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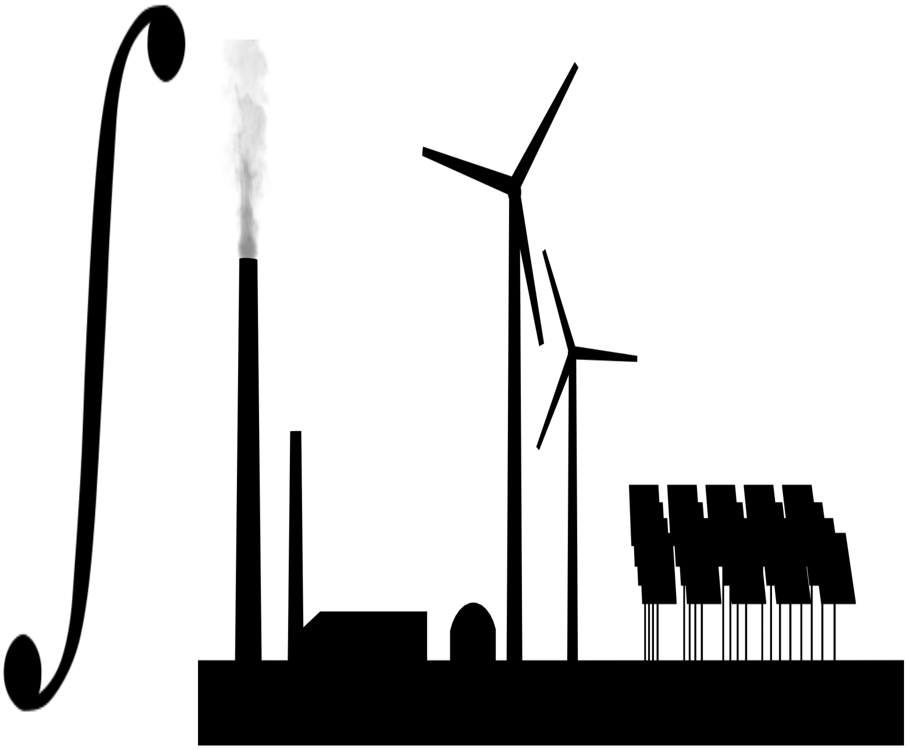
21 December 2017

Physics Education Project Report

For my project, I developed an introductory calculus handbook for the Energy, Climate and Sustainability 100 (ECS100) course at Amsterdam University College (AUC). The course is offered to first year bachelor students with no mathematics, physics or chemistry prerequisites. Therefore, some of the students have never been exposed to introductory calculus. However, the subjects covered in the course are based on physics and chemistry concepts that inevitably rely on basic understanding of integration and differentiation. From discussions with my project advisor, Judith van Santen and the course professor, Dr. Forrest Bradbury, we decided that developing an introductory calculus handbook tailored to the ECS100 course would be most beneficial for future students. The goal of this project was to generate a handbook that will provide students with the mathematical tools required for understanding the physics and chemistry concepts they encounter in the course.

From my course observations and brief conversations with students challenged by the calculus, I determined the most relevant mathematical tools to include in the introductory calculus handbook. Furthermore, the handbook will provide future generations of students with general integration and differentiation exercises (and solutions) in addition to exercises tailored specifically to the course. For example, students in the course need to perform integration calculations for thermodynamical processes such as the Carnot heat engine. Some students had trouble following the Carnot heat engine lectures because they struggled with integration, as it was the first or second time they had seen such calculations. Equipped with this handbook, students will have the opportunity to learn the essential calculus required for the course and will solve (in advance) the integration and differentiation exercises they will encounter in the lectures and laboratory assignments. Overall, this handbook will make the students' learning more efficient.

Introduction to Calculus For Energy Science



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1 FUNCTIONS

1.1 What is a Function?

A function f assigns to each variable x exactly one value called *f(x)*. You may think of a function as a machine which takes in a number of inputs defined by x and returns exactly one value for each input. You may be familiar with linear functions of the form $y = mx + b$, where m and b are constants and y is the same as *f(x)*. In this course we will encounter four types of functions, namely *polynomial*, *power*, *exponential*, and *logarithmic* functions. It will be helpful to become familiar with these functions and their properties.

1.2 Polynomial Functions

A *polynomial* is any function of the form

$$P(x) = a_c x^c + a_{c-1} x^{c-1} + ... + a_2 x^2 + a_1 x + a_0 \tag{1}$$

where c is a non-negative integer and the numbers $a_0, a_1, a_2, ...a_c$ are called the coefficients of the polynomial. Figure (1) shows several example graphs.

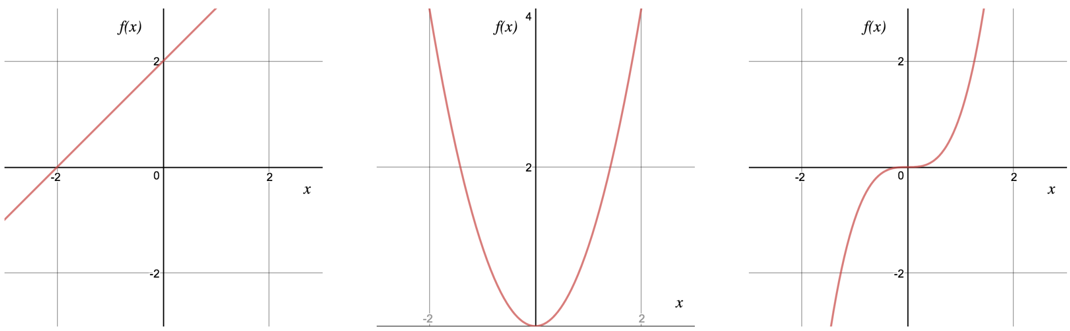

Figure 1: Left to right: $f(x) = x + 2$, $f(x) = x^2$, and $f(x) = x^3$.

1.3 Power Functions

A *power function* is any function of the form

$$f(x) = x^a \tag{2}$$

where a is a constant, not necessarily an integer and can be positive or negative. This type of function is known as the power function because the variable x is raised to the power of a. Figure (2) shows several example graphs.

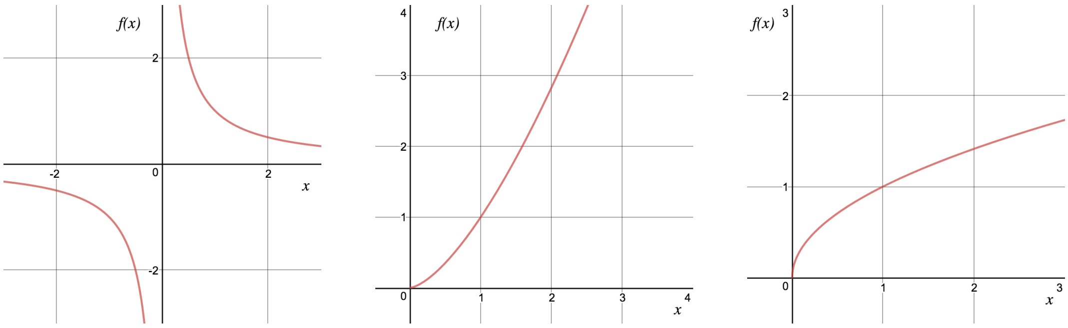

Figure 2: Left to right: $f(x) = x^{-1} = 1/x$, $f(x) = x^{3/2}$, and $f(x) = x^{1/2}$.

1.4 Exponential Functions

An *exponential function* is any function of the form

$$f(x) = a^x \tag{3}$$

where a, known as the *base*, is a positive constant. This type of function is known as the exponential function because the variable x serves as the exponent. In this course, the base for all of our exponential functions will be the natural number $e = 2.718...$, which will simplify our differentiation and integration computations in the following sections. Figure (3) shows two important example graphs.

When working with exponential functions, it is useful to know the laws of exponents. If a and b are positive numbers and x and y are any real numbers, the rules for treating exponents are

1. $a^{x+y} = a^x a^y$

2. $a^{x-y} = a^x a^{-y} = \frac{a^x}{a^y}$

3. $(a^x)^y = a^{xy}$

4. $(ab)^x = a^x b^x$

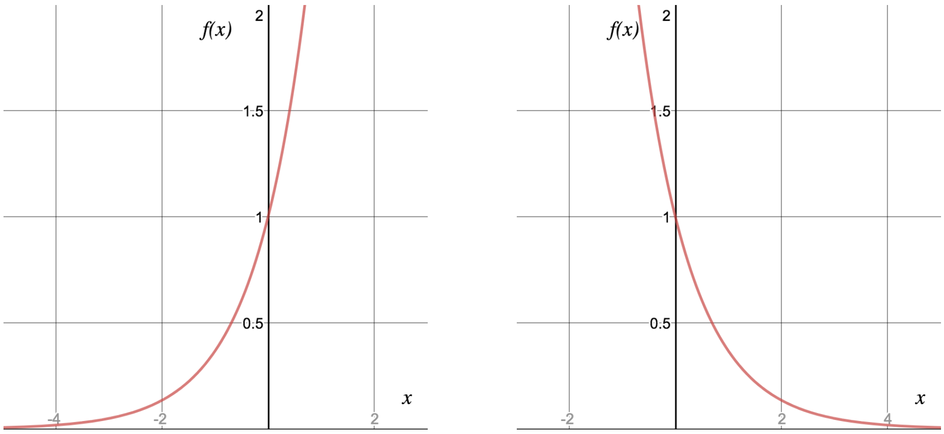

Figure 3: Left to right: $f(x) = e^x$, $f(x) = e^{-x}$.

1.5 Logarithmic Functions

A $logarithmic function$ is any function of the form

$$y = \log_a x \tag{4}$$

where a is a positive constant and is known as the *base* of the logarithmic function. The logarithmic function is related to the power function through

$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x \tag{5}$$

where you can think of the *log* going away when moving a over to the other side of the equation and raising it to the power of y. When working with logarithms, it is useful to know the laws of logarithms. If a, x and y are any positive numbers, and r is any real number, then

1. $\log_a(a^x) = x$ **2.** $a^{\log_a x} = x$

3. $\log_a(xy) = \log_a x + \log_a y$ **4.** $\log_a(x/y) = \log_a x - \log_a y$

5. $\log_a(x^r) = r \log_a x$ **6.** $\log_a(1) = 0$; for any a

It is worth mentioning that any number raised to the power of zero is 1. This explains the logarithm law **6**, which can be explicitly confirmed using law **1**:

$$\log_a(a^0) = \log_a(1) = 0 \tag{6}$$

A logarithmic function with base $e = 2.718...$ is known as the *natural logarithm* and has the special notation

$$log_e x = \ln(x) \tag{7}$$

The graph of this function is shown in figure (4). The natural logarithmic function is related to the power function through

$$\ln(x) = y \qquad \Leftrightarrow \qquad e^y = x \tag{8}$$

The laws for natural logarithms follow directly from the laws of general logarithms by specifying the base to be e.

1. $\ln(e^x) = x$ **2.** $e^{\ln(x)} = x$

3. $\ln(xy) = ln(x) + \ln(y)$ **4.** $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$

5. $\ln(x^r) = r\ln(x)$ **6.** $\ln(1) = 0$

No need to define this (again). However, it does not really matter....

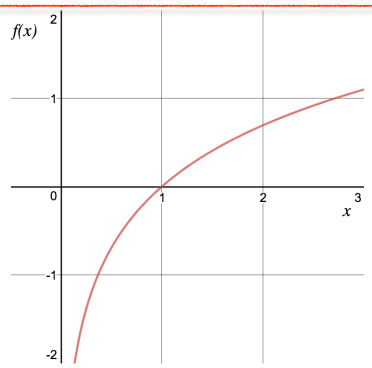

Figure 4: $f(x) = \ln(x)$.

1.6 Combination of Functions

The functions above may be combined to form ***composite functions***. A composite function is composed of 2 separate function. For example, the functions $f(a) = \sqrt{a}$ and $g(x) = x+1$ can be used to form a composite function $h(x) = f(g(x))$. This composite function is formed by substituting $g(x)$ for a in the function $f(a)$.

$$h(x) = f(g(x)) = f(x+1) = \sqrt{x+1} \tag{9}$$

EXERCISES

1. Simplify the following expressions:

a. $x^2 \cdot x^3$ **b.** $\frac{y^4}{y^8}$ **c.** $(a^3)^2$ **d.** $\left(\frac{g^3}{g}\right) \cdot g^4$

2. Simplify the following expressions:

a. $\log_5(x) + \log_5(y)$ **b.** $\log_{17}(w) - \log_{17}(z)$ **c.** $3^{\log_3(r)}$ **d.** $\log_6(6^t)$

3. Simplify the following expressions:

a. $\log_{56}(56^7)$ **b.** $\log_5(25^4) + \log_5(5)$ **c.** $\log_{10}(1000) - \log_{10}(10)$ **d.** $\log_{399}(1)$

4. Simplify the following expressions:

a. $e^{\ln(7)}$ **b.** $\ln(15) - \ln(3)$ **c.** $-[\ln(15) - \ln(3)]$ **d.** $\ln(\sqrt{e}) + \ln(\sqrt{e})$

2 DIFFERENTIATION

2.1 What is a Derivative?

The derivative of a function can be interpreted as the slope or the rate of change of the function. To find the derivative of a function we must ***differentiate*** the function. Differentiation can be used to determine quantities such as the velocity (rate of change of position with respect to time) or acceleration (rate of change of velocity with respect to time) of a system. Differentiation is at the core of modern science, and in this section we introduce you to several rules for differentiating the most commonly used functions in this course.

When differentiating a function, we must specify the variable that the function is to be differentiated with respect to. We are interested in how the function's rate of change depends on this variable. Notation commonly used to indicate the derivative of a function *f(x)* $= y$ with respect to the variable x is

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \cdot f(x) = f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} \cdot y \tag{10}$$

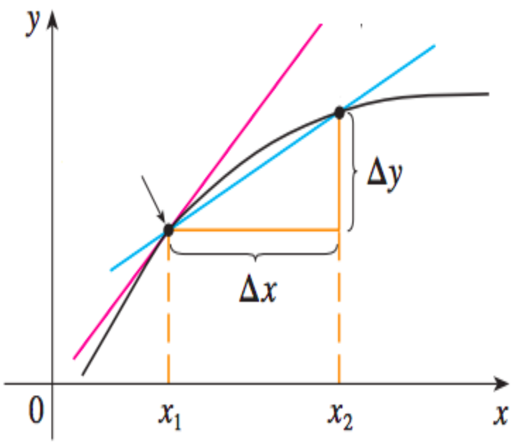

Figure 5: The derivative of a function y (black) at x_1 is the same as the slope of the tangent line (pink) to the function at x_1. An approximation is shown by what is known as a *secant line* (blue).

The letter d indicates a "small change", often represented by the symbol Δ as in figure (5).The slope (derivative) of a function is the fraction of the vertical change (the rise) over the horizontal change (the run). Notice that the notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ clearly represents a slope, since this is a fraction of the small change in y (dy) over the small change in x (dx). This is demonstrated in figure (5) figure (1). Throughout this book, we will use the different representations in equation (10) to refer to a derivative.

2.2 Differentiation Rules

Table 1 lists fundamental rules for differentiating polynomial, power, natural exponential and natural logarithmic functions, as well as sums and products of these. Understanding how to apply these rules will be essential for working with further rules in the following sections. The best way of learning how to differentiate functions is to see it in practice, so let's proceed with several examples.

Differentiation Rule	**Procedure (c is a constant)**
1. Constant Rule	$\frac{\mathrm{d}c}{\mathrm{d}x} = 0$
2. Power Rule	$\frac{\mathrm{d}}{\mathrm{d}x} \cdot (x^c) = (c) \cdot x^{c-1}$
3. Constant Multiple Rule	$\frac{\mathrm{d}}{\mathrm{d}x}[c \cdot f(x)] = c \cdot \left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right]$
4. Sum Rule	$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{\mathrm{d}g(x)}{\mathrm{d}x}$
5. Natural Exponential (e) Rule	$\frac{\mathrm{d}}{\mathrm{d}x}(e^{cx}) = c \cdot e^{cx}$
6. Natural Logarithmic (ln) Rule	$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$
7. Product Rule	$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \cdot g(x)] = f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} + g(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}$

Table 1: Differentiation rules.

The table heading should be on top of the table

Example 1. Differentiate the function $p(u) = u^3 + 5u^2 - 7$ with respect to u.

Solution:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}u} \cdot p(u) &= \tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (u^3 + 5u^2 - 7) && \text{(definition)} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (u^3) + \tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (5u^2) - \tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (7) && \text{(sum rule)} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (u^3) + 5\tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (u^2) - \tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (7) && \text{(constant multiple rule)} \\
&= 3u^2 + 5 \cdot 2u - 0 && \text{(power and constant rules)} \\
&= 3u^2 + 10u
\end{aligned}$$

In the first line, we apply the definition of the derivative with respect to the variable u on both sides of the equation. In the second line, we use the sum rule and apply the derivative to each term in the function separately. In the third line, we use the constant multiple rule to factor out the 5 in the second term. In the fourth line, we apply the power rule to the first and second terms and the constant rule to the third term. The derivative $p'(u)$ of the function $p(u)$ is given in the fifth line.

Example 2. Differentiate the function

$$f(x) = \frac{ax^5 + bx^2}{x}$$

with respect to x. Assume a and b are constants that do not depend on x.

Solution:

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot f(x) &= \tfrac{\mathrm{d}}{\mathrm{d}x} \cdot (ax^4 + bx) && \text{(definition)} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}x} \cdot (ax^4) + \tfrac{\mathrm{d}}{\mathrm{d}x} \cdot (bx) && \text{(sum rule)} \\
&= a\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot x^4 + b\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot x && \text{(constant multiple rule)} \\
&= 4ax^3 + b && \text{(power rule)}
\end{aligned}$$

Notice that we first simplify $f(x)$ because we have not provided you with the mathematical tools to differentiate fractions. For the purpose of this course, a good rule is to always simplify your functions to fit one of the differentiation rules above.

Remark: The mathematical tool for differentiating fractions is commonly known as the quotient rule. You will not need to know it in this course.

Example 3. Differentiate the function $k(r) = \frac{7}{r^3}$ with respect to r.

Solution:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}r} \cdot k(r) = & \tfrac{\mathrm{d}}{\mathrm{d}r} \cdot (7r^{-3}) && \text{(definition)} \\ = & 7\tfrac{\mathrm{d}}{\mathrm{d}r} \cdot (r^{-3}) && \text{(constant multiple rule)} \\ = & -21r^{-4} && \text{(power rule)} \end{aligned}$$

Notice we first rewrote the fraction $\frac{7}{r^3}$ as $7r^{-3}$ in order to use the power rule. When differentiating a function, it will be necessary to rewrite fractions this way whenever the variable of interest is raised to any power in the denominator.

Example 4. Differentiate the function $T(t) = 3e^{-2t} + 5\ln(t)$ with respect to t.

Solution:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \cdot T(t) = & \tfrac{\mathrm{d}}{\mathrm{d}t} \cdot (3e^{-2t} + 5\ln(t)) && \text{(definition)} \\ = & \tfrac{\mathrm{d}}{\mathrm{d}t} \cdot (3e^{-2t}) + \tfrac{\mathrm{d}}{\mathrm{d}t} \cdot (5\ln t) && \text{(sum rule)} \\ = & 3\tfrac{\mathrm{d}}{\mathrm{d}t} \cdot e^{-2t} + 5\tfrac{\mathrm{d}}{\mathrm{d}t} \cdot \ln(t) && \text{(constant multiple)} \\ = & 3 \cdot (-2)e^{-2t} + 5(\tfrac{1}{t}) && \text{(e and ln rules)} \\ = & -6e^{-2t} + \tfrac{5}{t} \end{aligned}$$

Example 5. Differentiate the function $p(t) = 2t^2 \ln(t)$ with respect to t.

Solution: This is an example where we need to use the product rule. Let's write $p(t) = f(t)g(t)$ and identify $f(t) = 2t^2$ and $g(t) = \ln(t)$.

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} \cdot p(t) &= f(t)\tfrac{\mathrm{d}g(t)}{\mathrm{d}t} + g(t)\tfrac{\mathrm{d}f(t)}{\mathrm{d}t} && \text{(definition)} \\
&= 2t^2\tfrac{\mathrm{d}}{\mathrm{d}t} \cdot \ln(t) + \ln(t)\tfrac{\mathrm{d}}{\mathrm{d}t} \cdot 2t^2 && \text{(substitution)} \\
&= 2t^2\tfrac{1}{t} + \ln(t)(4t) && \text{(ln and power rules)} \\
&= 2t + 4t\ln(t)
\end{aligned}$$

EXERCISES

1. Differentiate the following with respect to x:

a. $f(x) = 5x^3 + 4x^2 + 3x + 2$ **b.** $r(x) = \frac{3x^2+5}{x}$ **c.** $t(x) = 2x^{-5/3}$

2. Differentiate the following with respect to r:

a. $x(r) = r^{-2} + e^{-r/2} + 3\ln(r)$ **b.** $f(r) = \ln(r^2) - \ln(r)$ **c.** $j(r) = \frac{9}{r^5}$

3. Differentiate the following with respect to L:

a. $M(L) = -L^3 + \frac{1}{3}L + 1$ **b.** $w(L) = \frac{5}{L^{4/5}} + e^L + \frac{1}{3}\ln(L)$

c. $f(L) = aL^b + c\ln(L) + ge^{hL} + j$;with a, b, c, g, h, j = contants

4. Use the product rule to differentiate the following with respect to w:

a. $f(w) = we^w$ **b.** $p(w) = \sqrt{w}(2 + 3w)$ **c.** $r(w) = \ln(w)e^w$

2.2.1 Substitutions

It is possible to encounter functions that depend on 2 variables which depend on each other. For example, the function

$$v(P, T) = \frac{3P}{T} \tag{11}$$

depends on the variables P and T, and we are given the constraint that the variable P depends on the variable T through

$$P(T) = 2T^2 \tag{12}$$

Assume we are interested in determining how the rate of change of v depends on T. Therefore, we should differentiate v with respect to T. Notice we cannot treat P as a constant because it also depends on T, so first we can substitute $2T^2$ for P in equation (11). This reduces the function v to

$$v(T) = 6T \tag{13}$$

which we now can differentiate with respect to T to obtain

$$\frac{\mathrm{d}v(T)}{\mathrm{d}T} = v'(T) = 6 \tag{14}$$

EXERCISES

1. Differentiate the following with respect to x given the constraint on y:

a. $F(x, y) = 3x^3 + y^2$; $y(x) = 2x$ **b.** $W(x, y) = \ln\left(\frac{4x^4}{2y}\right)$; $y(x) = x^4$

2. Differentiate the following with respect to r given the constraint on s:

a. $P(r, s) = 3e^{2sr^3} + \ln(r)$; $s(r) = r^{-2}$ **b.** $R(r, s) = \ln(e^{rs})$; $s(r) = r$

2.2.2 Partial Differentiation

So far, we have only looked at functions that ultimately depend on a single variable. However, in real life processes, a function may depend on two or more variables that are independent of each other. For example, the pressure P of a gas in a sealed container may depend on the temperature T and volume V ; $P(T, V)$. We may be interested in how a function's rate of change depends on only one of the variables. Therefore, we must differentiate the function with respect to the variable of interest while treating all other variables as constants. This is known as a ***partial differentiation***. Notation commonly used to indicate partial differentiation of a function $f(x, y) = z$ with respect to the variable x is

$$\frac{\partial f(x, y)}{\partial x} = \frac{\partial}{\partial x} \cdot f(x, y) = \frac{\partial z}{\partial x} \tag{15}$$

Notice that in contrast to a full derivative, a partial derivative is indicated using a "curly" letter ∂ instead of a regular letter d. The differentiation rules apply equally to full and partial differentiation.

Example 1. Differentiate the function $P(T, V) = T^2 + 3V - 7$ with respect to T given that V is independent of T.

Solution:

$$\begin{aligned}
\tfrac{\partial}{\partial T} \cdot P(T, V) &= \tfrac{\partial}{\partial T} \cdot (T^2 + 3V - 7) && \text{(definition)} \\
&= \tfrac{\partial}{\partial T} \cdot (T^2) + \tfrac{\partial}{\partial T} \cdot (3V) - \tfrac{\partial}{\partial T} \cdot (7) && \text{(sum rule)} \\
&= 2T + 0 + 0 && \text{(power and constant rules)} \\
&= 2T
\end{aligned}$$

EXERCISES

1. Differentiate the following with respect to a given b is independent of a:

a. $G(a, b) = ab + b^2 + a^2 + 7$ **b.** $Z(a, b) = -\frac{1}{7}be^{-7a/2} + 3b\ln(a)$

2.3 The Chain Rule

We may be interested in differentiating composite functions, i.e. $f(g(x))$ with respect to x. The tool required for such procedure is the ***chain rule***.

The Chain Rule:

The derivative of a composite function $F(x) = f(g(x))$ with respect to x is given by

$$F'(x) = f'(g(x)) \cdot g'(x)$$

where $F'(x)$ refers to the derivative of F(x) with respect to x, $f'(g(x))$ refers to the derivative of the function f with respect to the function $g(x)$ and $g'(x)$ refers to the derivative of the function g with respect to x. This is explicitly shown through the notation

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}F}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

Notice that on the right hand side of the equation, the second derivative in the product of derivatives is always taken with respect to the variable of interest, in this case x.

Example 1. Differentiate the function $z(x) = \sqrt{x+1}$ with respect to x.

Solution: This is the composite function from equation (9), which is composed of the functions $f(a) = \sqrt{a}$ and $g(x) = x + 1$. Therefore, we can write

$$z(x) = f(g(x)) = \sqrt{g}$$

and apply the chain rule

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

The first term on the right is the derivative of z with respect to g. Therefore, if we want to differentiate with respect to g, we should use z in terms of g, $z(g)$. Likewise, the second term is the derivative of g with respect to x, so we should use $g(x)$.

definition

$$\begin{aligned}
z'(x) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}g} \cdot \sqrt{g}\right) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot (1+x)\right) && \text{(defitnition)} \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{d}g} \cdot g^{1/2}\right) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot 1 + \tfrac{\mathrm{d}}{\mathrm{d}x} \cdot x\right) && \text{(sum rule)} \\
&= \left(\tfrac{1}{2} g^{-1/2}\right) \cdot (0+1) && \text{(power and constant rules)} \\
&= \left(\tfrac{1}{2\sqrt{g}}\right) = \left(\tfrac{1}{2\sqrt{x+1}}\right)
\end{aligned}$$

Notice that because we are interested in the derivative of z with respect to x, our result should be in terms of x.

Example 2. Differentiate the function $r(u) = \ln(4u^3)$ with respect to u.

Solution: The function $r(u)$ includes the ln function. Looking at the natural logarithmic differentiation rule, we see that we can only apply this rule if the ln function is only dependent on a single variable "x". Therefore, in order to differentiate $r(u)$, we must must rewrite r as a function of some other variabe x (it does not matter which letter you choose, as long as it is not u)

$$r(u) = r(x) = \ln(x)$$

with $x(u) = 4u^3$ and apply the chain rule

$$\frac{\mathrm{d}r}{\mathrm{d}u} = \frac{\mathrm{d}r}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}u}$$

which yields

$$\begin{aligned}
r'(u) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot \ln(x)\right) \cdot \left(\tfrac{\mathrm{d}}{\mathrm{d}u} \cdot 4u^3\right) && \text{(definition)} \\
&= \left(\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot \ln(x)\right) \cdot \left(4\tfrac{\mathrm{d}}{\mathrm{d}u} \cdot (u^3)\right) && \text{(constant multiple rule)} \\
&= \left(\tfrac{1}{x}\right) \cdot (12u^2) && \text{(ln and power rules)} \\
&= \tfrac{12u^2}{4u^3} = \tfrac{3}{u}
\end{aligned}$$

EXERCISES

1. Differentiate the following with respect to s:

a. $f(s) = 3e^{s^3} + 2s^2$ **b.** $r(s) = \frac{7}{3}\ln(s^3 + 2s^2)$ **c.** $g(s) = 4e^{\sqrt{s}} - \ln(7s)$

2.3.1 Multivariate Differentation

The chain rule may be further generalized to differentiate functions that depend on 2 or more variables which may depend on each other. Assume we have a function $f(a, b, ..., k, ..., n)$ that depends on n variables. If we are interested in differentiating f with respect to one of the variables, say k, we can apply the general chain rule.

> The General Chain Rule:
>
> The derivative of a function $f(a, b, ..., k, ..., n)$, which depends on n variables, with respect to a single variable k is given by
>
> $$\frac{\mathrm{d}f}{\mathrm{d}k} = \frac{\partial f}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}k} + \frac{\partial f}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}k} + ... + \frac{\partial f}{\partial k}\frac{\mathrm{d}k}{\mathrm{d}k} + ... + \frac{\partial f}{\partial n}\frac{\mathrm{d}n}{\mathrm{d}k}$$
>
> where the variables $a, b, ..., n$ may depend on each other.

Notice on the right hand side of the equation, the number of terms added up is equal to the number of variables the function f depends on.

Example 1. Differentiate the function $R(a, b, c) = ba^2 + 3c$ with respect to b given the constraints

$$a(b) = (b + 2) \qquad \text{and} \qquad c(b) = ln(b)$$

Solution: The function R depends on 3 variables, therefore we need to apply the general chain rule with 3 terms in the sum on the right hands side. Since we are interested in the derivative of $R(a, b, c)$ with respect to b, this implies

$$b = k$$

in the general chain rule. We can therefore write the derivative of R with respect to b as

$$\frac{\mathrm{d}R}{\mathrm{d}b} = \frac{\partial R}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}b} + \frac{\partial R}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}b} + \frac{\partial R}{\partial c}\frac{\mathrm{d}c}{\mathrm{d}b}$$

Our task now is to compute each of the derivatives on the right hand side, multiply and add them accordingly.

$$
\begin{aligned}
\frac{\partial R}{\partial a} &= \frac{\partial}{\partial a}\cdot(ba^2+3c) &&= 2ba \\
\frac{\mathrm{d}a}{\mathrm{d}b} &= \frac{\mathrm{d}}{\mathrm{d}b}\cdot(b+2) &&= 1 \\
\frac{\partial R}{\partial b} &= \frac{\partial}{\partial b}\cdot(ba^2+3c) &&= a^2 \\
\frac{\mathrm{d}b}{\mathrm{d}b} &= \frac{\mathrm{d}}{\mathrm{d}b}\cdot b &&= 1 \\
\frac{\partial R}{\partial c} &= \frac{\partial}{\partial c}\cdot(ba^2+3c) &&= 3 \\
\frac{\mathrm{d}c}{\mathrm{d}b} &= \frac{\mathrm{d}}{\mathrm{d}b}\cdot ln(b) &&= \frac{1}{b}
\end{aligned}
$$

And therefore the partial derivative of the function $R(a, b, c)$ with respect to b is

$$
\frac{\mathrm{d}R}{\mathrm{d}a} = (2ba)(1) + (a^2)(1) + (3)\left(\frac{1}{b}\right) = 2ba + a^2 + \frac{3}{b}
$$

EXERCISES

1. Differentiate the following with respect to t given the constraints on x and y:

a. $B(x, y, t) = xy^2 + t$; $x(t) = 4t^2$ and $y(t) = \ln(t)$

b. $F(x, y, t) = 5e^{xyt}$; $x(t) = t^{-2/3}$ and $y(t) = e^{2t}$

2.4 Optimization

An important application of differentiation is ***optimization***. Optimization is used to determine the ***local maximum*** and/or ***local minimum*** output value(s) of a function. Recall that the derivative of a function can be interpreted as the function's slope. If the function has a local maximum and/or a minimum, these values will occur at the point where the function's slope is zero, as shown in figure 5. Notice these are known as local values because the function could have smaller or larger values, but optimization will only provide information about local maxima and minima.

figure (6)

To determine the value(s) of x at which a function $f(x)$ has a local maximum or a local minimum, we must find the derivative of the function with respect to x,

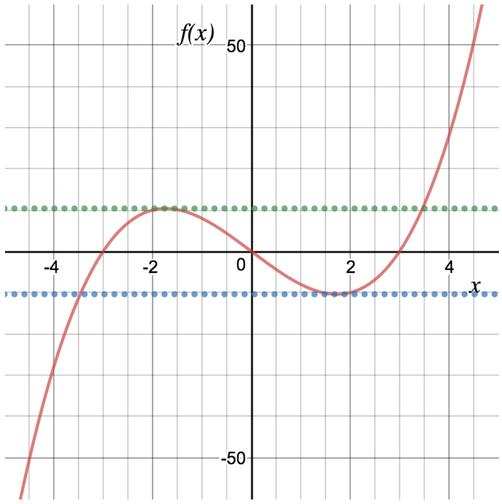

Figure 6: $f(x) = x^3 - 9x$ with $x_{crit,max} = -\sqrt{3}$ and $x_{crit,min} = \sqrt{3}$.

set it equal to zero and solve for x. The value of x at which the function has a local maximum or a local minimum is known as a ***critical point***, which we will refer to as x_{crit}. To determine whether the critical points represent a maximum or a minimum, we must test values of x to the right and left of x_{crit} for the function $f(x)$. Assume we test the points $a < x_{crit}$ and $b > x_{crit}$. The conditions to determine whether x produces a local maximum or a local minimum for $f(x)$ are:

1. If $\quad f(x_{crit}) > f(a)$ and $f(x_{crit}) > f(b); \quad f(x_{crit}) =$ maximum

2. If $\quad f(x_{crit}) < f(a)$ and $f(x_{crit}) < f(b); \quad f(x_{crit}) =$ minimum

Example 1. Determine the value of r at which the function $z = r^2$ has a local maximum or local minimum.

Solution: Because we are asked to determine the value of r at which the function $z = r^2$ has a local maximum or local minimum, we must differentiate the function z with respect to r, set the derivative equal to zero and solve for the critical value(s) of r, r_{crit}.

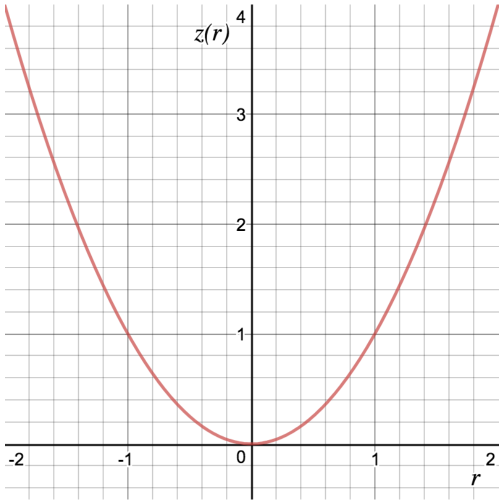

Figure 7: $z(r) = r^2$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} \cdot z(r) &= \frac{\mathrm{d}}{\mathrm{d}r} \cdot r^2 \quad \text{(differentiation)} \\
&= 2r
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} \cdot z(r_{crit}) &= 0 \qquad \text{(optimization)} \\
2r_{crit} &= 0 \\
r_{crit} &= 0
\end{aligned}$$

Is $z(r_{crit}) = z(0) = 0$ a local maximum or a local minimum?

$$\begin{aligned}
&r = -2 \text{ and } r = 3 && \text{(test values)} \\
&z(-2) = 4 \text{ and } z(3) = 9 \\
&z(0) < z(-2) \text{ and } z(0) < z(3) && \text{(condition } \mathbf{2} \text{ is satisfied)}
\end{aligned}$$

We conclude that the function $z = r^2$ has a local minimum value at $x = 0$. Looking at a graph of this function in figure (7), we observe that the function's minimum value indeed occurs at $r = 0$. Note that for this example the local minimum is also the (absolute) minimum of z(r).

Example 2. Determine the local maximum of the function $f(x) = x^3 - 27x$.

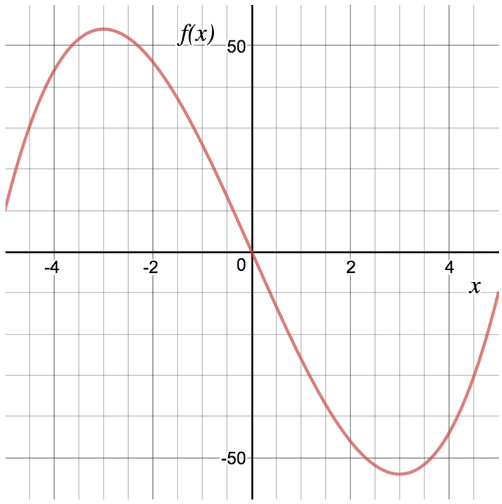

Figure 8: $f(x) = x^3 - 27x$

Solution: The function $f(x)$ is shown in figure (8). Because we are asked to determine the local maximum of $f(x)$, we must differentiate $f(x)$ with respect to x, set the derivative equal to zero and solve for the critical value(s) x_{crit}. We can then test these critical values to determine if they represent a local maximum or local minimum.

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x} \cdot f(x) &= \tfrac{\mathrm{d}}{\mathrm{d}r} \cdot (x^3 - 27x) \quad \text{(differentiation)} \\
&= 3x^2 - 27 \\
\tfrac{\mathrm{d}}{\mathrm{d}r} \cdot f(x_{crit}) &= 0 \qquad\qquad\qquad\qquad \text{(optimization)} \\
3x_{crit}^2 - 27 &= 0 \\
3x_{crit}^2 &= 27 \\
x_{crit}^2 &= 9 \\
x_{crit,1} = 3 &\quad \text{and} \quad x_{crit,2} = -3
\end{aligned}
$$

Is $f(x_{crit,1}) = f(3) = -54$ a local maximum or a local minimum?

$$
\begin{aligned}
& x = 2 \text{ and } x = 4 && \text{(test values)} \\
& f(2) = -46 \text{ and } z(4) = -44 \\
& f(3) < f(2) \text{ and } f(3) < f(4) && \text{(condition } \mathbf{2} \text{ is satisfied)}
\end{aligned}
$$

Therefore, $f(x)$ has a local minimum at $x = 3$, and the value is -54.

Is $f(x_{crit,2}) = f(-3) = 54$ a local maximum or a local minimum?

$$
\begin{aligned}
& x = -2 \text{ and } x = -4 && \text{(test values)} \\
& f(-2) = 46 \text{ and } f(-4) = 44 \\
& f(-3) > f(-2) \text{ and } f(-3) > f(-4) && \text{(condition } \mathbf{1} \text{ is satisfied)}
\end{aligned}
$$

Therefore, $f(x)$ has a local maximum at $x = -3$, and the value is 54.

EXERCISES

1. Find the local maxima and/or mimima value of the function:

a. $F(r) = 15 - 2r - r^2$ **b.** $R(z) = 4z^2 - 8z - 21$

c. $P(w) = w^3 + \frac{7}{2}w^2 - 6w - 10$ [hint: $3w^2 + 7w - 6 = (3w - 2)(w + 3)$]

3 INTEGRATION

3.1 Definite Integrals: Areas

The integral of a function $f(x)$ between two points a and b (with $a < b$) may be interpreted as the area enclosed by the lines $x = a$, $x = b$ and the function $f(x)$. Let's take for example the function $f(x) = y = 5$ and consider the lines $x = 2$ and $x = 6$. This is depicted in figure (8).

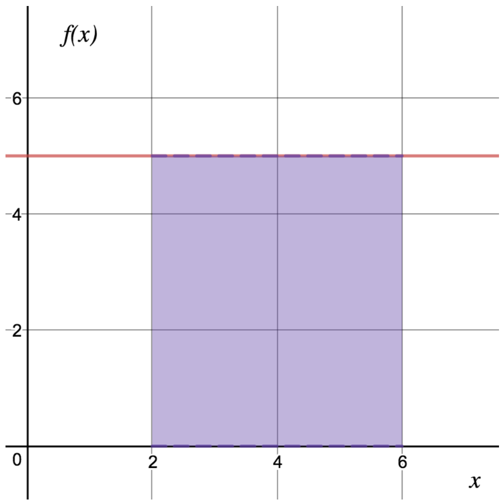

Figure 9: $\int_2^6 5dx$

It is clear that the area enclosed is $5 \cdot 4 = 20$. In the language of calculus, we say we could find the integral of, or ***integrate*** the function $f(x) = y = 5$ from $x = 2$ to $x = 6$ to determine the area enclosed.

Notation commonly used to indicate the integral of a function $f(x)$ between the points $x = a$ and $x = b$ is

$$Area = F(x) = \int_a^b f(x)dx \tag{16}$$

where $F(x)$ is the solutions to the integral (the total area) and a and b are known as the bounds of the integral. The term dx at the end of the integral belongs to the integral notation and indicates that $f(x)$ is to be integrated with respect to the variable x.
Applying the notation in equation (16) to our example in figure (9) would yield the integral

$$Area = F(x) = \int_2^6 5dx = 20 \tag{17}$$

Because the bounds a and b are specified, this is known as a ***definite integral***, which may be evaluated as

$$Area = \int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a) \tag{18}$$

Notice that the bounds are transferred onto a bar to the right of the integral solution $F(x)$, with the upper bound b on top and the lower bound a on the bottom. The bar indicates that the solution to the integral is to be evaluated at b (i.e. $F(b)$) F(b))
and subtracted by its value at a (i.e. $F(a)$). This yields the area enclosed by the bounds $x = a$, $x = b$, and the curve $f(x)$. The integral of $f(x) = 5$ with respect to x is $5x$ (this will be explained in section 3.2), so equation (17) can be more explicitly written as

$$Area = \int_2^6 5dx = 5x|_2^6 = 5 \cdot 6 - 5 \cdot 2 = 20 \tag{19}$$

You may be wondering why we use integrals to determine areas. In our example, we could have easily just multiplied the length by the width to obtain the area. The utility of integrals is that we can determine the area between 2 bounds and under any curve! Figure (10) provides two examples in which the precise area could not be determined without using an integral. Section 3.2 will introduce you to a set of integration rules for finding the area under curves necessary for this course.

Remark: Equation (18) is known as ***the fundamental theorem of calculus*** given that the function $f(x)$ is continuous on the interval [a,b]. This need not concern our integration interests, but is included for completeness.

3.2 Indefinite Integrals: +C

If the bounds are not specified, i.e.

$$F(x) = \int f(x)dx \tag{20}$$

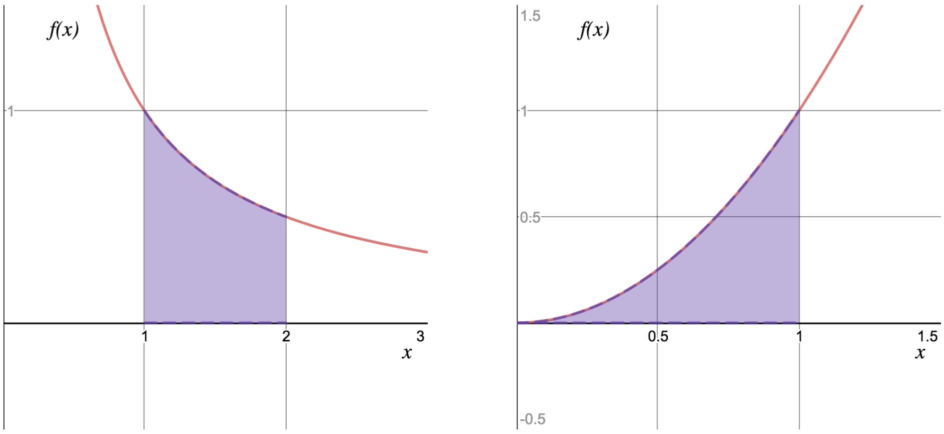

Figure 10: Left to right: $\int_1^2 \frac{1}{x}dx = \log(2)$ and $\int_0^1 x^2 dx = \frac{1}{3}$

the integral is known as an ***indefinite integral***. In contrast to definite integrals, which provide the area under the curve, indefinite integrals are used to determine potential antiderivatives of the function $f(x)$. This means that the solution to the integral in equation (20) will remain a function of x and in addition must include an arbitrary constant term $\boldsymbol{+C}$ attached to it. Notice that because the integral $F(x)$ is the anti-derivative of the function $f(x)$, differentiating the function $F(x)$ with respect to x would return the function $f(x)$ since the constant C is independent of x. Assuming we were not told the bounds of our example above, the solution to the integral would be

$$F(x) = \int 5dx = 5x + C. \tag{21}$$

Our goal now is to learn how to apply a set of integration rules in order to compute the integrals we will encounter in this course.

3.3 Integration Rules

Table 2 lists fundamental rules for integrating polynomial, natural exponential and natural logarithmic functions, as well as sums of these. Notice that rules 1-6 have the same name as the rules for differentiation. Because the integral of a function f(x) is the same as the function's anti-derivative, differentiating the solutions to integration rules 1, 2, 5 and 6 would return the function f(x) (try it!). In addition, we have included a rule for swapping the bounds of a definite integral. This can be done if the integral is multiplied by a factor of -1. As we did with differentiation, we will continue with several examples. Remember that you evaluate definite integrals using the given bounds and add a constant $+C$ to indefinite integrals.

Integration Rule	**Procedure (c is a constant)**
1. Constant Rule	$$\int c \cdot dx = c \cdot x + C$$
2. Power Rule	$$\int x^c \cdot dx = \frac{1}{c+1} \cdot x^{c+1} + C$$ (c $\neq$ -1, see rule 6)
3. Constant Multiple Rule	$$\int cf(x) \cdot dx = c \int f(x) \cdot dx$$
4. Sum Rule	$$\int [f(x) + g(x)] \cdot dx = \int f(x) \cdot dx + \int g(x) \cdot dx$$
5. Natural Exponential (e) Rule	$$\int e^{cx} \cdot dx = \frac{1}{c} \cdot e^{cx} + C$$
6. Natural Logarithmic (ln) Rule	$$\int x^{-1} = \int \frac{1}{x} = \ln(x) + C$$
7. Bound Swap Rule	$$-\int_a^b f(x) \cdot dx = \int_b^a f(x) \cdot dx$$

Table 2: Integration rules.

The table heading should be on top of the table

Example 1. Find

$$\int (3u^2 + 10u) \cdot du.$$

Solution:

$$\begin{aligned}
\int (3u^2 + 10u) \cdot du &= \int 3u^2 \cdot du + \int 10u \cdot du && \text{(sum rule)} \\
&= 3 \int u^2 \cdot du + 10 \int u \cdot du && \text{(constant multiple rule)} \\
&= 3 \cdot \tfrac{1}{3} u^3 + 10 \cdot \tfrac{1}{2} u^2 + C && \text{(power rule)} \\
&= u^3 + 5u^2 + C
\end{aligned}$$

Example 2. Find

$$\int \left(\frac{1}{x} + 4e^{-3x} \right) \cdot dx.$$

Solution:

$$\begin{aligned}
\int \left(\tfrac{1}{x} + 4e^{-3x} \right) \cdot dx &= \int \tfrac{1}{x} \cdot dx + \int 4e^{-3x} \cdot dx && \text{(sum rule)} \\
&= \int \tfrac{1}{x} \cdot dx + 4 \int e^{-3x} \cdot dx && \text{(constant multiple rule)} \\
&= \ln(x) + 4 \left(\tfrac{1}{-3} \right) e^{-3x} + C && \text{(ln and e rule)} \\
&= \ln(x) - \left(\tfrac{4}{3} \right) e^{-3x} + C
\end{aligned}$$

Example 3. Find

$$\int_1^4 \left(\frac{4}{w^{\frac{3}{2}}} + 3 \right) \cdot dw.$$

Solution:

$$\begin{aligned}
\int_1^4 \left(\frac{4}{w^{\frac{3}{2}}} + 3\right) \cdot dw &= \int_1^4 4w^{-\frac{3}{2}} \cdot dw + \int_1^4 3 \cdot dw && \text{(sum rule)} \\
&= 4\int_1^4 w^{-\frac{3}{2}} \cdot dw + \int_1^4 3 \cdot dw && \text{(constant multiple rule)} \\
&= 4\left(-\frac{2}{1}\right) w^{-\frac{1}{2}}|_1^4 + 3 \cdot w|_1^4 && \text{(power and constant rule)} \\
&= -8\left[\frac{1}{w^{\frac{1}{2}}}|_1^4\right] + 3\left[w|_1^4\right] \\
&= -8\left[\frac{1}{2} - 1\right] + 3\left[4 - 1\right] && \text{(evaluation)} \\
&= 13
\end{aligned}$$

Notice in this example we rewrote the fraction $\frac{4}{w^{\frac{3}{2}}}$ as $4w^{-\frac{3}{2}}$ in order to use the power rule. It will be necessary to rewrite fractions this way whenever the variable of interest is in the denominator and is raised to any power other than 1. Of course, when the variable is in the denominator and is raised to the power of 1, the ln rule applies for integration.

EXERCISES

1. Evaluate the following indefinite integrals:

a. $\int(3y^2 + 2e^y)dy$ **b.** $\int(\frac{3}{x} + 6x^2)dx$ **c.** $\int(\frac{1}{3}r^{-2/3} + 5r^{-1} + 2)dr$

2. Evaluate the following definite integrals:

a. $\int_1^e \left(\frac{5}{x}\right) dx$ **b.** $\int_0^2 (2w + 5)dw$ **c.** $\int_{z_{low}}^{z_{high}} (10z^4 + 8z^3 + 2)dz$

3. Evaluate the following indefinite integrals; a, b, c, and d are constants:

a. $\int \left(ax^b + \frac{c}{x} + de^{5x} + c\right) dx$ **b.** $\int \left(at^3 + 5e^{bt} + c + \frac{d}{t} + \frac{2a}{t^4}\right) dt$

4. Evaluate the following definite integrals; c_v, n, R, P and T are constants:

a. $\int_{T_{low}}^{T_{high}} c_v n dT$ **b.** $\int_{V_{low}}^{V_{high}} (nRT/V)dV$ **c.** $\int_{V_a}^{V_b} PdV$

3.3.1 Substitutions

2.2.1

Similar to differentiation substitutions in section 2.2.2, it is possible to encounter integrals that depend on 2 variables which depend on each other. For example, the indefinite integral

$$\int \frac{a^3}{b} da \tag{22}$$

depends on the variables a and b, and we are given the constraint

$$b(a) = a^2. \tag{23}$$

Because the term da at the end of the integral indicates integration with respect to a, we cannot treat b as a constant because it depends on a. If b was independent of the integration variable a, then it could be treated as a constant. Therefore, we first need to substitute $b = a^2$ to obtain the integral

$$\int a da. \tag{24}$$

This integral has the solution

$$\int a da = \frac{1}{2}a^2 + C. \tag{25}$$

.

EXERCISES

1. Evaluate the integrals given the constraints:

a. $\int_1^5 \left(\frac{-1}{ab^2}\right) da$; $b(a) = \frac{2}{a}$ **b.** $\int_0^{\ln(2)} (3re^{2t})dt$; $r(t) = e^{-t}$

c. $\int \left(\frac{c_1 T}{V}\right) dV$; $T(V) = c_2 V^{2/7}$, c_1 and c_2 are constants

d. $\int_{V_a}^{V_b} P dV$; $P(V) = \frac{(nRT)}{V}$, n, R and T are constants

SOLUTIONS TO EXERCISES

Section 1:

1a. x^5 **1b.** $y^{-4} = \frac{1}{y^4}$ **1c.** a^6 **1d.** g^6

2a. $\log_5(xy)$ **2b.** $\log_{17}(w/z)$ **2c.** r **2d.** t

3a. 7 **3b.** 9 **3c.** 2 **3d.** 0

4a. 7 **4b.** $\ln(5)$ **4c.** $\ln(1/5)$ **3d.** 1

Section 2.2:

1a. $f'(x) = 15x^2 + 8x + 3$ **1b.** $r'(x) = 3 - 5x^{-2}$ **1c.** $t'(x) = -\frac{10}{3}x^{-8/3}$

2a. $x'(r) = -2r^{-3} - \frac{1}{2}e^{-r/2}$ **2b.** $f'(r) = \frac{1}{r}$ **2c.** $j'(x) = -45/x^6$

3a. $M'(L) = -3L^2 + 1/3$ **3b.** $w'(L) = -4L^{-9/5} + e^L + \frac{1}{3L}$

3c. $f'(L) = abL^{b-1} + c/L + ghe^{hL}$

4a. $f'(w) = e^w(w+1)$ **4b.** $p'(w) = 3\sqrt{w} + \frac{2+3w}{2\sqrt{w}}$ **4c.** $r'(w) = e^w\left[\ln(w) + 1/w\right]$

Section 2.2.1:

1a. $F'(x) = 9x^2 + 8x$ **1b.** $W'(x) = \ln(2)$

2a. $P'(r) = 6e^{2r} + \frac{1}{r}$ **2b.** $R'(r) = 2r$

Section 2.2.2:

1a. $\frac{\partial G}{\partial a} = b + 2a$ **1b.** $\frac{\partial Z}{\partial a} = \frac{1}{2}be^{-7a/2} + 3b/a$

Section 2.3:

1a. $f'(s) = 9s^2e^{s^3}$ **1b.** $r'(s) = \frac{7}{3}\left(\frac{3s^2+4s}{s^3+2s^2}\right)$ **1c.** $g'(s) = \frac{7}{3}\left(\frac{3s^2+4s}{s^3+2s^2}\right)$

Section 2.3.1:

1a. $B'(t) = 8ty^2 + \frac{2xy}{t} + 1$

1b. $F'(t) = 5e^{xyt}\left(\frac{-2}{3}yt^{-2/3} + 2xte^{2t} + xy\right)$

Section 2.4:

1a. $F_{local,max} = 16$ **1b.** $R_{local,min} = -25$

1c. $P_{local,min} = -\frac{719}{54}$ and $P_{local,max} = 12.5$

Section 3.3:

1a. $y^3 + 2e^y + C$ **1b.** $3\ln(x) + 2x^3 + C$ **1c.** $r^{1/3} + 5\ln(r) + 2r + C$

2a. 5 **2b.** 14 **2c.** $2\left[(z_h^5 + z_h^4 + z_h) - (z_l^5 + z_l^4 + z_l)\right]$

3a. $\frac{a}{b+1}x^{b+1} + c\ln(x) + \frac{1}{5}de^{5x} + cx + C$ **3b.** $\frac{a}{4}t^4 + \frac{5}{b}e^{bt} + ct + d\ln(t) - \frac{2}{3}at^{-3} + C$

4a. $c_v n[T_h - T_l]$ **4b.** $nRT\ln(V_h/V_l)$ **4c.** $P[V_h - V_l]$

Section 3.3.1:

1a. 3 **1b.** 3 **1c.** $\frac{7}{2}c_1c_2V^{2/7} + C$ **1d.** $nRT\ln(V_b/V_a)$