Transverse momenta of partons in high-energy scattering processes

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Summary. — In these lectures we look how properties of hadrons like charge densities and parton distributions can be described within the framework of Quantum Chromodynamics and how they appear in high energy scattering cross sections. For the parton distributions, we specifically look at the role of transverse momenta of partons.

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1. – Introduction

The central topic of these lectures is the proton. We want to grasp its structure. On the one hand as part of our quest to understand the heart of the matter, at least the visible matter, in our universe. On the other hand because protons besides electrons are basic tools in experiments that probe the fundamental forces in nature. Therefore we want to use their properties, global ones like energy, momentum and spin, but also other indirect degrees of freedom. The latter is of interest in particular for the proton, which is a composite state. Although a number of global variables can be used in the characterization of the quantum state,

$$|\text{proton}\rangle = |M, S; \boldsymbol{P}, M_s; \ldots\rangle,$$

the full quantum state of a proton is much more complex than the corresponding electron state,

$$|m_e, s; \boldsymbol{p}, m_s; \ldots\rangle = b_e^\dagger(\boldsymbol{p}, m_s)|0\rangle,$$

which is created from the vacuum by a creation operator, satisfying elementary (anti-commutation) relations. The only properties remaining in the ... for the electron are charges determining the coupling of the electron to electroweak gauge bosons. Creation and annihilation operators appear in the free field expansion of the electron field. They are multiplied with plane-wave solutions including a spinor structure in Dirac space. Using these fields a nice description of electrons and their interactions through the coupling to photons is achieved within the framework of the U(1) gauge theory of Quantum Electrodynamics and its embedding in the larger framework of the Standard Model of Particle Physics.

Within the Standard Model the strong interactions are described by an SU(3) color gauge theory, Quantum Chromodynamics (QCD). A basic problem is that the constituents in this theory, quarks and gluons, do not appear as asymptotic states, so the starting point in scattering experiments, be it as the objects of study or as the tools, are hadrons, the bound states of QCD.

In these lectures, for which background material can also be found in Refs [1, 2]. I want to discuss a variety of aspects in the study of hadronic structure and focus on the physical meaning of form factors, and quark momentum distributions. Focus is on extending the concept of collinear quark distributions to transverse momentum dependent (TMD) parton distribution functions (PDFs), for which we also refer to reviews in Refs [3, 4].

2. – Basics

Quantum Mechanics. – When looking at the structure of a quantum mechanical state in coordinate or momentum space, it is important to realize that these are conjugate, making things quite different from classical phase space in which one can consider coordinates and momenta as independent degrees of freedom. For purpose of illustration consider a state described with a one-dimensional spatial wave function $\varphi(x) = \langle x|\varphi\rangle$ or the corresponding momentum space representation $\tilde{\varphi}(p) = \langle p|\varphi\rangle$, linked through $\langle x|p\rangle = e^{ipx}$,

$$\varphi(x) = \int \frac{dp}{2\pi}\, e^{ipx}\, \tilde{\varphi}(p) \quad \text{and} \quad \tilde{\varphi}(p) = \int dx\; e^{-ipx}\, \varphi(x). \tag{1}$$

In many applications a momentum q is transferred to the system and absorbed, described by a (charge) operator

$$\hat{Q}(q) = \int \frac{dp}{2\pi}\, |p + \tfrac{1}{2}q\rangle\, \langle p - \tfrac{1}{2}q| = \int dx\, |x\rangle\, e^{iqx}\, \langle x|. \tag{2}$$

One measures an elastic response $\sigma(q) \propto |F(q)|^2$, where $F(q)$ is the expectation value of the operator $\hat{Q}(q)$, known as the *form factor*

$$F(q) = \langle\varphi|\hat{Q}(q)|\varphi\rangle = \int \frac{dp}{2\pi}\, \tilde{\varphi}^*(p + \tfrac{1}{2}q)\, \tilde{\varphi}(p - \tfrac{1}{2}q) = \int dx\, e^{iqx}\, |\varphi(x)|^2, \tag{3}$$

which is the Fourier transform of the charge density $\rho(x) = |\varphi(x)|^2$. The form factor at $q = 0$ is just a number, in this case the normalization of the wave function. In other applications one knocks out a particle from the system, producing a plane-wave final state $|q\rangle$. The inelastic response is now $\sigma(q) \propto W(q)$, which reflects the momentum distribution $f(p)$, in the simplest case at $q = p$. Interchanging the roles of x and p in the above equation, this momentum distribution can be written as

$$f(p) = |\tilde{\varphi}(p)|^2 = \int dx\, dy\, e^{ipy}\, \varphi^*(x - \tfrac{1}{2}y)\, \varphi(x + \tfrac{1}{2}y). \tag{4}$$

It is important to note that the form factors are non-local in momentum space, the momentum distribution is non-local in coordinate space.

A convenient way to incorporate both is the concept of quantum phase-space (Wigner) distributions. For a nice introduction in the field of hadron physics we refer to Ref. [5]. The Wigner functions in quantum mechanics are defined as,

$$W(x,p) = \int dy\, e^{ipy}\, \varphi^*(x - \tfrac{1}{2}y)\, \varphi(x + \tfrac{1}{2}y) \tag{5}$$

$$= \int \frac{dq}{2\pi}\, e^{-iqx}\, \tilde{\varphi}^*(p + \tfrac{1}{2}q)\, \tilde{\varphi}(p - \tfrac{1}{2}q). \tag{6}$$

The densities are obtained after integration over positions or momenta, respectively

$$f(p) = \tilde{\varphi}^*(p)\, \tilde{\varphi}(p) = \int dx\, W(x,p), \tag{7}$$

$$\rho(x) = \varphi^*(x)\, \varphi(x) = \int \frac{dp}{2\pi}\, W(x,p). \tag{8}$$

The Wigner distributions, generalized to appropriate 3-dimensional variables, are useful tools in analyzing the 3-dimensional structure of nucleons.

Momenta and phase space. – In more dimensions and in relativistic situations we will use momentum four vectors with components p^μ, either expressed as time- and space-like components $p = (p^0, \boldsymbol{p})$ or light-cone components $p = [p^-, p^+, p_T]$, with $p^\pm = (p^0 \pm p^3)/\sqrt{2}$. We will often use tetrads of vectors n_α to formalize components. The components of n_α are $n_\alpha^\mu = g_\alpha^\mu$. In particular, we may look in a covariant treatment for a time-like and space-like set of vectors $n_0^\mu = \hat{t}^\mu$ and $n_3^\mu = \hat{z}^\mu$ to give a covariant meaning to $p^0 \equiv p \cdot n_0$. For light-cone components one needs two light-like vectors $n_\pm^\mu$ satisfying $n_+ \cdot n_- = 1$ and one has $p^\pm = p \cdot n_\mp$. In the transverse directions one works with the tensors

$$g_T^{\mu\nu} = g^{\mu\nu} - \hat{n}_+^{\{\mu}\hat{n}_-^{\nu\}} = g^{\mu\nu} + n_3^\mu n_3^\nu - n_0^\mu n_0^\nu \tag{9}$$

$$\epsilon_T^{\mu\nu} = \epsilon^{-+\mu\nu} = \epsilon^{03\mu\nu}, \tag{10}$$

where we use $\epsilon^{0123} = \epsilon^{-+12} = 1$. For light-cone components the scaling of plus-component with α and minus-component with $1/\alpha$ just corresponds with a boost. The well-known measure or phase space for momentum states (with $p^2 = m^2$) and positive energy ($p^0 > 0$ or $p^+ > 0$) is given by

$$\begin{aligned} \int d\tilde{p} &= \int \frac{d4p}{(2\pi)^4}\,(2\pi)\,\delta(p^2 - m^2)\,\theta(p^0) \\ &= \int \frac{d^3p}{(2\pi)^3\,2p^0} = \int \frac{dp^0\,d^2p_T}{(2\pi)^3\,2p^3} = \int \frac{dp^+\,d^2p_T}{(2\pi)^3\,2p^+} = \int \frac{dp^-\,d^2p_T}{(2\pi)^3\,2p^-}. \end{aligned} \tag{11}$$

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Exercise: Show that for on-shell momenta $p^2 = m^2$ one has

$$p = p^+ n_+ + p^- n_- + \boldsymbol{p}_T \quad \text{with} \quad p^- = (m^2 + \boldsymbol{p}_T^2)/2p^+, \tag{12}$$

$$p = p^0 n_0 + \boldsymbol{p} \quad \text{with} \quad p^0 = \pm\sqrt{\boldsymbol{p}^2 + m^2}. \tag{13}$$

Show that for an off-shell momentum $p = P - P_R$ with $P^2 = M^2$ and $P_R^2 = M_R^2$ one can write in terms of light-cone components

$$P = P^+ n_+ + (M^2/2P^+) n_- \quad \text{and} \quad p = xP^+ n_+ + \sigma\, n_- + \boldsymbol{p}_T$$

(x is the light-cone momentum fraction) with

$$2P^+\sigma = 2\,p \cdot P - x\,M^2 = p^2 - M_R^2 + (1-x)M^2 \tag{14}$$

$$(1-x)p^2 = x(1-x)\,M^2 - x\,M_R^2 - \boldsymbol{p}_T^2, \tag{15}$$

which is useful for applications to quarks in nucleons In terms of the usual four vector components one can write

$$P = Mn_0 \quad \text{and} \quad p = En_0 + \boldsymbol{p}$$

with

$$M\,E = p{\cdot}P = (M^2 + m^2 - M_R^2)/2 \approx M(m - B) \tag{16}$$

$$p^2 = (M^2 + M_R^2) - 2M\sqrt{\boldsymbol{p}^2 + M_X^2}. \tag{17}$$

where we have shown the results also for small binding energies, i.e. for $M_R = M - m - B$ with $B \ll \{m, M, M_R\}$, e.g. applicable to electrons in atoms or nucleons in nuclei.
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Quantum Field Theory. – In quantum field theory space-time dependence is treated in a different way, as only momentum operators as generator of space-time translations and the corresponding momentum eigenstates remain useful concepts. The fields $\phi(x)$ act as operators in Hilbert space with wave functions showing up as modes in the description of the space-time dependence multiplied with creation or annihilation operators for these modes, which in the standard free field expansions create or annihilate momentum eigenstates. Composite operators like the current for fermions, $J^\mu(x) = \overline{\psi}(x)\,\gamma^\mu\psi(x)$, become important because they describe the coupling to photons or electromagnetic fields.

Simplifying our treatment one has operators $O(x)$ which having in them creation and annihilation operators have in essence the same structure as the operator $\hat{Q}(q)$ in our quantum mechanical discussion. The field operator allows transitions between momentum eigenstates, $\langle P'|O(x)|P\rangle$. In the situation that the operator involves different degrees of freedom (say quarks) than the states (say hadrons), these states are like the states $|\varphi\rangle$ in the quantum mechanical case. One expresses the matrix element as

$$\langle P'|O(x)|P\rangle = e^{iq\cdot x}\left[G_1(q^2) - i\,q_\mu\,G_2^\mu(q^2)\right], \tag{18}$$

where translation invariance now is used and all unknowns are in the *form factors* depending on the relevant invariants which is $t = q^2$ with $q = P' - P$. Realize, actually, how much info there is in the vectors. If we use $n_0 \sim P + P'$ (orthogonal to the spacelike direction $q = \boldsymbol{q}$). Thus also G_2^μ is spacelike (orthogonal to n_0). We note the relation of form factors at $q^2 = 0$ to forward matrix elements,

$$G_1(0) = \langle P|O(x)|P\rangle \quad \text{and} \quad G_2^\mu(0) = \langle P|x^\mu\,O(x)|P\rangle, \tag{19}$$

This provides covariant definitions of vector or axial charges as well as magnetic moments and masses in quantum field theory. See Appendix A for the Nucleon formfactor.

Extending our analysis to non-local field combinations, we can look at off-forward expectation values of non-local correlators, in essence Wigner functions

$$\langle P'|O\left(x - \tfrac{1}{2}y, x + \tfrac{1}{2}y\right)|P\rangle = e^{i\Delta\cdot x}\,\langle P'|O\left(-\tfrac{1}{2}y, \tfrac{1}{2}y\right)|P\rangle, \tag{20}$$

where $\Delta = P' - P$. These types of matrix elements, both off-forward ($\Delta \neq 0$), as well as non-local ($y \neq 0$) are the topic of *generalized parton distribution functions*. We limit ourselves to forward matrix elements and look at the specific example that the non-local combination involves a 'square' combination of two fields $\phi(x)$, i.e. $O(x_1,x_2) = \phi^\dagger(x_1)\,\phi(x_2)$. In that case, one can easily see that the forward matrix element is a momentum density given by

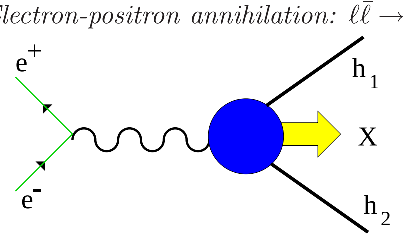
$$f(p) = \int d^4y\; e^{ip\cdot y}\langle P|\phi^\dagger\left(-\tfrac{1}{2}y\right)\,\phi\left(\tfrac{1}{2}y\right)|P\rangle = \int d^4y\; e^{ip\cdot y}\,\langle P|\phi^\dagger(0)\,\phi(y)|P\rangle$$
$$= \sum_X \int \frac{d^4P_X}{(2\pi)^4}\,(2\pi)^4\,\delta^4(P_X - P + p)\,\left|\langle P_X|\phi(0)|P\rangle\right|^2 \tag{21}$$

which represents the momentum distribution of ϕ-particles in the target, summed over all final states that remain after removing or adding a ϕ-particle.

3. – Electroweak scattering processes

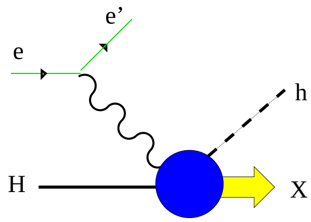
Electroweak scattering processes are in a sense ideal to study properties of hadrons. Only one specific process, namely lepton-hadron scattering will be dealt with in detail. A signal for an electroweak process is the presence of leptons which do not feel strong interactions. Characteristic processes are the following three types, the *annihilation* process, $\ell\bar{\ell} \to X$, the *lepton-hadron* scattering process, $\ell H \to \ell' X$, and the *lepton-pair production* process, $AB \to \ell\bar{\ell}X$ (Drell-Yan process). All of these processes involve electroweak currents, coupling to the leptons in a known way. The advantage of electromagnetic processes further lies in the fact that the process is accurately described in terms of the exchange of one photon, as the coupling, $\alpha = e^2/4\pi \approx 1/137$ is weak. The same is true for the weak vector bosons. On the hadronic side, the coupling to the quarks is known, but the structure of hadrons in terms of quarks and gluons is the unknown part. The fact that the coupling to the quarks is known, however, enables the study of quarks and (indirectly) gluons.

First let us consider the variables that are used to describe these scattering processes.
(1) Electron-positron annihilation: $\ell\bar{\ell} \to X$ *or* $\ell\bar{\ell} \to h(P_h)X$



$$q^2 = (\ell + \ell')^2 \equiv Q^2 \geq 0 \tag{22}$$
$$2\,P_{h_i}\cdot q \equiv z_i\,Q^2 \tag{23}$$
$$2\,P_1\cdot\ell \equiv y\,P_1\cdot q \tag{24}$$

In the case of production of hadrons with a timelike (virtual) photon one can consider the rest-frame of the virtual photon, in which case it is clear that Q is a measure of the excitation energy.
(2) Lepton-hadron scattering: $\ell H(P) \to \ell' X$ *or* $\ell H(P) \to \ell' h(P_h)X$



(25) $$q^2 = (\ell - \ell')^2 \equiv -Q^2 \le 0$$

(26) $$2P \cdot q \equiv 2M\,\nu \equiv \frac{Q^2}{x_{B}}$$

(27) $$2P_h \cdot q \equiv -z_h\,Q^2$$

(28) $$P \cdot \ell \equiv \frac{P \cdot q}{y} = \frac{Q^2}{2x_{B}y}$$

The variable x_{B} is the Bjorken scaling variable. In this scattering process a hadron is probed with a spacelike (virtual) photon, for which one can consider a frame in which the momentum only has a spatial component, from which it is clear that the resolving power of the probing photon is of the order $\lambda \approx 1/Q$. Roughly spoken one probes a nucleus (1 - 10 fm) with $Q \approx 10 - 100$ MeV, baryon or meson structure (with sizes in the order of 1 fm) with $Q \approx 0.1 - 1$ MeV and one probes deep into the nucleon (< 0.1 fm) with $Q > 2$ GeV.

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Exercise: Rewrite the invariant mass squared of the hadronic final state, W^2, in terms of invariants and use that $W^2 \ge M^2$ to show that $0 \le x_{B} \le 1$, with $x_{B} = 1$ corresponding to elastic scattering.

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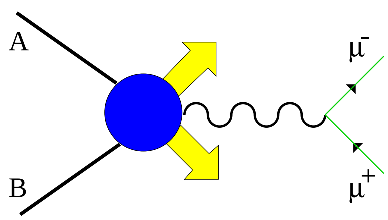
(3) Lepton-pair production or Drell-Yan scattering: $A(P_A)B(P_B) \to \ell\bar{\ell}X$



(29) $$q^2 = (\ell + \ell')^2 \equiv Q^2 \ge 0$$

(30) $$2P_A \cdot q \equiv \frac{Q^2}{x_{A}}$$

(31) $$2P_B \cdot q \equiv \frac{Q^2}{x_{B}}$$

(32) $$2\ell \cdot q \equiv \frac{P_A \cdot q}{y}$$

In Drell-Yan scattering one already has two hadrons to start with. Here Q^2 is the invariant mass of the lepton pair. The above processes are characteristic for a large number of other processes involving particles for which the interactions are fully known on the one side and hadrons on the other side.

Taking inclusive lepton-hadron scattering as an example, the cross section is given by

(33) $$E'\,\frac{d\sigma}{d^3\ell'} = \frac{1}{s - M^2}\,\frac{\alpha^2}{Q^4}\,L_{\mu\nu}\,2MW^{\mu\nu},$$

where $L_{\mu\nu}$ is the lepton tensor,

(34) $$L_{\mu\nu}(\ell,\ell';\lambda_e) = 2\ell_\mu\ell'_\nu + 2\ell_\nu\ell'_\mu - Q^2 g_{\mu\nu} + 2i\lambda_e\,\epsilon_{\mu\nu\rho\sigma}q^\rho\ell^\sigma.$$

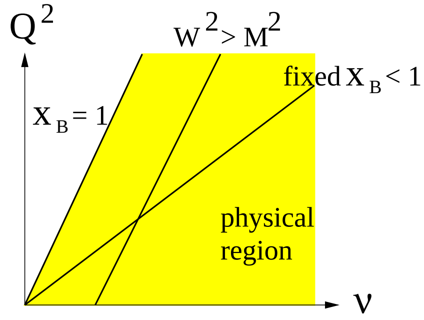


Fig. 1. – *The physical region in deep inelastic scattering in the Q^2-ν plane. The variable $\nu = P\cdot q/M$ is the photon energy in the target (momentum P) rest frame. Indicated are lines of constant invariant mass squared for the hadronic system, $W^2 = (P+q)^2$. The elastic limit $W^2 = M^2$ corresponds with $\nu = Q^2/2M$ or $x_B = 1$.*

and $W_{\mu\nu}$ is the hadron tensor, which depends on exchanged momentum q and target momentum P. It contains the information on the hadronic part of the scattering process,
(35)

$$2M\,W_{\mu\nu}(P,q) = \frac{1}{2\pi}\sum_n \int \frac{d^3P_n}{(2\pi)^3\,2E_n}\;\langle P|J_\mu^\dagger(0)|P_n\rangle\,\langle P_n|J_\nu(0)|P\rangle\;(2\pi)^4\,\delta^4(P+q-P_n).$$

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Exercise: Show, using $\delta^4(P+q-P_n) = \int d^4x\;\exp(iP\cdot x + iq\cdot x - iP_n\cdot x)$, shifting the argument of the current, $J_\mu(x) = \exp(iP_{op}\cdot x)J_\mu(0)\exp(-iP_{op}\cdot x)$ and using completeness for the intermediate states that the hadron tensor can be written as the expectation value of the product of currents $J_\mu(x)J_\nu(0)$. Then, adding a second term $\propto J_\nu(0)J_\mu^\dagger(0)\delta^4(P-q-P_n)$, which in the physical region ($\nu > 0$) is zero because of the spectral conditions of the intermediate states n ($P_n^0 > M$) one can after a similar procedure combine the terms to

$$2M\,W_{\mu\nu} = \frac{1}{2\pi}\int d^4x\;e^{i\,q\cdot x}\,\langle P|[J_\mu^\dagger(x),J_\nu(0)]|P\rangle, \tag{36}$$

where summation and averaging over spins is understood.

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In the leptoproduction $e + H \to e' + X$ the (unobserved) final state X can be the target (elastic scattering) or an excitation thereof. In Fig. 1) a plot of the two independent variables ν and Q^2 is given as well as a lines for fixed W^2 or fixed x_B. Elastic scattering corresponds to $W^2 = M^2$ or $x_B = 1$. For an elementary fermion that cannot be broken up or excited, this line relates ν and Q^2. For nucleons the elastic scattering cross section (on top of the $1/Q^4$ of the Mott cross section) is proportional to a form factor squared, measuring the expectation value of the electromagnetic current $\langle P'|J^\mu(x)|P\rangle$. It is also possible to excite the system. This gives rise to inelastic contributions in the cross section at $\nu > Q^2/2M$, starting at the threshold $W = M + M_\pi$. When Q^2 and the energy transfer ν are high enough the cross section will reflect elastic scattering off the pointlike constituents of the nucleon and the cross section will become equal to an incoherent sum of the electron-quark cross section. This is known as the deep inelastic scattering region, in which one finds Bjorken scaling. The cross section, more precisely the structure functions, become functions of one (kinematic) variable x_B, which is identified with the momentum fraction of the struck quark in the nucleon, enabling measurement of quark distributions. In zeroth order the cross section thus becomes constant along lines of fixed x_B (scaling).

It is also possible to consider in more detail the space-time correlations that are probed. As already indicated q is space-like. But from the kinematics of deep inelastic scattering one can see that the process probes the light-cone. Sitting in the nucleon rest-frame we note that both $q^0 = \nu$ and $q^3 = \sqrt{Q^2+\nu^2}$ go to infinity but working at finite x_B one sees that taking the sum and the difference only one of them goes to infinity. Choosing $\boldsymbol{q}$ along the negative z-axis one has $q^- = (\nu + |\boldsymbol{q}|)/\sqrt{2} \to \infty$ and $q^+ = (\nu - |\boldsymbol{q}|)/\sqrt{2} \approx -M_N x_B/\sqrt{2}$. This corresponds in the hadronic tensor which involves a Fourier transform of the product of currents to $|x^+| \approx 1/q^- \to 0$ and $|x^-| \approx 1/|q^+| \to 1/Mx_B$ or $|\boldsymbol{x}| \approx |t| \approx 1/Mx_B$. Thus, depending on the value of x_B the distances and times not necessarily are small, but one has $x^2 = x^+x^- - \boldsymbol{x}_\perp^2 \approx -\boldsymbol{x}_\perp^2 \le 0$, while on the other hand causality requires that $x^2 \ge 0$. Therefore, one sees that deep inelastic scattering probes the light-cone, $x^2 \approx 0$.

4. – Structure functions and cross sections

The simplest thing one can do with the hadron tensor is to express it in standard tensors and functions depending on the invariants, the *structure functions*. Instead of the traditional choice using tensors, $g_{\mu\nu}$, $P_\mu P_\nu$ and $\epsilon_{\mu\nu\rho\sigma}\, q^\rho P^\sigma$ multiplying structure functions W_1, W_2 and W_3 depending on ν and Q^2, we immediately go to a dimensionless representation. First we define a Cartesian basis of vectors [6], starting with the natural space-like momentum (defined by q). Using the target hadron momentum P^μ one can construct an orthogonal four vector $\tilde{P}^\mu = P^\mu - (P\cdot q/q^2)\, q^\mu$, which is timelike with length $\tilde{P}^2 = \kappa\, P\cdot q$ with

$$\kappa = 1 + \frac{M^2\, Q^2}{(P\cdot q)^2} = 1 + \frac{4\, M^2 x_B^2}{Q^2}. \tag{37}$$

The quantity κ takes into account mass corrections $\propto M^2/Q^2$ which will vanish for large Q^2 ($\kappa \to 1$). Thus we define

$$Z^\mu \equiv -q^\mu, \tag{38}$$

$$T^\mu \equiv -\frac{q^2}{P\cdot q}\, \tilde{P}^\mu = q^\mu + 2x_B\, P^\mu. \tag{39}$$

For these vectors we have $Z^2 = -Q^2$ and $T^2 = \kappa\, Q^2$ and we will often use the normalized vectors $\hat{z}^\mu = -\hat{q}^\mu = Z^\mu/Q$ and $\hat{t}^\mu = T^\mu/Q\sqrt{\kappa}$. With respect to these vectors one can use the transverse tensors, $g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu$ and $\epsilon_\perp^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma}\hat{t}_\rho\hat{q}_\sigma$. To get the parametrization of hadronic tensors, such as the one in Eq. 36, including for generality also an (axial) spin vector S (see section B), we use the general symmetry property,

$$W_{\mu\nu}(q,P,S) = W_{\nu\mu}(-q,P,S) \tag{40}$$

as well as properties following from hermiticity, parity and time-reversal invariance,

$$W^*_{\mu\nu}(q,P,S) = W_{\nu\mu}(q,P,S), \tag{41}$$

$$W_{\mu\nu}(q,P,S) = \overline{W}_{\nu\mu}(\bar{q},\bar{P},-\bar{S}) \qquad \text{[Parity]}, \tag{42}$$

$$W^*_{\mu\nu}(q,P,S) = \overline{W}_{\mu\nu}(\bar{q},\bar{P},\bar{S}) \qquad \text{[Time reversal]}, \tag{43}$$

where $\bar{p} = (p^0, -\boldsymbol{p})$. Finally we use current conservation implying $q^\mu W_{\mu\nu} = W_{\mu\nu} q^\nu = 0$. Note that depending on the situation not all constraints can be applied. For inclusive unpolarized leptoproduction one obtains as the most general form for the symmetric tensor,

$$\begin{aligned} M\,W^{\mu\nu\,(S)}(P,q) &= \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right) F_1(x_{\scriptscriptstyle B},Q^2) + \frac{\tilde{P}^\mu \tilde{P}^\nu}{P\cdot q}\, F_2(x_{\scriptscriptstyle B},Q^2) \\ &= -g_\perp^{\mu\nu}\, \underbrace{F_1(x_{\scriptscriptstyle B},Q^2)}_{F_T} + \hat{t}^\mu \hat{t}^\nu\, \underbrace{\left(-F_1 + \frac{\kappa}{2x_{\scriptscriptstyle B}}\, F_2\right)}_{F_L}, \end{aligned} \tag{44}$$

where the structure functions F_1, F_2 or the transverse and longitudinal structure functions, $F_T = F_1$ and F_L, depend only on the for the hadron part relevant invariants Q^2 and $x_{\scriptscriptstyle B}$. This is the structure for the electromagnic (photon exchange) part of the electroweak interaction. For the weak (W- or Z-exchange) part both vector and axial vector currents with different parity behavior come in. In that case also the following antisymmetric tensor is allowed,

$$M\,W^{\mu\nu\,(A)}(q,P) = \frac{i\epsilon^{\mu\nu\rho\sigma} P_\rho q_\sigma}{(P\cdot q)}\, F_3(x_{\scriptscriptstyle B},Q^2) = i\,\kappa\,\epsilon_\perp^{\mu\nu}\, F_3(x_{\scriptscriptstyle B},Q^2). \tag{45}$$

It appears in the part of the tensor in which one of the currents in the product is a vector current and the other an axial vector current.

The cross section is obtained from the contraction of lepton and hadron tensors. It is convenient to expand also the lepton momenta ℓ and $\ell' = \ell - q$ in other vectors. Examples using ℓ and P (and the scaling variables for leptoproduction) are

$$\ell' = x_{\scriptscriptstyle B} y\, P + (1-y)\,\ell + Q\sqrt{1-y}\,\hat{\ell}_{\perp(\ell,P)}, \tag{46}$$

or using P and q (or linear combinations like P and n or $\hat{t}$ and $\hat{z}$)

$$\begin{aligned} \ell' &= \frac{x_{\scriptscriptstyle B}}{y}\, P + \frac{1-y}{y}\,\frac{Q^2}{2x_{\scriptscriptstyle B}}\, n + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}_{\perp(q,P)} \\ &= \frac{(2-y)Q}{2y}\,\hat{t}^\mu + \frac{Q}{2}\,\hat{q}^\mu + \frac{Q\sqrt{1-y}}{y}\,\hat{\ell}_{\perp(q,P)}. \end{aligned} \tag{47}$$

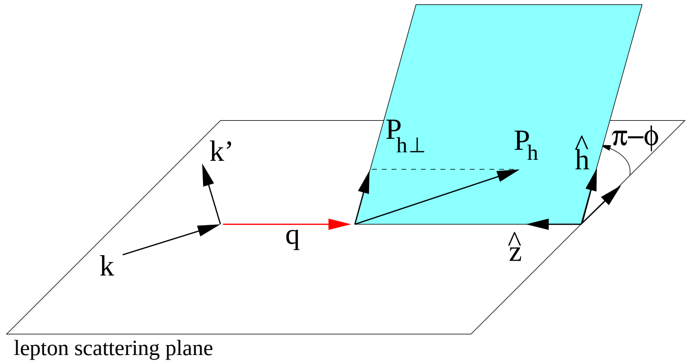


Fig. 2. – *Kinematics for lepton-hadron scattering. Transverse directions indicated with a* $\perp$ *index are orthogonal to* P *and* q*, e.g. the orthogonal component of the momentum of a produced hadron has been indicated as example. Similarly one can consider the orthogonal component of the spin vector of the target.*

where $n = (q + x_B P)/P{\cdot}q = (\hat{t} + \hat{z})/\sqrt{2}$, which is the (approximately) light-like vector $n^2 \approx 0$, conjugate to P, $P{\cdot}n = 1$. The unit vectors $\hat{\ell}_\perp$ define the transverse directions with between brackets following the $\perp$ the momenta with respect to which the perpendicular directions are being considered. We note that in particular when one considers azimuthal asymmetries one should be careful with the phase space,

$$d\tilde{\ell}' = \frac{d(\ell'{\cdot}P)\, d|\ell_{\perp(\ell,P)}|^2\, d\varphi_{\ell P}}{(2\pi)^3\, 2(\ell'{\cdot}P)} = \frac{y\, s}{8\pi^2}\, dx_B\, dy\, \frac{d\varphi_{\ell P}}{2\pi} \tag{48}$$

$$= \frac{d(\ell'{\cdot}n)\, d|\ell_{\perp(q,P)}|^2\, d\varphi_{qP}}{(2\pi)^3\, 2(\ell'{\cdot}n)} = \frac{s}{8\pi^2\, y}\, dx_B\, dy\, \frac{d\varphi_{qP}}{2\pi}. \tag{49}$$

The kinematics in the frame where virtual photon and target are collinear (including target rest frame) is illustrated in Fig. 2. With the definition of $\hat{\ell}_\perp$, we obtain neglecting mass corrections ($\kappa = 1$) for unpolarized leptons the symmetric leptonic tensor

$$\begin{aligned} L^{\mu\nu\,(S)} = \frac{Q^2}{y^2}\Bigg[& -2\left(1 - y + \frac{1}{2}\,y^2\right) g_\perp^{\mu\nu} + 4(1-y)\hat{t}^\mu \hat{t}^\nu \\ & + 4(1-y)\left(\hat{\ell}_\perp^\mu \hat{\ell}_\perp^\nu + \frac{1}{2}\,g_\perp^{\mu\nu}\right) + 2(2-y)\sqrt{1-y}\;\hat{t}^{\{\mu}\hat{\ell}_\perp^{\nu\}}\Bigg]. \end{aligned} \tag{50}$$

We will need this form when we look at explicit production of particles h in the final state. The explicit contraction of lepton and hadron tensors for inclusive scattering gives for electromagnetic scattering (only symmetric tensor and $\kappa = 1$) the result

$$
\begin{aligned}
(51)\qquad \frac{d\sigma^{ep}}{dx_{B}dy} &= \frac{4\pi\,\alpha^2\,x_{B}\,s}{Q^4}\left[\left(1-y+\frac{1}{2}\,y^2\right)F_T(x_{B},Q^2) + (1-y)\,F_L(x_{B},Q^2)\right]\\
&= \frac{2\pi\,\alpha^2\,s}{Q^4}\left[(1-y)\,F_2(x_{B},Q^2) + x_{B}\,y^2\,F_1(x_{B},Q^2)\right].
\end{aligned}
$$

We have now used the known photon coupling to the lepton and parametrized our ignorance for what happened with the hadron in a hadronic tensor. The fact that we know how the photon interacts with the (quark) constituents of the hadrons will be used later to relate the structure functions to quark properties. In the same way one also knows for the weak interaction processes leading to antisymmetric part containing F_3 in the tensor for unpolarized hadrons, how the Z^0 and W couple to quarks. To describe weak interactions also the antisymmetric part of the lepton tensor is needed, which is also encountered when one looks at polarization (see Appendix B)

5. – Virtual photon cross sections

Dressing the hadronic tensor $W_{\mu\nu}$ with photon polarization vectors one obtains the total cross section for $\gamma^* H \to$ everything, where γ^* indicates a virtual photon. For a given virtuality Q^2 of the photon this cross section depends on only one variable, $W^2 = (P+q)^2$, or equivalently on the variable $\nu = P\cdot q/M$,

$$
(52)\qquad \sigma^{\gamma^* H}(\nu) = \frac{4\pi^2\alpha}{K}\,\epsilon^{\mu *}W_{\mu\nu}\epsilon^{\nu},
$$

where $4M\,K$ is the photon flux factor. This flux factor only is physical for real photons ($Q^2 = 0$). One convention is to take $4M\,K = 4\sqrt{(P\cdot q)^2 - P^2 q^2}$, i.e. $K = \sqrt{\nu^2+Q^2}$. Other possibilities are to take the real photon result $4M\,K = 4\,P\cdot q$ or $K = \nu$ (Hand convention). Another convention that has been used is to equate the final state invariant mass squared $W^2 = (P+q)^2$, i.e. take the result $4M\,K = 2(W^2 - M^2)$ for a massless photon and equate W^2 to the invariant mass in the case of a virtual photon, $W^2 = 2\,P\cdot q + M^2 - Q^2$ or $K = \nu - Q^2/2M$.

Being a (total) cross section for (virtual) photoabsorption, the hadronic tensor is related to the forward (virtual) Compton amplitude through the optical theorem,

$$
(53)\qquad W_{\mu\nu} = \frac{1}{\pi}\,\mathrm{Im}\;T_{\mu\nu},
$$

where

$$
(54)\qquad 2M\,T_{\mu\nu}(P,q) = i\int d^4x\;e^{iq\cdot x}\langle P|\mathcal{T}\,J_\mu(x)J_\nu(0)|P\rangle,
$$

For a virtual photon we need three polarization vectors ϵ^μ_α, where α indicates the polarization directions (perpendicular to q^μ),

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}}\,(0,\mp 1,-i,0) = \mp\frac{1}{\sqrt{2}}(\epsilon_x \pm i\,\epsilon_y) \quad \text{and} \quad \epsilon^\mu_{\scriptscriptstyle L} = \frac{1}{\sqrt{Q^2}}\,(q^3,0,0,q^0). \tag{55}$$

One gets two transverse structure functions and one longitudinal structure function, $F_\alpha = \epsilon^{\mu *}_\alpha\, MW_{\mu\nu}\epsilon^\nu_\alpha$. Because of the fact that they are cross sections for (virtual) photons, the structure functions, F_+, F_- and F_L are positive. We have (as already indicated)

$$F_T = \frac{1}{2}\,(F_+ + F_-) = F_1, \quad F_L = \frac{F_2}{2x_{\scriptscriptstyle B}} - F_1, \quad \text{and} \quad F_3 = F_+ - F_-. \tag{56}$$

6. – Form factors

In the structure functions the currents $J_\mu(x)$ enter that describe for instance coupling to photons or other electroweak bosons. Even if in that case the currents are known to be quark-currents([1]),

$$J^{(\gamma)}_\mu = :\overline{\psi}(x)\,Q\,\gamma_\mu\,\psi(x):, \tag{57}$$

$$\begin{aligned} J^{(Z)}_\mu &= :\overline{\psi}(x)\,(I^3_W - Q\sin^2\theta_W)\,\gamma_{\mu L}\,\psi(x): - :\overline{\psi}(x)\,Q\sin^2\theta_W\,\gamma_{\mu R}\,\psi(x): \\ &= :\overline{\psi}(x)\,(I^3_W - 2\,Q\sin^2\theta_W)\,\gamma_\mu\,\psi(x): - :\overline{\psi}(x)\,I^3_W\,\gamma_\mu\gamma_5\,\psi(x):, \end{aligned} \tag{58}$$

$$J^{(W)}_\mu = :\overline{\psi}(x)\,I^\pm_W\,\gamma_{\mu L}\,\psi(x):, \tag{59}$$

where $\gamma_{\mu R/L} = \gamma_\mu(1\pm\gamma_5)$. Although one has known currents of which even the flavor decomposition is known, their expectation values are not known.

In the special case in which the final state is identical to the initial state (elastic scattering, which is part of the inclusive scattering) $P' = P + q$ and is fixed to be $(P+q)^2 = M^2$, i.e. $x_{\scriptscriptstyle B} = 1$. We can still use the formalism developed so far, but the hadronic tensor becomes becomes

$$2M\,W_{\mu\nu}(q,P) = \underbrace{\langle P|J_\mu(0)|P'\rangle\langle P'|J_\nu(0)|P\rangle}_{H_{\mu\nu}(P;P')}\,\frac{1}{Q^2}\,\delta(1-x_{\scriptscriptstyle B}). \tag{60}$$

One needs the hadronic current matrix elements, instead of those of the commutator or the time-ordered product. The current matrix elements are expressed in form factors, which are Fourier transforms of charge and current distributions. We will consider in more detail the vector and axial vector currents, V_μ and A_μ. For composite systems the

([1]) Note that the factors e, $e/2\sin\theta_W\cos\theta_W$ and $e/2\sqrt{2}\sin\theta_W$ that would be expected for the γ, Z^0 and W-couplings to the quarks are omitted here. In writing cross sections, they will be needed (like the e^2 in previous section) and are conventionally included in the fine structure constant α and the Fermi-coupling $G_F/\sqrt{2} = e^2/8M_W^2\sin^2\theta_W$

expectation values of these currents is in general unknown. One can however write down a most general form and use translation invariance. The most general form consistent with the requirements of Poincaré invariance, gauge invariance and invariance under parity and time reversal yields for a spin 0 system (e.g. a pion or a ^{12}C nucleus) the form

$$\langle P', S'|V_\mu(x)|P, S\rangle = (P_\mu + P'_\mu)\, F(Q^2) e^{i(P'-P)\cdot x}, \tag{61}$$

involving one invariant function (*form factor*) depending on the (only) invariant, $q^2 \equiv -Q^2$, where $q = P' - P$. For spin 1/2 systems (such as a nucleon, ^{3}H or ^{3}He) the most general expression for the expectation value of the current can be written using free Dirac spinors $U(P, S)$,

$$\langle P', S'|V_\mu(0)|P, S\rangle = \overline{U}(P', S') \left[\gamma_\mu\, F_1(Q^2) + \frac{i\sigma_{\mu\nu} q^\nu}{2M}\, F_2(Q^2)\right] U(P, S), \tag{62}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. The functions F_1 and F_2 are the Dirac and Pauli form factors.

The elastic cross section then is found by inserting the parametrizations, for instance in the case of electromagnetic scattering one obtains the tensor $H_{\mu\nu}$

$$\begin{aligned} H_{\mu\nu}(P, P') &= \frac{1}{2} \sum_{S,S'} \langle P, S|J_\mu(0)|P', S'\rangle\langle P', S'|J_\nu(0)|P, S\rangle \\ &= \frac{1}{2}\,\mathrm{Tr}\;\left[\Gamma_\mu(\not{P}' + M)\Gamma_\nu(\not{P} + M)\right] \\ &= \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) Q^2 (F_1 + F_2)^2 + 4\,\tilde{P}_\mu \tilde{P}_\nu \left(F_1^2 + \frac{Q^2}{4M^2}\, F_2^2\right) \\ &= -g_{\perp\,\mu\nu}\, Q^2 G_M^2 + \hat{t}_\mu \hat{t}_\nu\, 4M^2\, G_E^2 \end{aligned} \tag{63}$$

One thus sees the following *elastic* contribution in the transverse and longitudinal structure functions

$$F_T(x_B, Q^2) = G_M^2(Q^2)\, \delta(1 - x_B), \tag{64}$$

$$F_L(x_B, Q^2) = \frac{4M^2}{Q^2}\, G_E^2(Q^2)\, \delta(1 - x_B). \tag{65}$$

In Appendix A, we have given some basic background information on the Nucleon form factors.

7. – Partons in QCD

The most intuitive way to discuss hard scattering processes is the parton model. One assumes a momentum distribution of quarks in the nucleon and folds it with the γ^*-quark cross section, given by

$$\hat\sigma(\gamma^* q) = \frac{4\pi^2\alpha}{2p\cdot q}\,\epsilon_\mu^*\widehat{W}^{\mu\nu}\epsilon_\nu$$
$$2M\,\widehat{W}_{\mu\nu}(p,q) = \left[\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)Q^2 + 4\,\tilde p_\mu \tilde p_\nu\right]\delta(2\,p\cdot q + q^2)$$

(note that there are in principle ambiguities here because of the flux factor for virtual photons). One finds

$$\hat\sigma_T(\gamma^* q) = 4\pi^2\alpha\; e_q^2\;\delta(2p\cdot q - Q^2), \tag{66}$$

$$\hat\sigma_L(\gamma^* q) = 4\pi^2\alpha\; e_q^2\;\frac{4m^2}{Q^2}\,\delta(2p\cdot q - Q^2) \ll \hat\sigma_T. \tag{67}$$

These partonic cross sections are folded with a probability function for finding partons in the target. It is convenient to use here light-cone components

$$p = xP + p_T + (p\cdot P - xM^2)n,$$

where n is a (at present) arbitrary light-like vector satisfying $P\cdot n = 1$. This vector plays the role of $n = n_-$, while in that case $n_+ = P - \frac{1}{2}M^2\,n$. Within the hard scattering process the dominant contraction of P with other vectors are much larger than M^2, so the second term in n_+ is irrelevant or in other words P can be considered a light-like vector itself. This Sudakov decomposition of the parton model is useful since it separates the components of p into a 'large' one $p \approx xP$ with $x = p\cdot n/P\cdot n = p^+/P^+$ being a momentum fraction, a 'normal-sized' component p_T and a small component involving the light-cone energy $p^- = p\cdot P - \frac{1}{2}xM^2$. For the leading order results one can integrate over p^- and usually also over p_T and look at the momentum distribution $f_i(x)$ for parton i. Although most results are independent of n one can construct it from other vectors in the problem, in the case of inclusive deep inelastic scattering the vector $n = (q + xP)/P\cdot q$ satisfies the requirements.
==============

Exercise: Derive the light-cone expansion for the (external) vectors P and q,

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} q = \frac{Q}{\sqrt2}\,n_- \;-\; \frac{Q}{\sqrt2}\,n_+ \\ \\ P = \frac{x_B M^2}{Q\sqrt2}\,n_- + \frac{Q}{x_B\sqrt2}\,n_+ \end{array}\right. ,$$

or in light-cone components ($n_+ \equiv [0, A, 0_\perp]$ and $n_- \equiv [1/A, 0, 0_\perp]$)

$$q = \left[\frac{Q^2}{A\sqrt2}, -\frac{A}{\sqrt2}, \mathbf{0}_\perp\right], \quad \text{and} \quad P = \left[\frac{x_B M^2}{A\sqrt2}, \frac{A}{x_B\sqrt2}, \mathbf{0}_\perp\right] \;\approx\; \left[0, \frac{A}{x_B\sqrt2}, \mathbf{0}_\perp\right].$$

What is the effect of a boost? What is A in the nucleon rest-frame. $A \to \infty$ is referred to as the *infinite momentum frame*. In particular the first representation with light-like vectors shows that when Q^2 becomes large, the nucleon momentum is 'on the scale Q in essence light-like. While the hard momentum has both components proportional to Q, this is not the case for P and one has $P^- \ll q^-$. In deep-inelastic scattering the plus components are of the same order with ratio being the scaling variable $x_B = -q^+/P^+$.
==============
Under the assumption that all invariants $p \cdot P \sim M_R^2 \sim p^2 \sim P^2 = M^2 \ll Q^2$ one sees that for the expansion in terms of $n_\pm$ one has for a quark in a hadron (as one would have for an on-shell quark) that $p^+ \sim P^+ \sim Q$, while $p^- \sim M^2/Q$ and $p_T^2 \sim M^2$ (see Eqs 14 and 15. This is sufficient to derive the parton model results.

Using the cross section for $\gamma^* q$ (elastic scattering) given above,

$$\hat{\sigma}_T = \frac{4\pi^2\alpha}{Q^2}\,\delta(1-x_p). \tag{68}$$

where $x_p = -q^+/p^+$ and introducing probabilities $f_i(x)$ for finding partons carrying momentum fraction $x = p^+/P^+ = x_B/x_p$ of the target light-cone momentum, leads to([2])

$$\sigma_T = \sum_i e_i^2 \int dx\, f_i(x)\,\frac{4\pi^2\alpha}{Q^2}\,\delta\left(1-\frac{x_B}{x}\right) = \frac{4\pi^2\alpha}{Q^2}\,\sum_i e_i^2\,x_B\,f_i(x_B). \tag{69}$$

Comparing with

$$\sigma_T = \frac{8\pi^2\alpha}{Q^2}\,x_B\,F_1, \tag{70}$$

we get

$$F_1(x_B) = \frac{1}{2}\sum_i e_i^2\,f_i(x_B). \tag{71}$$

As $\hat{\sigma}_L \propto 1/Q^2 \to 0$ one obtains $F_L = 0$ or the Callan-Gross relation,

$$F_2(x_B) = 2x_B\,F_1(x_B). \tag{72}$$

8. – The diagrammatic approach

The diagrammatic approach has a different starting point. It realizes the absence of wave functions for the quarks and gluons. The situation thus differs from that of QED

([2]) We note that the probability involves in fact $f_i(x)\,dx/x$, but we need to fold conting rates which requires that we need the cross section multiplied with flux factors. The ratio of the flux factors for quarks and hadrons is $p \cdot q/P \cdot q \approx p^+/P^+ = x$. Hence we need to weigh the cross section with $f_i(x)\,dx$.

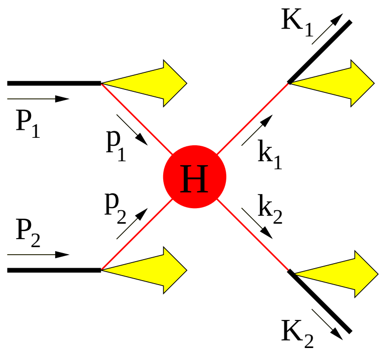


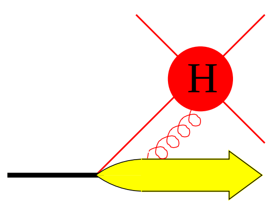
Fig. 3. – Schematic illustration of the contribution of a hard subprocess, parton (p_1) + parton (p_2) $\rightarrow$ parton (k_1) + parton (k_2), to the (2-particle inclusive) scattering process hadron (P_1) + hadron (P_2) $\rightarrow$ hadron (K_1) + hadron (K_2) + X, at the level of the amplitude. The process being hard implies for the hadronic momenta $P_1 \cdot P_2 \sim P_1 \cdot K_1 \sim Q^2$, etc.

with physical electrons and photons. In the latter case one knows how in the calculation of an S-matrix element contraction of annihilation and creation operator in the field and particle state lead to the spinor wave function. For positive times $\xi^0 = t > 0$ one has

$$\langle 0|\psi_i(\xi)|p,s\rangle = \langle 0|\psi_i(\xi)\, b^\dagger(\boldsymbol{p},s)|0\rangle = \langle 0|\psi_i(0)|p,s\rangle\, e^{-i\,p\cdot\xi} = u_i(\boldsymbol{p},s)\, e^{-i\,p\cdot\xi}, \tag{73}$$

with $p^0 = +E_p = +\sqrt{\boldsymbol{p}^2+m^2}$. Such a matrix element is 'untruncated' as seen e.g. from

$$\langle 0|\psi_i(\xi)|p,s\rangle\,\theta(t) = \theta(t)\int \frac{d^4k}{(2\pi)^4}\; e^{-ik\cdot\xi}\; \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\;\frac{u_i(\boldsymbol{p},s)}{2m}\,(2\pi)^3\,2E_p\,\delta^3(\boldsymbol{k}-\boldsymbol{p}). \tag{74}$$



In a process involving a composite hadronic state $|P\rangle$, contractions with one or several of the quark and gluon operators may be involved, leading to nonzero matrix elements for a quark between the hadron state and a remainder, but also for nonzero matrix elements involving multi-parton field combinations,

$$\langle X|\psi_i(\xi)|P\rangle,\ \langle X|A^\mu(\eta)\,\psi(\xi)|P\rangle, \dots\ .$$

Correlators, describing parton distributions. – For a particular hadron and a parton field combination, one may collect those operators that involve hadron $|P\rangle$ into (distribution) correlators

$$\begin{aligned}\Phi_{ij}(p;P) &= \sum_X \int \frac{d^3P_X}{(2\pi)^3\,2E_X}\;\langle P|\overline{\psi}_j(0)|X\rangle\,\langle X|\psi_i(0)|P\rangle\,\delta^4(p+P_X-P)\\ &= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle,\end{aligned} \tag{75}$$

where a summation over color indices is understood. It is often convenient to use momentum space fields, $\psi(p) \equiv \int d^4x\ e^{i\,p\cdot x}\,\psi(x)$, for a free field expansion leading to

$$\psi_i(\boldsymbol{p},t) = \sum_s \left(u_i(\boldsymbol{p},s)\, b(\boldsymbol{p},s) \frac{e^{-iE_p t}}{2E_p} + v_i(-\boldsymbol{p},s)\, d^\dagger(-\boldsymbol{p},s) \frac{e^{iE_p t}}{2E_p} \right), \tag{76}$$

$$\begin{aligned}\psi_i(p) = \sum_s \Big(& u_i(\boldsymbol{p},s)\, b(\boldsymbol{p},s)\, (2\pi)\delta(p^2-m^2)\,\theta(p^0) \\ & + v_i(-\boldsymbol{p},s)\, d^\dagger(-\boldsymbol{p},s)\, (2\pi)\delta(p^2-m^2)\,\theta(-p^0)\Big).\end{aligned} \tag{77}$$

For the correlator we have

$$(2\pi)^4\delta^4(p-p')\,\Phi_{ij}(p;P) = \frac{1}{(2\pi)^4}\,\langle P|\overline{\psi}_j(p')\,\psi_i(p)|P\rangle. \tag{78}$$

This latter form is convenient for interpretation of the nature of the correlators because for a free field we have

$$\begin{aligned}\langle X|\psi_i(p)|P\rangle = {} & \langle P_X|b(\boldsymbol{p},s)|P\rangle\, u_i(\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(p^0) \\ & + \langle P_X|d^\dagger(-\boldsymbol{p},s)|P\rangle\, v_i(-\boldsymbol{p},s)\,(2\pi)\delta(p^2-m^2)\,\theta(-p^0).\end{aligned}$$

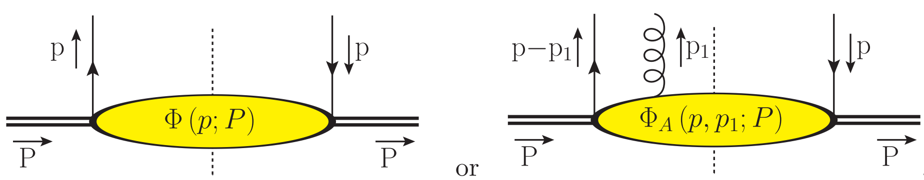
One also encounters correlators involving matrix elements including gluon fields, of the form

$$\Phi^\mu_{A\,ij}(p,p_1;P) = \frac{1}{(2\pi)^8}\int d^4\xi\, d^4\eta\ e^{i\,(p-p_1)\cdot\xi}\ e^{i\,p_1\cdot\eta}\ \langle P|\overline{\psi}_j(0)\,A^\mu(\eta)\,\psi_i(\xi)|P\rangle, \tag{79}$$

or with momentum space operators (for gluons $A_\mu(p) \equiv \int d^4x\ e^{i\,p\cdot x}\,A_\mu(x)$) one has

$$(2\pi)^4\delta^4(p-p')\,\Phi^\mu_{A\,ij}(p,p_1;P) = \frac{1}{(2\pi)^8}\,\langle P|\overline{\psi}_j(p')\,A^\mu(p_1)\,\psi_i(p-p_1)|P\rangle. \tag{80}$$

Pictorially one has for the correlators,



We will not attempt to calculate these, but leave them as the soft parts, requiring nonperturbative QCD methods to calculate them. In particular, although being 'untruncated' in the quark legs, they will no longer exhibit poles corresponding to free quarks. These are fully unintegrated parton correlators for initial state hadrons, in general quite problematic quantities. For example, they are by themselves not even color gauge-invariant, an issue to be discussed below. We will later also discuss similar correlators for final state hadrons. When more hadrons are involved, one needs to consider two-hadron correlators, involving two-hadron states (or correlators involving hadronic states in initial and final state), etc. If the hadrons are well-separated in momentum phase-space with $P_i \cdot P_j \sim Q^2$, one expects on dimensional grounds that incoherent contributions are suppressed by $1/(P_i - P_j)^2 \sim 1/Q^2$. Such a separation in momentum space requires a hard inclusive scattering process ($Q^2 \sim s$), which then at high energy and/or for large momentum transfer still can be factorized into forward correlators. The inclusive character is needed to assure that partons originate from *one* hadron, leaving a (target) jet. In turn, partons decay into a jet in which we limit ourselves to the consideration of an identified hadronic state (which could in principle also be a multi-particle, e.g. two-pion, state). In all of the hadronic states mentioned before one can also consider polarized hadronic states (see Appendices B and C). The spin of quarks is contained in Dirac structure and that of gluons in the Lorentz structure of correlators.

The basic idea in the diagrammatic approach is to realize that the correlator involves hadronic states and quark and gluon operators. The correlators can be studied independent from the hard process, provided we have dealt with the issue of color gauge invariance. The correlator is the Fourier transform in the space-time arguments of the quark and gluon fields. In the correlators, all momenta of hadrons *and* quarks and gluons (partons) inside the hadrons are soft which means that $p^2 \sim p\cdot P \sim P^2 = M_N^2 \ll Q^2 \sim s$. The *off-shellness* being of hadronic order implies that in the hard process partons are in essence *on-shell*. Consistency of this may be checked by using QCD interactions to give partons a large off-shellness of $\mathscr{O}(Q)$ and check the behavior as a function of the momenta. In these considerations one must also realize that beyond tree-level one has to distinguish bare and renormalized fields.

The simplest expression that is needed in scattering a photon from the target, is the correlator and the scattering of the photon from a quark. Restricting us to the quark part is

$$
\begin{aligned}
2M\, W^{\mu\nu}(P,q) &= \sum_q e_q^2 \int dp^-\, dp^+\, d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu(\not p + \not q + m)\gamma^\nu\right)\,\delta((p+q)^2 - m^2) \\
&\approx \sum_q e_q^2 \int dp^-\, dp^+\, d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\Phi(p)\,\gamma^\mu\,\frac{\not q}{2q^-}\,\gamma^\nu\right)\,\delta(p^+ + q^+) \\
&\approx -g_\perp^{\mu\nu}\,\frac{1}{2}\,\int dp^-\, d^2\boldsymbol{p}_\perp\ \mathrm{Tr}\left(\gamma^+\,\Phi(p)\right)\Big|_{p^+ = x_B\, P^+} + \ldots, \qquad (81)
\end{aligned}
$$

where $\Phi(p)$ is the correlator discussed above. The relevant soft part then is a particular Dirac trace of the quantity

$$
\Phi_{ij}(x) = \int dp^-\, d^2p_T\ \Phi_{ij}(p,P,S) = \int \frac{d\xi^-}{2\pi}\, e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \qquad (82)
$$

depending on the lightcone fraction $x = p^+/P^+$.

Comparing with the general form of the hadronic tensor, we read off (including now also the antiquark part)

$$2\,F_1(x_B) = 2M\,W_1(x_B,Q^2) = \sum_q e_q^2\,\left[f_1^q(x_B) + \bar{f}_1^q(x_B)\right], \tag{83}$$

with

$$f_1^q(x) = \int \frac{d\xi\cdot P}{4\pi}\;e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\not{n}\psi(\xi)|P,S\rangle\Big|_{\xi\cdot n=\xi_\perp=0} = \frac{1}{2}\,\mathrm{Tr}\,(\Phi(x)\not{n})\,, \tag{84}$$

$$\bar{f}_1^q(x) = \int \frac{d\xi\cdot P}{4\pi}\;e^{-ip\cdot\xi}\langle P,S|\overline{\psi}(0)\not{n}\psi(\xi)|P,S\rangle\Big|_{\xi\cdot n=\xi_\perp=0}\,, \tag{85}$$

often simply denoted as $q(x) = f_1^q(x)$, satisfying $\overline{q}(x) = -q(-x)$ (Exercise!). The result is (as expected) a light-cone correlation function of quark fields.

The operator in coordinate space. – The parton result for the structure functions can also be derived by inserting free currents in the hadronic tensor for the current commutator and using the expression for the free field commutator.

===============

Exercise: Use the anticommutation relations for free quark fields, $\{\psi(\xi),\overline{\psi}(0)\} = \frac{1}{2\pi}\,\not{\partial}\,\delta(\xi^2)\,\epsilon(\xi^0)$ to derive for the $g_{\mu\nu}$ contribution in the current-current commutator for quarks

$$\begin{aligned}[J_\mu(\xi),J_\nu(0)] &= [:\overline{\psi}(\xi)\gamma_\mu\psi(\xi):,:\overline{\psi}(0)\gamma_\nu\psi(0):]\\ &= \frac{-g_{\mu\nu}}{2\pi}\left[\partial_\rho\,\delta(\xi^2)\,\epsilon(\xi^0)\right]:\overline{\psi}(\xi)\gamma^\rho\psi(0) - \overline{\psi}(0)\gamma^\rho\psi(\xi):\,.\end{aligned} \tag{86}$$

An important feature, evident in the free-current commutator, is the light-cone dominance. By sandwiching the commutator between physical states and taking the Fourier transform, it is a straightforward calculation to obtain again the hadron tensor and the same result as in the diagrammatic approach above. Details can be found in Ref. [7].

===============

Interpretation as densities. – To convince oneself that the above expressions for $f_1(x)$ and $bar f_1(x)$ actually can be interpreted as quark momentum density one needs to realize that $\overline{\psi}(\xi)\gamma^+\psi(0) = \sqrt{2}\,\psi_+^\dagger(\xi)\psi_+(0)$ where $\psi_\pm = P_\pm\,\psi$ are projections obtained with projection operators $P_\pm = \frac{1}{2}\gamma^\mp\gamma^\pm$. One then can insert a complete set of states and obtain

$$\begin{aligned}f_1(x) &= \int \frac{d\xi^-}{2\pi\sqrt{2}}\;e^{ip\cdot\xi}\,\langle P,S|\psi_+^\dagger(0)\psi_+(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}\\ &= \frac{1}{\sqrt{2}}\sum_n |\langle P_n|\psi_+|P\rangle|^2\;\delta\left(P_n^+ - (1-x)P^+\right),\end{aligned} \tag{87}$$

which represents the probability that a quark is annihilated from $|P\rangle$ giving a state $|n\rangle$ with $P_n^+ = (1-x)P^+$. As a further note, we mention that ψ_+ fields are actually the good fields of the light-cone quantized field theory, so the expansion in plane wave states with specified p^+ and p_T components makes sense even in the interacting theory (see Appendices D and quantumforms).

Relation to forward amplitudes. – The full hadronic tensor as a squared amplitude is also the imaginary part of a forward amplitude (optical theorem). The same relation holds for the hard (QCD) amplitude. A similar property also holds for the soft part after integration over the light-cone momentum $p^- = p{\cdot}P$. All operators in the correlator now are evaluated at light-cone time $\xi^+ = \xi{\cdot}n = 0$. This implies that time-ordering has become automatic. Thus not only the full hadronic tensor, but also the correlators $\Phi(x,p_T)$ and $\Phi(x)$, besides being densities, can be seen as time-ordered products of fields, which means related to the forward antiquark-hadron scattering amplitudes [8, 9]. This is important because amplitudes in a field theoretical setting have various analytic properties, e.g. related to unitarity. That the time-ordering is only evident in light-cone time is not a problem, because the ordering as such is covariant.

9. – Quark correlation functions in 1PI leptoproduction

We now consider the case in which one particle is detected in coincidence with the scattered lepton, *one-particle-inclusive* or 1PI leptoproduction. The kinematics of this process is already in the picture given before (Fig. 2). With a target hadron (momentum P) and a detected hadron h in the final state (momentum P_h) one has a situtation in which two hadrons are involved and the operator product expansion cannot be used. Within the framework of QCD and knowing that the photon or Z^0 current couples to the quarks, it is possible to write down a diagrammatic expansion for leptoproduction, with in the deep inelastic limit ($Q^2 \to \infty$) as relevant diagrams only the ones given in Fig. 4 for 1-particle inclusive scattering.

In analogy with the case of inclusive scattering, we also in 1-particle inclusive scattering parametrize the momenta with the help of two lightlike vectors, which are choosen now along the hadron momenta,

$$\left.\begin{array}{l} q^2 = -Q^2 \\ P^2 = M^2 \\ P_h^2 = M_h^2 \\ 2\,P\cdot q = \frac{Q^2}{x_B} \\ 2\,P_h\cdot q = -z_h\,Q^2 \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} P_h = \frac{z_h\,Q}{\sqrt{2}}\,n_- + \frac{M_h^2}{z_h\,Q\sqrt{2}}\,n_+ \\ \\ q = \frac{Q}{\sqrt{2}}\,n_- - \frac{Q}{\sqrt{2}}\,n_+ + q_T \\ \\ P = \frac{x_B M^2}{Q\sqrt{2}}\,n_- + \frac{Q}{x_B\sqrt{2}}\,n_+ \end{array}\right.$$

An additional invariants z_h comes in. Note that the expansion is appropriate for the socalled current fragmentation, in which case the produced hadron is *hard* with respect to the target momentum, i.e. $P\cdot P_h \sim Q^2$. The minus component p^- is irrelevant in the lower soft part, while the plus component k^+ is irrelevant in the upper soft part. Note

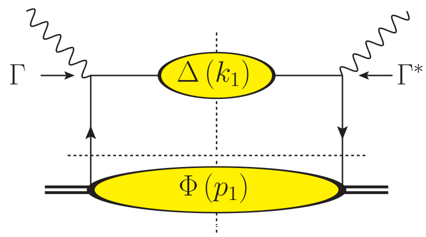


Fig. 4. – *The simplest (parton-level) diagrams for semi-inclusive scattering, of which we consider one-particle inclusive leptoproduction. Note that also the diagram with opposite fermion flow has to be added.*

that after the choice of P and P_h one can no longer omit a transverse component in the other vector, in the consideration above put in the momentum transfer q. One sees that one has (up to mass effects) the relation

(88) $$q_T^\mu = q^\mu + x_B\, P^\mu - \frac{P_h^\mu}{z_h} \equiv -Q_T\,\hat{h}^\mu.$$

This relation allows the experimental determination of the 'transverse momentum' effect from the external vectors q, P and P_h which are in general not collinear. The vector $\hat{h}$ defines the orientation of the hadronic plane in Fig. 2.

An important consequence in the theoretical approach (Fig. 4) is that one can no longer simply integrate over the transverse components of the quark momenta.

Structure functions and cross sections. – For an unpolarized (or spin 0) hadron in the final state the symmetric part of the tensor is given by

(89) $$\begin{aligned} M\mathcal{W}_S^{\mu\nu}(q,P,P_h) = & -g_\perp^{\mu\nu}\,\mathcal{H}_T + \hat{t}^\mu\hat{t}^\nu\,\mathcal{H}_L \\ & + \hat{t}^{\,\{\mu}\hat{h}^{\nu\}}\,\mathcal{H}_{LT} + \left(2\,\hat{h}^\mu\hat{h}^\nu + g_\perp^{\mu\nu}\right)\mathcal{H}_{TT}. \end{aligned}$$

Noteworthy is that also an antisymmetric term in the tensor is allowed,

(90) $$M\mathcal{W}_A^{\mu\nu}(q,P,P_h) = -i\hat{t}^{\,[\mu}\hat{h}^{\nu]}\,\mathcal{H}'_{LT}.$$

Clearly the lepton tensor in Eq. 51 or B.3 is able to distinguish all the structures in the semi-inclusive hadron tensor.

The symmetric part gives the cross section for unpolarized leptons,

(91) $$\begin{aligned} \frac{d\sigma_{OO}}{dx_B\,dy\,dz_h\,d^2q_T} = \frac{4\pi\,\alpha^2\,s}{Q^4}\,x_B\,z_h \Bigg\{ & \left(1 - y + \frac{1}{2}\,y^2\right)\mathcal{H}_T + (1-y)\,\mathcal{H}_L \\ & - (2-y)\sqrt{1-y}\,\cos\phi_h^\ell\;\mathcal{H}_{LT} \\ & + (1-y)\,\cos 2\phi_h^\ell\;\mathcal{H}_{TT} \Bigg\} \end{aligned}$$

while the antisymmetric part gives the cross section for a polarized lepton (note the target is not polarized!)

$$\frac{d\sigma_{LO}}{dx_B\,dy\,dz_h d^2q_T} = \lambda_e\,\frac{4\pi\,\alpha^2}{Q^2}\,z_h\,\sqrt{1-y}\,\sin\phi_h^\ell\;\mathcal{H}'_{LT}. \tag{92}$$

Of course many more structure functions appear for polarized targets or if one considers polarimetry in the final state. In this case the (theoretically) most convenient way to describe the spin vector of the target is via an expansion of the form

$$S^\mu = -S_L\,\frac{Mx_B}{Q\sqrt{2}}\,n_- + S_L\,\frac{Q}{Mx_B\sqrt{2}}\,n_+ + S_T. \tag{93}$$

One has up to $\mathcal{O}(1/Q^2)$ corrections $S_L \approx M\,(S\cdot q)/(P\cdot q)$ and $S_T \approx S_\perp$, where the subscript $\perp$ still refers to perpendicular to q and P. For a pure state one has $S_L^2+\boldsymbol{S}_T^2=1$, in general this quantity being less or equal than one.

The parton model approach. – The expression for $\mathcal{W}_{\mu\nu}$ can be rewritten as a nonlocal product of currents and it is a straightforward exercise to show by inserting the currents $j_\mu(x) =: \overline{\psi}(x)\gamma_\mu\psi(x) :$ that for 1-particle inclusive scattering one obtains in tree approximation

$$\begin{aligned}
2M\mathcal{W}_{\mu\nu}(q;PS;P_hS_h) \hspace{-3em}&\\
&= \frac{1}{(2\pi)^4}\int d^4x\;e^{iq\cdot x}\,\langle PS|:\overline{\psi}_j(x)(\gamma_\mu)_{jk}\psi_k(x):\sum_X|X;P_hS_h\rangle\\
&\qquad\qquad\times\langle X;P_hS_h|:\overline{\psi}_l(0)(\gamma_\nu)_{li}\psi_i(0):|PS\rangle\\
&= \frac{1}{(2\pi)^4}\int d^4x\;e^{iq\cdot x}\,\langle PS|\overline{\psi}_j(x)\psi_i(0)|PS\rangle(\gamma_\mu)_{jk}\\
&\qquad\langle 0|\psi_k(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle(\gamma_\nu)_{li}\\
&\quad+\frac{1}{(2\pi)^4}\int d^4x\;e^{iq\cdot x}\,\langle PS|\psi_k(x)\overline{\psi}_l(0)|PS\rangle(\gamma_\nu)_{li}\\
&\qquad\langle 0|\overline{\psi}_j(x)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\psi_i(0)|0\rangle(\gamma_\mu)_{jk}+\dots\\
&= \int d^4p\,d^4k\,\delta^4(p+q-k)\;\mathrm{Tr}\left(\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right)+\left\{\begin{array}{c} q\leftrightarrow -q\\ \mu\leftrightarrow\nu\end{array}\right\},
\end{aligned} \tag{94}$$

where we encounter not only a correlator describing quark distributions but also a correlator describing the fragmentation process,

$$\Phi_{ij}(p) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{ip\cdot\xi}\,\langle PS|\overline{\psi}_j(0)\psi_i(\xi)|PS\rangle, \tag{95}$$

$$\Delta_{kl}(k) = \frac{1}{(2\pi)^4}\int d^4\xi\;e^{ik\cdot\xi}\,\langle 0|\psi_k(\xi)\sum_X|X;P_hS_h\rangle\langle X;P_hS_h|\overline{\psi}_l(0)|0\rangle. \tag{96}$$

Note that in Φ (quark production) a summation over colors is assumed, while in Δ (quark decay) an averaging over colors is assumed. The quantities Φ and Δ correspond to the blobs in Fig. 4 and parametrize the soft physics, leading to the definitions of distribution and fragmentation functions [10, 11, 12]. Soft refers to all invariants of momenta being small as compared to the hard scale, i.e. for $\Phi(p)$ one has $p^2 \sim p\cdot P \sim P^2 = M^2 \ll Q^2$.

The fact that $p{\cdot}P$ and $k{\cdot}P_h$ are soft, allows rewriting of

$$\begin{aligned}\delta^4(p+q-k) &= P{\cdot}P_h \delta(p{\cdot}P_h + q{\cdot}P_h)\,\delta(P{\cdot}q - k{\cdot}P)\,\delta^2(p_T + q_T - k_T)\\ &= P{\cdot}P_h\,\delta(x-x_B)\,\delta(1/z_h - z^{-1})\,\delta^2(p_T+q_T-k_T),\end{aligned}$$

where we used that $x_B \equiv Q^2/2P{\cdot}q \approx -q{\cdot}P_h/P{\cdot}P_h$ and $z_h = -2P_h{\cdot}q/Q^2 = P{\cdot}P_h/P{\cdot}q$. Thus the semi-inclusive tensor becomes

$$2M\mathcal{W}_{\mu\nu}(q;PS;P_hS_h) = \int d^2p_T\,d^2k_T\,\delta^2(p_T+q_T-k_T)\;\mathrm{Tr}\left(\Phi(x,p_T)\gamma_\mu\Delta(z,k_T)\gamma_\nu\right), \tag{97}$$

with convoluted transverse momentum dependent (TMD) correlators for distributions and fragmentation [12],

$$\Phi_{ij}(x,p_T;n) = \int \frac{d(\xi^-)\,d^2\xi_T}{(2\pi)^3}\;e^{i\,p\cdot\xi}\;\langle P|\overline{\psi}_j(0)\,\psi_i(\xi)|P\rangle\Big|_{\xi^+=0}, \tag{98}$$

$$\Delta_{ij}(z,k_T;n) = \sum_X \int \frac{d\xi^+ d^2\xi_T}{(2\pi)^3}\,e^{ik\cdot\xi}\,Tr\langle 0|\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\overline{\psi}_j(0)|0\rangle\Big|_{\xi^-=0}, \tag{99}$$

where we have suppressed the hadron momenta.

In general many more diagrams have to be considered in evaluating the hadron tensors, but in the deep inelastic limit they can be neglected or considered as corrections to the soft blobs. We return to this later. One situation we want to mention here are target fragmentation parts involving matrix elements of the form $\langle PS|\psi_j(x)\sum_X|X;P_hS_h\rangle$ $\langle X;P_hS_h|\psi_i(0)|PS\rangle$, known as *fracture functions*. They are relevant in the situation where $P\cdot P_h \sim M^2$ (target fragmentation region), which can be distinguished experimentally from the region we are interested in, $P\cdot P_h \sim Q^2$ (current fragmentation region).

10. – Collinear parton distributions

The form of Φ is constrained by hermiticity, parity and time-reversal invariance. The quantity depends besides the quark momentum p on the target momentum P and the spin vector S and one must have

$$[\text{Hermiticity}] \quad \Rightarrow \quad \Phi^\dagger(p,P,S) = \gamma_0\,\Phi(p,P,S)\,\gamma_0, \tag{100}$$

$$[\text{Parity}] \quad \Rightarrow \quad \Phi(p,P,S) = \gamma_0\,\Phi(\bar p,\bar P,-\bar S)\,\gamma_0, \tag{101}$$

$$[\text{Time reversal}] \;\Rightarrow\; \Phi^*(p,P,S) = (-i\gamma_5 C)\,\Phi(\bar p,\bar P,\bar S)\,(-i\gamma_5 C), \tag{102}$$

where $C = i\gamma^2\gamma_0$, $-i\gamma_5 C = i\gamma^1\gamma^3$ and $\bar{p} = (p^0, -\boldsymbol{p})$.

To obtain the leading contribution in inclusive deep inelastic scattering one can integrate over the component p^- and the transverse momenta (see discussion in the section where the parton model has been derived). This integration restricts the nonlocality in $\Phi(p)$. When one wants to calculate the leading order in $1/Q$ for a hard process, one looks for leading parts in M/P^+ because $P^+ \propto Q$. The leading contribution [13] in the integrated part turns out to be proportional to $(M/P^+)^0$, given by

$$\Phi(x) = \frac{1}{2}\left\{f_1(x)\,\not{n}_+ + S_L\,g_1(x)\,\gamma_5\,\not{n}_+ + h_1(x)\,\frac{\gamma_5\,[\not{S}_\perp, \not{n}_+]}{2}\right\}. \tag{103}$$

The precise expression of the functions $f_1(x)$, etc. as integrals over the amplitudes can be easily written down after tracing with the appropriate Dirac matrix,

$$f_1(x) = \int \frac{d\xi^-}{4\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma^+\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{104}$$

$$S_L\,g_1(x) = \int \frac{d\xi^-}{4\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{105}$$

$$S_T^i\,h_1(x) = \int \frac{d\xi^-}{4\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\,i\sigma^{i+}\gamma_5\,\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{106}$$

Including flavor indices, the functions $f_1^q(x) = q(x)$ and $g_1^q(x) = \Delta q(x)$ are precisely the functions that we encountered before.

The third function in the above parametrization is known as *transversity* or *transverse spin distribution* [14, 15]. Including flavor indices one also denotes $h_1^q(x) = \delta q(x)$. In the same way as we have seen for $f_1(x)$ and $g_1(x)$, the function h_1 can be interpreted as a density, but one needs instead of the projectors on quark chirality states, $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$, those on quark transverse spin states, $P_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^i\gamma_5)$. One has

$$f_1(x) = f_{1R}(x) + f_{1L}(x) = f_{1\uparrow}(x) + f_{1\downarrow}(x), \tag{107}$$

$$g_1(x) = f_{1R}(x) - f_{1L}(x), \tag{108}$$

$$h_1(x) = f_{1\uparrow}(x) - f_{1\downarrow}(x). \tag{109}$$

This results in some trivial bounds such as $f_1(x) \geq 0$ and $|g_1(x)| \leq f_1(x)$. We already did discuss the support and charge conjugation properties of $f_1(x)$. The analysis for all these functions shows that the support is in all cases $-1 \leq x \leq 1$, while the charge conjugation properties of the functions are $\overline{f}(x) = -f(-x)$ (C-even) for f_1 and h_1 and $\overline{f}(x) = +f(-x)$ (C-odd) for g_1.

==============

Exercise: Show that the Dirac structure for h_1 in terms of chirality states is $\overline{\psi}_R\psi_L$ and $\overline{\psi}_L\psi_R$. Such functions are called *chiral-odd*. Explain why chiral-odd functions cannot be

measured in inclusive deep inelastic scattering.
==============

11. – Bounds on the distribution functions

The trivial bounds on the distribution functions ($|h_1(x)| \leq f_1(x)$ and $|g_1(x)| \leq f_1(x)$) can be sharpened. For instance one can look explicitly at the structure in Dirac space of the correlation function Φ_{ij}. Actually, we will look at the correlation functions $(\Phi\,\gamma_0)_{ij}$, which involves at leading order matrix elements $\psi^\dagger_{+j}(0)\psi_{+i}(\xi)$. One has in Weyl representation ($\gamma^0 = \rho^1$, $\gamma^i = -i\rho^2\sigma^i$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \rho^3$) the matrices

$$P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_+\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad P_+\gamma^1\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The good projector only leaves two (independent) Dirac spinors, one righthanded (R), one lefthanded (L). On this basis of good R and L spinors the for hard scattering processes relevant matrix $(\Phi\not{n}_-)$ is given by

$$(\Phi\,\not{n}_-)_{ij}(x) = \begin{pmatrix} f_1 + S_L\, g_1 & (S_T^1 + i\,S_T^2)\,h_1 \\ & \\ (S_T^1 - i\,S_T^2)\,h_1 & f_1 - S_L\, g_1 \end{pmatrix} \tag{110}$$

One can also turn the S-dependent correlation function Φ into a matrix in the nucleon spin space via the standard spin 1/2 density matrix $\rho(P,S)$. The relation is $\Phi(x;P,S) = \mathrm{Tr}\,[\Phi(x;P)\,\rho(P,S)]$. Writing

$$\Phi(x;P,S) = \Phi_O + S_L\,\Phi_L + S_T^1\,\Phi_T^1 + S_T^2\,\Phi_T^2, \tag{111}$$

one has on the basis of spin 1/2 target states with $S_L = +1$ and $S_L = -1$ respectively

$$\Phi_{ss'}(x) = \begin{pmatrix} \Phi_O + \Phi_L & \Phi_T^1 - i\,\Phi_T^2 \\ & \\ \Phi_T^1 + i\,\Phi_T^2 & \Phi_O - \Phi_L \end{pmatrix} \tag{112}$$

==============

Exercise: Show by generalizing $\Phi(p)$ to a matrix elements between states $\langle P,s|$ and $|P,s'\rangle$ that for the matrix $M = (\Phi\not{n}_-)^T$ (transposed in Dirac space) one has $v^\dagger M v \geq 0$ for any direction v in Dirac space.

==============

On the basis $+R$, $-R$, $+L$ and $-L$ the matrix in quark $\otimes$ nucleon spin-space becomes

$$
(\Phi(x)\,\not{n}_-)^T = \begin{pmatrix} f_1+g_1 & 0 & 0 & 2\,h_1 \\ 0 & f_1-g_1 & 0 & 0 \\ 0 & 0 & f_1-g_1 & 0 \\ 2\,h_1 & 0 & 0 & f_1+g_1 \end{pmatrix} \tag{113}
$$

Of this matrix any diagonal matrix element must always be positive, hence the eigenvalues must be positive, which gives a bound on the distribution functions stronger than the trivial bounds, namely

$$
|h_1(x)| \le \frac{1}{2}\,(f_1(x)+g_1(x)) \tag{114}
$$

known as the Soffer bound [16].

==============

Exercise: Show that a change to the transverse quark spin basis gives the quark production matrix

$$
(\Phi(x)\,\not{n}_-)^T = \begin{pmatrix} f_1+h_1 & 0 & 0 & g_1+h_1 \\ 0 & f_1-h_1 & g_1-h_1 & 0 \\ 0 & g_1-h_1 & f_1-h_1 & 0 \\ g_1+h_1 & 0 & 0 & f_1+h_1 \end{pmatrix} \tag{115}
$$

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12. – Transverse momentum dependent correlation functions

For the TMD correlators for distributions in Eq. 98 one can write down parametrizations which for the parts involving unpolarized targets (O), longitudinally polarized targets (L) and transversely polarized targets (T) up to parts proportional to M/P^+ take the form [17, 18]

$$\Phi_O(x,\boldsymbol{p}_T) = \frac{1}{2}\left\{ f_1(x,\boldsymbol{p}_T)\,\not n_+ + h_1^\perp(x,\boldsymbol{p}_T)\,\frac{i\,[\not p_T,\not n_+]}{2M}\right\} \tag{116}$$

$$\Phi_L(x,\boldsymbol{p}_T) = \frac{1}{2}\left\{ S_L\,g_{1L}(x,\boldsymbol{p}_T)\,\gamma_5\,\not n_+ + S_L\,h_{1L}^\perp(x,\boldsymbol{p}_T)\frac{\gamma_5\,[\not p_T,\not n_+]}{2M}\right\} \tag{117}$$

$$\begin{aligned}\Phi_T(x,\boldsymbol{p}_T) = \frac{1}{2}\Bigg\{ & f_{1T}^\perp(x,\boldsymbol{p}_T)\,\frac{\epsilon_{\mu\nu\rho\sigma}\gamma^\mu n_+^\nu p_T^\rho S_T^\sigma}{M} \\ & + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\,g_{1T}(x,\boldsymbol{p}_T)\,\gamma_5\,\not n_+ + h_{1T}(x,\boldsymbol{p}_T)\,\frac{\gamma_5\,[\not S_T,\not n_+]}{2} \\ & + \frac{\boldsymbol{p}_T\cdot\boldsymbol{S}_T}{M}\,h_{1T}^\perp(x,\boldsymbol{p}_T)\,\frac{\gamma_5\,[\not p_T,\not n_+]}{2M}\Bigg\}.\end{aligned} \tag{118}$$

All functions appearing here have a natural interpretation as densities. This is seen as discussed before for the $\boldsymbol{p}_T$-integrated functions. Now it includes densities such as the density of longitudinally polarized quarks in a transversely polarized nucleon (g_{1T}) and the density of transversely polarized quarks in a longitudinally polarized nucleon ($h_{1L}^\perp$).

Upon integration over p_T not all functions survive. We are then left with the collinear correlator in Eq. 103 with $f_1(x) = \int d^2p_T\ f_1(x,p_T)$, $g_1(x) = \int d^2p_T\ g_{1L}(x,p_T)$ and $h_1(x) = \int d^2p_T\left[h_{1T}(x) + \frac{\boldsymbol{p}_T^2}{2M^2}\,h_{1T}^\perp(x,p_T)\right]$. The explicit treatment of transverse momenta also provides also a way to include the evolution of quark distribution and fragmentation functions. The assumption that soft parts vanish sufficiently fast as a function of the invariants $p\cdot P$ and p^2, which at constant x implies a sufficiently fast vanishing as a function of $\boldsymbol{p}_T^2$, simply turns out not to be true. Assuming that the result for $\boldsymbol{p}_T^2 \ge \mu^2$ is given by the emission of an additional gluon one finds that the extra distribution written in terms of p_T becomes

$$f_1(x,\boldsymbol{p}_T^2) \overset{\boldsymbol{p}_T^2\ge\mu^2}{\Longrightarrow} \frac{1}{\pi\,\boldsymbol{p}_T^2}\,\frac{\alpha_s(\mu^2)}{2\pi}\int_x^1 \frac{dy}{y}\,P_{qq}\left(\frac{x}{y}\right)\,f_1(y;\mu^2), \tag{119}$$

which gives $f_1(x;\mu^2) \equiv \pi\int_0^{\mu^2} d\boldsymbol{p}_T^2\ f_1(x,\boldsymbol{p}_T^2)$ a logarithmic scale dependence. A more detailed discussion of problems in matching small and large p_T is given in Ref. [19].

Actually we find that different functions survive when one integrates over $\boldsymbol{p}_T$ weighting with p_T^α, e.g.

$$\Phi_\partial^\alpha(x) \equiv \int d^2p_T\, \frac{p_T^\alpha}{M}\,\Phi(x,\boldsymbol{p}_T) \tag{120}$$
$$= \frac{1}{2}\left\{ -g_{1T}^{(1)}(x)\, S_T^\alpha\, \not{n}_+\gamma_5 - S_L\, h_{1L}^{\perp(1)}(x)\, \frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2} \right.$$
$$\left. - f_{1T}^{\perp(1)}\,\epsilon^\alpha{}_{\mu\nu\rho}\gamma^\mu n_-^\nu S_T^\rho - h_1^{\perp(1)}\,\frac{i[\gamma^\alpha,\not{n}_+]}{2}\right\},$$

involving *transverse moments* defined as

$$g_{1T}^{(1)}(x) = \int d^2p_T\, \frac{\boldsymbol{p}_T^2}{2M^2}\, g_{1T}(x,\boldsymbol{p}_T), \tag{121}$$

and similarly for the other functions. The functions $h_1^\perp$ and $f_{1T}^\perp$ are *T-odd*. As we will explain in the section on color gauge invariance they do not to vanish because time reversal invariance cannot be used for the transverse moments. Also for fragmentation functions they will not vanish. The T-odd functions correspond to unpolarized quarks in a transversely polarized nucleon ($f_{1T}^\perp$) or transversely polarized quarks in an unpolarized hadron ($h_1^\perp$). The easiest way to interpret the functions is by considering their place in the quark production matrix $(\Phi(x,p_T)\,\not{n}_-)^T$, which becomes [20]

$$\begin{pmatrix} f_1 + g_{1L} & \frac{|p_T|}{M}\,e^{i\phi}\,g_{1T} & \frac{|p_T|}{M}\,e^{-i\phi}\,h_{1L}^\perp & 2\,h_1 \\ \frac{|p_T|}{M}\,e^{-i\phi}\,g_{1T}^* & f_1 - g_{1L} & \frac{|p_T|^2}{M^2}\,e^{-2i\phi}\,h_{1T}^\perp & -\frac{|p_T|}{M}\,e^{-i\phi}\,h_{1L}^{\perp *} \\ \frac{|p_T|}{M}\,e^{i\phi}\,h_{1L}^{\perp *} & \frac{|p_T|^2}{M^2}\,e^{2i\phi}\,h_{1T}^\perp & f_1 - g_{1L} & -\frac{|p_T|}{M}\,e^{i\phi}\,g_{1T}^* \\ 2\,h_1 & -\frac{|p_T|}{M}\,e^{i\phi}\,h_{1L}^\perp & -\frac{|p_T|}{M}\,e^{-i\phi}\,g_{1T} & f_1 + g_{1L} \end{pmatrix}.$$

In this representation T-odd functions can be incorporated as imaginary parts, $f_{1T}^\perp = -\mathrm{Im}\,g_{1T}$ and $h_1^\perp = \mathrm{Im}\,h_{1L}^\perp$.

13. – Fragmentation functions

Just as for the distribution functions one can perform an analysis of the soft part describing the quark fragmentation in Eq. 99. For the production of unpolarized (or spin 0) hadrons h in hard processes one needs to leading order in $1/Q$ the $(M_h/P_h^-)^0$ part of the correlation function,

$$\Delta_O(z, \boldsymbol{k}_T) = z\, D_1(z, \boldsymbol{k}_T')\, \not n_- + z\, H_1^\perp(z, \boldsymbol{k}_T')\, \frac{i\,[\not k_T, \not n_-]}{2M_h}. \tag{122}$$

The arguments of the fragmentation functions D_1 and $H_1^\perp$ are $z = P_h^-/k^-$ and $\boldsymbol{k}_T' = -z\boldsymbol{k}_T$. The first is the (lightcone) momentum fraction of the produced hadron, the second is the transverse momentum of the produced hadron with respect to the quark. The fragmentation function D_1 is the equivalent of the distribution function f_1. It can be interpreted as a quark decay function, giving the probability of finding a hadron h in a quark. The quantity $n_h = \int dz\; D_1(z)$ is the number of hadrons.
==============
Exercise: Show that the normalization of the fragmentation functions is given by $\sum_h \int dz\; z\, D_1^{q\to h}(z)$ by relating this quantity to a local operator. One needs to eliminate the hadrons in the intermediate state via the momentum operator

$$P^\mu = \sum_{h,X} |P_h, X\rangle\, P_h^\mu\, \langle P_h, X|.$$

==============
The function $H_1^\perp$, interpretable as the difference between the numbers of unpolarized hadrons produced from a transversely polarized quark depending on the hadron's transverse momentum, is allowed because of the non-applicability of time reversal invariance [21]. This is natural for the fragmentation functions [22, 23] because of the appearance of out-states $|P_h, X\rangle$ in the definition of Δ, in contrast to the plane wave states appearing in Φ. The function $H_1^\perp$ is of interest because it is chiral-odd. This means that it can be used to probe the chiral-odd quark distribution function h_1, which can be achieved e.g. by measuring a particular azimuthal asymmetry of produced pions in the current fragmentation region.

The spin structure of fragmentation functions is also conveniently summarized by explicitly giving it on a R and L chiral quark basis, for which we find for decay into spin zero hadrons,

$$(\Delta(z, k_T)\not n_+)^T = \begin{array}{cc} \begin{pmatrix} D_1 & i\frac{|k_T|\, e^{-i\phi}}{M_h}\, H_1^\perp \\ -i\frac{|k_T|\, e^{+i\phi}}{M_h}\, H_1^\perp & D_1 \end{pmatrix} & \begin{array}{c} R \\ \\ L \end{array} \\ \begin{array}{cc} R & \qquad\qquad L \end{array} & \end{array} \tag{123}$$

Examples of azimuthal asymmetries. – Transverse momentum dependence shows up in the azimuthal dependence in the SIDIS cross section (via $\hat{h}$ or transverse spin vectors), in most cases requiring polarization of beam and/or target or requiring polarimetry [24, 25, 3]. Examples of leading azimuthal asymmetries, appearing for polarized leptoproduction are

$$\left\langle \frac{Q_T}{M}\,\sin(\phi_h^\ell - \phi_S^\ell)\right\rangle_{OT} = \frac{2\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_T|\,\left(1-y+\frac{1}{2}\,y^2\right)\sum_{a,\bar a} e_a^2\,x_B\,f_{1T}^{\perp(1)a}(x_B)D_1^a(z_h). \tag{124}$$

$$\left\langle \frac{Q_T}{M_h}\,\sin(\phi_h^\ell + \phi_S^\ell)\right\rangle_{OT} = \frac{4\pi\alpha^2\,s}{Q^4}\,|\boldsymbol{S}_T|\,(1-y)\sum_{a,\bar a} e_a^2\,x_B\,h_1^a(x_B)H_1^{\perp(1)a}(z_h). \tag{125}$$

The notation $\langle W\rangle$ is the q_T-integrated cross section including a weight W. The factor Q_T is included, because it together with the direction $\hat h$ combines to q_T, allowing a defolding of the cross section in distribution and fragmentation parts (one of them weighted with transverse momentum). Note that both of these asymmetries involve T-odd functions, which can only appear in single spin asymmetries. The latter can easily be checked from the conditions on the hadronic tensor, which are the same as those in Eq. 41 to 43. They require an odd number of spins vectors entering in the symmetric part and an even number of spins entering in the antisymmetric part of the hadron tensor. The results of single spin asymmetries in SIDIS measurements on a transversely polarized target from HERMES [26] are shown in Fig. 5. An extended review of transverse momentum dependent functions and transversity can be found in Ref. [27, 4]

14. – Inclusion of subleading contributions

If one proceeds up to order $1/Q$ one also needs terms in the parametrization of the soft part proportional to M/P^+. Limiting ourselves to the $\boldsymbol{p}_T$-integrated correlations one needs [13]

$$\Phi(x) = \frac{1}{2}\left\{f_1(x)\,\not n_+ + S_L\,g_1(x)\,\gamma_5\,\not n_+ + h_1(x)\,\frac{\gamma_5\,[\not S_T,\not n_+]}{2}\right\} + \frac{M}{2P^+}\left\{e(x) + g_T(x)\,\gamma_5\,\not S_T + S_L\,h_L(x)\,\frac{\gamma_5\,[\not n_+,\not n_-]}{2}\right\} \tag{126}$$

We will use inclusive scattering off a transversely polarized nucleon ($|\boldsymbol{S}_\perp| = 1$) as an example to show how higher twist effects can be incorporated in the cross section. The hadronic tensor for a transversely polarized nucleon is zero in leading order in $1/Q$. At order $1/Q$ one obtains a contribution from the handbag diagram, which turns out to involve the transverse moments in Φ_∂^α in Eq. 121. There is a second contribution at order $1/Q$, however, coming from diagrams as the one shown in Fig. 6. For these gluon diagrams one needs bilocal matrix elements containing $1/Q$ one only needs the matrix

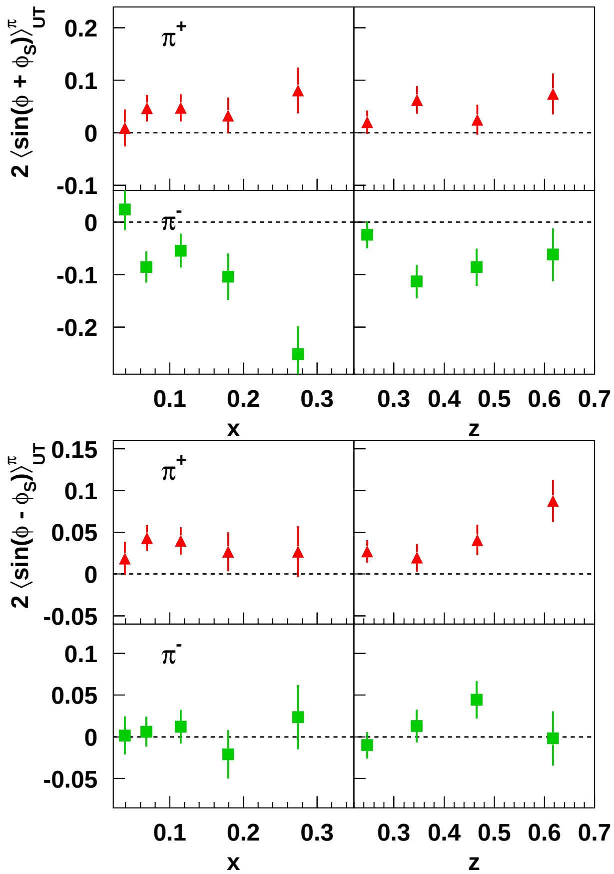


Fig. 5. – Weighted asymmetries for the Collins and Sivers angles (see Eqs 124 and 125) obtained in semi-inclusive single spin asymmetries measured on a transversely polarized Hydogen target by the HERMES collaboration at DESY [26]. The error bars represent the statistical uncertainties.

element of the bilocal combinations $\overline{\psi}(0)\, gA_T^\alpha(\xi)\,\psi(\xi)$ and $\overline{\psi}(0)\, gA_T^\alpha(0)\,\psi(\xi)$. The $\Phi_A^\alpha(x)$ and $\Phi_\partial^\alpha(x)$ contributions sum to $\Phi_D^\alpha(x)$ involving matrix elements of bilocal combinations $\overline{\psi}(0)\, iD_T^\alpha\,\psi(\xi)$ for which one can Use the QCD equations of motion to relate them to the

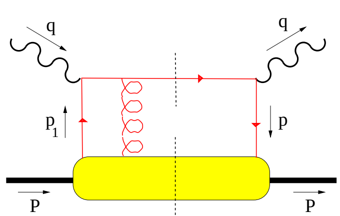


Fig. 6. – Example of a contribution involving a quark-gluon matrix element that must be included at sub-leading order in lepton hadron inclusive scattering.

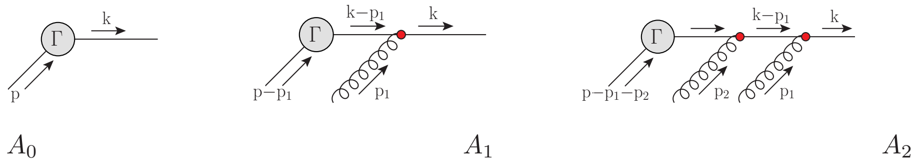


Fig. 7. – Inclusion of collinear gluons from $\Phi_{A\ldots A}(p - p_1 \ldots - p_N, p_1, \ldots, p_N)$ coupling to an outgoing (colored) quark line with momentum k results in a Wilson line running from the point ξ in field $\psi(\xi)$ to the point where $\xi^- = \infty$.

functions appearing in Φ [13, 28],

$$\Phi_D^\alpha(x) = \frac{M}{2}\left\{-\left(x\,g_T - \frac{m}{M}\,h_1\right)\,S_T^\alpha\,\not{n}_+\gamma_5 - S_L\left(x\,h_L - \frac{m}{M}\,g_1\right)\,\frac{[\gamma^\alpha,\not{n}_+]\gamma_5}{2}\right\}. \tag{127}$$

The distribution function g_T e.g. shows up in the corresponding structure function of polarized inclusive deep inelastic scattering

$$2M\,W_A^{\mu\nu}(q,P,S_T) = i\,\frac{2Mx_B}{Q}\,\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}\,g_T(x_B), \tag{128}$$

leading for the structure function $g_T(x_B,Q^2)$ defined in Eq. B.9 to the result

$$g_T(x_B,Q^2) = \frac{1}{2}\sum_q e_q^2\left(g_T^q(x_B) + g_T^{\bar{q}}(x_B)\right). \tag{129}$$

15. – Color gauge invariance

We have sofar disregarded two issues. The first issue is that the correlation function Φ discussed in previous sections involve two quark fields at different space-time points and hence are not color gauge invariant. The second issue are the gluonic diagrams similar as the ones we have discussed in the previous section (see Fig. 6), among which also correlation functions appear involving matrix elements with longitudinal (A^+) gluon fields,

$$\overline{\psi}_j(0)\,gA^+(\eta)\,\psi_i(\xi).$$

These do not lead to any suppression. The reason is that because of the +-index in the gluon field the matrix element is proportional to P^+, p^+ or $M\,S^+$ rather than the proportionality to $M\,S_T^\alpha$ or p_T^α that one gets for a gluonic matrix element with transverse gluons.

A straightforward calculation, however, shows that the gluonic diagrams with one or more longitudinal gluons involve matrix elements (soft parts) of operators $\overline{\psi}\psi$, $\overline{\psi}\,A^+\,\psi$,

$\overline{\psi}\, A^+ A^+\, \psi$, etc. (see Fig. 7) can be resummed into a correlation function

$$\Phi_{ij}(x) = \int \frac{d\xi^-}{2\pi}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,U_{[0,\xi]}\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=\xi_T=0}, \tag{130}$$

where U is a gauge link operator [29, 28, 30, 31, 32]

$$U_{[0,\xi]} = \mathcal{P}\exp\left(-i\int_0^{\xi^-} d\zeta^-\, A^+(\zeta)\right) \tag{131}$$

(path-ordered exponential with path along $-$-direction). Et voila, the unsuppressed gluonic diagrams combine into a color gauge invariant correlation function [33, 34]. We note that at the level of operators, one expands

$$\overline{\psi}(0)\psi(\xi) = \sum_n \frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\psi(0), \tag{132}$$

in a set of local operators, but only the expansion of the non-local combination with a gauge link

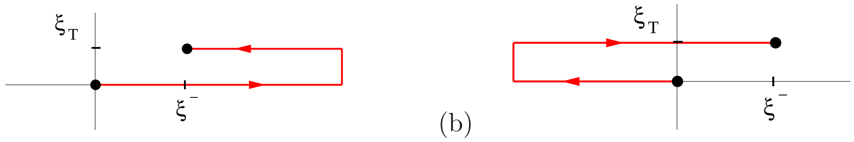
$$\overline{\psi}(0)\,U_{[0,\xi]}\,\psi(\xi) = \sum_n \frac{\xi^{\mu_1}\ldots\xi^{\mu_n}}{n!}\,\overline{\psi}(0)D_{\mu_1}\ldots D_{\mu_n}\psi(0), \tag{133}$$

is an expansion in terms of local gauge invariant operators. The latter operators are precisely the local (quark) operators that appear in the operator product expansion applied to inclusive deep inelastic scattering.

For the p_T-dependent functions, one finds that inclusion of A^+ gluonic diagrams leads to a color gauge invariant matrix element with links running via $\xi^= = \pm\infty$ [35, 36] where the direction depends on the coupling of the gluons to an final state or initial state color line. For instance in lepton-hadron scattering one has a situation like the one in Fig. 7 and one finds

$$\Phi^{[+]}(x,\boldsymbol{p}_T) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_T}{(2\pi)^3}\; e^{ip\cdot\xi}\,\langle P,S|\overline{\psi}(0)\,U^{[+]}_{[0,\xi]}\,\psi(\xi)|P,S\rangle\Big|_{\xi^+=0}, \tag{134}$$

where the link $U^{[+]}$ is shown in Fig. 8a. In contrast, for Drell-Yan scattering one finds a gauge link that runs via minus infinity, involving the link in Fig. 8b. We note that the gauge link involves transverse gluons [29, 28] showing that one in processes involving more hadrons the effects of transverse gluons are not necessarily suppressed, as has also been shown in explicit model calculations [37]. Integration over transverse momenta implies in the correlator, which is a Fourier transform that $\xi_T = 0_T$, in which case the $U^{[+]}$ and $U^{[-]}$ links reduce to a unique collinear link connecting 0 and ξ^-.



(a) (b)

Fig. 8. – The gauge link structure in the quark-quark correlator Φ in SIDIS (a) and DY (b) respectively

The transverse momentum dependent distribution functions, however, do depend on the gauge link structure, which is still tractable in simple processes like DIS or DY with a simple color flow, but the color structure of various correlators become entangled if the color flow is more complicated [30, 31, 32].

Even if the color flow is simple, like SIDIS or DY, there are effects. Most notable is a sign change in single spin asymmetries going from SIDIS to DY. To understand this sign change, one notes that TMDs are no longer constrained by time-reversal, as the time reversal operation interchanges the $U^{[+]}$ and $U^{[-]}$ links, leading to the appearance of T-odd functions in Eqs 116 - 118 and also occuring in Eq. 121. One has e.g.

$$\Phi_O^{[\pm]}(x,\boldsymbol{p}_T) = \frac{1}{2}\left\{ f_1(x,\boldsymbol{p}_T)\,\not{n}_+ \pm h_1^\perp(x,\boldsymbol{p}_T)\,\frac{i\,[\not{p}_T,\not{n}_+]}{2M}\right\}. \tag{135}$$

Looking back at our example of an azimuthal asymmetry, we have seen the $\sin(\phi_h^\ell - \phi_S^\ell)$ asymmetry proportional to $f_{1T}^\perp\,D_1$. The corresponding asymmetry in Drell-Yan proportional to $f_{1T}^\perp\,\overline{f}_1$ would get an additional minus sign. Actually for the fragmentation correlator, such as Δ_O in Eq. 122 there is no such dependence on the gauge link [38, 39, 40]. The function $H_1^\perp$, however, is nonzero because the states $|P_h, X\rangle$ in the case of fragmentation are out-states and time reversal (changing out- into in-states) simply cannot be used as a constraint. The asymmetry in Eq. 125 thus will not change sign going from SIDIS to the corresponding DY asymmetry. In general situations in which one has a convolution of TMD distribution functions of two hadrons in the initial state, factorization is, already at tree-level hampered by the entanglement of Wilson lines [41, 42]. entanglement of Wilson lines. At the level of the weighted asymmetries, it implies more complicated factors than just a sign change in the appearance of the weighted functions.

Gluonic pole matrix elements. – The inclusion of gauge links also allows us to study in some more detail the operator structure of the T-odd parts in the correlators. Given the full operator structure for a TMD correlator including a gauge links $U^{[\pm]}$ one can explicitly calculate the transverse moments $\Phi_\partial^\alpha(x)$ of which we have given the parametrization in Eq. 121. One finds that they, depending on the gauge links, can be related to color gauge invariant quark-quark-gluon matrix elements Φ_G^α and Φ_D^α,

$$\begin{aligned}\left(\Phi_\partial^{[\pm]\alpha}\right)_{ij}(x) &= \int \mathrm{d}^2p_T \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{(2\pi)^3}\; e^{i\,p\cdot\xi}\,\langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\pm\infty]}U^T_{[0_T,\pm\infty_T]}\\ &\qquad\qquad i\partial_\xi^\alpha\,U^T_{[\pm\infty_T,\xi_T]}U^-_{[\pm\infty,\xi]}\,\psi_i(\xi)|P,S\rangle\Big|_{\xi^+=0}\\ &= \int \frac{d\xi^-}{(2\pi)}\;e^{ip\cdot\xi}\Big\{\langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\xi]}\,iD_T^\alpha\psi_i(\xi)|P,S\rangle\Big|_{LC}\\ &\quad -\langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\pm\infty]}\int_{\pm\infty}^{\xi^-}d\eta^-\,U^-_{[\pm\infty,\eta]}\,g\,G^{+\alpha}(\eta)\,U^-_{[\eta,\xi]}\,\psi_i(\xi)|P,S\rangle\Big|_{LC}\Big\},\end{aligned}$$

or

$$\Phi_\partial^{[\pm]\,\alpha}(x) = \Phi_D^\alpha(x) - \int_{-\infty}^{\infty}\mathrm{d}p_1^+\;\frac{i}{p_1^+\mp i\epsilon}\,\Phi_G^\alpha(p^+,p^+-p_1^+), \tag{136}$$

where

$$\Phi^\alpha_{D\,ij}(x) = \int\frac{\mathrm{d}\xi^-}{2\pi}\;e^{i\,p\cdot\xi}\langle P,S|\overline{\psi}_j(0)\,U^-_{[0,\xi]}\,iD^\alpha(\xi)\psi_i(\xi)|P,S\rangle\Big|_{LC}. \tag{137}$$

The difference between correlation functions with links running to $\pm\infty$, respectively, is related to a collinear quark-gluon correlator,

$$\Phi_\partial^{[+]\alpha}(x) - \Phi_\partial^{[-]\alpha}(x) = 2\pi\,\Phi_G^\alpha(x,x), \tag{138}$$

the latter being given the name *gluonic-pole* matrix element since it corresponds to the soft-gluon point $p_1^+ = 0$. It is a collinear matrix element which has been extensively studied [43, 44, 45, 46, 47, 48, 49, 50] as a collinear mechanism to generate single spin asymmetries. With the definition

$$\Phi_A^\alpha(x) = \mathrm{PV}\int_{-\infty}^{\infty}\mathrm{d}p_1^+\;\frac{i}{p_1^+}\,\Phi_G^\alpha(p^+,p^+-p_1^+), \tag{139}$$

one gets a separation

$$\Phi_\partial^{[\pm]\,\alpha}(x) = \underbrace{\Phi_D^\alpha(x) - \Phi_A^\alpha(x)}_{\widetilde{\Phi}_\partial^\alpha} \pm \pi\,\Phi_G^\alpha(x,x), \tag{140}$$

into T-even and T-odd parts.

16. – Gluon TMDs

Except in higher twist matrix elements and in gauge links, gluon fields also appear in the gluon distribution functions. To better understand the treatment of gluon fields, we also make for gluon fiels $A^\mu(\eta)$ a Sudakov expansion,

$$A^\mu(\eta) = A^n(\eta)\,P^\mu + A_T^\mu(\eta) + \left(A^P(\eta) - A^n(\eta)\,M^2\right)n^\mu. \tag{141}$$

A similar expansion can be written down for $A^\mu(p)$. It will be convenient to look at the (collinear) gluon field component along parton momentum p^μ, hence we write

$$\begin{aligned} A^\mu(p) &= \int d^4\eta\; e^{i\,p\cdot\eta}\,A^\mu(\eta) \\ &= \int d^4\eta\; e^{ip\cdot\eta}\left[\frac{A^n(\eta)}{p{\cdot}n}\,p^\mu + \frac{(p{\cdot}n)\,A_T^\mu(\eta) - p_T^\mu\,A^n(\eta)}{p{\cdot}n}\right. \\ &\qquad\left. + \frac{(p{\cdot}n)\,A^P(\eta) - (p{\cdot}P)\,A^n(\eta)}{p{\cdot}n}\,n^\mu\right]. \end{aligned} \tag{142}$$

In the correlator the momentum $p^\mu \longrightarrow i\partial^\mu(\eta)$, so

$$\begin{aligned} A^\mu(p) &= \frac{1}{p{\cdot}n}\int d^4\eta\; e^{i\,p\cdot\eta}\Big[A^n(\eta)\,p^\mu + i\partial^n(\eta)\,A_T^\mu(\eta) - i\partial_T^\mu(\eta)\,A^n(\eta) \\ &\qquad + \left(i\partial^n(\eta)A^P(\eta) - i\partial^P(\eta)A^n(\eta)\right)n^\mu\Big] \end{aligned} \tag{143}$$

$$= \frac{1}{p{\cdot}n}\left[A^n(p)\,p^\mu + i\,G_T^{n\mu}(p) + i\,G^{nP}(p)\,n^\mu\right]. \tag{144}$$

Although the latter appears to be only true for the Abelian case, we will find the same result in the non-abelian case, but to complete that proof, we first need to incorporate the collinear gluons into the matrix elements as gauge links. For that we need the $A^n(p)$ gluons and actually also some boundary terms (shown very explicitly in Ref. [42]). Using the expansion in Eq. 144 rather than the one in Eq. 141 streamlines the inclusion of collinear gluons, because one can immediately make use of Ward identities. It circumvents the explicit treatment of transverse momentum dependent parts (as done in Ref. [28]). The results are of course identical.

Rather than transverse gluon fields one thus encounters $G^{n\alpha}$ field strengths in the matrix elements, as we saw explicitly already in the previous section. Transverse momentum dependent gluon distribution functions are projections of the TMD correlator

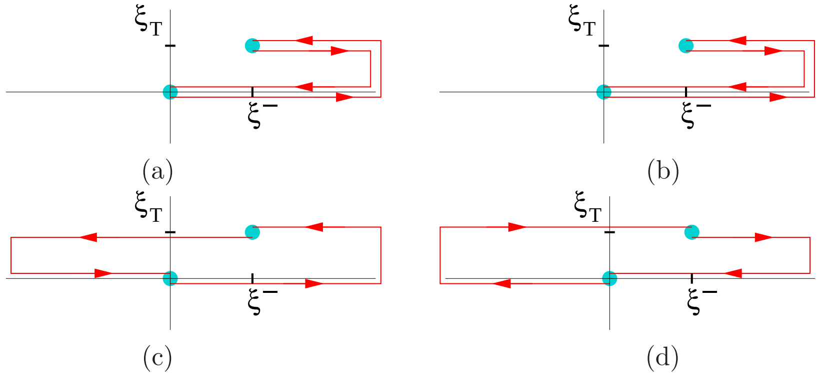


Fig. 9. – The gauge links for gluon TMDs

Γ (often also referred to as Φ^g)

$$(145)\quad \Gamma^{[U,U']\,\mu\nu}(x,p_T) = \frac{1}{(p{\cdot}n)^2}\int \frac{d(\xi{\cdot}P)\,d^2\xi_T}{(2\pi)^3}\,e^{ip\cdot\xi} \times \mathrm{Tr}\,\langle P,S|\,F^{n\mu}(0)\,U_{[0,\xi]}\,F^{n\nu}(\xi)\,U'_{[\xi,0]}\,|P,S\rangle\,\big]_{\mathrm{LF}}\ .$$

Here the field-operators are written in the color-triplet representation requires the inclusion of *two* Wilson lines $U_{[0,\xi]}$ and $U'_{[\xi,0]}$. They again arise from the resummation of gluon initial and final-state interactions. In general this will lead to two unrelated Wilson lines U and U'. In the particular case that $U' = U^\dagger$, the gluon correlator can also be written as the product of two gluon fields with the Wilson line U in the adjoint representation of $SU(N)$. This is for instance the case for the gluon correlators which acquire gaugelinks as in Figs. 9a and b, but not for the gluon correlators in Figures. 9c and d.

In the p_T-integrated correlator on the lightcone the process dependence of the TMD gluon correlator disappears,

$$(146)\quad \Gamma^{\mu\nu}(x) = \int d^2p_T\ \ \Gamma^{[U,U']\,\mu\nu}(x,p_T) = \frac{1}{(p{\cdot}n)^2}\int \frac{d(\xi{\cdot}P)}{2\pi}\,e^{ix(\xi\cdot P)}\ \mathrm{Tr}\,\langle P,S|\,F^{n\mu}(0)\,U^n_{[0,\xi]}\,F^{n\nu}(\xi)\,U^n_{[\xi,0]}|P,S\rangle\,\big]_{\mathrm{LC}}\ .$$

However, as for the quark correlator, a subprocess-dependence due to the Wilson lines in the TMD gluon correlators remains in the transverse moments. The analogue of the process-dependent decomposition in the case of the gluon correlator is (with for each diagrammatic contribution d a TMD correlator $\Gamma^{[\mathcal{U},\mathcal{U}']}(x,p_T) = \Gamma^{[\mathcal{U}(D),\mathcal{U}'(D)]}(x,p_T) \equiv \Gamma^{[D]}(x,p_T)$ and omitting the gluon field indices μ and ν) [51],

$$(147)\quad \Gamma_\partial^{[D]\,\alpha}(x) = \widetilde{\Gamma}_\partial^\alpha(x) + C_G^{(f)\,[D]}\,\pi\Gamma^\alpha_{G_f}(x,x) + C_G^{(d)\,[D]}\,\pi\Gamma^\alpha_{G_d}(x,x)\ .$$

The matrix elements Γ_{G_f} and Γ_{G_d} are the two gluonic pole matrix elements that correspond to the two possible ways to construct color-singlets from three gluon fields [52, 51]. They involve the antisymmetric f and symmetric d structure constants of $SU(3)$, respectively. The only process dependence coming from the Wilson lines in the TMD correlators is contained in the gluonic pole strengths $C_G^{(f/d)\,[D]} \equiv C_G^{(f/d)\,[\mathcal{U}(D),\mathcal{U}'(D)]}$. The collinear correlators are

$$\begin{aligned}
\Gamma_D^{\mu\nu;\alpha}(x) &= \frac{1}{(p{\cdot}n)^2}\int \frac{d(\xi{\cdot}P)}{2\pi}\; e^{ix(\xi\cdot P)} \\
&\qquad \times \mathrm{Tr}\,\langle P,S|\, F^{n\mu}(0)\, U^n_{[0;\xi]}\left[iD^\alpha(\xi), F^{n\nu}(\xi)\right] U^n_{[\xi;0]}\,|P,S\rangle\,\big\rfloor_{\mathrm{LC}}\ , \\
\Gamma_{G_f}^{\mu\nu;\alpha}(x,x{-}x') &= \frac{1}{(p{\cdot}n)^2}\int \frac{d(\xi{\cdot}P)}{2\pi}\frac{d(\eta{\cdot}P)}{2\pi}\; e^{ix'(\eta\cdot P)}e^{i(x-x')(\xi\cdot P)} \\
&\qquad \times \mathrm{Tr}\,\langle P,S|\, F^{n\mu}(0)\left[U^n_{[0,\eta]} gF^{n\alpha}(\eta)U^n_{[\eta,0]}, U^n_{[0,\xi]}F^{n\nu}(\xi)U^n_{[\xi,0]}\right]|P,S\rangle\,\big\rfloor_{\mathrm{LC}}\ , \\
\Gamma_{G_d}^{\mu\nu;\alpha}(x,x{-}x') &= \frac{1}{(p{\cdot}n)^2}\int \frac{d(\xi{\cdot}P)}{2\pi}\frac{d(\eta{\cdot}P)}{2\pi}\; e^{ix'(\eta\cdot P)}e^{i(x-x')(\xi\cdot P)} \\
&\qquad \times \mathrm{Tr}\,\langle P,S|\, F^{n\mu}(0)\left\{U^n_{[0,\eta]} gF^{n\alpha}(\eta)U^n_{[\eta,0]}, U^n_{[0,\xi]}F^{n\nu}(\xi)U^n_{[\xi,0]}\right\}|P,S\rangle\,\big\rfloor_{\mathrm{LC}}\ ,
\end{aligned}$$

and

$$\widetilde{\Gamma}^\alpha_\partial(x) = \Gamma^\alpha_D(x) - \int dx'\; P\frac{i}{x'}\;\Gamma^\alpha_{G_f}(x,x{-}x')\ . \tag{148}$$

For gluon distribution functions we follow the naming convention discussed in Ref. [53] and use the parameterizations of the TMD gluon correlators in Ref. [54]

$$\begin{aligned}
\Gamma^{(T\text{-even})\,\mu\nu}(x,p_T) &= \frac{1}{2x}\Bigg\{ -g_T^{\mu\nu}\; f_1^g(x,p_T^2) + \left(\frac{p_T^\mu p_T^\nu}{M^2} + g_T^{\mu\nu}\frac{\boldsymbol{p}_T^2}{2M^2}\right) h_1^{\perp g}(x,p_T^2) \\
&\qquad + i\epsilon_T^{\mu\nu} S_L\; g_{1L}^g(x,p_T^2) + i\epsilon_T^{\mu\nu}\,\frac{\boldsymbol{p}_T{\cdot}\boldsymbol{S}_T}{M}\; g_{1T}^g(x,p_T^2)\Bigg\}\ , \\
\Gamma^{(T\text{-odd})\,\mu\nu}_{(f/d)}(x,p_T) &= \frac{1}{2x}\Bigg\{ g_T^{\mu\nu}\frac{\epsilon_T^{p_T S_T}}{M}\; f_{1T}^{\perp g\,(f/d)}(x,p_T^2) - \frac{\epsilon_T^{p_T\{\mu}S_T^{\nu\}} + \epsilon_T^{S_T\{\mu}p_T^{\nu\}}}{4M}\; h_{1T}^{g\,(f/d)}(x,p_T^2) \\
&\qquad - \frac{\epsilon_T^{p_T\{\mu}p_T^{\nu\}}}{2M^2} S_L\; h_{1L}^{\perp g\,(f/d)}(x,p_T^2) - \frac{\epsilon_T^{p_T\{\mu}p_T^{\nu\}}}{2M^2}\frac{\boldsymbol{p}_T{\cdot}\boldsymbol{S}_T}{M}\; h_{1T}^{\perp g\,(f/d)}(x,p_T^2)\Bigg\}\ .
\end{aligned}$$

In particular one has two distinct gluon-Sivers distribution functions $f_{1T}^{\perp g\,(f)}(x,p_T^2)$ and $f_{1T}^{\perp g\,(d)}(x,p_T^2)$ corresponding to the two ways to construct T-odd gluon correlators.

17. – Concluding remarks

The material in these lectures is just a selection from the work that is going on in the field of TMDs. We have tried to focus on some basics, which is the structure of the matrix elements. We have not addressed the rich phenomenology of azimuthal asymmetries [24,

$$\begin{pmatrix} f_1 + g_{1L} & \frac{p_T e^{-i\phi}}{M}\, g_{1T}^* & e^{-2i\phi}\, h_1^{\perp\,\prime} & i\frac{k_T e^{-3i\phi}}{M} h_{1T}^{\perp(1)} \\ \frac{k_T e^{+i\phi}}{M}\, g_{1T} & f_1 - g_{1L} & \frac{i k_T e^{-i\phi}}{M}\, h_1 & e^{-2i\phi}\, h_1^{\perp(1)\,*} \\ e^{+2i\phi}\, h_1^{\perp(1)\,*} & \frac{-i k_T e^{+i\phi}}{M}\, h_1 & f_1 - g_{1L} & \frac{k_T e^{-i\phi}}{M}\, g_{1T}^* \\ -i\frac{k_T e^{+3i\phi}}{M} h_{1T}^{\perp(1)} & e^{+2i\phi}\, h_1^{\perp(1)} & \frac{k_T e^{+i\phi}}{M}\, g_{1T} & f_1 + g_{1L} \end{pmatrix}$$

Fig. 10. – Illustration of the rich structure when one includes spin and transverse momentum dependence for gluon distribution (or fragmentation) functions. Given are matrix elements involving gluonic fields with different helicities (two possibilities) between proton states with different helicities (two possibilities). On top the basis entities are given, with inside the circle the gluon helicity. In this representation T-odd functions appear as imaginary entries, which are only allowed in off-diagonal elements, in particular $f_{1T}^\perp = -\mathrm{Im} g_{1T}$ and $h_{1L}^\perp = -\mathrm{Im} h_1^\perp$. All functions depend on longitudinal momentum fraction (x) and absolute value of transverse momentum (k_T). The angle ϕ is the azimuthal angle of the transverse momentum vector. The notation $h_1^{\perp(1)}(x,k_T^2) \equiv (k_T^2/2M^2)\, h_1^\perp(x,k_T^2)$ is used and $h_1 \equiv h_{1T} + h_{1T}^{\perp(1)}$. A gluon index ($g$) used in the text is omitted from all functions. The functions $f_{1..}$ and $g_{1..}$ are sum and difference of matrix elements of gluon fields with different circular polarization, while the functions $h_{1..}$ involve gluon fields with different polarizations (becoming diagonal elements if one uses linear polarization). Finally the subscripts L and T indicate the polarization of the nucleon states between which the gluon field operators are evaluated. On the chosen helicity basis matrix elements between transversely polarized nucleons are off-diagonal.

25, 3, 4]. Neither have we addressed aspects going beyond tree-level, which is important to get to the full field theoretical treatment of TMDs including evolution [55, 56].

* * *

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APPENDIX A.

What information is in the form factors

We look here specifically at the space-like form factors discussed above, with $q = P' - P$ and $q^2 = -Q^2$. We have already introduced the time-like vector $\hat{t}^\mu$, which in this case is proportional to $T^\mu = P^\mu + P'^\mu$, thus $q \cdot \hat{t} = 0$. Any four-vector can be expanded as $a = (a \cdot \hat{t})\,\hat{t} + \boldsymbol{a}$ and one has specifically $q = \boldsymbol{q}$. A further specification of the spacelike direction (z-direction or transverse) is at this stage not needed, but may for specific applications be useful. The frame in which $\hat{t} = (1, 0, 0, 0)$ is known as the Breit frame or the Brick-Wall frame.

Working out the current expression for the nucleon with the above definition of zero and space-like components (or if one likes in the Breit frame) with $P'^0 = P^0$, $\boldsymbol{P} = -\boldsymbol{q}/2$, $\boldsymbol{P}' = +\boldsymbol{q}/2$, gives

$$\langle P', S'|J_0^{\text{em}}(0)|P, S\rangle = 2M\left[F_1 - \frac{Q^2}{4M^2}F_2\right] \equiv 2M\,G_E(Q^2), \tag{A.1}$$

$$\langle P', S'|\boldsymbol{J}^{\text{em}}(0)|P, S\rangle = [F_1 + F_2]\;(i\boldsymbol{\sigma}_N \times \boldsymbol{q}) \equiv G_M(Q^2)\,(i\boldsymbol{\sigma}_N \times \boldsymbol{q}), \tag{A.2}$$

where $\chi_{S'}^\dagger\,\boldsymbol{\sigma}\,\chi_S \equiv \boldsymbol{\sigma}_N$. These expressions show the relevance of the Sachs form factors G_E and G_M. The quantity $e\,G_E(0)$ is the charge Q of the nucleon, $e\,G_M(0)/M$ is the magnetic moment of the nucleon. The quantity $\kappa = G_M(0) - Q = G_M(0) - G_E(0)$ is the anomalous magnetic moment. Following Eqs 19 one has the forward matrix elements

$$\langle P, S'|J_0^{\text{em}}(0)|P, S\rangle = 2M\,F_1(0) = 2M\,G_E(0), \tag{A.3}$$

$$\langle P, S'|(\boldsymbol{r} \times \boldsymbol{J}^{\text{em}})(0)|P, S\rangle = [F_1(0) + F_2(0)]\;\boldsymbol{\sigma}_N = G_M(0)\,\boldsymbol{\sigma}_N, \tag{A.4}$$

For an elementary fermion one has $F_1(Q^2) = 1$ and $F_2(Q^2) = 0$. For the nucleon one has for the proton $F_1^p(0) = Q_p = 1$ and $F_2^p(0) = \kappa_p = 1.793$ and for the neutron $F_1^n(0) = Q_n = 1$ and $F_2^n(0) = \kappa_n =$ -1.913. Roughly one has for nucleons the dipole-like behavior

$$G_E^p(Q^2) \approx \frac{G_M^p}{\mu_p} \approx \frac{G_M^n}{\mu_n} \approx \frac{1}{(1 + Q^2/\Lambda^2)^2}, \tag{A.5}$$

with $\Lambda^2 \approx 0.71$ GeV^2 and $\mu_N = G_M^N(0) = Q_N + \kappa_N$ is the nucleon magnetic moment.

We note that for the (diagonal) vector currents for the quarks, $V_\mu^q = \overline{\psi}^q\gamma_\mu\psi^q$, the quantities $F_1^q(0)$ are (by definition) the quark numbers $F_1^q(0) = n_q$. The matrix element of the electromagnetic current J_μ^{em} in Eq. 57 then indeed satisfying $F_1^{em}(0) = Q$ (the charge of the hadron state).

Appendix B.

Polarized leptoproduction

For spin-polarized leptons in the initial state we have

$$
\begin{aligned}
L_{\mu\nu}^{(s)} &= \mathrm{Tr}\ \left[\gamma_\mu(\not{k}' + m)\gamma_\nu(\not{k} + m)\,\frac{1 \pm \gamma_5\,\not{s})}{2}\right] \\
&= 2\,k_\mu k'_\nu + 2\,k'_\mu k_\nu - Q^2\,g_{\mu\nu} \pm 2i\,m\,\epsilon_{\mu\nu\rho\sigma}q^\rho s^\sigma .
\end{aligned} \tag{B.1}
$$

Note that for light particles or particles at high energies helicity states ($\hat{s} = \hat{\boldsymbol{k}}$) become chirality eigenstates. For $L_{\mu\nu}$ the equivalence is easily seen because for $s^\mu = (|\boldsymbol{k}|/m, E\hat{\boldsymbol{k}}/m)$ ($s^2 = -1$ and $s \cdot p = 0$) one obtains in the limit $E \approx |\boldsymbol{k}|$ the result $m\,s^\mu \approx k^\mu$. Then the leptonic tensor for helicity states ($\lambda_e = \pm$) becomes

$$
L_{\mu\nu}^{(\lambda_e = \pm 1/2)} \approx L_{\mu\nu}^{(R/L)} = L_{\mu\nu}^{(S)} + \lambda_e\,L_{\mu\nu}^{(A)}, \tag{B.2}
$$

where the antisymmetric lepton tensor is given by

$$
L_{\mu\nu}^{(A)}(k,k') = \mathrm{Tr}\ [\gamma_\mu\gamma_5\not{k}'\gamma_\nu\not{k}] = 2i\,\epsilon_{\mu\nu\rho\sigma}q^\rho k^\sigma . \tag{B.3}
$$

Expanding in the Cartesian set $\hat{t}$, $\hat{z}$ and the vector $\hat{\ell}$ in the same way as for the symmetric part, we have for the antisymmetric part of the leptonic tensor([3]) the result

$$
L^{\mu\nu\,(A)} = \frac{Q^2}{y^2}\left[-i\,y(2-y)\,\epsilon_\perp^{\mu\nu} - 2i\,y\sqrt{1-y}\;\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}\hat{\ell}_\rho\right]. \tag{B.4}
$$

One can use polarized leptons in deep inelastic $\vec{e}p \to X$ to probe the antisymmetric tensor for unpolarized hadrons, containing the F_3 structure function. This contribution comes in via the interference between the γ and Z interference term.

In the situation where the target is polarized, one has several more structure functions as compared to the case of an unpolarized target. For a spin 1/2 particle the initial state is described by a 2-dimensional spin density matrix $\rho = \sum_\alpha |\alpha\rangle p_\alpha \langle\alpha|$ describing the probabilities p_α for a variety of spin possibilities. This density matrix is hermitean with

---

([3]) A useful relation is

$$
\epsilon_{\mu\nu\rho\sigma}\,g_{\alpha\beta} = \epsilon_{\alpha\nu\rho\sigma}\,g_{\mu\beta} + \epsilon_{\mu\alpha\rho\sigma}\,g_{\nu\beta} + \epsilon_{\mu\nu\alpha\sigma}\,g_{\rho\beta} + \epsilon_{\mu\nu\rho\alpha}\,g_{\sigma\beta}
$$

or for a vector $a_\perp$ orthogonal to $\hat{t}$ and $\hat{q}$,

$$
\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}\,\hat{z}_\rho a_{\perp\sigma} &= \hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho}, \\
\epsilon^{\mu\nu\rho\sigma}\,\hat{t}_\rho a_{\perp\sigma} &= -\hat{z}^{\,[\mu}\epsilon_\perp^{\nu]\rho}a_{\perp\rho}.
\end{aligned}
$$

$\mathrm{Tr}\,\rho = 1$. It can in the target rest frame be expanded in terms of the unit matrix and the Pauli matrices,

$$\rho_{ss'} = \frac{1}{2}\left(1 + \boldsymbol{S}\cdot\boldsymbol{\sigma}_{ss'}\right), \tag{B.5}$$

where $\boldsymbol{S}$ is the spin vector. When $|\boldsymbol{S}| = 1$ one has a pure state (only one state $|\alpha\rangle$ and $\rho^2 = \rho$), when $|\boldsymbol{S}| \leq 1$ one has an ensemble of states. For the case $|\boldsymbol{S}| = 0$ one has simply an averaging over spins, corresponding to an unpolarized ensemble. To include spin one could generalize the hadron tensor to a matrix in spin space, $\tilde{W}^{\mu\nu}_{s's}(q,P) \propto \langle P,s'|J^\mu|X\rangle\langle X|J^\nu|P,s>$ depending only on the momenta or one can look at the tensor $\sum_\alpha p_\alpha \tilde{W}^{\mu\nu}_{\alpha\alpha}(q,P)$. The latter is given by

$$W^{\mu\nu}(q,P,S) = \mathrm{Tr}\left(\rho(P,S)\tilde{W}^{\mu\nu}(q,P)\right), \tag{B.6}$$

with the spacelike spin vector S appearing *linearly* and in an arbitrary frame satisfying $P\cdot S = 0$. It has invariant length $-1 \leq S^2 \leq 0$. It is convenient to write

$$S^\mu = \frac{S_L}{M}\left(P^\mu - \frac{M^2}{P\cdot q}\,q^\mu\right) + S_\perp^\mu, \tag{B.7}$$

with

$$S_L \equiv \frac{M\,(S\cdot q)}{(P\cdot q)}. \tag{B.8}$$

For a pure state one has $S_L^2 + \boldsymbol{S}_\perp^2 = 1$. Parity requires that the polarized part of the tensor, i.e. the part containing the spin vector, enters in an antisymmetric tensor of the form

$$\begin{aligned} M\,W^{\mu\nu\,(A)}(q,P,S) &= S_L\,\frac{i\epsilon^{\mu\nu\rho\sigma}q_\rho P_\sigma}{(P\cdot q)}\,g_1 + \frac{M}{P\cdot q}\,i\epsilon^{\mu\nu\rho\sigma}q_\rho S_{\perp\sigma}\,(g_1+g_2) \\ &= -iS_L\,\epsilon_\perp^{\mu\nu}\,g_1 - i\,\frac{2M}{Q}\,\hat{t}^{\,[\mu}\epsilon_\perp^{\nu]\rho}S_{\perp\rho}\,x_B(g_1+g_2). \end{aligned} \tag{B.9}$$

It contains two structure functions $g_1(x_B,Q^2)$ and $g_2(x_B,Q^2)$. One also uses $g_T \equiv g_1+g_2$. The resulting cross section is

$$\frac{d\Delta\sigma_{LL}}{dx_B\,dy} = \frac{4\pi\,\alpha^2}{Q^2}\,\lambda_e\left[S_L\,(2-y)\,g_1 - |\boldsymbol{S}_\perp|\,\cos\phi_S^\ell\,\frac{2M}{Q}\,\sqrt{1-y}\,x_B(g_1+g_2)\right]. \tag{B.10}$$

(Note that in all of the above formulas mass corrections proportional to M^2/Q^2 have been neglected).

Appendix C.

Polarized parton densities

Analogously to the unpolarized structure functions one can obtain for the polarized structure functions

$$2\,g_1(x_B) = \sum_q e_q^2\,[g_1^q(x_B) + \bar{g}_1^q(x_B)] \tag{C.1}$$

where

$$S_L g_1^q(x) = \frac{1}{4\pi}\int d\xi^-\; e^{ixP^+\xi^-}\,\langle P,S|\overline{\psi}(0)\gamma^+\gamma_5\psi(\xi)|P,S\rangle\Big|_{\xi^+=\xi_\perp=0}\,, \tag{C.2}$$

often denoted as $\Delta q(x) = g_1^q(x)$. This correlation exists in a hadron with the light-cone component of the spin vector $S_L \neq 0$. It represents the difference of chiral even and odd quarks (in infinite momentum frame quarks parallel or antiparallel to proton spin). The corresponding quark fields are projected out by

$$P_{R/L} = \frac{1}{2}(1 \pm \gamma_5), \tag{C.3}$$

which commute with the projectors $P_\pm$. In this way one obtains distributions $q_R(x)$ and $q_L(x)$ for which $q(x) = q_R(x) + q_L(x)$ and $\Delta q(x) = q_R(x) - q_L(x)$.

==============

Exercise: Show that the antiquark distributions are given by $\bar{q}(x) = -q(-x)$ and $\Delta\bar{q}(x) = \Delta q(-x)$. To do this start with the 'proper' definition of antiquark distributions,

$$\Phi^c_{ij}(p) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{ip\cdot\xi}\,\langle PS|\overline{\psi}^c_j(0)\psi^c_i(\xi)|PS\rangle, \tag{C.4}$$

with $\psi^c(\xi) = C\overline{\psi}^T(\tilde{\xi})$. Show that one finds $\overline{\Phi}(p) = -C(\Phi^c)^T C^\dagger$. One also needs to use the anticommutation relations for fermions, to obtain $\overline{\Phi}_{ij}(p) = -\Phi_{ij}(-p)$, which leads to the crossing relations for quark and antiquark distributions.

==============

Appendix D.

Forms of quantization

As discussed by Dirac in his seminal paper [P.A.M. Dirac, *Forms of Relativistic Dynamics*, Rev. Mod. Phys. 21 (1949) 392] there are more *forms* of quantization. One can for instance start on a hypersurface x^0 = constant, known as *instant form* or on a hypersurface $x^+ = (x^0 + x^3)/\sqrt{2}$ = constant, known as *front form.* Depending on the choice, there are different sets of *dynamical generators* out of the ten Poincaré generators that leave the hypersurface invariant. These kinematical generators can be choosen to be free of interactions.

In **instant form** there are six kinematical generators, three translations P^i and three rotations $J^i = \frac{1}{2}\,\epsilon_{ijk}\,M^{jk}$. Quantization in instant form starts with (interaction independent) canonical commutation relations $[x^i,p^j] = i\,\delta^{ij}$ and $[s^i,s^j] = i\,\epsilon^{ijk}\,s^k$, with on the $x^0 = t$ hypersurface the kinematical generators given by

$$\boldsymbol{P} = \boldsymbol{p}, \tag{D.1}$$
$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{s}. \tag{D.2}$$

In **front form** there are seven kinematical generators, three translations P^+ and P_T^i, and furthermore $J^3 = M^{12}$, M^{-+}, M^{1+}, M^{2+}. Quantization in front form starts with (interaction independent) canonical commutation relations $[x^+,p^-] = i$, $[x^i,p^j] = i\,\delta_T^{ij}$ (i = 1,2), with on the $x^+ = \tau$ hypersurface the (simple) kinematical generators given by

$$P^+ = p^+, \tag{D.3}$$
$$\boldsymbol{P}_T = \boldsymbol{p}_T, \tag{D.4}$$
$$J^3 = M^{12} = x^1p^2 - x^2p^1 + \lambda \tag{D.5}$$
$$M^{-+} = x^-p^+ - \tau p^-, \tag{D.6}$$
$$M^{i+} = x^ip^+ - \tau p^i \qquad (i = 1,2). \tag{D.7}$$

Spin $\boldsymbol{s}$ and light-cone helicity λ are allowed extensions with vanishing commutators with the x and p operators.

==============

Exercise: Check that the kinematical operators satisfy the commutation relations of the Poincaré group,

$$[P^\mu,P^\nu] = 0, \tag{D.8}$$
$$[M^{\mu\nu},P^\rho] = -i\left(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu\right), \tag{D.9}$$
$$[M^{\mu\nu},M^{\rho\sigma}] = -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}\right) - i\left(g^{\mu\sigma}M^{\rho\nu} - g^{\nu\sigma}M^{\rho\mu}\right). \tag{D.10}$$

==============

Appendix E.

Field quantization in front form

In order to discuss the quantization of the QCD field-theoretical lagrangian we need the 'momenta'

$$\overline{\Pi}_\psi^\mu = \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\psi} = \frac{i}{2}\,\overline{\psi}\gamma^\mu, \tag{E.1}$$

$$\Pi_\psi^\mu = \frac{\delta\mathcal{L}}{\delta\,\partial_\mu\overline{\psi}} = -\frac{i}{2}\,\gamma^\mu\psi, \tag{E.2}$$

$$\Pi_{A_\nu}^\mu = \frac{\delta\mathcal{L}}{\delta\,\partial_\mu A_\nu} = -F^{\mu\nu}. \tag{E.3}$$

In **instant form** fields are quantized by imposing equal time commutation relations with the conjugate momenta. These momenta can be obtained from the above general momenta. We find for the fermion field $\overline{\Pi}_\psi = \delta\mathcal{L}/\delta\,\partial_0\psi = \frac{i}{2}\,\psi^\dagger$. The canonical momentum of the A^μ fields and the remaining components of the field tensor are given by

$$F^{0i} = -E^i = \dot{A}^i + \partial_i A^0 - i\,g\,[A^0, A^i] = \dot{A}^i + D_i A^0, \tag{E.4}$$

$$F^{ij} = -\partial_i\,A^j + \partial_j\,A^i - i\,g\,[A^i, A^j] = -\epsilon^{ijk}\,B^k \tag{E.5}$$

where $\dot{A}^\mu = \partial_0 A^\mu$. The absence of a canonical momentum for A^0 implies a constraint

$$\frac{\delta\mathcal{L}}{\delta\,A^0} \;=\; -D_i E^i + j^0 = 0, \tag{E.6}$$

where $D_i E^i = [D_i, E^i] = \partial_i E^i + ig\,[A^i, E^i]$ and $j^0 = \psi^\dagger\psi$. This equation is the non-abelian Gauss' law. The other equations of motion are

$$\left(i\,\gamma^0 D_0 + i\,\gamma^i D_i - m\right)\psi = 0, \tag{E.7}$$

$$D_0 F^{0i} + D_j F^{ji} = j^i. \tag{E.8}$$

Rewriting slightly, we have as equations of motion

$$i\,D_0\psi = -\gamma_0\left(i\,\gamma^j D_j - m\right)\psi = \left(i\,\gamma^j D_j + m\right)\gamma_0\,\psi, \tag{E.9}$$

$$D_0 E^i = \epsilon^{ikl} D_k B^l - j^i, \tag{E.10}$$

$$D_j E^j = j^0. \tag{E.11}$$

Note that the last of these equations is just the constraint implied by the absence of a canonical momentum for A^0. The constraint involves interactions, in this case through $\dot{A}^i$ in E^i. To proceed, one still can make a gauge choice. We just want to mention that the (obvious) choice $A^0 = 0$ in combination with $\partial_\mu\,A^\mu = 0$ has problems (Gribov ambiguity).
[The classical reference on constraints in quantum mechanics is P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer graduate school of Science, 1964]

In **front form quantization**, fields are quatized by imposing equal time commutation relations with the conjugate momenta using the light-cone time x^+, i.e. $\dot{\psi} = \partial_+\psi = \partial^-\psi$, etc. In terms of the light-cone coordinates one has for the fermion field $\overline{\Pi}_\psi = \delta\mathcal{L}/\delta\,\partial_+\psi = \frac{i}{\sqrt{2}}\,\psi_+^\dagger$, where the *good/bad* fields $\psi_\pm = P_\pm\psi$ are introduced with the projectors $P_\pm \equiv \frac{1}{2}\gamma^\mp\gamma^\pm$. We note that there is no canonical momentum for the bad field ψ_-. For the gluon fields one has (using $i, j, \ldots$ for transverse indices!)

$$F^{-+} = \dot{A}^+ - \partial_- A^- + ig\,[A^+, A^-] = \dot{A}^+ - D_- A^-, \tag{E.12}$$

$$F^{-i} = \dot{A}^i + \partial_i A^- + ig\,[A_T^i, A^-] = \dot{A}_T^i + D_{Ti} A^- \equiv -E_T^i, \tag{E.13}$$

$$F^{+i} = \partial_- A^i + \partial_i A^+ - ig\,[A_T^i, A^+] = \partial_- A_T^i + D_i A^+ \equiv B_T^i, \tag{E.14}$$

$$F^{ij} = \partial_j A_T^i - \partial_i A_T^j - ig\,[A_T^i, A_T^j] \equiv -\epsilon_T^{ij}\,B, \tag{E.15}$$

the first two being canonical momenta. Note that there is no canonical momentum for A^-. We can simplify by going to the light-cone gauge $A^+ = 0$. In that case the first and third expressions simplify

$$F^{-+} = -\partial_- A^-, \tag{E.16}$$

$$F^{+i} = B_T^i = \partial_- A_T^i. \tag{E.17}$$

These equations actually also indicate that choosing $A^+ = 0$ not removes all gauge freedom. Inverting the equations shows that one also needs to specify a boundary condition at infinity. To be precise one needs a prescription for $(\partial_-)^{-1}$, for which we use

$$(\partial_-)^{-1} f(x^-) = \frac{1}{2}\int dy^-\,\epsilon(x^- - y^-) f(y^-), \tag{E.18}$$

i.e. one imposes an antisymmetric boundary condition $f(-\infty) = -f(\infty)$. The equations of motion made explicit in light-cone coordinates and using the gauge $A^+ = 0$ are

$$\begin{aligned}
&\left(i\gamma^+ D_+ + i\gamma^- D_- + i\boldsymbol{\gamma}^i D_i - m\right)\psi = 0,\\
&D_- F^{-+} + D_i F^{i+} = \partial_- F^{-+} + \partial_i F^{i+} + ig\,[A_T^i, F^{i+}] = j^+,\\
&D_+ F^{+-} + D_i F^{i-} = \dot F^{+-} - ig\,[A^-, F^{+-}] + \partial_i F^{i-} + ig\,[A_T^i, F^{i-}] = j^-,\\
&D_+ F^{+i} + D_- F^{-i} + D_j F^{ji} = \dot F^{+i} - ig\,[A^-, F^{+i}] + \partial_- F^{-i} + \partial_j F^{ji} + ig\,[A_T^j, F^{ji}] = j^i.
\end{aligned}$$

Using the fields $\psi_\pm$, the equations of motion become

$$2i\,D_+\psi_+ = (i\,\not{D}_T + m\,)\,(\gamma^-\,\psi_-), \tag{E.19}$$

$$i\,\partial_-(\gamma^-\psi_-) = (i\,\not{D}_T - m)\,\psi_+, \tag{E.20}$$

$$-(\partial_-)^2 A^- - D_T\cdot B_T = j^+ = g\sqrt{2}\,\psi_+^\dagger\psi^+, \tag{E.21}$$

$$D_+(\partial_- A^-) + D_T\cdot E_T = j^- = \frac{g}{\sqrt{2}}\,(\gamma^-\psi_-)^\dagger(\gamma^-\psi_-), \tag{E.22}$$

$$D_+ B_T^i - \partial_- E_T^i + \epsilon_T^{ij} D_j\,B = j^i = \frac{g}{\sqrt{2}}\left[(\gamma^-\psi_-)^\dagger\gamma^i\psi_+ + \psi_+^\dagger\gamma^i(\gamma^-\psi_-)\right]. \tag{E.23}$$

The second equation is a constraint not involving interactions, which can be solved as $\psi_-[A_T, \psi_+]$, since there are no time-derivatives involved. The same is true for the third equation, which can be solved as $A^-[A_T, \psi_+]$. To be precise

$$(\gamma^-\psi_-) = (i\partial_-)^{-1}(i\not{D}_T - m)\,\psi_+ \tag{E.24}$$

$$A^- = (i\partial_-)^{-2}\left[g\,\psi_+^\dagger\psi_+ + D_T\cdot B_T\right]. \tag{E.25}$$

One is left with two (independent) equations governing the time dependence of a minimal set of fields, ψ_+ and A_T^i. For these (dynamical) fields canonical commutation relations can be imposed independent of the interactions [J.B. Kogut and D.E. Soper, *Quantum* *electrodynamics in the infinite-momentum frame*, Phys. Rev. D 1 (1970) 2901]. The lagrangian left after gauge fixing can also be written as

$$\begin{aligned}\mathcal{L}_G &= -\frac{1}{2}\,F^{+-}F^{-+} + \frac{1}{2}\,F^{+i}F^{-i} - \frac{1}{2}\,B^2 \\ &= \frac{1}{2}\left(\partial_- A^-\right)^2 - \frac{1}{2}\,E_T\cdot B_T - \frac{1}{2}\,B^2.\end{aligned} \tag{E.26}$$

The fermionic part (including the quark-gluon interaction terms) is

$$\begin{aligned}\mathcal{L}_F = \frac{i}{\sqrt{2}}\left(\psi_+^\dagger\dot{\psi}_+ - \dot{\psi}_+^\dagger\psi_+\right) + g\sqrt{2}\,\psi_+^\dagger A^-\psi_+ + \frac{1}{2\sqrt{2}}\,(\gamma^-\psi_-)^\dagger\,\overleftrightarrow{\partial}_-\,(\gamma^-\psi_-) \\ + \frac{1}{\sqrt{2}}\left[(\gamma^-\psi_-)^\dagger\,(i\not{D}_T - m)\,\psi_+ + \psi_+^\dagger\,(i\not{D}_T + m)\,(\gamma^-\psi_-)\right].\end{aligned} \tag{E.27}$$

The nice thing of the front form quantization is the fact that the fields ψ_+ and A_T are dynamical fields, for which canonical equal (light-cone) time commutation relations can be imposed, of which the right hand side is interaction-independent. That means that a free field expansion for ψ_+ is a sensible thing, even if the quanta are confined. This is the basis behind the successful parton model in deep inelastic scattering.