Mathematical Methods of Physics

You may use your lecture notes and the two textbooks (*Mathematics of Classical and Quantum Physics*, by Byron and Fuller and *Theoretical Mechanics of Particles and Continua*, by Fetter and Walecka) for reference.

The individual parts of the questions below are weighed equally.

Problem 1
Consider the integral

$$I[y(x)] = \int_a^b f(y, y')dx \quad \text{where} \quad f(y, y') = \frac{\sqrt{1 + y'^2}}{y}.$$

(a) Show that for I to be stationary (i.e., a minimum or maximum),

$$\frac{d}{dx}\left[y'\frac{\partial f}{\partial y'} - f\right] = 0.$$

Express the resulting differential equation for $y(x)$ in the form

$$y \times h(y') = \text{const.},$$

where h depends only on $y' = dy/dx$.

(b) Assuming $y > 0$, explain whether the solution in part (a) represents a minimum or a maximum of I.

Problem 2
In a reactor, neutrons propagate diffusively, with diffusion constant D. In addition, the medium absorbs neutrons at a rate αn per unit volume per unit time, where $n(\vec{r}, t)$ is the number density of neutrons at point $\vec{r}$ and time t. Thus, you may assume that the neutron density satisfies

$$\frac{\partial n}{\partial t} = D\nabla^2 n - \alpha n.$$

(a) A thin rod of length L of such a material is impenetrable to neutrons (i.e., $\vec{j} \cdot d\vec{A} = 0$) on its sides and at one end. The other end is open to a neutron bath at constant $n = n_0$. Find the equilibrium (steady-state) neutron density at the closed end of the rod. Discuss the limiting cases of small and large $\alpha L^2/D$.

(b) Find the equilibrium value of neutron flux into the rod through the open end, and verify explicitly that this equals the total rate of absorption within the rod. Again, discuss the limiting cases of small and large $\alpha L^2/D$.

Problem 3

(a) For any $n \times n$ Hermitian matrix H, let

$$U = \sum_{n=0}^{\infty} \frac{(iH)^n}{n!} = I + iH + \frac{(iH)^2}{2} + \cdots$$

Assuming that this series converges, show that U is unitary.

(b) Show that for any matrix A (Hermitian or not!), both

$$A_+ = \frac{1}{2}\left(A + A^\dagger\right)$$

and

$$A_- = \frac{1}{2i}\left(A - A^\dagger\right)$$

are Hermitian. Thus, any matrix can be decomposed into two Hermitian parts:

$$A = A_+ + iA_-$$

This is analogous to

$$z = x + iy$$

where $x = (z + z^*)/2$ and $y = (z - z^*)/(2i)$ for complex numbers z.

Problem 4

Consider the ordinary differential equation

$$y''(x) + y(x) = 0$$

for

$$y(x) = \sum_{\ell=0}^{\infty} a_\ell x^\ell.$$

(a) Find the series solution for $a_0 = 1$ and $a_1 = 0$.

(b) Find the series solution for $a_0 = 0$ and $a_1 = 1$.