Mathematical Methods of Physics

You may use your lecture notes and the two textbooks (*Mathematics of Classical and Quantum Physics*, by Byron and Fuller and *Theoretical Mechanics of Particles and Continua*, by Fetter and Walecka) for reference.

The individual parts of the questions below are weighed equally.

Problem 1

(a) Find the Euler-Lagrange equation satisfied by $y(x)$ that minimizes

$$I[y(x)] = \int_{-1}^{1} \left[\frac{\alpha}{2} y^2 + \frac{\gamma}{2} (y')^2 \right]$$

for positive constants α and γ.

(b) Assuming $y(\pm 1) = 1$, find and sketch an example solution to this equation. Also, explain why such solutions must correspond to minima of I.

Problem 2
Consider the diffusion of particles characterized by the diffusion equation

$$D \nabla^2 n = \frac{\partial n}{\partial t}$$

in a cubical box of size $L \times L \times L$, defined by $0 \leq x, y, z \leq L$. No particles are allowed to escape, so that all boundaries, at $x, y, z = 0$ or L, are reflecting with vanishing flux.

(a) Explain why all solutions for $n(\vec{r}, t)$ can be described in the form

$$n(x, y, z, t) = \sum_{m,n,p} c_{m,n,p} \cos(k_x x) \cos(k_y y) \cos(k_z z) e^{-k^2 D t},$$

where $k_x = m\pi/L$, $k_y = n\pi/L$, $k_z = p\pi/L$ and $k^2 = k_x^2 + k_y^2 + k_z^2$. What values of m, n, p are possible here? Find the solution for an initial distribution of particles $n(x, y, z, 0)$ given by $n = 1$ for $0 \leq x \leq L/2$ and $n = 0$ for $L/2 < x \leq L$. In other words, all the particles are initially in one half of the box.

(b) The final, steady-state solution of this problem corresponds to a uniform density $n = 1/2$ throughout the box. Find the dominant long-time correction to this and use it to estimate the time required for the distribution to become uniform to within 1%.

Problem 3

(a) Consider the vector $\vec{x}(t) = \exp(At)\,\vec{x}(0)$, where A is some constant $n \times n$ matrix, and

$$\exp(At) = \sum_{\ell=0}^{\infty} \frac{(At)^\ell}{\ell!}.$$

Show that $\vec{x}(t)$ satisfies the equation

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t).$$

(b) Find $f(t)$ and $g(t)$ such that

$$\frac{df}{dt} = 3f + 4g$$

and

$$\frac{dg}{dt} = f + 3g,$$

where $f(0) = g(0) = 1$.

Problem 4
Consider the 3D vector space of polynomials of the form $p(x) = a_0 + a_1 x + a_2 x^2$, where $a_{0,1,2}$ can be complex.

(a) Show that

$$(p, q) = \int_0^\infty p(x)^* q(x) e^{-x} dx$$

is an inner product in this space.

(b) Starting from the obvious basis $\left\{1, x, x^2\right\}$ for this space, construct an orthonormal basis $\{q_0(x), q_1(x), q_3(x)\}$. You may use, without showing, the fact that

$$\int_0^\infty x^n e^{-x} dx = n!$$