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Review

# Polarized structure functions

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## Abstract

We discuss the spin structure of quarks in hadrons, in particular the transverse spin polarization or transversity. The most direct way to probe transversity appears to be via azimuthal spin asymmetries. This brings in the role of intrinsic transverse momenta of quarks in hadrons and the study of  $T$ -odd phenomena, which are connected to single spin asymmetries.

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## 1. Introduction

The topic of this lecture is the spin structure of hadrons and the way it is probed in high-energy scattering processes. We specifically study electroweak scattering processes. One specific process, namely lepton–hadron scattering will be dealt with in detail. A signal for an electroweak process is the presence of leptons which do not feel strong interactions. They allow a separation of the scattering amplitude for the process into a leptonic part and a hadronic part, where the leptonic part, involving elementary particles is known. The structure of the hadronic part is constrained by its (Lorentz) structure and fundamental symmetries and can be parametrized in terms of a number of structure functions. The emerging expression for the scattering amplitude can be used to calculate the cross sections

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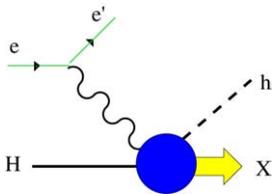
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in terms of these structure functions. In turn one can make a theoretical study of the structure functions. Part of this can be done rigorously with as only input (assumption if one wants) the known interactions of the hadronic constituents, quarks and gluons, within the standard model. For this both the electroweak couplings of the quarks needed to describe the interactions with the leptonic part via the exchange of photon,  $Z^0$  or  $W^\pm$  bosons as well as the strong interactions of the quarks among themselves via the exchange of gluons described in the QCD part of the standard model are important. For a general reference we refer to the book of Roberts [1].

### 2. Leptoproduction

In this section we discuss the basic kinematics of particular hard electroweak processes, namely the scattering of a high-energy lepton, e.g. an electron, muon or neutrino from a hadronic target,  $\ell(k)H(P) \rightarrow \ell'(k')X$ . In this process at least one hadron is involved. If one does not care about the final state, counting every event irrespective of what is happening in the scattering process, one talks about an inclusive measurement. If one detects specific hadrons in coincidence with the scattered lepton one talks about semi-inclusive measurements or more specifically 1-particle inclusive, 2-particle inclusive, depending on the number of particles that are detected.



$$\begin{aligned}
 q^2 &= (k - k')^2 \equiv -Q^2 \leq 0 \\
 2P \cdot q &\equiv 2M v \equiv \frac{Q^2}{x_B} \\
 2P_h \cdot q &\equiv -z_h Q^2 \\
 P \cdot k &= \frac{P \cdot q}{y} = \frac{Q^2}{2x_B y}
 \end{aligned}$$

The variable  $x_B$  is the Bjorken scaling variable, by the kinematics constrained to  $0 \leq x_B \leq 1$ , and  $x_B = 1$  corresponding to elastic scattering. In this scattering process a hadron is probed with a spacelike (virtual) photon, for which one could consider a frame in which the momentum only has a spatial component. This shows that the spatial resolving power of the probing photon is of the order  $\lambda \approx 1/Q$ . Roughly speaking one probes a nucleus (1–10 fm) with  $Q \approx 10\text{--}100$  MeV, baryon or meson structure (with sizes in the order of 1 fm) with  $Q \approx 0.1\text{--}1$  GeV and one probes deep into the nucleon ( $<0.1$  fm) with  $Q > 2$  GeV. Leptoproduction is characteristic for a large number of other processes involving particles (leptons) for which the interactions are fully known together with hadrons. The electroweak interactions with the constituents of the hadrons (quarks), however, are also known. This opens the way to study how quarks are embedded in the hadrons (e.g. in leptoproduction or in the Drell–Yan process,  $A(P_A)B(P_B) \rightarrow \ell(k)\bar{\ell}(k')X$ ) or to study how quarks fragment into hadrons (in leptoproduction and  $e^+e^-$  annihilation into hadrons).

For inclusive unpolarized electron scattering the cross section, assuming one-photon exchange, is given by

$$E' \frac{d\sigma}{d^3k'} = \frac{1}{s - M^2} \frac{\alpha^2}{Q^4} L_{\mu\nu}^{(S)} 2M W^{\mu\nu}, \tag{1}$$

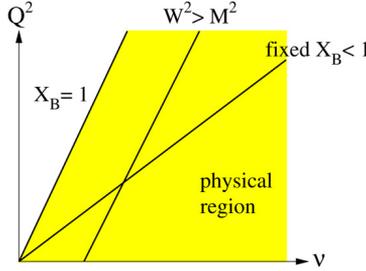


Fig. 1. The physical region in deep inelastic scattering.

where  $L_{\mu\nu}^{(S)}$  is the symmetric lepton tensor,

$$L_{\mu\nu}^{(S)}(k, k') = \text{Tr} [\gamma_\mu(\not{k}' + m)\gamma_\nu(\not{k} + m)] = 2k_\mu k'_\nu + 2k_\nu k'_\mu - Q^2 g_{\mu\nu} \quad (2)$$

and  $W_{\mu\nu}$  is the hadron tensor, which contains the information on the hadronic part of the scattering process,

$$2M W_{\mu\nu}(P, q) = \frac{1}{2\pi} \sum_n \int \frac{d^3 P_n}{(2\pi)^3 2E_n} \times \langle P | J_\mu^\dagger(0) | P_n \rangle \langle P_n | J_\nu(0) | P \rangle (2\pi)^4 \delta^4(P + q - P_n), \quad (3)$$

where  $|P\rangle$  represents a target with momentum  $P$ . For inclusive scattering the tensor can be rewritten to

$$2M W_{\mu\nu} = \frac{1}{2\pi} \int d^4 x e^{iq \cdot x} \langle P | [J_\mu^\dagger(x), J_\nu(0)] | P \rangle. \quad (4)$$

Including polarization this expression remains valid if summation and averaging over spins is understood.

What is one actually probing in leptonproduction? In the leptonproduction  $e + H \rightarrow e' + X$  the (unobserved) final state  $X$  can be the target (elastic scattering) or an excitation thereof. In a plot of the two independent variables  $\nu$  and  $Q^2$  (see Fig. 1) elastic scattering corresponds to a line given by  $\nu = Q^2/2M$ , where  $M$  is the mass of the target. As we have seen the behavior of the cross section along this line, where the ratio  $x_B = 1$ , is (besides the  $1/Q^4$  of the Mott cross section) proportional to a form factor squared, measuring the expectation value of the electromagnetic current  $\langle P' | J^\mu(x) | P \rangle$ . Exciting the nucleon gives rise to inelastic contributions in the cross section at  $\nu > Q^2/2M$ , starting at the threshold  $W = M + M_\pi$ . Note that as a function of  $x_B$  any resonance contribution will move closer to the elastic limit when  $Q^2$  increases. When  $Q^2$  and the energy transfer  $\nu$  are high enough the cross section will reflect elastic scattering off the pointlike constituents of the nucleon and the cross section will become equal to an incoherent sum of the electron–quark cross section. This is known as the deep inelastic scattering region, in which one finds Bjorken scaling. The cross section, or more precisely the structure functions, become functions of one (kinematic) variable  $x_B$ , which is identified with the momentum fraction of the struck quark in the nucleon, enabling measurement of quark distributions. We will make this explicit below. The picture will break down in the limiting cases, such as  $x_B \rightarrow 1$ , where it

becomes dual to the summation over resonances or  $x_B \rightarrow 0$ , corresponding (for fixed  $Q^2$ ) to  $\nu \rightarrow \infty$ . In this region one may employ Regge theory.

It is also possible to consider in more detail the space–time correlations that are probed. As already indicated  $q$  is space-like. But from the kinematics of deep inelastic scattering one can see that the process probes the lightcone. Sitting in the nucleon rest-frame we note that both  $q^0 = \nu$  and  $q^3 = \sqrt{Q^2 + \nu^2}$  go to infinity but working at finite  $x_B$  one sees that taking the sum and the difference only one of them goes to infinity.<sup>1</sup> Choosing  $q$  along the negative  $z$ -axis one has  $q^- = (\nu + |q|)/\sqrt{2} \rightarrow \infty$  and  $q^+ = (\nu - |q|)/\sqrt{2} \approx -M_N x_B/\sqrt{2}$ . This corresponds in the hadronic tensor which involves a Fourier transform of the product of currents to  $|x^+| \approx 1/q^- \rightarrow 0$  and  $|x^-| \approx 1/|q^+| \rightarrow 1/M_N x_B$  or  $|x| \approx |t| \approx 1/M_N x_B$ . Thus, depending on the value of  $x_B$  the distances and times are not necessarily small, but one has  $x^2 = x^+ x^- - \mathbf{x}_\perp^2 \approx -\mathbf{x}_\perp^2 \leq 0$ , while on the other hand causality requires that  $x^2 \geq 0$ . Therefore, one sees that deep inelastic scattering probes the lightcone,  $x^2 \approx 0$ .

### 3. Structure functions and cross section

The simplest thing one can do with the hadron tensor is to express it in standard tensors and functions depending on the invariants, the *structure functions*. Instead of the traditional choice using tensors,  $g_{\mu\nu}$ ,  $P_\mu P_\nu$  and  $\epsilon_{\mu\nu\rho\sigma} q^\rho P^\sigma$  multiplying structure functions  $W_1$ ,  $W_2$  and  $W_3$  depending on  $\nu$  and  $Q^2$ , we immediately go to a dimensionless representation. First we define a Cartesian basis of vectors [2], starting with the natural space-like momentum (defined by  $q$ ). Using the target hadron momentum  $P^\mu$  one can construct an orthogonal four vector  $\tilde{P}^\mu = P^\mu - (P \cdot q/q^2) q^\mu$ , which is timelike with length  $\tilde{P}^2 = \kappa P \cdot q$  with

$$\kappa = 1 + \frac{M^2 Q^2}{(P \cdot q)^2} = 1 + \frac{4 M^2 x_B^2}{Q^2}. \quad (5)$$

The quantity  $\kappa$  takes into account mass corrections  $\propto M^2/Q^2$  which will vanish for large  $Q^2$  ( $\kappa \rightarrow 1$ ). Thus we define

$$Z^\mu \equiv -q^\mu \quad \text{and} \quad T^\mu \equiv -\frac{q^2}{P \cdot q} \tilde{P}^\mu = q^\mu + 2x_B P^\mu. \quad (6)$$

For these vectors we have  $Z^2 = -Q^2$  and  $T^2 = \kappa Q^2$  and we will often use the normalized vectors  $\hat{z}^\mu = -\hat{q}^\mu = Z^\mu/Q$  and  $\hat{t}^\mu = T^\mu/Q\sqrt{\kappa}$ . With respect to these vectors one can also define transverse tensors,

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu \hat{q}^\nu - \hat{t}^\mu \hat{t}^\nu \quad \text{and} \quad \epsilon_\perp^{\mu\nu\rho\sigma} \equiv \epsilon^{\mu\nu\rho\sigma} \hat{t}_\rho \hat{q}_\sigma. \quad (7)$$

To get the parametrization of hadronic tensors, such as the one in Eq. (4), including for generality also an (axial) spin vector  $S$  (see Section 4), we use the general symmetry property,

$$W_{\mu\nu}(q, P, S) = W_{\nu\mu}(-q, P, S) \quad (8)$$

<sup>1</sup> Lightcone coordinates are defined  $a^\pm = (a^0 \pm a^3)/\sqrt{2}$ . The scalar product of two vectors is given by  $a \cdot b = a^+ b^- + a^- b^+ - a^\perp b^\perp - a^2 b^2$ .

as well as properties following from hermiticity, parity and time-reversal invariance,

$$W_{\mu\nu}^*(q, P, S) = W_{\nu\mu}(q, P, S), \tag{9}$$

$$W_{\mu\nu}(q, P, S) = \overline{W}_{\nu\mu}(\bar{q}, \bar{P}, -\bar{S}) \quad [\text{Parity}], \tag{10}$$

$$W_{\mu\nu}^*(q, P, S) = \overline{W}_{\mu\nu}(\bar{q}, \bar{P}, \bar{S}) \quad [\text{Time reversal}], \tag{11}$$

where  $\bar{p} = (p^0, -\mathbf{p})$ . Finally we use current conservation implying  $q^\mu W_{\mu\nu} = W_{\mu\nu} q^\nu = 0$ . Note that depending on the situation not all constraints can be applied. For inclusive unpolarized leptonproduction one obtains as the most general form for the symmetric tensor,

$$\begin{aligned} M W^{\mu\nu(S)}(P, q) &= \left( \frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} \right) F_1(x_B, Q^2) + \frac{\tilde{P}^\mu \tilde{P}^\nu}{P \cdot q} F_2(x_B, Q^2) \\ &= -g_{\perp}^{\mu\nu} \underbrace{F_1(x_B, Q^2)}_{F_T} + \hat{t}^\mu \hat{t}^\nu \underbrace{\left( -F_1 + \frac{\kappa}{2x_B} F_2 \right)}_{F_L}, \end{aligned} \tag{12}$$

where the structure functions  $F_1, F_2$  or the transverse and longitudinal structure functions,  $F_T = F_1$  and  $F_L$ , depend only on the hadron part relevant invariants  $Q^2$  and  $x_B$ . This is the structure for the electromagnetic (photon exchange) part of the electroweak interaction. For the weak ( $W$ - or  $Z$ -exchange) part both vector and axial vector currents with different parity behavior come in. In that case also the following antisymmetric tensor is allowed,

$$M W^{\mu\nu(A)}(q, P) = \frac{i\epsilon^{\mu\nu\rho\sigma} P_\rho q_\sigma}{(P \cdot q)} F_3(x_B, Q^2) = i\kappa \epsilon_{\perp}^{\mu\nu} F_3(x_B, Q^2). \tag{13}$$

It appears in the part of the tensor in which one of the currents in the product is a vector current and the other an axial vector current.

The cross section is obtained from the contraction of lepton and hadron tensors. It is convenient to expand also the lepton momenta  $k$  and  $k' = k - q$  in  $\hat{t}, \hat{z}$  and a perpendicular component using the scaling variable  $y = P \cdot q / P \cdot k$  (in the target restframe reducing to  $y = v/E$ ). The result (including target mass corrections) is

$$\begin{aligned} k^\mu &= \frac{2-y}{y} \frac{1}{\kappa} T^\mu - \frac{1}{2} Z^\mu + k_{\perp}^\mu \\ &= \frac{Q}{2} \hat{q}^\mu + \frac{(2-y)}{2y} \frac{Q}{\sqrt{\kappa}} \hat{t}^\mu + \frac{\sqrt{1-y + \frac{1}{4}(1-\kappa)y^2}}{y} \frac{Q}{\sqrt{\kappa}} \hat{\ell}^\mu \\ &\xrightarrow{Q^2 \rightarrow \infty} \frac{Q}{2} \hat{q}^\mu + \frac{(2-y)}{2y} \frac{Q}{\sqrt{\kappa}} \hat{t}^\mu + \frac{Q\sqrt{1-y}}{y} \hat{\ell}^\mu, \end{aligned} \tag{14}$$

where  $\hat{\ell}^\mu = k_{\perp}^\mu / |\mathbf{k}_{\perp}|$ , is a spacelike unit-vector in the perpendicular direction lying in the (lepton) scattering plane. The kinematics in the frame where virtual photon and target are collinear (including target rest frame) is illustrated in Fig. 2. With the definition of  $\hat{\ell}$ , we obtain neglecting mass corrections ( $\kappa = 1$ ) for unpolarized leptons the symmetric leptonic tensor

$$L^{\mu\nu(S)} = \frac{Q^2}{y^2} \left[ -2 \left( 1 - y + \frac{1}{2} y^2 \right) g_{\perp}^{\mu\nu} + 4(1-y) \hat{t}^\mu \hat{t}^\nu \right]$$

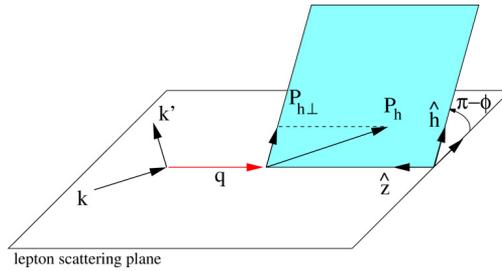


Fig. 2. Kinematics for lepton–hadron scattering. Transverse directions indicated with a  $\perp$  index are orthogonal to  $P$  and  $q$ , e.g. the orthogonal component of the momentum of a produced hadron has been indicated as example. Similarly one can consider the orthogonal component of the spin vector of the target.

$$+ 4(1 - y) \left( \hat{\ell}^\mu \hat{\ell}^\nu + \frac{1}{2} g_{\perp}^{\mu\nu} \right) + 2(2 - y) \sqrt{1 - y} \hat{t}^{\{\mu} \hat{\ell}^{\nu\}} \Big]. \tag{15}$$

The explicit contraction of lepton and hadron tensors gives for electromagnetic scattering (only symmetric tensor) the result

$$\begin{aligned} \frac{d\sigma^{ep}}{dx_B dy} &= \frac{4\pi \alpha^2 x_B s}{Q^4} \left[ \left( 1 - y + \frac{1}{2} y^2 \right) F_T(x_B, Q^2) + (1 - y) F_L(x_B, Q^2) \right] \\ &= \frac{2\pi \alpha^2 s}{Q^4} \left[ (1 - y) F_2(x_B, Q^2) + x_B y^2 F_1(x_B, Q^2) \right]. \end{aligned} \tag{16}$$

We have now used the known photon coupling to the lepton and parametrized our ignorance for what happened with the hadron in a hadronic tensor. The fact that we know how the photon interacts with the (quark) constituents of the hadrons will be used later to relate the structure functions to quark properties. In the same way one also knows for the weak interaction processes leading to an antisymmetric part containing  $F_3$  in the tensor for unpolarized hadrons, how the  $Z^0$  and  $W$  couple to quarks. To describe weak interactions also the antisymmetric part of the lepton tensor is needed, which we will also encounter when we discuss polarization.

### 4. Polarized leptonproduction

For spin-polarized leptons in the initial state we have

$$\begin{aligned} L_{\mu\nu}^{(s)} &= \text{Tr} \left[ \gamma_\mu (\not{k}' + m) \gamma_\nu (\not{k} + m) \frac{(1 \pm \gamma_5 \not{s})}{2} \right] \\ &= 2 k_\mu k'_\nu + 2 k'_\mu k_\nu - Q^2 g_{\mu\nu} \pm 2i m \epsilon_{\mu\nu\rho\sigma} q^\rho s^\sigma. \end{aligned} \tag{17}$$

Note that for light particles or particles at high energies helicity states ( $\hat{s} = \hat{k}$ ) become chirality eigenstates. For  $L_{\mu\nu}$  the equivalence is easily seen because for  $s^\mu = (|\mathbf{k}|/m, E\hat{k}/m)$  ( $s^2 = -1$  and  $s \cdot p = 0$ ) one obtains in the limit  $E \approx |\mathbf{k}|$  the result  $m s^\mu \approx k^\mu$ . Then the leptonic tensor for helicity states ( $\lambda_e = \pm$ ) becomes

$$L_{\mu\nu}^{(\lambda_e=\pm 1/2)} \approx L_{\mu\nu}^{(R/L)} = L_{\mu\nu}^{(S)} + \lambda_e L_{\mu\nu}^{(A)}. \tag{18}$$

where the antisymmetric lepton tensor is given by

$$L_{\mu\nu}^{(A)}(k, k') = \text{Tr}[\gamma_\mu \gamma_5 \not{k}' \gamma_\nu \not{k}] 2i \epsilon_{\mu\nu\rho\sigma} q^\rho k^\sigma. \tag{19}$$

Expanding in the Cartesian set  $\hat{t}$ ,  $\hat{z}$  and the vector  $\hat{\ell}$  in the same way as for the symmetric part, we have for the antisymmetric part of the leptonic tensor<sup>2</sup> the result

$$L^{\mu\nu(A)} = \frac{Q^2}{y^2} [-iy(2-y) \epsilon_{\perp}^{\mu\nu} - 2iy\sqrt{1-y} \hat{t}^{[\mu} \epsilon_{\perp}^{\nu]\rho} \hat{\ell}_\rho]. \tag{20}$$

One can use polarized leptons in deep inelastic  $\vec{e}p \rightarrow X$  to probe the antisymmetric tensor for unpolarized hadrons, containing the  $F_3$  structure function. This contribution comes in via the interference between the  $\gamma$  and  $Z$  interference term.

In the situation where the target is polarized, one has several more structure functions as compared to the case of an unpolarized target. For a spin 1/2 particle the initial state is described by a 2-dimensional spin density matrix  $\rho = \sum_\alpha |\alpha\rangle p_\alpha \langle\alpha|$  describing the probabilities  $p_\alpha$  for a variety of spin possibilities. This density matrix is hermitean with  $\text{Tr} \rho = 1$ . It can in the target rest frame be expanded in terms of the unit matrix and the Pauli matrices,

$$\rho_{ss'} = \frac{1}{2} (1 + \mathbf{S} \cdot \boldsymbol{\sigma}_{ss'}), \tag{21}$$

where  $\mathbf{S}$  is the spin vector. When  $|\mathbf{S}| = 1$  one has a pure state (only one state  $|\alpha\rangle$  and  $\rho^2 = \rho$ ), when  $|\mathbf{S}| \leq 1$  one has an ensemble of states. For the case  $|\mathbf{S}| = 0$  one has simply an averaging over spins, corresponding to an unpolarized ensemble. To include spin one could generalize the hadron tensor to a matrix in spin space,  $\tilde{W}_{s's}^{\mu\nu}(q, P) \propto \langle P, s' | J^\mu | X \rangle \langle X | J^\nu | P, s \rangle$  depending only on the momenta or one can look at the tensor  $\sum_\alpha p_\alpha \tilde{W}_{\alpha\alpha}^{\mu\nu}(q, P)$ . The latter is given by

$$W^{\mu\nu}(q, P, S) = \text{Tr}(\rho(P, S) \tilde{W}^{\mu\nu}(q, P)), \tag{22}$$

with the spacelike spin vector  $S$  appearing linearly and in an arbitrary frame satisfying  $P \cdot S = 0$ . It has invariant length  $-1 \leq S^2 \leq 0$ . It is convenient to write

$$S^\mu = \frac{S_L}{M} \left( P^\mu - \frac{M^2}{P \cdot q} q^\mu \right) + S_\perp^\mu, \tag{23}$$

with

$$S_L \equiv \frac{M(S \cdot q)}{(P \cdot q)}. \tag{24}$$

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<sup>2</sup> A useful relation is

$$\epsilon_{\mu\nu\rho\sigma} g_{\alpha\beta} = \epsilon_{\alpha\nu\rho\sigma} g_{\mu\beta} + \epsilon_{\mu\alpha\rho\sigma} g_{\nu\beta} + \epsilon_{\mu\nu\alpha\sigma} g_{\rho\beta} + \epsilon_{\mu\nu\rho\alpha} g_{\sigma\beta}$$

or for a vector  $a_\perp$  orthogonal to  $\hat{t}$  and  $\hat{q}$ ,

$$\epsilon^{\mu\nu\rho\sigma} \hat{z}_\rho a_{\perp\sigma} = \hat{t}^{[\mu} \epsilon_{\perp}^{\nu]\rho} a_{\perp\rho},$$

$$\epsilon^{\mu\nu\rho\sigma} \hat{t}_\rho a_{\perp\sigma} = -\hat{z}^{[\mu} \epsilon_{\perp}^{\nu]\rho} a_{\perp\rho}.$$

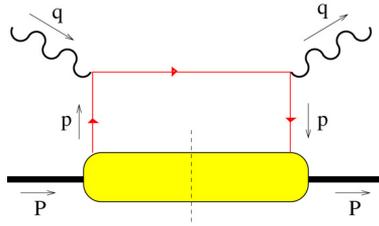


Fig. 3. The handbag diagram for inclusive deep inelastic scattering off a hadronic target.

For a pure state one has  $S_L^2 + S_\perp^2 = 1$ . Parity requires that the polarized part of the tensor, i.e. the part containing the spin vector, enters in an antisymmetric tensor of the form

$$\begin{aligned}
 M W^{\mu\nu(A)}(q, P, S) &= S_L \frac{i\epsilon^{\mu\nu\rho\sigma} q_\rho P_\sigma}{(P \cdot q)} g_1 + \frac{M}{P \cdot q} i\epsilon^{\mu\nu\rho\sigma} q_\rho S_{\perp\sigma} (g_1 + g_2) \\
 &= -iS_L \epsilon_\perp^{\mu\nu} g_1 - i \frac{2M}{Q} \hat{t}^{[\mu} \epsilon_\perp^{\nu]\rho} S_{\perp\rho} x_B (g_1 + g_2). \tag{25}
 \end{aligned}$$

It contains two structure functions  $g_1(x_B, Q^2)$  and  $g_2(x_B, Q^2)$ . One also uses  $g_T \equiv g_1 + g_2$ . The resulting cross section is

$$\frac{d\Delta\sigma_{LL}}{dx_B dy} = \frac{4\pi \alpha^2}{Q^2} \lambda_e \left[ S_L (2 - y) g_1 - |S_\perp| \cos\phi_S^\ell \frac{2M}{Q} \sqrt{1 - y} x_B (g_1 + g_2) \right]. \tag{26}$$

(Note that in all of the above formulas mass corrections proportional to  $M^2/Q^2$  have been neglected.)

### 5. The parton model

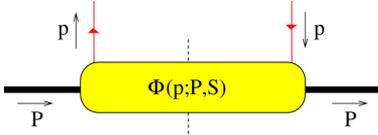
We will not follow the standard intuitive approach that folds the  $\gamma^*$ -quark cross section with a quark momentum distribution, but use a diagrammatic approach. The parton model is obtained by just considering the so-called quark handbag diagram (see Fig. 3) and its antiquark equivalent. These diagrams turn out to be the leading ones out of a full set in which the connection to hadrons is left as an unknown quantity [3,4]. The basic expression corresponding to the handbag diagram restricting us to the quark part is

$$\begin{aligned}
 2M W^{\mu\nu}(P, q) &= \sum_q e_q^2 \int dp^- dp^+ d^2p_\perp \text{Tr}(\Phi(p) \gamma^\mu (\not{p} + \not{q} + m) \gamma^\nu) \\
 &\quad \times \delta((p + q)^2 - m^2) \\
 &\approx \sum_q e_q^2 \int dp^- dp^+ d^2p_\perp \text{Tr} \left( \Phi(p) \gamma^\mu \frac{\not{q}}{2q^-} \gamma^\nu \right) \delta(p^+ + q^+) \\
 &\approx -g_\perp^{\mu\nu} \frac{1}{2} \int dp^- d^2p_\perp \text{Tr}(\gamma^+ \Phi(p)) \Big|_{p^+ = x_B P^+} + \dots, \tag{27}
 \end{aligned}$$

where  $\Phi(p)$  is the forward antiquark-target scattering amplitude,

$$\Phi_{ij}(p, P, S) = \frac{1}{(2\pi)^4} \int d^4\xi e^{i p \cdot \xi} \langle P, S | \bar{\psi}_j(0) \psi_i(\xi) | P, S \rangle, \quad (28)$$

diagrammatically represented by



Comparing with the general form of the hadronic tensor, we read off (including now also the antiquark part)

$$2 F_1(x_B) = 2M W_1(x_B, Q^2) = \sum_q e_q^2 [q(x_B) + \bar{q}(x_B)], \quad (29)$$

with

$$q(x) = \frac{1}{4\pi} \int d\xi^- e^{ix P^+ \xi^-} \langle P, S | \bar{\psi}(0) \gamma^+ \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_\perp = 0}, \quad (30)$$

$$\bar{q}(x) = \frac{1}{4\pi} \int d\xi^- e^{-ix P^+ \xi^-} \langle P, S | \bar{\psi}(0) \gamma^+ \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_\perp = 0}. \quad (31)$$

The result is (as expected) a lightcone correlation function of quark fields. It is easy to show that the antiquark distributions are given by  $\bar{q}(x) = -q(-x)$ . To do this start with the ‘proper’ definition of antiquark distributions,

$$\Phi_{ij}^c(p) = \frac{1}{(2\pi)^4} \int d^4\xi e^{i p \cdot \xi} \langle PS | \bar{\psi}_j^c(0) \psi_i^c(\xi) | PS \rangle, \quad (32)$$

with  $\psi^c(\xi) = C \bar{\psi}^T(\xi)$ . Show that one finds  $\bar{\Phi}(p) = -C(\Phi^c)^T C^\dagger$ . One also needs to use the anticommutation relations for fermions, to obtain  $\bar{\Phi}_{ij}(p) = -\Phi_{ij}(-p)$ , which leads to the crossing relations for quark and antiquark distributions,  $\bar{q}(x) = -q(-x)$ .

### 5.1. The operator in coordinate space

The parton result for the structure functions can also be derived by inserting free currents in the hadronic tensor for the current commutator and using the expression for the free field commutator. Using the anticommutation relations for free quark fields, given by  $\{\psi(\xi), \bar{\psi}(0)\} = \frac{1}{2\pi} \not{\partial} \delta(\xi^2) \epsilon(\xi^0)$  one derives for the  $g_{\mu\nu}$  contribution in the current–current commutator for quarks

$$\begin{aligned} [J_\mu(\xi), J_\nu(0)] &= [ : \bar{\psi}(\xi) \gamma_\mu \psi(\xi) :, : \bar{\psi}(0) \gamma_\nu \psi(0) : ] \\ &= \frac{-g_{\mu\nu}}{2\pi} \left[ \partial_\rho \delta(\xi^2) \epsilon(\xi^0) \right] : \bar{\psi}(\xi) \gamma^\rho \psi(0) - \bar{\psi}(0) \gamma^\rho \psi(\xi) :. \end{aligned} \quad (33)$$

An important feature, evident in the free-current commutator, is the lightcone dominance. By sandwiching the commutator between physical states and taking the Fourier transform, it is a straightforward calculation to obtain again the hadron tensor and the same result as in the diagrammatic approach above. Details can be found in [5].

## 5.2. Flavor dependence

The explicit flavor and spin dependence of the structure functions in electroweak processes depend on the probe being a  $\gamma$ ,  $Z^0$  or  $W^\pm$  boson. We are in the situation, however, that we know the currents in terms of the quark fields. Omitting the coupling constants  $e$  or  $\sqrt{G_F}$ , the standard model currents coupling to fermions are

$$J_\mu^{(\gamma)} = : \bar{\psi}(x) Q \gamma_\mu \psi(x) :, \quad (34)$$

$$\begin{aligned} J_\mu^{(Z)} &= : \bar{\psi}(x) (I_W^3 - Q \sin^2 \theta_W) \gamma_{\mu L} \psi(x) : - : \bar{\psi}(x) Q \sin^2 \theta_W \gamma_{\mu R} \psi(x) : \\ &= : \bar{\psi}(x) (I_W^3 - 2 Q \sin^2 \theta_W) \gamma_\mu \psi(x) : - : \bar{\psi}(x) I_W^3 \gamma_\mu \gamma_5 \psi(x) :, \end{aligned} \quad (35)$$

$$J_\mu^{(W)} = : \bar{\psi}(x) I_W^\pm \gamma_{\mu L} \psi(x) :, \quad (36)$$

where  $\gamma_{\mu R/L} = \gamma_\mu (1 \pm \gamma_5)$ .

Using the electromagnetic current, one obtains for the one-photon exchange contribution to  $ep$  scattering the following expression in terms of the distribution functions for the quarks with various flavors,  $u(x)$ ,  $d(x)$ , etc. and the corresponding antiquark distributions,  $\bar{u}(x)$ , etc.,

$$\begin{aligned} \frac{F_2^{ep}(x)}{x} &= 2 F_1^{ep}(x) = \frac{4}{9} (u(x) + \bar{u}(x)) + \frac{1}{9} (d(x) + \bar{d}(x)) \\ &\quad + \frac{1}{9} (s(x) + \bar{s}(x)) + \dots \end{aligned} \quad (37)$$

By definition quark distributions in the proton are used as reference. For a neutron one has using isospin symmetry,  $u_p = d_n$ ,  $d_p = u_n$ ,  $s_p = s_n$  and one has for  $en$  scattering

$$\begin{aligned} \frac{F_2^{en}(x)}{x} &= 2 F_1^{en}(x) = \frac{1}{9} (u(x) + \bar{u}(x)) + \frac{4}{9} (d(x) + \bar{d}(x)) \\ &\quad + \frac{1}{9} (s(x) + \bar{s}(x)) + \dots \end{aligned} \quad (38)$$

As the difference between quarks and antiquarks contributes to the quantum numbers, it is convenient to divide the quark distribution in a valence part and a sea part

$$q(x) = q_v(x) + q_s(x) \quad \text{with} \quad q_v(x) \equiv q(x) - \bar{q}(x). \quad (39)$$

The quark distributions are positive definite (see also Section 6), leading to specific inequalities. For instance the experimental observation that  $F_2^{ep}/F_2^{en}$  tends towards 1/4 for  $x \rightarrow 1$  shows dominance of  $u(x)$  over  $d(x)$  near  $x \rightarrow 1$ . It is clear that in order to determine the quark distributions, several processes are needed. For compilations we refer to Ref. [6].

## 6. Properties of quark distributions

### 6.1. Interpretation as densities

To convince oneself that the above expressions for  $q(x)$  and  $\bar{q}(x)$  actually can be interpreted as quark momentum density one needs to realize that  $\bar{\psi}(\xi) \gamma^+ \psi(0) = \sqrt{2} \psi_+^\dagger(\xi) \psi_+(0)$  where  $\psi_\pm = P_\pm \psi$  are projections obtained with projection operators

onto *good* quark states [7],  $P_{\pm} = \frac{1}{2}\gamma^{\mp}\gamma^{\pm}$ . One then can insert a complete set of states and obtain

$$\begin{aligned}
 q(x) &= \int \frac{d\xi^-}{2\pi\sqrt{2}} e^{iP\cdot\xi} \langle P, S | \psi_{+}^{\dagger}(0) \psi_{+}(\xi) | P, S \rangle \Big|_{\xi^{+}=\xi_{T}=0} \\
 &= \frac{1}{\sqrt{2}} \sum_n |(P_n | \psi_{+} | P)|^2 \delta(P_n^{+} - (1-x)P^{+}),
 \end{aligned}
 \tag{40}$$

which represents the probability that a quark is annihilated from  $|P\rangle$  giving a state  $|n\rangle$  with  $P_n^{+} = (1-x)P^{+}$ . Since  $P_n^{+} \geq 0$  one sees that  $x \leq 1$ . From the antiquark distribution  $\bar{q}(x)$  and its relation to  $-q(-x)$  one obtains  $x \geq -1$ , thus showing that the support of the functions is  $-1 \leq x \leq 1$  [8,9].

### 6.2. Polarized parton densities

Analogously to the unpolarized structure functions one can obtain for the polarized structure functions

$$2 g_1(x_B) = \sum_q e_q^2 [\Delta q(x_B) + \Delta \bar{q}(x_B)]
 \tag{41}$$

where

$$S_L \Delta q(x) = \frac{1}{4\pi} \int d\xi^- e^{ixP^{+}\xi^-} \langle P, S | \bar{\psi}(0) \gamma^{+} \gamma_5 \psi(\xi) | P, S \rangle \Big|_{\xi^{+}=\xi_{\perp}=0},
 \tag{42}$$

$$S_L \Delta \bar{q}(x) = \frac{1}{4\pi} \int d\xi^- e^{-ixP^{+}\xi^-} \langle P, S | \bar{\psi}(0) \gamma^{+} \gamma_5 \psi(\xi) | P, S \rangle \Big|_{\xi^{+}=\xi_{\perp}=0},
 \tag{43}$$

a correlation existing in a hadron with the lightcone component of the spin vector  $S_L \neq 0$ . This represents the difference of chiral even and odd quarks (in infinite momentum frame quarks parallel or antiparallel to proton spin). The corresponding quark fields are projected out by  $P_{R/L} = (1 \pm \gamma_5)/2$ , which commute with the projectors  $P_{\pm}$ . In this way one obtains distributions  $q_R(x)$  and  $q_L(x)$  for which  $q(x) = q_R(x) + q_L(x)$  and  $\Delta q(x) = q_R(x) - q_L(x)$ .

### 6.3. Sum rules

As probability distributions the quark distribution functions satisfy a number of obvious sum rules, such as

$$\int_0^1 dx u_v(x) = n_u = 2, \quad \int_0^1 dx d_v(x) = n_d = 1, \quad \int_0^1 dx s_v(x) = n_s = 0,
 \tag{44}$$

corresponding to the (net) number of each of these quark species in the proton. This also leads to sum rules for the structure functions.

Weighing the distribution functions with  $x$ , one can determine the quantity

$$\begin{aligned}
 \int_0^1 dx x \Sigma(x) &\equiv \int_0^1 dx x [u(x) + \bar{u}(x) + d(x) + \bar{d}(x) \\
 &\quad + s(x) + \bar{s}(x) + \dots] = \epsilon_q.
 \end{aligned}
 \tag{45}$$

It represents the total momentum fraction of the proton carried by quarks. It must obviously be smaller than one.

For the polarized structure functions one can obtain similar estimates using the naive flavor-spin structure of the proton based on  $SU(6)$  symmetry,

$$|p \uparrow\rangle = \frac{1}{\sqrt{18}} (2 u_\uparrow u_\uparrow d_\downarrow - u_\uparrow u_\downarrow d_\uparrow - u_\downarrow u_\uparrow d_\uparrow + [d \text{ at places 1 and 2}]). \quad (46)$$

From this wave function one finds in terms of a normalized one-quark distribution  $q(x)$  the ‘naive’ results

$$u_\uparrow(x) = \frac{5}{3} q(x), \quad u_\downarrow(x) = \frac{1}{3} q(x), \quad d_\uparrow(x) = \frac{1}{3} q(x), \quad d_\downarrow(x) = \frac{2}{3} q(x), \quad (47)$$

or

$$u(x) = 2q(x), \quad \Delta u(x) = \frac{4}{3} q(x), \quad d(x) = q(x), \quad \Delta d(x) = -\frac{1}{3} q(x), \quad (48)$$

and all other distributions (strange quarks or antiquarks) are zero. In this case one obtains naive sum rule results for the polarized distributions,

$$\int_0^1 dx [\Delta u(x) + \Delta \bar{u}(x)] = \Delta u = \frac{4}{3},$$

$$\int_0^1 dx [\Delta d(x) + \Delta \bar{d}(x)] = \Delta d = -\frac{1}{3}. \quad (49)$$

Note that the sum  $\Delta\Sigma = \Delta u + \Delta d + \Delta s + \dots$  represents (as probability distributions) the total number of quarks parallel to the proton spin. If the proton spin comes from the quark spins, as is the case for the above  $SU(6)$  wave function multiplying a spherically symmetric spatial wave function, one expects this to be one. It leads to

$$\Gamma_1^p = \int_0^1 dx g_1^p(x) = \frac{1}{2} \left[ \frac{4}{9} \Delta u + \frac{1}{9} \Delta d + \frac{1}{9} \Delta s \right] = \frac{5}{18} \approx 0.28, \quad (50)$$

$$\Gamma_1^n = \int_0^1 dx g_1^n(x) = \frac{1}{2} \left[ \frac{4}{9} \Delta d + \frac{1}{9} \Delta u + \frac{1}{9} \Delta s \right] = 0, \quad (51)$$

which is in disagreement with the experimental result [10],  $\Gamma_1^p \approx 0.15$  and the result  $\Gamma_1^n \approx -0.04$ , obtained from the deuteron sum rule  $\Gamma_1^d = (\Gamma_1^p + \Gamma_1^n) \left(1 + \frac{3}{2}\omega_D\right)$  with  $\omega_D \approx 0.05$  being the  $D$ -wave probability in the deuteron.

The importance of these sum rules becomes clearer when one starts with the expressions for the distribution functions in terms of matrix elements of bilocal operator combinations. One has

$$\int_{-1}^1 dx q(x) = \int_0^1 dx [q(x) - \bar{q}(x)] = \frac{\langle P | \bar{\psi}(0) \gamma^+ \psi(0) | P \rangle}{2P^+} = n_q, \quad (52)$$

where  $n_q$  is the coefficient in the expectation value  $\langle P | \bar{\psi}(x) \gamma^\mu \psi(0) | P \rangle = 2 n_q P^\mu$ . The coefficient  $n_q$  is precisely the quark number because the vector currents are used to obtain the quantum numbers for flavor (upness, downness, strangeness, etc.). In general one

obtains

$$\int_{-1}^1 dx x^{n-1} q(x) = \int_0^1 dx x^{n-1} [q(x) + (-)^n \bar{q}(x)] = \frac{1}{2P^+} \langle P | \bar{\psi}(0) \gamma^+ \left( \frac{i\partial^+}{P^+} \right)^{n-1} \psi(0) | P \rangle. \tag{53}$$

The moments of the structure functions are related to expectation values of particular quark operators. In a field theory these matrix elements depend on a renormalization scale  $\mu^2$  and thus a similar renormalization scale dependence must be present for the structure functions. In some cases such as the rule in Eq. (52) the result is scale independent. This is true if the operator combination corresponds to a conserved current. The situation is different for the second moment that appears in the momentum sumrule in Eq. (45),

$$\int_{-1}^1 dx x q(x, \ln \mu^2) = \int_0^1 dx x [q(x, \ln \mu^2) + \bar{q}(x, \ln \mu^2)] = \frac{\langle P | \bar{\psi}(0) i\gamma^+ \partial^+ \psi(0) | P \rangle_{(\mu^2)}}{2(P^+)^2} = \epsilon_q(\mu^2), \tag{54}$$

where  $\epsilon_q$  is defined in  $\langle P | \bar{\psi}(0) i\gamma_\mu \partial_\nu \psi(0) | P \rangle = 2\epsilon_q P_\mu P_\nu + \dots$  and is the relative contribution of quarks to the energy momentum tensor of the proton (the dots indicate trace terms  $\propto M^2 g_{\mu\nu}$ ). Only the first moment of the sum including quark *and* gluon distributions in the proton is scale independent as the local operator turns out to be the energy momentum stress tensor of QCD.

For polarized structure functions the lowest moment is given by

$$\int_{-1}^1 dx \Delta q(x, \ln \mu^2) = \int_0^1 dx [\Delta q(x, \ln \mu^2) + \Delta \bar{q}(x, \ln \mu^2)] = \frac{\langle P | \bar{\psi}(0) \gamma^+ \gamma_5 \psi(0) | P \rangle_{(\mu^2)}}{2P^+} = \Delta q(\mu^2). \tag{55}$$

The quantity  $\Delta q$  appears in  $\langle P | \bar{\psi}(0) \gamma^+ \gamma_5 \psi(0) | P \rangle = 2M \Delta q S^\mu$ , the matrix element of the axial current, where  $S^\mu$  is the spin vector for the nucleon. Also this is in general not independent of the renormalization scale. In particular the flavor singlet axial current is not conserved (because of the Adler–Bardeen–Jackiw-anomaly). It implies, however, a breaking independent of the flavor of the quarks. For the non-singlet axial currents that are important in the flavor-changing weak decays of baryons, e.g. for the neutron  $\propto \tau_+ \gamma^\mu \gamma_5$ , the current is conserved and the corresponding matrix elements are scale independent. From the neutron decay one deduces the (scale independent) flavor non-singlet combinations

$$\Delta q^3 = \Delta u(\mu^2) - \Delta d(\mu^2) = G_A = 1.259, \tag{56}$$

while from hyperon decays one finds (using  $SU(3)$  symmetry),

$$\Delta q^8 = \Delta u(\mu^2) + \Delta d(\mu^2) - 2\Delta s(\mu^2) \approx 0.6, \tag{57}$$

in terms of these combinations and the scale-dependent singlet combination,

$$\Delta\Sigma(\mu^2) = \Delta u(\mu^2) + \Delta d(\mu^2) + \Delta s(\mu^2), \quad (58)$$

one has

$$\Gamma_1^{p/n}(x) = \frac{1}{9} \Delta\Sigma(\mu^2) \pm \frac{1}{12} \Delta q^3 + \frac{1}{36} \Delta q^8. \quad (59)$$

A sum rule involving only flavor non-singlet combinations is for example the *polarized Bjorken sum rule* [11],

$$\int_0^1 dx [g_1^{ep}(x) - g_1^{en}(x)] = \frac{1}{6} \Delta q^3 = \frac{G_A}{6} \approx 0.21, \quad (60)$$

in reasonable agreement with experiment. Note that this result is a factor 0.75 smaller than the naive expectation for which  $G_A = \Delta q^3 = 5/3$  instead of  $G_A \approx 1.26$ . A long-known explanation for this reduction is the relativistic nature of quarks in hadrons implying a sizable  $p$ -wave contribution in the lower components of the quark spinor that reduces the spinor densities  $\bar{\psi}\gamma_5\gamma_3\psi = \psi^\dagger\sigma_z\psi$ .

Using the result for  $\Gamma_1^p$  or  $\Gamma_1^n$  as the third input one can solve for  $\Delta\Sigma$ , leading to  $\Delta\Sigma \approx 0.2$ , very small compared to the naive expectations  $\Delta\Sigma$  being of the order of 0.75 (taking the same reduction factor for relativistic quarks as for  $G_A$ ). This was known as the ‘proton spin puzzle’. At present we know that the scale dependence is important and that the deep inelastic measurements imply  $\Delta\Sigma(20 \text{ GeV}^2) \approx 0.2$ .

## 7. QCD corrections to structure functions and quark distributions

There are two sorts of corrections that appear when higher orders in the strong coupling are included. The first concerns the relation of structure functions and quark distributions. The structure functions or sum rules constructed from them get parts that are proportional to  $\alpha_s$ . For instance, the longitudinal structure function no longer remains zero, violating the Callan–Gross relation at order  $\alpha_s$ . Another example is the spin sum rule, which gets a correction  $(1 - \alpha_s/\pi)$ , or explicitly (including also the next order [12]),

$$\int_0^1 dx [g_1^p(x, Q^2) - g_1^n(x, Q^2)] = \frac{G_A}{6} \left( 1 - \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2} (-4.5833 + f/3) \right), \quad (61)$$

where  $f$  is the number of (active) flavors. This gives an excellent explanation of the experimental result  $\Gamma_1^p - \Gamma_1^n \approx 0.19$  being somewhat smaller than  $G_A/6 = 0.21$ .

The second type of corrections give the scale dependence of the quark and gluon matrix elements appearing in the moments of structure functions, e.g.  $\Delta u(Q^2)$ , or they can be obtained immediately for the full  $x$ -dependent quark distributions, known as evolution equations,

$$\frac{\partial q(x, \ln Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} q(z) P_{qq} \left( \frac{x}{z} \right). \quad (62)$$

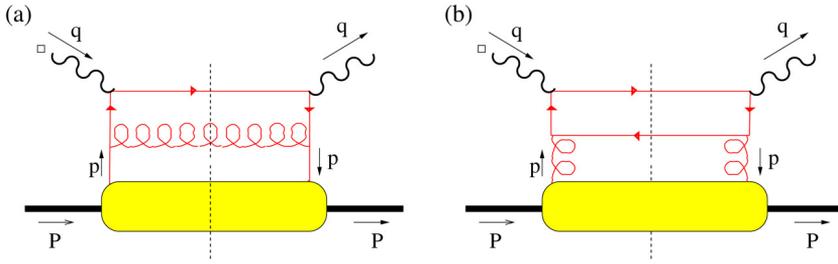


Fig. 4. Ladder diagrams used to find the evolution equations.

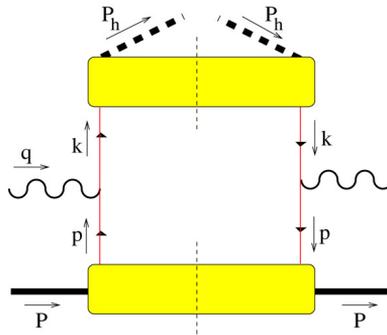


Fig. 5. The simplest (parton-level) diagrams for semi-inclusive scattering, of which we consider one-particle inclusive leptoproduction. Note that also the diagram with opposite fermion flow has to be added.

where

$$P_{qq}(\xi) = \frac{4}{3} \frac{1 + \xi^2}{(1 - \xi)_+} + 2 \delta(1 - \xi) \tag{63}$$

with

$$\int_0^1 dx \frac{f(x)}{(1 - x)_+} \equiv \int_0^1 dx \frac{f(x) - f(1)}{1 - x} \tag{64}$$

while this splitting function describes how QCD corrections arising from  $q \rightarrow qG$  splitting are incorporated into the parton distributions (Fig. 4(a)), one needs in addition other splitting functions such as  $P_{qG}$  describing how quark and gluon distribution functions mix (Fig. 4(b)). Since gluons are flavor-blind, non-singlet and valence distribution functions are not affected by such corrections.

The splitting function for the polarized distribution functions is given by

$$\Delta P_{qq}(x) = P_{qq}(x) = \frac{4}{3} \frac{1 + \xi^2}{(1 - \xi)_+} + 2 \delta(1 - \xi). \tag{65}$$

### 8. Quark correlation functions in 1PI leptonproduction

We now consider the case in which one particle is detected in coincidence with the scattered lepton, *one-particle inclusive* or 1PI leptonproduction. The kinematics of this process is already in the picture given before (Fig. 2). With a target hadron (momentum  $P$ ) and a detected hadron  $h$  in the final state (momentum  $P_h$ ) one has a situation in which two hadrons are involved and the operator product expansion cannot be used. Within the framework of QCD and knowing that the photon or  $Z^0$  current couples to the quarks, it is possible to write down a diagrammatic expansion for leptonproduction, within the deep inelastic limit ( $Q^2 \rightarrow \infty$ ) as relevant diagrams only the ones given in Fig. 5 for 1-particle inclusive scattering.

In analogy with the case of inclusive scattering, we also in 1-particle inclusive scattering parametrize the momenta with the help of two lightlike vectors, which are chosen now along the hadron momenta,

$$\left. \begin{aligned} q^2 &= -Q^2 \\ P^2 &= M^2 \\ P_h^2 &= M_h^2 \\ 2 P \cdot q &= \frac{Q^2}{x_B} \\ 2 P_h \cdot q &= -z_h Q^2 \end{aligned} \right\} \longleftrightarrow \left\{ \begin{aligned} P_h &= \frac{z_h Q}{\sqrt{2}} n_- + \frac{M_h^2}{z_h Q \sqrt{2}} n_+ \\ q &= \frac{Q}{\sqrt{2}} n_- - \frac{Q}{\sqrt{2}} n_+ + q_T \\ P &= \frac{x_B M^2}{Q \sqrt{2}} n_- + \frac{Q}{x_B \sqrt{2}} n_+ \end{aligned} \right.$$

An additional invariant  $z_h$  comes in. Note that the expansion is appropriate for the so-called current fragmentation, in which case the produced hadron is *hard* with respect to the target momentum, i.e.  $P \cdot P_h \sim Q^2$ . The minus component  $p^-$  is irrelevant in the lower soft part, while the plus component  $k^+$  is irrelevant in the upper soft part. Note that after the choice of  $P$  and  $P_h$  one can no longer omit a transverse component in the other vector, in the consideration above put in the momentum transfer  $q$ . One sees that one has (up to mass effects) the relation

$$q_T^\mu = q^\mu + x_B P^\mu - \frac{P_h^\mu}{z_h} \equiv -Q_T \hat{h}^\mu. \tag{66}$$

This relation allows the experimental determination of the ‘transverse momentum’ effect from the external vectors  $q$ ,  $P$  and  $P_h$  which are in general not collinear. The vector  $\hat{h}$  defines the orientation of the hadronic plane in Fig. 2.

An important consequence in the theoretical approach (Fig. 5) is that one can no longer simply integrate over the transverse components of the quark momenta.

#### 8.1. Structure functions and cross sections

For an unpolarized (or spin 0) hadron in the final state the symmetric part of the tensor is given by

$$\begin{aligned} M\mathcal{W}_S^{\mu\nu}(q, P, P_h) &= -g_\perp^{\mu\nu} \mathcal{H}_T + \hat{t}^\mu \hat{t}^\nu \mathcal{H}_L \\ &\quad + \hat{t}^{\{\mu} \hat{h}^{\nu\}} \mathcal{H}_{LT} + (2 \hat{h}^\mu \hat{h}^\nu + g_\perp^{\mu\nu}) \mathcal{H}_{TT}. \end{aligned} \tag{67}$$

Noteworthy is that also an antisymmetric term in the tensor is allowed,

$$M\mathcal{W}_A^{\mu\nu}(q, P, P_h) = -i\hat{t}^{\mu}\hat{h}^{\nu 1}\mathcal{H}'_{LT}. \tag{68}$$

Clearly the lepton tensor in Eq. (15) or (19) is able to distinguish all the structures in the semi-inclusive hadron tensor.

The symmetric part gives the cross section for unpolarized leptons,

$$\frac{d\sigma_{OO}}{dx_B dy dz_h d^2q_T} = \frac{4\pi\alpha^2 s}{Q^4} x_B z_h \left\{ \left(1 - y + \frac{1}{2}y^2\right) \mathcal{H}_T + (1 - y) \mathcal{H}_L - (2 - y)\sqrt{1 - y} \cos\phi_h^\ell \mathcal{H}_{LT} + (1 - y) \cos 2\phi_h^\ell \mathcal{H}_{TT} \right\} \tag{69}$$

while the antisymmetric part gives the cross section for a polarized lepton (note the target is not polarized!)

$$\frac{d\sigma_{LO}}{dx_B dy dz_h d^2q_T} = \lambda_e \frac{4\pi\alpha^2}{Q^2} z_h \sqrt{1 - y} \sin\phi_h^\ell \mathcal{H}'_{LT}. \tag{70}$$

Of course many more structure functions appear for polarized targets or if one considers polarimetry in the final state. In this case the (theoretically) most convenient way to describe the spin vector of the target is via an expansion of the form

$$S^\mu = -S_L \frac{Mx_B}{Q\sqrt{2}} n_- + S_L \frac{Q}{Mx_B\sqrt{2}} n_+ + S_T. \tag{71}$$

One has up to  $\mathcal{O}(1/Q^2)$  corrections  $S_L \approx M(S \cdot q)/(P \cdot q)$  and  $S_T \approx S_\perp$ , where the subscript  $\perp$  still refers to perpendicular to  $q$  and  $P$ . For a pure state one has  $S_L^2 + S_T^2 = 1$ , in general this quantity being less or equal than one.

### 8.2. The parton model approach

The expression for  $\mathcal{W}_{\mu\nu}$  can be rewritten as a nonlocal product of currents and it is a straightforward exercise to show by inserting the currents  $j_\mu(x) = : \bar{\psi}(x)\gamma_\mu\psi(x) :$  that for 1-particle inclusive scattering one obtains in tree approximation

$$\begin{aligned} 2M\mathcal{W}_{\mu\nu}(q; PS; P_h S_h) &= \frac{1}{(2\pi)^4} \int d^4x e^{iq \cdot x} \langle PS | : \bar{\psi}_j(x)(\gamma_\mu)_{jk} \psi_k(x) : \\ &\quad \times \sum_X |X; P_h S_h\rangle \langle X; P_h S_h | : \bar{\psi}_l(0)(\gamma_\nu)_{li} \psi_i(0) : |PS\rangle \\ &= \frac{1}{(2\pi)^4} \int d^4x e^{iq \cdot x} \langle PS | \bar{\psi}_j(x) \psi_i(0) |PS\rangle (\gamma_\mu)_{jk} \\ &\quad \times \langle 0 | \psi_k(x) \sum_X |X; P_h S_h\rangle \langle X; P_h S_h | \bar{\psi}_l(0) |0\rangle (\gamma_\nu)_{li} \\ &\quad + \frac{1}{(2\pi)^4} \int d^4x e^{iq \cdot x} \langle PS | \psi_k(x) \bar{\psi}_l(0) |PS\rangle (\gamma_\nu)_{li} \\ &\quad \times \langle 0 | \bar{\psi}_j(x) \sum_X |X; P_h S_h\rangle \langle X; P_h S_h | \psi_i(0) |0\rangle (\gamma_\mu)_{jk}, \end{aligned}$$

$$\begin{aligned}
 &= \int d^4 p d^4 k \delta^4(p + q - k) \text{Tr} (\Phi(p) \gamma_\mu \Delta(k) \gamma_\nu) \\
 &\quad + \left\{ \begin{array}{l} q \leftrightarrow -q \\ \mu \leftrightarrow \nu \end{array} \right\}, \tag{72}
 \end{aligned}$$

where

$$\begin{aligned}
 \Phi_{ij}(p) &= \frac{1}{(2\pi)^4} \int d^4 \xi e^{ip \cdot \xi} \langle PS | \bar{\psi}_j(0) \psi_i(\xi) | PS \rangle, \\
 \Delta_{kl}(k) &= \frac{1}{(2\pi)^4} \int d^4 \xi e^{ik \cdot \xi} \langle 0 | \psi_k(\xi) \sum_X |X; P_h S_h \rangle \langle X; P_h S_h | \bar{\psi}_l(0) | 0 \rangle.
 \end{aligned}$$

Note that in  $\Phi$  (quark production) a summation over colors is assumed, while in  $\Delta$  (quark decay) an averaging over colors is assumed. The quantities  $\Phi$  and  $\Delta$  correspond to the blobs in Fig. 5 and parametrize the soft physics, leading to the definitions of distribution and fragmentation functions [13,14]. Soft refers to all invariants of momenta being small as compared to the hard scale, i.e. for  $\Phi(p)$  one has  $p^2 \sim p \cdot P \sim P^2 = M^2 \ll Q^2$ .

In general many more diagrams have to be considered in evaluating the hadron tensors, but in the deep inelastic limit they can be neglected or considered as corrections to the soft blobs. We return to this later.

### 9. Collinear parton distributions

The form of  $\Phi$  is constrained by hermiticity, parity and time-reversal invariance. The quantity depends besides the quark momentum  $p$  on the target momentum  $P$  and the spin vector  $S$  and one must have

$$\text{[Hermiticity]} \quad \Rightarrow \quad \Phi^\dagger(p, P, S) = \gamma_0 \Phi(p, P, S) \gamma_0, \tag{73}$$

$$\text{[Parity]} \quad \Rightarrow \quad \Phi(p, P, S) = \gamma_0 \Phi(\bar{p}, \bar{P}, -\bar{S}) \gamma_0, \tag{74}$$

$$\text{[Time reversal]} \quad \Rightarrow \quad \Phi^*(p, P, S) = (-i\gamma_5 C) \Phi(\bar{p}, \bar{P}, \bar{S}) (-i\gamma_5 C), \tag{75}$$

where  $C = i\gamma^2\gamma_0$ ,  $-i\gamma_5 C = i\gamma^1\gamma^3$  and  $\bar{p} = (p^0, -\mathbf{p})$ .

To obtain the leading contribution in inclusive deep inelastic scattering one can integrate over the component  $p^-$  and the transverse momenta (see discussion in the section where the parton model has been derived). This integration restricts the nonlocality in  $\Phi(p)$ . The relevant soft part then is a particular Dirac trace of the quantity

$$\begin{aligned}
 \Phi_{ij}(x) &= \int dp^- d^2 p_T \Phi_{ij}(p, P, S) \\
 &= \int \frac{d\xi^-}{2\pi} e^{ip \cdot \xi} \langle P, S | \bar{\psi}_j(0) \psi_i(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_T = 0}, \tag{76}
 \end{aligned}$$

depending on the lightcone fraction  $x = p^+ / P^+$ . When one wants to calculate the leading order in  $1/Q$  for a hard process, one looks for leading parts in  $M/P^+$  because  $P^+ \propto Q$ . The leading contribution [15] turns out to be proportional to  $(M/P^+)^0$ ,

$$\Phi(x) = \frac{1}{2} \left\{ f_1(x) \not{n}_+ + S_L g_1(x) \gamma_5 \not{n}_+ + h_1(x) \frac{\gamma_5 [\not{\beta}_\perp, \not{n}_+]}{2} \right\}. \tag{77}$$

The precise expression of the functions  $f_1(x)$ , etc. as integrals over the amplitudes can be easily written down after tracing with the appropriate Dirac matrix,

$$f_1(x) = \int \frac{d\xi^-}{4\pi} e^{ip \cdot \xi} \langle P, S | \bar{\psi}(0) \gamma^+ \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_T = 0}, \quad (78)$$

$$S_L g_1(x) = \int \frac{d\xi^-}{4\pi} e^{ip \cdot \xi} \langle P, S | \bar{\psi}(0) \gamma^+ \gamma_5 \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_T = 0}, \quad (79)$$

$$S_T^i h_1(x) = \int \frac{d\xi^-}{4\pi} e^{ip \cdot \xi} \langle P, S | \bar{\psi}(0) i\sigma^{i+} \gamma_5 \psi(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_T = 0}, \quad (80)$$

including flavor indices, the functions  $f_1^q(x) = q(x)$  and  $g_1^q(x) = \Delta q(x)$  are precisely the functions that we encountered before.

The third function in the above parametrization is known as *transversity* or *transverse spin distribution* [16]. Including flavor indices one also denotes  $h_1^q(x) = \delta q(x)$ . In the same way as we have seen for  $f_1(x)$  and  $g_1(x)$ , the function  $h_1$  can be interpreted as a density, but one needs instead of the projectors on quark chirality states,  $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$ , those on quark transverse spin states,  $P_{\uparrow/\downarrow} = \frac{1}{2}(1 \pm \gamma^i \gamma_5)$ . One has

$$f_1(x) = f_{1R}(x) + f_{1L}(x) = f_{1\uparrow}(x) + f_{1\downarrow}(x), \quad (81)$$

$$g_1(x) = f_{1R}(x) - f_{1L}(x), \quad (82)$$

$$h_1(x) = f_{1\uparrow}(x) - f_{1\downarrow}(x). \quad (83)$$

This results in some trivial bounds such as  $f_1(x) \geq 0$  and  $|g_1(x)| \leq f_1(x)$ . We already did discuss the support and charge conjugation properties of  $f_1(x)$ . The analysis for all these functions shows that the support is in all cases  $-1 \leq x \leq 1$ , while the charge conjugation properties of the functions are  $\bar{f}(x) = -f(-x)$  (C-even) for  $f_1$  and  $h_1$  and  $\bar{f}(x) = +f(-x)$  (C-odd) for  $g_1$ .

The Dirac structure for  $h_1$  in terms of chirality states is  $\bar{\psi}_R \psi_L$  and  $\bar{\psi}_L \psi_R$ . Such functions are called *chiral-odd*. Since the QCD-interactions preserve chirality, this function cannot be measured in inclusive deep inelastic scattering.

While the evolution equations for  $q(x)$  and  $\Delta q(x)$  require quark–quark and quark–gluon splitting functions, the evolution for  $\delta q(x)$  does not involve mixing with gluon distributions because of the chiral-odd nature of  $\delta q(x)$ . The splitting function is given by

$$\delta P_{qq}(\xi) = \frac{4}{3} \frac{2\xi}{(1-\xi)_+} + 2\delta(1-\xi). \quad (84)$$

### 10. Bounds on the distribution functions

The trivial bounds on the distribution functions ( $|h_1(x)| \leq f_1(x)$  and  $|g_1(x)| \leq f_1(x)$ ) can be sharpened. For instance one can look explicitly at the structure in Dirac space of the correlation function  $\Phi_{ij}$ . Actually, we will look at the correlation functions  $(\Phi \gamma_0)_{ij}$ , which involve at leading order matrix elements  $\psi_{+j}^\dagger(0) \psi_{+i}(\xi)$ . One has in Weyl representation

( $\gamma^0 = \rho^1, \gamma^i = -i\rho^2\sigma^i, \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \rho^3$ ) the matrices

$$P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_+\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad P_+\gamma^1\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The good projector only leaves two (independent) Dirac spinors, one righthanded ( $R$ ), one lefthanded ( $L$ ). On this basis of good  $R$  and  $L$  spinors the for hard scattering processes relevant matrix ( $\Phi \not{n}_-$ ) is given by

$$(\Phi \not{n}_-)_{ij}(x) = \begin{pmatrix} f_1 + S_L g_1 & (S_T^1 - i S_T^2) h_1 \\ (S_T^1 + i S_T^2) h_1 & f_1 - S_L g_1 \end{pmatrix}. \tag{85}$$

One can also turn the  $S$ -dependent correlation function  $\Phi$  into a matrix in the nucleon spin space via the standard spin 1/2 density matrix  $\rho(P, S)$ . The relation is  $\Phi(x; P, S) = \text{Tr} [\Phi(x; P) \rho(P, S)]$ . Writing

$$\Phi(x; P, S) = \Phi_O + S_L \Phi_L + S_T^1 \Phi_T^1 + S_T^2 \Phi_T^2, \tag{86}$$

one has on the basis of spin 1/2 target states with  $S_L = +1$  and  $S_L = -1$  respectively

$$\Phi_{ss'}(x) = \begin{pmatrix} \Phi_O + \Phi_L & \Phi_T^1 - i \Phi_T^2 \\ \Phi_T^1 + i \Phi_T^2 & \Phi_O - \Phi_L \end{pmatrix}. \tag{87}$$

Generalizing  $\Phi(p)$  to matrix elements between states  $\langle P, s |$  and  $|P, s' \rangle$  one has for the matrix  $M = (\Phi \not{n}_-)^T$  (transposed in Dirac space)  $v^\dagger M v \geq 0$  for any direction  $v$  in Dirac space. On the basis  $+R, -R, +L$  and  $-L$  the matrix in quark  $\otimes$  nucleon spin-space becomes

$$(\Phi(x) \not{n}_-)^T = \begin{pmatrix} f_1 + g_1 & 0 & 0 & 2 h_1 \\ 0 & f_1 - g_1 & 0 & 0 \\ 0 & 0 & f_1 - g_1 & 0 \\ 2 h_1 & 0 & 0 & f_1 + g_1 \end{pmatrix} \begin{matrix} \textcircled{R} \rightarrow \\ \leftarrow \textcircled{R} \\ \textcircled{L} \rightarrow \\ \leftarrow \textcircled{L} \end{matrix} \tag{88}$$

$\textcircled{R} \rightarrow \quad \leftarrow \textcircled{R} \quad \textcircled{L} \rightarrow \quad \leftarrow \textcircled{L}$

Of this matrix any diagonal matrix element must always be positive, hence the eigenvalues must be positive, which gives a bound on the distribution functions stronger than the trivial bounds, namely

$$|h_1(x)| \leq \frac{1}{2}(f_1(x) + g_1(x)) \tag{89}$$

known as the Soffer bound [17].

### 11. Transverse momentum dependent correlation functions

Without integration over  $p_T$ , the soft part is

$$\Phi(x, \mathbf{p}_T) = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3} e^{iP \cdot \xi} \langle P, S | \bar{\psi}(0) \psi(\xi) | P, S \rangle \Big|_{\xi^+ = 0}. \tag{90}$$

For the leading order results one can write down parametrizations which for the parts involving unpolarized targets ( $O$ ), longitudinally polarized targets ( $L$ ) and transversely polarized targets ( $T$ ) up to parts proportional to  $M/P^+$  take the form [18,19]

$$\Phi_O(x, \mathbf{p}_T) = \frac{1}{2} \left\{ f_1(x, \mathbf{p}_T) \not{n}_+ + h_1^\perp(x, \mathbf{p}_T) \frac{i[\not{p}_T, \not{n}_+]}{2M} \right\} \tag{91}$$

$$\Phi_L(x, \mathbf{p}_T) = \frac{1}{2} \left\{ S_L g_{1L}(x, \mathbf{p}_T) \gamma_5 \not{n}_+ + S_L h_{1L}^\perp(x, \mathbf{p}_T) \frac{\gamma_5 [\not{p}_T, \not{n}_+]}{2M} \right\} \tag{92}$$

$$\begin{aligned} \Phi_T(x, \mathbf{p}_T) = \frac{1}{2} \left\{ f_{1T}^\perp(x, \mathbf{p}_T) \frac{\epsilon_{\mu\nu\rho\sigma} \gamma^\mu n_+^\nu p_T^\rho S_T^\sigma}{M} + \frac{\mathbf{p}_T \cdot \mathbf{S}_T}{M} g_{1T}(x, \mathbf{p}_T) \gamma_5 \not{n}_+ \right. \\ \left. + h_{1T}(x, \mathbf{p}_T) \frac{\gamma_5 [\not{S}_T, \not{n}_+]}{2} \right. \\ \left. + \frac{\mathbf{p}_T \cdot \mathbf{S}_T}{M} h_{1T}^\perp(x, \mathbf{p}_T) \frac{\gamma_5 [\not{p}_T, \not{n}_+]}{2M} \right\}. \end{aligned} \tag{93}$$

All functions appearing here have a natural interpretation as densities. This is seen as discussed before for the  $\mathbf{p}_T$ -integrated functions. Now it includes densities such as the density of longitudinally polarized quarks in a transversely polarized nucleon ( $g_{1T}$ ) and the density of transversely polarized quarks in a longitudinally polarized nucleon ( $h_{1L}^\perp$ ).

Upon integration over  $p_T$  not all functions survive. We are then left with  $f_1 g_1(x)$  obtained after integration of  $g_{1L}(x, p_T)$  and  $h_1(x)$  with contributions from  $h_{1T}(x, p_T)$  and  $h_{1T}^\perp(x, p_T)$ .

Evolution is directly related to the asymptotic behavior of the functions. The assumption that the functions vanish sufficiently fast as a function of the invariants  $p \cdot P$  and  $p^2$ , which at constant  $x$  implies a sufficiently fast vanishing as a function of  $\mathbf{p}_T^2$ , simply turns out not to be true. Assuming that the result for  $\mathbf{p}_T^2 \geq \mu^2$  is given by the diagram shown in Fig. 4 one finds that the extra distribution written in terms of  $p_T$  becomes

$$f_1(x, \mathbf{p}_T^2) \xrightarrow{\mathbf{p}_T^2 \geq \mu^2} \frac{1}{\pi \mathbf{p}_T^2} \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dy}{y} P_{qq} \left( \frac{x}{y} \right) f_1(y; \mu^2), \tag{94}$$

which gives  $f_1(x; \mu^2) \equiv \pi \int_0^{\mu^2} d\mathbf{p}_T^2 f_1(x, \mathbf{p}_T^2)$  a logarithmic scale dependence.

Actually we find that different functions survive when one integrates over  $\mathbf{p}_T$  weighting with  $p_T^\alpha$ , e.g.

$$\begin{aligned} \Phi_3^\alpha(x) \equiv \int d^2 p_T \frac{p_T^\alpha}{M} \Phi(x, \mathbf{p}_T) = \frac{1}{2} \left\{ -g_{1T}^{(1)}(x) S_T^\alpha \not{n}_+ \gamma_5 - S_L h_{1L}^{\perp(1)}(x) \right. \\ \left. \times \frac{[\gamma^\alpha, \not{n}_+] \gamma_5}{2} - f_{1T}^{\perp(1)} \epsilon_{\mu\nu\rho}^\alpha \gamma^\mu n_-^\nu S_T^\rho - h_1^{\perp(1)} \frac{i[\gamma^\alpha, \not{n}_+]}{2} \right\}, \end{aligned} \tag{95}$$

involving *transverse moments* defined as  $g_{1T}^{(1)}(x) = \int d^2 p_T (\mathbf{p}_T^2/2M^2) g_{1T}(x, \mathbf{p}_T)$ , and similarly for the other functions. The functions  $h_1^\perp$  and  $f_{1T}^\perp$  are *T-odd*. As we will explain in the section on color gauge invariance they do not vanish because time reversal invariance cannot be used for the transverse moments. Also for fragmentation functions they will not vanish. The *T-odd* functions correspond to unpolarized quarks in a transversely polarized nucleon ( $f_{1T}^\perp$ ) or transversely polarized quarks in an unpolarized hadron ( $h_1^\perp$ ). The easiest way to interpret the functions is by considering their place in the quark production matrix  $(\Phi(x, p_T) \not{n}_-)^T$ , which becomes [20]

$$\begin{array}{cccc}
 \textcircled{R} \rightarrow & \leftarrow \textcircled{R} & \textcircled{L} \rightarrow & \leftarrow \textcircled{L} \\
 \left( \begin{array}{cccc}
 f_1 + g_{1L} & \frac{|p_T|}{M} e^{i\phi} g_{1T} & \frac{|p_T|}{M} e^{-i\phi} h_{1L}^\perp & 2h_1 \\
 \frac{|p_T|}{M} e^{-i\phi} g_{1T} & f_1 - g_{1L} & \frac{|p_T|^2}{M^2} e^{-2i\phi} h_{1T}^\perp & -\frac{|p_T|}{M} e^{-i\phi} h_{1L}^\perp \\
 \frac{|p_T|}{M} e^{i\phi} h_{1L}^\perp & \frac{|p_T|^2}{M^2} e^{2i\phi} h_{1T}^\perp & f_1 - g_{1L} & -\frac{|p_T|}{M} e^{i\phi} g_{1T} \\
 2h_1 & -\frac{|p_T|}{M} e^{i\phi} h_{1L}^\perp & -\frac{|p_T|}{M} e^{-i\phi} g_{1T} & f_1 + g_{1L}
 \end{array} \right) .
 \end{array}$$

In this representation *T-odd* functions appear as imaginary parts,  $f_{1T}^\perp = \text{Im } g_{1T}$  and  $h_1^\perp = \text{Im } h_{1L}^\perp$ .

### 12. Fragmentation functions

Just as for the distribution functions one can perform an analysis of the soft part describing the quark fragmentation [14]. One needs

$$\Delta_{ij}(z, \mathbf{k}_T) = \sum_X \int \frac{d\xi^+ d^2 \xi_T}{(2\pi)^3} e^{ik \cdot \xi} \text{Tr} \langle 0 | \psi_i(\xi) | P_h, X \rangle \langle P_h, X | \bar{\psi}_j(0) | 0 \rangle \Big|_{\xi^- = 0} . \tag{96}$$

For the production of unpolarized (or spin 0) hadrons  $h$  in hard processes one needs to leading order in  $1/Q$  the  $(M_h/P_h^-)^0$  part of the correlation function,

$$\Delta_O(z, \mathbf{k}_T) = z D_1(z, \mathbf{k}'_T) \not{n}_- + z H_1^\perp(z, \mathbf{k}'_T) \frac{i[\not{k}_T, \not{n}_-]}{2M_h} . \tag{97}$$

The arguments of the fragmentation functions  $D_1$  and  $H_1^\perp$  are  $z = P_h^-/k^-$  and  $\mathbf{k}'_T = -z\mathbf{k}_T$ . The first is the (lightcone) momentum fraction of the produced hadron, the second is the transverse momentum of the produced hadron with respect to the quark. The fragmentation function  $D_1$  is the equivalent of the distribution function  $f_1$ . It can be interpreted as a quark decay function, giving the probability of finding a hadron  $h$  in a quark. The quantity  $n_h = \int dz D_1(z)$  is the number of hadrons. The normalization of the fragmentation functions is given by  $\sum_h \int dz z D_1^{q \rightarrow h}(z) = 1$ .

The function  $H_1^\perp$ , interpretable as the difference between the numbers of unpolarized hadrons produced from a transversely polarized quark depending on the hadron's

transverse momentum, is allowed because of the non-applicability of time reversal invariance [21]. This is natural for the fragmentation functions [22,23] because of the appearance of out-states  $|P_h, X\rangle$  in the definition of  $\Delta$ , in contrast to the plane wave states appearing in  $\Phi$ . The function  $H_1^\perp$  is of interest because it is chiral-odd. This means that it can be used to probe the chiral-odd quark distribution function  $h_1$ , which can be achieved e.g. by measuring a particular azimuthal asymmetry of produced pions in the current fragmentation region.

The spin structure of fragmentation functions is also conveniently summarized by explicitly giving it on a  $R$  and  $L$  chiral quark basis, for which we find for decay into spin zero hadrons,

$$(\Delta(z, k_T) \hat{n}_+)^T = \begin{pmatrix} D_1 & i \frac{|k_T| e^{-i\phi}}{M_h} H_1^\perp \\ -i \frac{|k_T| e^{+i\phi}}{M_h} H_1^\perp & D_1 \end{pmatrix} \begin{matrix} \textcircled{R} \\ \textcircled{L} \end{matrix} \quad (98)$$

### 12.1. Examples of azimuthal asymmetries

Transverse momentum dependence shows up in the azimuthal dependence in the SIDIS cross section (via  $\hat{h}$  or transverse spin vectors), in most cases requiring polarization of beam and/or target or requiring polarimetry [24,25]. Examples of leading azimuthal asymmetries, appearing for polarized leptonproduction are

$$\left\langle \frac{Q_T}{M} \sin(\phi_h^\ell - \phi_S^\ell) \right\rangle_{OT} = \frac{2\pi\alpha^2 s}{Q^4} |S_T| \left( 1 - y + \frac{1}{2} y^2 \right) \times \sum_{a,\bar{a}} e_a^2 x_B f_{1T}^{\perp(1)a}(x_B) D_1^a(z_h). \quad (99)$$

$$\left\langle \frac{Q_T}{M_h} \sin(\phi_h^\ell + \phi_S^\ell) \right\rangle_{OT} = \frac{4\pi\alpha^2 s}{Q^4} |S_T| (1 - y) \sum_{a,\bar{a}} e_a^2 x_B h_1^a(x_B) H_1^{\perp(1)a}(z_h). \quad (100)$$

The notation  $\langle W \rangle$  is the  $q_T$ -integrated cross section with weight  $W$ . The factor  $Q_T$  is included, because it together with the direction  $\hat{h}$  combines to  $q_T$ , allowing a defolding of the cross section in distribution and fragmentation parts (one of them weighted with transverse momentum). Note that both of these asymmetries involve  $T$ -odd functions, which can only appear in single spin asymmetries. The latter can easily be checked from the conditions on the hadronic tensor, which are the same as those in Eq. (9)–(11). They require an odd number of spin vectors entering in the symmetric part and an even number of spins entering in the antisymmetric part of the hadron tensor. An extended review of transverse momentum dependent functions and transversity can be found in Ref. [26].

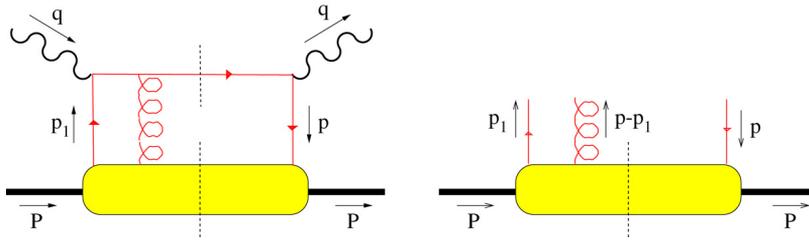


Fig. 6. Examples of gluonic diagrams that must be included at subleading order in lepton hadron inclusive scattering (left) and the soft part entering this process (right).

### 13. Inclusion of subleading contributions

If one proceeds up to order  $1/Q$  one also needs terms in the parametrization of the soft part proportional to  $M/P^+$ . Limiting ourselves to the  $\mathbf{p}_T$ -integrated correlations one needs [15]

$$\begin{aligned} \Phi(x) = & \frac{1}{2} \left\{ f_1(x) \not{n}_+ + S_L g_1(x) \gamma_5 \not{n}_+ + h_1(x) \frac{\gamma_5 [\not{\beta}_T, \not{n}_+]}{2} \right\} \\ & + \frac{M}{2P^+} \left\{ e(x) + g_T(x) \gamma_5 \not{\beta}_T + S_L h_L(x) \frac{\gamma_5 [\not{n}_+, \not{n}_-]}{2} \right\}. \end{aligned} \tag{101}$$

The function  $g_T$ , for instance, shows up in inclusive scattering off a transversely polarized nucleon ( $|\mathbf{S}_\perp| = 1$ ). One needs to include subleading ( $1/Q$ ) in the handbag diagram, as well as gluonic diagrams as shown in Fig. 6. For these gluon diagrams one needs bilocal matrix elements containing  $1/Q$ ; one only needs the matrix element of the bilocal combinations  $\overline{\psi}(0) g A_T^\alpha(\xi) \psi(\xi)$  and  $\overline{\psi}(0) g A_T^\alpha(0) \psi(\xi)$ . The sum of the contributions produces matrix elements  $\Phi_D^\alpha(x)$  involving matrix elements of bilocal combinations  $\overline{\psi}(0) iD_T^\alpha \psi(\xi)$  for which one can use the QCD equations of motion to relate them to the functions appearing in  $\Phi$ ,

$$\begin{aligned} \Phi_D^\alpha(x) = & \frac{M}{2} \left\{ - \left( x g_T - \frac{m}{M} h_1 \right) S_T^\alpha \not{n}_+ \gamma_5 \right. \\ & \left. - S_L \left( x h_L - \frac{m}{M} g_1 \right) \frac{[\gamma^\alpha, \not{n}_+]}{2} \gamma_5 \right\}. \end{aligned} \tag{102}$$

The distribution function  $g_T$  e.g. shows up in the corresponding structure function of polarized inclusive deep inelastic scattering

$$2M W_A^{\mu\nu}(q, P, S_T) = i \frac{2Mx_B}{Q} \hat{t}^{\mu} \epsilon_\perp^{\nu\rho} S_{\perp\rho} g_T(x_B), \tag{103}$$

leading for the structure function  $g_T(x_B, Q^2)$  defined in Eq. (25) to the result

$$g_T(x_B, Q^2) = \frac{1}{2} \sum_q e_q^2 \left( g_T^q(x_B) + g_T^{\bar{q}}(x_B) \right). \tag{104}$$

In the process of integrating the correlation functions over  $p^-$ ,  $p_T$  and finally over  $p^+$ , consecutively restraining the nonlocality to lightfront-separated fields, lightcone-separated fields and local fields, interesting relations can be derived. For instance, the correlators  $\int dx \Phi^{[\gamma^\mu \gamma_5]}(x)$  must yield  $g_A S_T^\mu$  (for any  $\mu$ ), which means that the functions in the nonlocal correlators  $\Phi^{[\gamma^+ \gamma_5]}(x)$  and  $\Phi^{[\gamma^\alpha \gamma_5]}(x)$  ( $\alpha$  transverse) yield the same result after integration over  $x$ ,  $\int dx g_1(x) = \int dx g_T(x)$  or  $\int dx g_2(x) = 0$ , known as the Burkhardt–Cottingham sumrule [27]. For quark–quark correlators, similar considerations yield relations between the subleading functions and the transverse momentum dependent leading functions, referred to as *Lorentz invariance relations*, such as [28,24]

$$g_T = g_1 + \frac{d}{dx} g_{1T}^{(1)}. \tag{105}$$

An interesting result is obtained by combining this relation with an often used approximation, in which the interaction-dependent part  $\Phi_A^\alpha$  is set to zero. In that case the difference  $\Phi_D^\alpha - \Phi_\partial^\alpha$  vanishes. Using Eqs. (95) and (102) this gives

$$x \tilde{g}_T = x g_T - g_{1T}^{(1)} - \frac{m}{M} h_1 = 0. \tag{106}$$

One then can eliminate  $g_{1T}^{(1)}$  and obtain a relation between  $g_T$ ,  $g_1$  and  $\tilde{g}_T$  (assuming sufficient neat behavior of the functions). For  $g_2 = g_T - g_1$  this relation takes the form

$$g_2(x) = - \left[ g_1(x) - \int_x^1 dy \frac{g_1(y)}{y} \right] + \frac{m}{M} \left[ \frac{h_1(x)}{x} - \int_x^1 dy \frac{h_1(y)}{y^2} \right], \tag{107}$$

the Wandzura–Wilczek approximation [29] for  $g_2$ , which in particular when one neglects the quark mass term provides a simple and often used estimate for  $g_2$ . It has become the standard with which experimentalists compare the results for  $g_2$ .

### 14. Color gauge invariance

We have so far disregarded two issues. The first issue is that the correlation function  $\Phi$  discussed in previous sections involves two quark fields at different space–time points and hence is not color gauge invariant. The second issue is the gluonic diagrams similar to the ones we have discussed in the previous section (see Fig. 6), among which also correlation functions appear involving matrix elements with longitudinal ( $A^+$ ) gluon fields,

$$\overline{\psi}_j(0) g A^+(\eta) \psi_i(\xi).$$

These do not lead to any suppression. The reason is that because of the  $+$ -index in the gluon field the matrix element is proportional to  $P^+$ ,  $p^+$  or  $M S^+$  rather than the proportionality to  $M S_T^\alpha$  or  $p_T^\alpha$  that one gets for a gluonic matrix element with transverse gluons.

A straightforward calculation, however, shows that the gluonic diagrams with one or more longitudinal gluons involve matrix elements (soft parts) of operators  $\overline{\psi} \psi$ ,  $\overline{\psi} A^+ \psi$ ,  $\overline{\psi} A^+ A^+ \psi$ , etc. that can be resummed into a correlation function

$$\Phi_{ij}(x) = \int \frac{d\xi^-}{2\pi} e^{i p \cdot \xi} \langle P, S | \overline{\psi}_j(0) \mathcal{U}(0, \xi) \psi_i(\xi) | P, S \rangle \Big|_{\xi^+ = \xi_T = 0}, \tag{108}$$

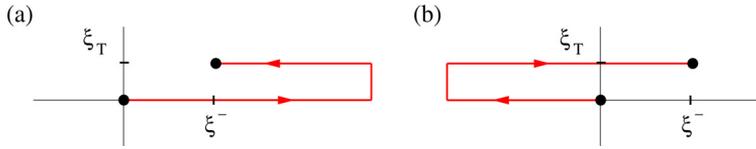


Fig. 7. The gauge link structure in the quark–quark correlator  $\Phi$  in SIDIS (a) and DY (b) respectively

where  $\mathcal{U}$  is a gauge link operator

$$\mathcal{U}(0, \xi) = \mathcal{P} \exp \left( -i \int_0^{\xi^-} d\zeta^- A^+(\zeta) \right) \quad (109)$$

(path-ordered exponential with path along  $--$ -direction). Et voila, the unsuppressed gluonic diagrams combine into a color gauge invariant correlation function [30]. We note that at the level of operators, one expands

$$\bar{\psi}(0) \psi(\xi) = \sum_n \frac{\xi^{\mu_1} \dots \xi^{\mu_n}}{n!} \bar{\psi}(0) \partial_{\mu_1} \dots \partial_{\mu_n} \psi(0), \quad (110)$$

in a set of local operators, but only the expansion of the nonlocal combination with a gauge link

$$\bar{\psi}(0) \mathcal{U}(0, \xi) \psi(\xi) = \sum_n \frac{\xi^{\mu_1} \dots \xi^{\mu_n}}{n!} \bar{\psi}(0) D_{\mu_1} \dots D_{\mu_n} \psi(0), \quad (111)$$

is an expansion in terms of local gauge invariant operators.

For the  $p_T$ -dependent functions, one finds that inclusion of  $A^+$  gluonic diagrams leads to a color gauge invariant matrix element with links running via  $\xi^\pm = \pm\infty$  [31,32]. For instance in lepton–hadron scattering one finds

$$\Phi(x, \mathbf{p}_T) = \int \frac{d\xi^- d^2 \xi_T}{(2\pi)^3} e^{i p \cdot \xi} \langle P, S | \bar{\psi}(0) \mathcal{U}^{[+]}(0, \xi) \psi(\xi) | P, S \rangle \Big|_{\xi^+ = 0}, \quad (112)$$

where the link  $\mathcal{U}^{[+]}$  is shown in Fig. 7(a). We note that the gauge link involves transverse gluons [33,34], showing that in processes involving more hadrons the effects of transverse gluons are not necessarily suppressed, as has also been shown in explicit model calculations [35].

Moreover, depending on the process the gauge link can also run via minus infinity, involving the link in Fig. 7(b). This is for instance the case in Drell–Yan processes. The transverse momentum dependent distribution functions also are no longer constrained by time-reversal, as the time reversal operation interchanges the  $U^{[+]}$  and  $U^{[-]}$  links, leading to the appearance of  $T$ -odd functions in Eq. (95), such as the Sivers function  $f_{1T}^{(1)}$  [36]. The process dependence of the gauge link, however, points to particular sign changes when single spin azimuthal asymmetries in semi-inclusive leptoproduction are compared to those in for instance Drell–Yan scattering. For such effects the measurement of transverse momentum dependence is a must, since the specific link structure [37] does not matter in

$p_T$ -integrated functions, in which both links in Fig. 7 reduce to the same straight-line link connecting 0 and  $\xi$ .

## References

- [1] R.G. Roberts, The structure of the proton, Cambridge University Press, 1990.
- [2] R. Meng, F.I. Olness, D.E. Soper, Nucl. Phys. B 371 (1992) 79.
- [3] R.K. Ellis, W. Furmanski, R. Petronzio, Nucl. Phys. B 212 (1983) 29; Nucl. Phys. B 207 (1982) 1.
- [4] A.V. Efremov, O.V. Teryaev, Sov. J. Nucl. Phys. 39 (1984) 962.
- [5] R.L. Jaffe, in: M.B. Johnson, A. Picklesheimer (Eds.), Relativistic Dynamics and Quark Nuclear Physics, Wiley, New York, 1986.
- [6] Durham-RAL databases, <http://durpdg.dur.ac.uk/HEPDATA/>;  
Particle Data Group, <http://pdg.web.cern.ch/pdg/>.
- [7] J.B. Kogut, D.E. Soper, Phys. Rev. D 1 (1970) 2901.
- [8] R.L. Jaffe, Nucl. Phys. B 229 (1983) 205.
- [9] M. Diehl, T. Gousset, Phys. Lett. B 428 (1998) 359.
- [10] The first surprising results were reported by the EMC Collaboration, J. Ashman et al., Phys. Lett. B 206 (1988) 364; Nucl. Phys. B 328 (1989) 1.
- [11] J.D. Bjorken, Phys. Rev. 179 (1969) 1547.
- [12] J. Kodaira et al., Phys. Rev. D 20 (1979) 627;  
S.A. Larin, F.V. Tkachev, J.A.M. Vermaseren, Phys. Lett. B 259 (1991) 345;  
S.A. Larin, J.A.M. Vermaseren, Phys. Rev. Lett. 66 (1991) 862.
- [13] D.E. Soper, Phys. Rev. D 15 (1977) 1141; Phys. Rev. Lett. 43 (1979) 1847.
- [14] J.C. Collins, D.E. Soper, Nucl. Phys. B 194 (1982) 445.
- [15] R.L. Jaffe, X. Ji, Nucl. Phys. B 375 (1992) 527.
- [16] X. Artru, M. Mekhfi, Z. Phys. 45 (1990) 669;  
J.L. Cortes, B. Pire, J.P. Ralston, Z. phys. C 55 (1992) 409.
- [17] J. Soffer, Phys. Rev. Lett. 74 (1995) 1292.
- [18] J.P. Ralston, D.E. Soper, Nucl. Phys. B 152 (1979) 109.
- [19] R.D. Tangerman, P.J. Mulders, Phys. Rev. D 51 (1995) 3357.
- [20] A. Bacchetta, M. Boglione, A. Henneman, P.J. Mulders, Phys. Rev. Lett. 85 (2000) 712.
- [21] J. Collins, Nucl. Phys. B 396 (1993) 161.
- [22] K. Hagiwara, K. Hikasa, N. Kai, Phys. Rev. D 27 (1983) 84.
- [23] R.L. Jaffe, X. Ji, Phys. Rev. Lett. 71 (1993) 2547; Phys. Rev. D 57 (1998) 3057.
- [24] P.J. Mulders, R.D. Tangerman, Nucl. Phys. B 461 (1996) 197; Nucl. Phys. B 484 (1997) 538.
- [25] D. Boer, P.J. Mulders, Phys. Rev. D 57 (1998) 5780.
- [26] V. Barone, A. Drago, P.G. Ratcliffe, Phys. Rep. 359 (2002) 1. Transverse polarisation of quarks in hadrons.
- [27] H. Burkhardt, W.N. Cottingham, Ann. Phys. (N.Y.) 56 (1976) 453.
- [28] A.P. Bukhvostov, E.A. Kuraev, L.N. Lipatov, Sov. Phys. JETP 60 (1984) 22.
- [29] S. Wandzura, F. Wilczek, Phys. Rev. D 16 (1977) 707.
- [30] A.V. Efremov, A.V. Radyushkin, Theoret. and Math. Phys. 44 (1981) 774.
- [31] D. Boer, P.J. Mulders, Nucl. Phys. B 569 (2000) 505.
- [32] J.C. Collins, Phys. Lett. B 536 (2002) 43.
- [33] A.V. Belitsky, X. Ji, F. Yuan, Nucl. Phys. B 656 (2003) 165.
- [34] D. Boer, P.J. Mulders, F. Pijlman, Nucl. Phys. B 667 (2003) 201.
- [35] S.J. Brodsky, D.S. Hwang, I. Schmidt, Phys. Lett. B 530 (2002) 99; Nucl. Phys. B 642 (2002) 344.
- [36] D. Sivers, Phys. Rev. D 41 (1990) 83; Phys. Rev. D 43 (1991) 261.
- [37] C. Bomhof, P.J. Mulders, F. Pijlman, Phys. Lett. B 596 (2004) 277.