, one has $\psi_R^C = \psi_L$ and $\psi_L^C = \psi_R$ and we can write

$$\mathcal{L} = \frac{1}{2}\,\overline{\psi}_L i\,\overleftrightarrow{\not\partial}\,\psi_L - \frac{M}{2}\,\left(\overline{\psi_L^C}\,\psi_L + \overline{\psi}_L\,\psi_L^C\right) = \frac{1}{2}\,\overline{\psi}_R i\,\overleftrightarrow{\not\partial}\,\psi_R - \frac{M}{2}\,\left(\overline{\psi_R^C}\,\psi_R + \overline{\psi}_R\,\psi_R^C\right). \tag{92}$$

Exercise
In this exercise we investigate the two-Majorana case with only an off-diagonal mass term, showing that it enables us to write any Dirac field in terms of two Majorana fields.

Calculate for the special choice $M_L = M_R = 0$ and M_D real, the mass eigenvalues and show that the mixing matrix is

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

This enables us to rewrite the Dirac field in terms of Majorana spinors. Give the explicit expressions that relate ψ and ψ^c with Υ_1 and Υ_2.

(solution)
One finds $M_1 = M_2 = M_D$. For both left- and righthanded fields the relations between ψ, ψ^c and Υ_1 and Υ_2 are the same,

$$\psi = \frac{1}{\sqrt{2}}\,(\Upsilon_1 + i\,\Upsilon_2)\,, \qquad \psi^c = \frac{1}{\sqrt{2}}\,(\Upsilon_1 - i\,\Upsilon_2)\,.$$

XI. THE STANDARD MODEL: $SU(2)_W \otimes U(1)_Y$

Symmetry plays an essential role in the standard model that describes the elementary particles, the quarks (up, down, etc.), the leptons (elektrons, muons, neutrinos, etc.) and the gauge bosons responsible for the strong, electromagnetic and weak forces. In the standard model one starts with a very simple basic lagrangian for (massless) fermions which exhibits more symmetry than observed in nature. By introducing gauge fields and breaking the symmetry a more complex lagrangian is obtained, that gives a good description of the physical world. The procedure, however, implies certain nontrivial relations between masses and mixing angles that can be tested experimentally and sofar are in excellent agreement with experiment.

The lagrangian for the leptons consists of three families each containing an elementary fermion (electron e^-, muon μ^- or tau τ^-), its corresponding neutrino (ν_e, ν_μ and ν_τ) and their antiparticles. As they are massless, left- and righthanded particles, $\psi_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\psi$ decouple. For the neutrino only a lefthanded particle (and righthanded antiparticle) exist. Thus

$$\mathcal{L}^{(f)} = i\,\overline{e_R}\partial\!\!\!/ e_R + i\,\overline{e_L}\partial\!\!\!/ e_L + i\,\overline{\nu_{eL}}\partial\!\!\!/ \nu_{eL} + (\mu, \tau). \tag{93}$$

One introduces a (weak) $SU(2)_W$ symmetry under which e_R forms a singlet, while the lefthanded particles form a doublet, i.e.

$$L = \begin{pmatrix} L^0 \\ L^- \end{pmatrix} = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \qquad \text{with } T_W = \frac{1}{2} \text{ and } T_W^3 = \begin{cases} +1/2 \\ -1/2 \end{cases}$$

and

$$R = R^- = e_R \qquad\qquad \text{with } T_W = 0 \text{ and } T_W^3 = 0.$$

Thus the basic lagrangian density is a Dirac lagrangian with massless (independent left- and right-handed species)

$$\mathcal{L}^{(f)} = i\,\overline{L}\partial\!\!\!/ L + i\,\overline{R}\partial\!\!\!/ R, \tag{94}$$

which has an $SU(2)_W$ symmetry under transformations $e^{i\vec{\alpha}\cdot\vec{T}_W}$, explicitly

$$L \overset{SU(2)_W}{\longrightarrow} e^{i\,\vec{\alpha}\cdot\vec{\tau}/2} L, \tag{95}$$

$$R \overset{SU(2)_W}{\longrightarrow} R. \tag{96}$$

One notes that the charges of the leptons can be obtained as $Q = T_W^3 - 1/2$ for lefthanded particles and $Q = T_W^3 - 1$ for righthanded particles. The difference between charge and 3-component of isospin is called *weak hypercharge* and one writes

$$Q = T_W^3 + \frac{Y_W}{2}. \tag{97}$$

The weak hypercharge Y_W is an operator that generates a $U(1)_Y$ symmetry with for the lefthanded and righthanded particles different hypercharges, $Y_W(L) = -1$ and $Y_W(R) = -2$. The particles transform according to $e^{i\beta Y_W/2}$, explicitly

$$L \overset{U(1)_Y}{\longrightarrow} e^{-i\,\beta/2} L, \tag{98}$$

$$R \overset{U(1)_Y}{\longrightarrow} e^{-i\,\beta} R. \tag{99}$$

Next the $SU(2)_W \otimes U(1)_Y$ symmetry is made into a local symmetry introducing gauge fields $\vec{W}_\mu$ and B_μ in the covariant derivative $D_\mu = \partial_\mu - i\,g\,\vec{W}_\mu \cdot \vec{T}_W - i\,g'\,B_\mu Y_W/2$, explicitly

$$D_\mu L \;=\; \partial_\mu L - \frac{i}{2}\,g\,\vec{W}_\mu \cdot \vec{\tau}\,L + \frac{i}{2}\,g'\,B_\mu\,L, \tag{100}$$

$$D_\mu R \;=\; \partial_\mu R + i\,g'\,B_\mu\,R, \tag{101}$$

where $\vec{W}_\mu$ is a triplet of gauge bosons with $T_W = 1$, $T_W^3 = \pm 1$ or 0 and $Y_W = 0$ (thus $Q = T_W^3$) and B_μ is a singlet under $SU(2)_W$ ($T_W = T_W^3 = 0$) and also has $Y_W = 0$. Putting this in leads to

$$\mathcal{L}^{(f)} = \mathcal{L}^{(f1)} + \mathcal{L}^{(f2)}, \tag{102}$$

$$
\begin{aligned}
\mathcal{L}^{(f1)} &= i\,\overline{R}\gamma^\mu(\partial_\mu + ig'B_\mu)R + i\,\overline{L}\gamma^\mu(\partial_\mu + \frac{i}{2}\,g'B_\mu - \frac{i}{2}\,g\vec{W}_\mu\cdot\vec{\tau})L \\
\mathcal{L}^{(f2)} &= -\frac{1}{4}(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + g\,\vec{W}_\mu\times\vec{W}_\nu)^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2.
\end{aligned}
$$

In order to break the symmetry to the symmetry of the physical world, the $U(1)_Q$ symmetry (generated by the charge operator), a complex Higgs field

$$
\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\theta_2 + i\theta_1) \\ \frac{1}{\sqrt{2}}(\theta_4 - i\theta_3) \end{pmatrix} \tag{103}
$$

with $T_W = 1/2$ and $Y_W = 1$ is introduced, with the following lagrangian density consisting of a symmetry breaking piece and a coupling to the fermions,

$$
\mathcal{L}^{(h)} = \mathcal{L}^{(h1)} + \mathcal{L}^{(h2)}, \tag{104}
$$

where

$$
\begin{aligned}
\mathcal{L}^{(h1)} &= (D_\mu\phi)^\dagger(D^\mu\phi)\underbrace{-m^2\,\phi^\dagger\phi - \lambda\,(\phi^\dagger\phi)^2}_{-V(\phi)}, \\
\mathcal{L}^{(h2)} &= -G_e(\overline{L}\phi R + \overline{R}\phi^\dagger L),
\end{aligned}
$$

and

$$
D_\mu\phi = (\partial_\mu - \frac{i}{2}\,g\vec{W}_\mu\cdot\vec{\tau} - \frac{i}{2}\,g'\,B_\mu)\phi. \tag{105}
$$

The Higgs potential $V(\phi)$ is choosen such that it gives rise to spontaneous symmetry breaking with $\varphi^\dagger\varphi = -m^2/2\lambda \equiv v^2/2$. For the classical field the choice $\theta_4 = v$ is made, which assures with the choice of Y_W for the Higgs field that Q generates the remaining $U(1)$ symmetry. Using *local* gauge invariance θ_i for $i = 1$, 2 and 3 may be eliminated (the necessary $SU(2)_W$ rotation is precisely $e^{-i\vec{\theta}(x)\cdot\tau}$), leading to the parametrization

$$
\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{106}
$$

and

$$
D_\mu\phi = \begin{pmatrix} -\frac{ig}{2}\left(\frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}\right)(v+h) \\ \frac{1}{\sqrt{2}}\,\partial_\mu h + \frac{i}{2}\left(\frac{gW_\mu^3 - g'B_\mu}{\sqrt{2}}\right)(v+h) \end{pmatrix}. \tag{107}
$$

Up to cubic terms, this leads to the lagrangian

$$
\begin{aligned}
\mathcal{L}^{(h1)} &= \frac{1}{2}(\partial_\mu h)^2 + m^2\,h^2 + \frac{g^2\,v^2}{8}\left[(W_\mu^1)^2 + (W_\mu^2)^2\right] \\
&\quad + \frac{v^2}{8}\left(gW_\mu^3 - g'B_\mu\right)^2 + \ldots \qquad (108) \\
&= \frac{1}{2}(\partial_\mu h)^2 + m^2\,h^2 + \frac{g^2\,v^2}{8}\left[(W_\mu^+)^2 + (W_\mu^-)^2\right] \\
&\quad + \frac{(g^2 + g'^2)\,v^2}{8}(Z_\mu)^2 + \ldots, \qquad (109)
\end{aligned}
$$

where the quadratically appearing gauge fields that are furthermore eigenstates of the charge operator are

$$
\begin{aligned}
W_\mu^\pm &= \frac{1}{\sqrt{2}}\left(W_\mu^1 \pm i\,W_\mu^2\right), &(110)\\
Z_\mu &= \frac{g\,W_\mu^3 - g'\,B_\mu}{\sqrt{g^2 + g'^2}} \equiv \cos\theta_W\,W_\mu^3 - \sin\theta_W\,B_\mu, &(111)\\
A_\mu &= \frac{g'\,W_\mu^3 + g\,B_\mu}{\sqrt{g^2 + g'^2}} \equiv \sin\theta_W\,W_\mu^3 + \cos\theta_W\,B_\mu, &(112)
\end{aligned}
$$

and correspond to three massive particle fields ($W^{\pm}$ and Z^0) and one massless field (photon γ) with

$$M_W^2 = \frac{g^2\,v^2}{4}, \tag{113}$$

$$M_Z^2 = \frac{g^2\,v^2}{4\,\cos^2\theta_W} = \frac{M_W^2}{\cos^2\theta_W}, \tag{114}$$

$$M_\gamma^2 = 0. \tag{115}$$

The weak mixing angle is related to the ratio of coupling constants, $g'/g = \tan\theta_W$.

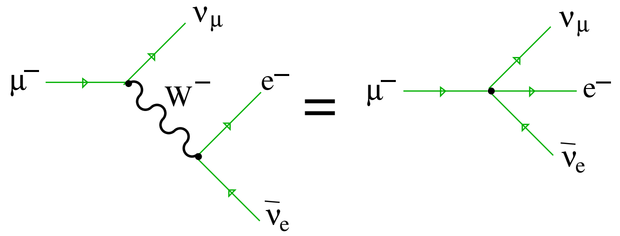
The coupling of the fermions to the physical gauge bosons are contained in $\mathcal{L}^{(f1)}$ giving

$$\begin{aligned}\mathcal{L}^{(f1)} &= i\,\bar{e}\gamma^\mu\partial_\mu e + i\,\overline{\nu_e}\gamma^\mu\partial_\mu\nu_e - g\sin\theta_W\,\bar{e}\gamma^\mu e\,A_\mu \\ &\quad + \frac{g}{\cos\theta_W}\left(\sin^2\theta_W\,\overline{e_R}\gamma^\mu e_R - \frac{1}{2}\,\cos 2\theta_W\,\overline{e_L}\gamma^\mu e_L + \frac{1}{2}\,\overline{\nu_e}\gamma^\mu\nu_e\right) Z_\mu \\ &\quad + \frac{g}{\sqrt{2}}\left(\overline{\nu_e}\gamma^\mu e_L\,W_\mu^- + \overline{e_L}\gamma^\mu\nu_e\,W_\mu^+\right).\end{aligned} \tag{116}$$

From the coupling to the photon, we can read off

$$e = g\,\sin\theta_W = g'\cos\theta_W. \tag{117}$$

The coupling of electrons or muons to their respective neutrinos, for instance in the amplitude for the decay of the muon



is given by

$$\begin{aligned}-i\,\mathscr{M} &= -\frac{g^2}{2}(\overline{\nu_\mu}\gamma^\rho\mu_L)\frac{-i\,g_{\rho\sigma}+\ldots}{k^2+M_W^2}(\overline{e_L}\gamma^\sigma\nu_e) \\ &\approx i\,\frac{g^2}{8\,M_W^2}\,\underbrace{(\overline{\nu_\mu}\gamma_\rho(1-\gamma_5)\mu)}_{(j_L^{(\mu)\dagger})_\rho}\underbrace{(\bar{e}\gamma^\rho(1-\gamma^5)\nu_e)}_{(j_L^{(e)})^\rho}\end{aligned} \tag{118}$$

$$\equiv i\,\frac{G_F}{\sqrt{2}}\,(j_L^{(\mu)\dagger})_\rho\,(j_L^{(e)})^\rho, \tag{119}$$

the good old four-point interaction introduced by Fermi to explain the weak interactions, i.e. one has the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\,M_W^2} = \frac{e^2}{8\,M_W^2\,\sin^2\theta_W} = \frac{1}{2\,v^2}. \tag{120}$$

In this way the parameters g, g' and v determine a number of experimentally measurable quantities, such as

$$\alpha = e^2/4\pi \approx 1/127, \tag{121}$$

$$G_F = 1.166\,4\times 10^{-5}\ \mathrm{GeV}^{-2}, \tag{122}$$

$$\sin^2\theta_W = 0.231\,2, \tag{123}$$

$$M_W = 80.40\ \mathrm{GeV}, \tag{124}$$

$$M_Z = 91.19\ \mathrm{GeV}. \tag{125}$$

The value of α deviates from the known value $\alpha \approx 1/137$ because of higher order contributions, giving rise to a *running coupling constant* after renormalization of the field theory. The coupling of the Z^0 to fermions is given by $(g/\cos\theta_W)\gamma^\mu$ multiplied with

$$T_W^3\,\frac{1}{2}(1-\gamma_5) - \sin^2\theta_W\,Q \equiv \frac{1}{2}\,C_V - \frac{1}{2}\,C_A\,\gamma_5, \tag{126}$$

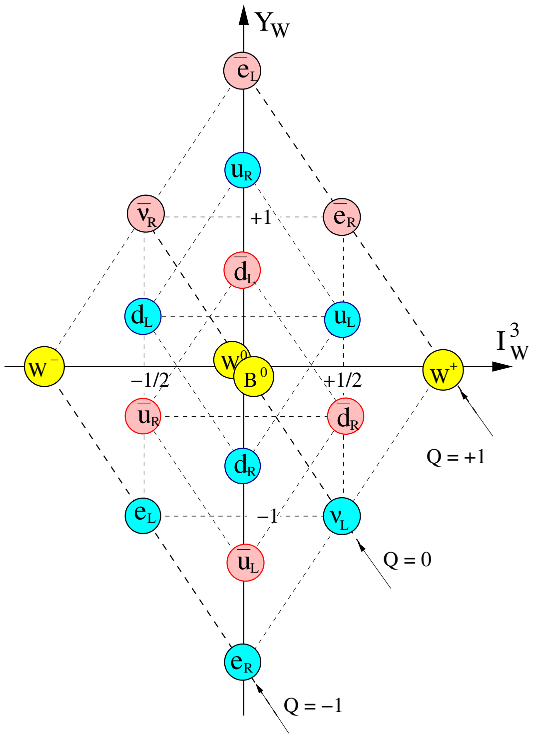


FIG. 1: *Appropriate Y_W and T_W^3 assignments of quarks, leptons, their antiparticles and the electroweak gauge bosons as appearing in each family. The electric charge Q is then fixed, $Q = T_W^3 + Y_W/2$ and constant along specific diagonals as indicated in the figure. The pattern is actually intriguing, suggesting an underlying larger unifying symmetry group, for which $SU(5)$ or $SO(10)$ are actually nice candidates. We will not discuss this any further in this chapter.*

with

$$C_V = T_W^3 - 2\sin^2\theta_W\, Q, \tag{127}$$

$$C_A = T_W^3. \tag{128}$$

From this coupling it is straightforward to calculate the partial width for Z^0 into a fermion-antifermion pair,

$$\Gamma(Z^0 \to f\overline{f}) = \frac{M_Z}{48\pi}\,\frac{g^2}{\cos^2\theta_W}\,(C_V^2 + C_A^2). \tag{129}$$

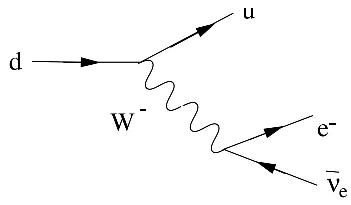
For the electron, muon or tau, leptons with $C_V = -1/2 + 2\sin^2\theta_W \approx -0.05$ and $C_A = -1/2$ we calculate $\Gamma(e^+e^-)$ $\approx$ 78.5 MeV (exp. $\Gamma_e \approx \Gamma_\mu \approx \Gamma_\tau \approx$ 83 MeV). For each neutrino species (with $C_V = 1/2$ and $C_A = 1/2$ one expects $\Gamma(\overline{\nu}\nu) \approx$ 155 MeV. Comparing this with the total width into (invisible!) channels, $\Gamma_{invisible} =$ 480 MeV one sees that three families of (light) neutrinos are allowed. Actually including corrections corresponding to higher order diagrams the agreement for the decay width into electrons can be calculated much more accurately and the number of allowed (light) neutrinos turns to be even closer to three.

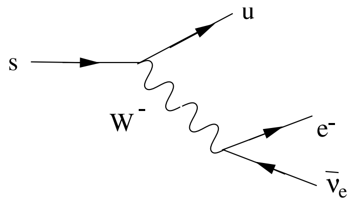
The masses of the fermions and the coupling to the Higgs particle are contained in $\mathcal{L}^{(h2)}$. With the choosen vacuum expectation value for the Higgs field, one obtains

$$\begin{aligned}\mathcal{L}^{(h2)} &= -\frac{G_e\,v}{\sqrt{2}}\,(\overline{e_L}e_R + \overline{e_R}e_L) - \frac{G_e}{\sqrt{2}}\,(\overline{e_L}e_R + \overline{e_R}e_L)\,h \\ &= -m_e\,\overline{e}e - \frac{m_e}{v}\,\overline{e}eh.\end{aligned} \tag{130}$$

First, the mass of the electron comes from the spontaneous symmetry breaking but is not predicted (it is in the coupling G_e). The coupling to the Higgs particle is weak as the value for v calculated e.g. from the M_W mass is about 250 GeV, i.e. m_e/v is extremely small.

Finally we want to say something about the weak properties of the quarks, as appear for instance in the decay of the neutron or the decay of the Λ (quark content uds),





$$n \longrightarrow pe^-\overline{\nu_e} \quad \Longleftrightarrow \quad d \longrightarrow ue^-\overline{\nu_e}, \qquad\qquad \Lambda \longrightarrow pe^-\bar{\nu}_e \quad \Longleftrightarrow \quad s \longrightarrow ue^-\bar{\nu}_e.$$

The quarks also turn out to fit into doublets of $SU(2)_W$ for the lefthanded species and into singlets for the righthanded quarks. As shown in Fig. 1, this requires particular Y_W-T_W^3 assignments to get the charges right.

A complication arises for quarks (and as we will discuss in the next section in more detail also for leptons) as it are not the 'mass' eigenstates that appear in the weak isospin doublets but linear combinations of them,

$$\left(\begin{array}{c} u \\ d' \end{array}\right)_L \qquad \left(\begin{array}{c} c \\ s' \end{array}\right)_L \qquad \left(\begin{array}{c} t \\ b' \end{array}\right)_L,$$

where

$$\left(\begin{array}{c} d' \\ s' \\ b' \end{array}\right)_L = \left(\begin{array}{ccc} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{array}\right) \left(\begin{array}{c} d \\ s \\ b \end{array}\right)_L \tag{131}$$

This mixing allows all quarks with $T_W^3 = -1/2$ to decay into an up quark, but with different strength. Comparing neutron decay and Λ decay one can get an estimate of the mixing parameter V_{us} in the socalled Cabibbo-Kobayashi-Maskawa mixing matrix. Decay of B-mesons containing b-quarks allow estimate of V_{ub}, etc. In principle one complex phase is allowed in the most general form of the CKM matrix, which can account for the (observed) CP violation of the weak interactions. This is only true if the mixing matrix is at least three-dimensional, i.e. CP violation requires three generations. The magnitudes of the entries in the CKM matrix are nicely represented using the socalled Wolfenstein parametrization

$$V = \left(\begin{array}{ccc} 1 - \frac{1}{2}\,\lambda^2 & \lambda & \lambda^3\,A(\rho - i\,\eta) \\ -\lambda & 1 - \frac{1}{2}\,\lambda^2 & \lambda^2\,A \\ \lambda^3\,A(1 - \rho - i\,\eta) & -\lambda^2\,A & 1 \end{array}\right) + \mathscr{O}(\lambda^4)$$

with $\lambda \approx 0.227$, $A \approx 0.82$ and $\rho \approx 0.22$ and $\eta \approx 0.34$. The imaginary part $i\,\eta$ gives rise to CP violation in decays of $\overline{K}$ and B-mesons (containing s and b quarks, respectively).

XII. FAMILY MIXING IN THE HIGGS SECTOR AND NEUTRINO MASSES

The quark sector

Allowing for the most general (Dirac) mass generating term in the lagrangian one starts with

$$\mathcal{L}^{\,(\mathrm{h2,q})} = -\overline{Q_L}\phi\Lambda_d D_R - \overline{D_R}\Lambda_d^\dagger\phi^\dagger Q_L - \overline{Q_L}\phi^c\Lambda_u U_R - \overline{U_R}\Lambda_u^\dagger\phi^{c\dagger}Q_L \tag{132}$$

where we include now the three lefthanded quark doublets in Q_L, the three righthanded quarks with charge $+2/3$ in U_R and the three righthanded quarks with charges $-1/3$ in D_R, each of these containing the three families, e.g. $\overline{U_R} = \left(\begin{array}{ccc} \overline{u_R} & \overline{c_R} & \overline{t_R} \end{array}\right)$. The Λ_u and Λ_d are complex matrices in the 3×3 family space. The Higgs field is still limited to one complex doublet. Note that we need the conjugate Higgs field to get a $U(1)_Y$ singlet in the case of the charge $+2/3$ quarks, for which we need the appropriate weak isospin doublet

$$\phi^c = \left(\begin{array}{c} \phi^{0*} \\ -\phi^- \end{array}\right) = \frac{1}{\sqrt{2}}\left(\begin{array}{c} v + h \\ 0 \end{array}\right).$$

For the (squared) complex matrices we can find positive eigenvalues,

$$\Lambda_u\,\Lambda_u^\dagger = V_u\,G_u^2\,V_u^\dagger, \quad \text{and} \quad \Lambda_d\,\Lambda_d^\dagger = V_d\,G_d^2\,V_d^\dagger, \tag{133}$$

where V_u and V_d are unitary matrices, allowing us to write

$$\Lambda_u = V_u\, G_u\, W_u^\dagger \quad \text{and} \quad \Lambda_d = V_d\, G_d\, W_d^\dagger, \tag{134}$$

with G_u and G_d being real and positive and W_u and W_d being different unitary matrices. Thus one has

$$\mathcal{L}^{\,(\mathrm{h2,q})} \Longrightarrow -\overline{D_L} V_d\, M_d\, W_d^\dagger D_R - \overline{D_R} W_d\, M_d\, V_d^\dagger D_L - \overline{U_L} V_u\, M_u\, W_u^\dagger U_R - \overline{U_R} W_u\, M_u\, V_u^\dagger U_L \tag{135}$$

with $M_u = G_u v/\sqrt{2}$ (diagonal matrix containing m_u, m_c and m_t) and $M_d = G_d v/\sqrt{2}$ (diagonal matrix containing m_d, m_s and m_b). One then reads off that starting with the family basis as defined via the left doublets that the mass eigenstates (and states coupling to the Higgs field) involve the righthanded states $U_R^{\mathrm{mass}} = W_u^\dagger U_R$ and $D_R^{\mathrm{mass}} = W_d^\dagger D_R$ and the lefthanded states $U_L^{\mathrm{mass}} = V_u^\dagger U_L$ and $D_L^{\mathrm{mass}} = V_d^\dagger D_L$. Working with the mass eigenstates one simply sees that the weak current coupling to the $W^\pm$ becomes $\overline{U}_L\, \gamma^\mu\, D_L = \overline{U}_L^{\mathrm{mass}}\, \gamma^\mu\, V_u^\dagger V_d\, D_L^{\mathrm{mass}}$, i.e. the weak mass eigenstates are

$$D_L' = D_L^{\mathrm{weak}} = V_u^\dagger V_d\, D_L^{\mathrm{mass}} = V_{\mathrm{CKM}}\, D_L^{\mathrm{mass}}, \tag{136}$$

the unitary CKM-matrix introduced above in an ad hoc way.

The lepton sector (massless neutrinos)

For a lepton sector with a lagrangian density of the form

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} = -\overline{L}\phi\Lambda_e E_R - \overline{E_R}\Lambda_e^\dagger \phi^\dagger L, \tag{137}$$

in which

$$L = \begin{pmatrix} N_L \\ E_L \end{pmatrix}$$

is a weak doublet containing the three families of neutrinos (N_L) and charged leptons (E_L) and E_R is a three-family weak singlet, we find massless neutrinos. As before, one can write $\Lambda_e = V_e\, G_e\, W_e^\dagger$ and we find

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} \Longrightarrow -M_e \left(\overline{E_L} V_e W_e^\dagger E_R - \overline{E_R} W_e V_e^\dagger E_L\right), \tag{138}$$

with $M_e = G_e v/\sqrt{2}$ the diagonal mass matrix with masses m_e, m_μ and m_τ. The mass fields $E_R^{\mathrm{mass}} = W_e^\dagger E_R$, $E_L^{\mathrm{mass}} = V_e^\dagger E_L$. For the (massless) neutrino fields we just can redefine fields into $N_L^{\mathrm{mass}} = V_e^\dagger N_L$, since the weak current is the only place where they show up. The W-current then becomes $\overline{E}_L\, \gamma^\mu\, N_L = \overline{E}_L^{\mathrm{mass}}\, \gamma^\mu\, N_L^{\mathrm{mass}}$, i.e. there is no family mixing for massless neutrinos.

The lepton sector (massive Dirac neutrinos)

In principle a massive Dirac neutrino could be accounted for by a lagrangian of the type

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} = -\overline{L}\phi\Lambda_e E_R - \overline{E_R}\Lambda_e^\dagger \phi^\dagger L - \overline{L}\phi^c \Lambda_n N_R - \overline{N_R}\Lambda_n^\dagger \phi^{c\dagger} L \tag{139}$$

with three righthanded neutrinos added to the previous case, decoupling from all known interactions. Again we continue as before now with matrices $\Lambda_e = V_e\, G_e\, W_e^\dagger$ and $\Lambda_n = V_n\, G_n\, W_n^\dagger$, and obtain

$$\mathcal{L}^{\,(\mathrm{h2},\ell)} \Longrightarrow -\overline{E_L} V_e\, M_e\, W_e^\dagger E_R - \overline{E_R} W_e\, M_e\, V_e^\dagger E_L - \overline{N_L} V_n\, M_n\, W_n^\dagger N_R - \overline{N_R} W_n\, M_n\, V_n^\dagger N_L. \tag{140}$$

We note that there are mass fields $E_R^{\mathrm{mass}} = W_e^\dagger E_R$, $E_L^{\mathrm{mass}} = V_e^\dagger E_L$, $N_L^{\mathrm{mass}} = V_n^\dagger N_L$ and $N_R^{\mathrm{mass}} = W_n^\dagger N_R$ and the weak current becomes $\overline{E_L}\, \gamma^\mu\, N_L = \overline{E_L^{\mathrm{mass}}}\, \gamma^\mu\, V_e^\dagger V_n\, N_L^{\mathrm{mass}}$. Working with the mass eigenstates for the charged leptons we see that the weak eigenstates for the neutrinos are $N_L^{\mathrm{weak}} = V_e^\dagger\, N_L$ with the relation to the mass eigenstates for the lefthanded neutrinos given by

$$N_L' = N_L^{\mathrm{weak}} = V_e^\dagger\, V_n\, N_L^{\mathrm{mass}} = U_{\mathrm{PMNS}}\, N_L^{\mathrm{mass}}, \tag{141}$$

with $U_{\rm PMNS} = V_e^\dagger V_n$ known as the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix.

For neutrino's this matrix is parametrized in terms of three angles θ_{ij} with $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ and one angle δ,

$$U_{\rm PMNS} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{array}\right) \left(\begin{array}{ccc} c_{13} & 0 & s_{13}\,e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\,e^{i\delta} & 0 & c_{13} \end{array}\right) \left(\begin{array}{ccc} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{array}\right), \tag{142}$$

a parametrization that in principle also could have been used for quarks. In this case, it is particularly useful because θ_{12} is essentially determined by solar neutrino oscillations, $\sin^2\theta_{12} \approx \sin^2\theta_\odot \approx 0.31$ and $|\Delta m^2_{12}| \approx \Delta m^2_\odot \approx 7.6\times 10^{-5}$ eV^2 (convention $m_2 > m_1$). The angle θ_{23} then is determined by atmospheric neutrino oscillations, $\sin^2\theta_{23} \approx \sin^2\theta_A \approx 0.42$ and $|\Delta m^2_{23}| \approx \Delta m^2_A \approx 2.4\times 10^{-3}$ eV^2. The mixing is intriguingly close to the Harrison-Perkins-Scott tri-bimaximal mixing matrix

$$U_{\rm HPS} = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \sqrt{1/2} & \sqrt{1/2} \\ 0 & -\sqrt{1/2} & \sqrt{1/2} \end{array}\right) \left(\begin{array}{ccc} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/3} & \sqrt{2/3} & 0 \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & -\sqrt{1/3} & \sqrt{1/2} \end{array}\right). \tag{143}$$

In the case of a tri-bimaximal mixing, having only two 2-dimensional mixing, there are no complex phases and no CP-violation.

The lepton sector (massive Majorana fields)

An even simpler option than sterile righthanded Dirac neutrinos, is to add in Eq. 138 a Majorana mass term for the (lefthanded) neutrino mass eigenstates,

$$\mathcal{L}^{\,\rm mass,\nu} = -\frac{1}{2}\left(M_L\,\overline{N_L^c}\,N_L + M_L^*\,\overline{N_L}\,N_L^c\right), \tag{144}$$

although this option is not attractive as it violates the electroweak symmetry. The way to circumvent this is to introduce as in the previous section righthanded neutrinos, with for the righthanded sector a mass term M_R,

$$\mathcal{L}^{\,\rm mass,\nu} = -\frac{1}{2}\left(M_R\,\overline{N_R}\,N_R^c + M_R^*\,\overline{N_R^c}\,N_R\right). \tag{145}$$

In order to have more than a completely decoupled sector, one must for the neutrinos as well as charged leptons, couple the right- and lefthanded species through Dirac mass terms coming from the coupling to the Higgs sector as in the previous section. Thus (disregarding family structure) one has two Majorana neutrinos, one being massive. For the charged leptons there cannot exist a Majorana mass term as this would break the U(1) electromagnetic symmetry. For the leptons, the left- and righthanded species then just form a Dirac fermion.

For the neutrino sector, the massless and massive Majorana neutrinos, coupled by a Dirac mass term, are equivalent to two decoupled Majorana neutrinos (see below). If the Majorana mass $M_R \gg M_D$ one actually obtains in a natural way one Majorana neutrino with a very small mass. This is called the see-saw mechanism (outlined below).

For these light Majorana neutrinos one has, as above, a unitary matrix relating them to the weak eigenstates. Absorption of phases in the states is not possible for Majorana neutrinos, however, hence the mixing matrix becomes

$$V_{\rm PMNS} = U_{\rm PMNS}\,K \quad \text{with} \quad K = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & e^{i\alpha_3/2} \end{array}\right). \tag{146}$$

containing three (CP-violating) phases (α_2, α_3 and δ).

The see-saw mechanism

Consider (for one family $N = n$) the most general Lorentz invariant mass term for two independent Majorana spinors, Υ_1' and Υ_2' (satisfying $\Upsilon^c = \Upsilon$ and as discussed in chapter 6, $\Upsilon_L^c \equiv (\Upsilon_L)^c = \Upsilon_R$ and $\Upsilon_R^c = \Upsilon_L$). We use here the primes starting with the weak eigenstates. Actually, it is easy to see that this incorporates the Dirac case by considering the lefthanded part of Υ_1' and the righthanded part of Υ_2' as a Dirac spinor ψ. Thus

$$\Upsilon_1' = n_L^c + n_L, \qquad \Upsilon_2' = n_R + n_R^c, \qquad \psi = n_R + n_L. \tag{147}$$

As the most general mass term in the lagrangian density we have

$$\begin{aligned}
\mathcal{L}^{\,\rm mass} &= -\frac{1}{2}\left(M_L\,\overline{n_L^c}\,n_L + M_L^*\,\overline{n_L}\,n_L^c\right) - \frac{1}{2}\left(M_R\,\overline{n_R}\,n_R^c + M_R^*\,\overline{n_R^c}\,n_R\right) \\
&\quad -\frac{1}{2}\left(M_D\,\overline{n_L^c}\,n_R^c + M_D^*\,\overline{n_L}\,n_R\right) - \frac{1}{2}\left(M_D\,\overline{n_R}\,n_L + M_D^*\,\overline{n_R^c}\,n_L^c\right) \qquad (148)\\
&= -\frac{1}{2}\begin{pmatrix}\overline{n_L^c} & \overline{n_R}\end{pmatrix}\begin{pmatrix} M_L & M_D \\ M_D & M_R\end{pmatrix}\begin{pmatrix} n_L \\ n_R^c\end{pmatrix} + \text{ h.c.} \qquad (149)
\end{aligned}$$

which for $M_D = 0$ is a pure Majorana lagrangian and for $M_L = M_R = 0$ and real M_D represents the Dirac case. The mass matrix can be written as

$$M = \begin{pmatrix} M_L & |M_D|\,e^{i\phi} \\ |M_D|\,e^{i\phi} & M_R \end{pmatrix} \qquad (150)$$

taking M_L and M_R real and non-negative. This choice is possible without loss of generality because the phases can be absorbed into Υ_1' and Υ_2' (real must be replaced by hermitean if one includes families). This is a mixing problem with a symmetric (complex) mass matrix leading to two (real) mass eigenstates. The diagonalization is analogous to what was done for the Λ-matrices and one finds $U\,M\,U^T = M_0$ with a (unitary) matrix U, which implies $U^*\,M^\dagger\,U^\dagger = U^*\,M^*\,U^\dagger = M_0$. This 'normal' diagonalization of the (hermitean) matrix $MM^\dagger$, $U\,(MM^\dagger)\,U^\dagger = M_0^2$, gives (as discussed for Majorana fermions) the eigenvalues

$$M_{1/2}^2 = \frac{1}{2}\left[M_L^2 + M_R^2 + 2|M_D|^2 \pm \sqrt{(M_L^2 - M_R^2)^2 + 4|M_D|^2\,\left(M_L^2 + M_R^2 + 2M_L M_R\,\cos(2\phi)\right)}\right], \qquad (151)$$

leaving two decoupled Majorana fields Υ_1 and Υ_2, related via

$$\begin{pmatrix}\Upsilon_{1L} \\ \Upsilon_{2L}\end{pmatrix} = U^*\begin{pmatrix} n_L \\ n_R^c\end{pmatrix}, \qquad \begin{pmatrix}\Upsilon_{1R} \\ \Upsilon_{2R}\end{pmatrix} = U\begin{pmatrix} n_L^c \\ n_R\end{pmatrix}. \qquad (152)$$

for each of which one finds the lagrangians

$$\mathscr{L} = \frac{1}{4}\,\overline{\Upsilon_i}\,i\,\overset{\leftrightarrow}{\not\partial}\,\Upsilon_i - \frac{1}{2}\,M_i\,\overline{\Upsilon_i}\,\Upsilon_i \qquad (153)$$

for $i = 1, 2$ with real masses M_i. For the situation $M_L = 0$ and $M_R \gg M_D$ (taking M_D real) one finds $M_1 \approx M_D^2/M_R$ and $M_2 \approx M_R$.

Exercise
In this exercise we make the seesaw mechanism explicit. We look at the situation of two Majorana's where $0 = M_L < |M_D| \ll M_R$, which leads to the socalled seesaw mechanism. Calculate the eigenvalues of the two-Majarano case for $M_L = 0$ and $M_R = M_X$. Given that neutrino masses are of the order of 1/20 eV, what is the mass M_X if we take for M_D the electroweak symmetry breaking scale v (about 250 GeV).

(solution)
The eigenvalues are $M_1 \approx M_D^2/M_X\sqrt{2}$ and $M_2 \approx M$. For a neutrino mass of the order of 1/20 eV, and a fermion mass of the order of the electroweak breaking scaling 250 GeV, this leads to $M_X \sim 10^{15}$ GeV. The recoupling matrix in this case is

$$U = \begin{pmatrix} i\,\cos\theta_S & -i\,\sin\theta_S \\ \sin\theta_S & \cos\theta_S \end{pmatrix},$$

with $\sin\theta_S \approx M_D/M_X$. The weak current couples to $n_L = \sin\theta_S\,\Upsilon_2 - i\,\cos\theta_S\,\Upsilon_1$, where Υ_1 is the light neutrino (mass) eigenstate.

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[1] J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012) and 2013 partial update for the 2014 edition, http://pdg.lbl.gov.