Approaching the classical limit in the diamagnetic helium atom

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We studied the diamagnetic helium atom under conditions of constant scaled energy over an extended range of Rydberg levels \( n = 77 \rightarrow 160 \). Experimental spectra are compared with extensive quantum calculations. In the studied range the effective Planck constant \( \hbar_e \) varies a factor of 2 (from 0.01 to 0.005). When comparing Fourier-transformed spectra in different \( \hbar_e \) regimes, the resulting action spectra show a large similarity, in accordance with the invariance of the classical dynamics. Prominent variations in the recurrence strength of peaks in the action spectra with \( \hbar_e \) could be attributed to interferences between orbits with nearly the same scaled action. In particular, the predicted difference in \( \hbar_e \) scaling of the orbit parallel to the magnetic field when compared with the off-axis orbits is confirmed experimentally. [S1050-2947(99)00509-0]

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I. INTRODUCTION

In the limit where Planck’s constant approaches zero, quantum mechanics evolves into classical mechanics. The scaling properties of a Rydberg atom can be used to study this classical limit. The increase in the density of states when closing in on the ionization limit, corresponding to a rapidly increasing size of the atom, reflects this transition to classical behavior.

The importance of the underlying classical structure in atoms close to their ionization limit was pointed out by Edmonds [1] to explain the barium spectrum in a 2.4 T magnetic field [2]. The development of closed-orbit theory greatly facilitated the interpretation of complex photoabsorption spectra of atoms in an external field [3]. According to this semiclassical theory oscillations in the absorption spectra relate to the recurrence time associated with classical electron orbits closing at the nucleus. In the Fourier transform of an absorption spectrum, such orbits show up as peaks in a time spectrum.

The concept of scaled energy as a parameter widened the scope of this so-called recurrence spectroscopy. In an external field, the scaling properties of the classical Hamiltonian provide transformation rules that give a prescription for how to perform experiments with the classical dynamics fixed. In such experiments energy and field strength are varied simultaneously so that the classical parameters of topological identical orbits remain constant. The diamagnetic Rydberg atom provides a case of special interest as the classical dynamics of a hydrogen atom in a strong magnetic field shows chaotic behavior [4–7]. Constant-scaled-energy experiments were performed in the group of Welge in Bielefeld [8,9]. Scanning a pulsed vacuum ultraviolet (vuv) laser the absorption spectrum of the hydrogen atom in a magnetic field B was recorded as a function of \( \gamma^{-1/3} \) \( (\gamma = B/2.35 \times 10^5 \text{ T}) \) at constant scaled energy \( e = E \gamma^{-2/3} \) \( (E \text{ is the energy relative to the ionization limit}) \). Resonances up to action 5 in the Fourier-transformed spectra could be connected to classical orbits. The limited resolution of their vuv source prevented the observation of phenomena related to the transition from regular to chaotic classical motion.

High resolution experiments with continuous lasers allow for the observation of trajectories connected to considerable higher action values. For instance, an extended action spectrum recorded at \( e = -0.7 \) in helium could be reproduced very well using hydrogenic standard closed-orbit theory, both in position and strength up to scaled action values of 20 [10]. One significant result was that an orbit already appeared in the action spectrum, although the scaled energy was below its classical bifurcation energy. Such so-called ghost orbits were not predicted by standard closed-orbit theory. They were, however, confirmed by quantum calculations on hydrogen. In the Fourier transform of a calculated hydrogen spectrum such a recurrence peak indeed appeared at the position of the ghost orbit. A quantum calculation for helium using the \( R \)-matrix formalism [11] showed excellent agreement with experimental spectra [12]. In these helium action spectra (experimental as well as theoretical) small additional peaks showed up at the sum of actions of two (hydrogenic) closed orbits. This provided the first evidence that a core gives rise to scattering from one orbit into another, forbidden in hydrogen. A subsequent modification of closed-orbit theory now incorporates these core effects in nonhydrogenic atoms [13–15]. In a recent paper this modified theory was applied to reproduce helium scaled-energy spectra in an electric field [16], demonstrating that theory and experiment converge when taking into account an increasing number of core scattering processes for larger actions. For atoms with large quantum defects the effects of core scattering become more pronounced, which was demonstrated in magnetic field experiments on rubidium atoms [17]. The varying probability of core scattering as a function of the quantum defect was studied for helium Rydberg atoms in an electric field [18], where the singlet and triplet action spectra were compared with calculated hydrogen spectra. Pronounced modulations observed in the triplet spectrum could be ascribed to interferences between hydrogenic and core-scattered orbits.

In this paper we present constant-scaled-energy absorption measurements in (hydrogenlike) odd-parity triplet helium in a magnetic field over a range of Rydberg levels \( n = 77 \rightarrow 160 \) much larger than ever reported. Although the spectra are recorded under fixed classical conditions, the field strength is not fully eliminated from the quantum problem: an effective Planck constant \( (\hbar_e = \gamma^{1/3}) \) has to be intro-
duced. This provides a handle to study the semiclassical limit ($\hbar \to 0$) by decreasing the magnetic field strength while keeping the scaled energy constant. Fourier transforming different parts of our extended absorption spectrum allows a comparison of action spectra corresponding to different values of $\hbar \gamma$. According to closed-orbit theory, the recurrence amplitude of a closed orbit in hydrogen varies with $\sqrt{\hbar \gamma}$, except for the parallel orbit which scales linearly with $\hbar \gamma$. It reflects the decrease in spectral modulations when approaching the semiclassical limit. Recurrences associated with quantum phenomena such as core scattering and ghost orbits show different $\hbar \gamma$ dependencies. In quantum spectra weak recurrences connected to classical orbits in the node of a wave function have been identified, a feature that also obeys a different scaling law [19]. In our experiment we have investigated these different dependencies on $\hbar \gamma$ in some detail.

This paper is organized as follows. In Sec. II we briefly review the basic principles of scaled-energy spectroscopy, closed-orbit theory, and $R$-matrix quantum calculations. The calculated $\hbar \gamma$ dependence of the recurrence strength of different types of closed orbits will be discussed. In Sec. III the experimental setup and results of laser cooling of the atomic beam are presented. In Sec. IV the experimental results are given, whereas in Sec. V differences between recurrence spectra recorded in different quantum regimes are discussed using closed-orbit theory. Finally, in Sec. VI we give some conclusions.

II. THEORY

A. Scaling transformations

The dynamics of a hydrogen atom in a uniform magnetic field (directed along the z axis) is described by the Hamiltonian (in atomic units $m = 1$, $e = 1$, $\hbar = 1$)

$$H = \frac{p^2}{2} - \frac{1}{r} + L_z \gamma + \frac{1}{8} \gamma^2 p^2.$$  

(1)

Here $\gamma = B/2.35 \times 10^5$ T, the magnetic field in atomic units, and $p^2 = (x^2 + y^2)$. When exciting $M_L = 0$ transitions the Zeeman term ($L_z \gamma$) vanishes. The potential then consists of a cylindrical diamagnetic term and the spherical Coulomb contribution. Using the scaling transformations $\tilde{r} = \gamma^{2/3} r$ and $\tilde{p} = \gamma^{-1/3} p$ a scaled Hamiltonian results:

$$\tilde{H} = H \gamma^{-2/3} \frac{\tilde{p}^2}{2} - \frac{1}{\tilde{r}} + \frac{\tilde{p}^2}{8}.$$  

(2)

This scaled Hamiltonian shows that the classical dynamics does not depend on the energy and magnetic field separately, but is determined by a single parameter, the scaled energy $E = E \gamma^{-2/3}$. When in an experiment this scaled energy is kept constant, the classical parameters (i.e., scaled actions) of topological identical trajectories remain the same. Experiments on a large Rydberg atom (with high principal quantum number $n$) in a low magnetic field and on a small atom (low-$n$ value) in a high magnetic field are thus equivalent in a classical sense.

The scaling transformation has important consequences for the quantum system. Inspection of the commutation relation for the scaled quantum mechanical operators associated with the classical variables shows that the quantum problem depends on the field strength [6]:

$$[\tilde{p}, \tilde{r}] = i \gamma^{1/3} = i \hbar \gamma.$$  

(3)

For experiments under fixed classical conditions, the magnetic field strength obviously defines different quantum regimes. The commutation relation (3) shows that a decrease of the magnetic field strength at constant scaled energy provides an opportunity for an experimental exploration of the classical limit $\hbar \gamma \to 0$.

B. Hydrogenic closed-orbit theory

The classical interpretation of a scaled-energy spectrum involves closed-orbit theory. This theory, developed by Delos and co-workers [3,9], has been extensively discussed in various papers. As the relevant quantum defect of odd-parity helium is small ($\delta = 0.068$), hydrogenic theory is to a large extent sufficient to describe our measurement. Therefore here we will only give a brief summary of this theory.

Closed-orbit theory connects (deformed) sine oscillations in a photoabsorption spectrum in a magnetic field with classical electron orbits closing at the nucleus. This semiclassical theory uses a quantum description near the nucleus ($r < 50a_0$) where photon absorption creates a near zero-energy outgoing Coulomb wave. In the outer region ($r > 50a_0$) the propagation of the wave front can be represented by a pencil of classical electron trajectories, determined by the Hamilton equations of motion. In the combined Coulomb magnetic field potential some of the trajectories will curve back exactly to the nucleus. A semiclassical approximation for the waves associated with a closed orbit $k$ is given by

$$\psi(q) = A_k(q) \exp \left( i S_k(q) - \frac{\pi}{2} \hbar_k \right).$$  

(4)

The classical amplitude $A_k$ relates to the spreading of this pencil of trajectories. The phase of the wave involves the Maslov index $\mu_k$ and the field-strength-dependent classical action $S = \oint p dq$. Here the advantage of performing experiments at constant scaled energy becomes clear. Under these conditions the classical parameters of topological identical trajectories remain the same. Inserting the scaling transformation results in the scaled action $\tilde{S}$:

$$\tilde{S} = \frac{1}{2} \int \tilde{p} dq \tilde{r} = \frac{\hbar \gamma^{1/3}}{2} S(\varepsilon, \hbar \gamma).$$  

(5)

It follows that the scaled classical action is determined by the scaled energy only. This definition separates the field-dependent quantum problem from the invariant classical dynamics.

The returning wave interferes with the outgoing Coulomb wave and is scattered back along its trajectory (repeated traversal). In an absorption experiment the average oscillator-
strength density is measured. It can be expressed in terms of a smooth background on which a series of oscillations is superimposed:

\[ f(\varepsilon, \hbar_e) = \sum_{n,k} \sqrt{\hbar_e} D_k^\varepsilon(\varepsilon) \sin[\Delta_k^\varepsilon(\varepsilon, \hbar_e)] \]

\[ + \sum_n \hbar_e D_0^n(\varepsilon) \sin[\Delta_0^n(\varepsilon, \hbar_e)]. \]  

Each oscillation in the spectrum is connected to a closed orbit \( k \) of the electron. Repeated traversals of an orbit are labeled with \( n \). The subscript \( k=0 \) is used for the parallel orbit. The amplitude of the oscillations connected to this orbit shows a different scaling with \( \hbar_e \) as the nonparallel orbits \((\theta \neq 0)\). The reduced amplitudes \( D_k^n \) introduced by Main et al. [9] are independent of \( \hbar_e \). \( D_k^n \) contains information on the stability of the orbit, via the classical amplitude \( A_k^n \), its initial and final angles \((\theta_f^k, \theta_f^k)\), and the geometry of the transition:

\[ D_k^n(\varepsilon) = 2^{19/4} \pi^{3/4} \frac{(\varepsilon)^{3/4}}{r_0} - 1/4 \]

\[ \times (\sin \theta_f^k \sin \theta_f^{k,n})^{1/2} Y(\theta_f^k) Y^*(\theta_f^{k,n}) A_n^k, \]  

\[ D_0^n(\varepsilon) = 2^{9/4} \pi r_0^{1/2} \times Y(0) Y^*(0) A_0^n. \]

The recurrence phases \( \Delta_k^n \) of Eq. (6) are given by

\[ \Delta_k^n(\varepsilon, \hbar_e) = 2 \pi \frac{\tilde{S}_k^n(\varepsilon)}{\hbar_e} - \frac{\pi}{2} \mu_k^n - \frac{3 \pi}{4}, \]

\[ \Delta_0^n(\varepsilon, \hbar_e) = 2 \pi \frac{\tilde{S}_0^n(\varepsilon)}{\hbar_e} - \frac{\pi}{2} \mu_0^n(\varepsilon) - \frac{\pi}{2}. \]

The varying value of \( 1/\hbar_e \) during a constant-scaled-energy experiment gives rise to oscillations with a period \( \tilde{S}_k^n \). The phase accumulated along an orbit also depends on the Maslov index \( \mu_k^n \) and an additional phase factor that depends on the type of orbit. The Maslov index is an integer determined by the number of caustics and foci along a trajectory. For the parallel orbit this index changes with \( \varepsilon \) each time a new orbit bifurcates from it. For nonhydrogenic atoms, the existence of the core results in a phase shift of the outgoing Coulomb wave. This shift depends on the quantum defect \( \delta \) (i.e., for odd-parity triplet helium \( \delta = 0.068 \)) and gives rise to an additional term in Eq. (8).

Recurrences associated with quantum phenomena show different scaling behavior. Ghost orbits are examples of such recurrences. These orbits, that show up at an energy below their classical bifurcation energy and thus cannot exist classically, show only a weak dependence on \( \hbar_e \). The existence of these orbits did not follow from standard closed-orbit theory. Recent improvements of the theory, which correct for the unphysical divergence of the classical amplitude near a bifurcation of an orbit do explain this phenomenon.

Another example of a different scaling rule can be found in the perpendicular orbit. These recurrences, which are classically forbidden as well, can be attributed to classical orbits in the node of a quantum wave function. In this case the outgoing Coulomb wave of the pencil of trajectories cannot be approximated by the amplitude at the central orbit, which would give zero amplitude. However, quantum calculations do show a small amplitude (100 times weaker than other orbits at \( \hbar_e \sim 0.01 \)), produced by the adjacent trajectories. Improved semiclassical relations result in a correct description of the amplitude and predict in this case a scaling of the reduced recurrence strength with \( \hbar_e^2 \) [19]. In previous experiments amplitude variations over a recorded spectrum were so small that they could be neglected for practical purposes. Our motivation to record an extended spectrum was to search for differences between recurrence spectra taken under varying quantum conditions.

C. Scaled-energy \( R \)-matrix quantum calculations

We performed numerical calculations for hydrogen and helium at \( e = -0.7 \) using the computer code of Ref. [11]. Here we will only give a brief review of the basic principles; more details can be found elsewhere [11,12]. The \( R \)-matrix method for nonhydrogenic atoms divides configuration space into two regions, where the Schrödinger equation is separately solved. The wave functions found are matched at the boundary (in helium radius \( R \sim 2.5a_0 \)). In the inner region \((r < a)\) the diamagnetic potential is negligible. Here a general solution for a complicated multielectron system is unknown. Outside the ionic core (but still inside the inner region) the solution of the problem is expressed as a linear combination of regular and irregular Coulomb functions, which has, at large distance, an additional phase shift \( (\pi \delta) \). In the outer region \((r > a)\) the ionic core is considered strictly Coulombic. Then the problem reduces to that of the hydrogen atom and the one-electron wave functions satisfy the hydrogenic diamagnetic Schrödinger equation.

The computational results for helium were used to check the value of the scaled energy during the experiments. A Fourier transformation of a calculated absorption spectrum (in the \( \hbar_e^{-1} \) scale) gives an action spectrum as well. To compensate for the decreasing oscillator-strength-density when transferring the spectra to a \( \hbar_e^{-1} \) scale, absorption peak intensities are first divided by \( \hbar_e^3 \) [19]. The power spectra (\( \text{[Fourier Transform]}^2 \)) of the helium and hydrogen calculations are used to deduce the variation of recurrence strength with \( \hbar_e \). Normalizing the recurrence spectra on the intensity of the \( R_1^6 \) orbit (dominant in the action spectrum at \( e = -0.7 \)) yields reduced action spectra, thus removing the \( \hbar_e \) dependence from the most prominent recurrences. The calculated variations in the recurrence strengths of several isolated hydrogenic orbits as a function of \( \hbar_e \) are plotted in Fig. 1 for an interval of interest of the present experiment. The scaling of the \( V^n \) and \( R^n \) orbits follows predictions. Figure 1 shows that the strongest variations may be expected for weak recurrences such as \( R^4 \). The recurrence strength of the ghost peak \( R_1^1 \) is much stronger; however, close to its bifurcation only a weak \( \hbar_e \) dependence is predicted.

III. EXPERIMENTAL SETUP

A. General

In Fig. 2 a schematic of the experimental setup is shown. In a crossed-beam experiment a single-photon transition is
used to excite $2^3S$ metastable helium atoms to a Rydberg state in the presence of a magnetic field. The beam of metastable atoms is produced in a dc discharge. In a supersonic ($v \sim 2000$ m/s) expansion through a boron-nitride nozzle the atoms enter a transversal laser cooling section. The diverging beam is collimated here to reduce residual Doppler widths and to enhance beam flux. Manipulation of atoms with laser light, which commonly is the subject of investigation itself, is here used as a tool to increase the signal strength and reduce the linewidth. The experimental details and the performances of the laser cooling section will be discussed in Sec. III B.

Downstream from the cooling section a pinhole (diameter $\sim 1$ mm) is inserted to prevent metastable atoms from hitting the excitation box. In the excitation region the atomic beam is perpendicularly intersected with UV radiation generated by intracavity frequency doubling of a cw ring dye laser (3 mW at 260 nm). The excitation region is shielded from stray electric fields by a carbon-coated stainless-steel box kept at a positive voltage of 30 V. Additional capacitor plates mounted outside the shielding box are used to minimize the residual electric fields to values below 5 mV/cm. In a magnetic field some levels show a large sensitivity to electric fields parallel to the magnetic field axis. A second compensation routine reduces this field component to values below 0.5 mV/cm. The magnetic field direction is chosen parallel to the atomic beam to minimize motional Stark effects. The atoms are field ionized downstream from the excitation region. The helium ions pass a mass selective filter and are counted with an electron multiplier. In our previous setup [10], a Lorentz force on the ions prevented them from reaching the electron multiplier when the magnetic field strength exceeded 0.2 T. Transport of Rydberg atoms to a low magnetic field region before ionization occurs is possible due to their long lifetimes ($\tau > 0.5$ ms) [21,22]. For this purpose we mounted a 10 cm long stainless-steel tube at the end of the shielding box and no decrease in detection efficiency was observed up to a field strength of 0.35 T.

For experiments at constant scaled energy the field strength is adjusted to the laser frequency during a scan. This requires an on-line frequency calibration. Each frequency scan starts with the recording of a zero-field resonance for an absolute calibration before switching on the magnetic field. Counting the fringes with a 150 MHz confocal etalon then determines the laser frequency during a scan. The current through the electromagnet is adjusted to the laser frequency at each frequency step of the laser. The uncertainty in the scaled energy value is below 0.0001 over a 30 GHz scan. The stray electric field may slightly decrease the energy of the zero-field reference peak. As this shift is larger at higher Rydberg states this effect may introduce a slow increase of $\epsilon$ when connecting overlapping spectra. Comparing our experimental spectra with quantum calculations for helium we found only small increases in $\epsilon$ of the order of 0.0005 over the total scanning range.

B. Laser cooling section

Applying laser cooling as a tool in a spectroscopic experiment starts from a different perspective than commonly encountered in cooling or trapping experiments. The size of the UV laser beam, the requirements on the homogeneity of the magnetic field, and the minimization of stray electric fields in our experimental setup put constraints on the size of the interaction volume ($\sim 1$ mm$^3$). This size eventually sets limits on the laser cooling performance.

The atoms are transversally cooled in two dimensions using the curved wave front technique [20]. The light for the $2^3S_1 \rightarrow 2^3P_2$ cooling transition is provided by a DBR diode laser (SDL 6702). This narrow band single mode laser generates 30 mW output power at 1083 nm. To prevent optical
feedback in the diode we use an optical isolator. Short-term frequency fluctuations (up to ~10 kHz) are compensated by locking the diode laser to the fringe of a stable 750 MHz étalon (made of Invar). For absolute frequency stabilization the length of the étalon is subsequently locked to the saturated absorption signal produced in a helium rf discharge. Modulation of the current at 400 Hz with a top-top excursion of the optical frequency of 1 MHz generates the error signal. The laser stayed in resonance for several hours during our experiments.

A 1 mm pinhole is positioned at a distance $d = 30$ cm of the exit nozzle of the discharge chamber. In the curved wave front technique, atoms in resonance with the radiation field follow the curvature of the light field until they move parallel to the magnetic field axis. Using the full optical access of the laser cooling region ($l = 18$ cm) and a radius of curvature $R$ of 32 m for the wave front a desired spot size of 0.5 mm radius is achieved ($2r - l^2 / R$). Atoms with an angle smaller than the capture angle $\beta = l / R$ interact with the light and will be collimated. Atoms with a velocity $v > v_{\text{max}}$ experience a centrifugal force exceeding the radiation pressure force $[v_{\text{max}} = \sqrt{V R / (2M)} \approx 3700$ m/s; $V$ natural linewidth; $M$ mass of the atom]. Since the most probable velocity of the metastable atoms is 2000 m/s nearly all atoms within the capture angle will be collimated. Theoretically an increase in particle number of $\beta d l r \approx 3$ for cooling from one side is expected. The capture angle is increased by a factor 2 by backreflecting the wave front. Therefore an increase of a factor 36 in two dimensions is the theoretical maximum that can be achieved.

A Faraday cup mounted behind the mass filter is used to optimize the laser cooling section and for diagnostics purposes. The increase of atomic density as an effect of laser cooling in the horizontal plane gives rise to a current increase on the Faraday cup by a factor 4. The collimation of the atomic beam follows from the narrowing of the Doppler broadened linewidth [23]. Our linewidth of 8 MHz is now limited by the divergence of the collimated UV laser beam. Adding the curved wave fronts in the vertical dimension does not affect the linewidth, but enhances the atomic density. Applying laser cooling in two dimensions typically results in a current increase on the Faraday cup by a factor 10. Laser cooling performance was originally strongly frustrated by the presence of the strong, nearby magnetic field. After mounting a $\mu$-metal cone with minimal optical access the laser cooling section could be adequately shielded. No field-dependent decrease of atomic density is observed even at the highest field strengths.

IV. RESULTS

We recorded a series of overlapping spectra at $\epsilon = -0.700$ varying the energy from $-18.6$ cm$^{-1}$ ($n = 77$) to $-4.3$ cm$^{-1}$ ($n = 160$). The magnetic field was varied over one order of magnitude from 0.31 T to 0.03 T. This corresponds to a variation in the $h^{-1}_c$ range from 91 to 189 (a factor $2 \ln h_c$). Comparison of the experimental spectra with $R$-matrix helium calculations in different $h^{-1}_c$ ranges shows a smooth increase in $\epsilon$ from $-0.7002$ to $-0.6997$. The connected spectra show an overlap accuracy comparable to that of the experimental resolution. The intensities of adjacent spectra are scaled on the integrated intensity of the zero magnetic field peak, thereby automatically accounting for the decreasing oscillator strength with increasing $n$. This procedure also reduces the effect of slow variations in laser power and atomic density.

Part of the squared Fourier transform of the full frequency spectrum, the action spectrum, is shown in Fig. 3 (recurrence peaks up to scaled-action values $\tilde{S} = 250$ were observed). The effect of spectral leakage in the Fourier transform is reduced by multiplying the spectrum with a smooth window function (see Appendix). The effect of using a sine window is a broadening of the central resonance, but due to the reduction of side lobes a better resolution than with a rectangular window is obtained. The extended energy interval results in a resonance width of 0.012 (in units of scaled action) in the experiment as well as in quantum calculations. This narrow width reflects the constant-scaled-energy conditions of our experiment. The experimental recurrence strength spectrum up to scaled action $\tilde{S} = 15$ in Fig. 3 is mirror imaged by the spectrum calculated with closed-orbit theory. The recurrences are normalized on the strongest peak in the experimental action spectrum. The position of recurrence peaks matches with the scaled action of calculated classical closed orbits within the width of the experimental peak ($\Delta \tilde{S} = 0.012$). We can identify recurrence peaks as vibrator orbits $V_k^n$, which are bifurcated from the parallel orbit $V^n$. Another class of recurrence peaks is connected to rotator orbits $R_k^n$, which bifurcate from the perpendicular orbit $R^n$. Small shifts in scaled action of typically 0.003 between the central position of the experimental recurrence peaks and the calculated classical scaled action are found (see the Appendix). The step size of the numerical integration routine which calculates the classical action is chosen to have an accuracy up to the fifth digit. The classical scaled actions perfectly match the central positions of the quantum mechanically calculated

![FIG. 3. Squared Fourier transform of the experimental spectrum covering the range $h^{-1}_c = 91$–189, normalized on the $R_1^1$ peak, mirror imaged by the calculated recurrence strength of the orbits up to action 15. The recurrence strength of the $V^n_k$ orbits (bold) is calculated for the average value $h^{-1}_c = 140$. The experimental peaks marked with an arrow relate to the $R_1^1$ ghost orbit and its repeated traversal.](image)
hydrogenic recurrence peaks. Model calculations show that stray electric fields in the order of 50 mV/cm are required to explain such small action differences. In our experiment, however, the perpendicular field resulting from the motional Stark effect is estimated to be smaller than 15 mV/cm, whereas the parallel field is below 0.5 mV/cm. An effect of the core on the calculated action is excluded by overlapping the recurrence peak obtained from quantum calculations for helium and hydrogen ($\delta S < 0.0001$). Therefore we attribute small action differences to experimental imperfections other than the accuracy in $\varepsilon$ and residual electric fields. The calculated recurrence strengths of the $R_n^5$ and $V_n^5$ orbits show good agreement with the experimental values. The intensity (bold) of the parallel orbit $V^1$ and its returns $V^n$, which depend on $\hbar^{-1}$, are calculated for $\hbar^{-1} = 140$.

In Fig. 3 the vertical scale of the first part (0.5 < $\bar{S} < 3.5$) of the experimental action spectrum is enlarged by a factor of 10. Despite its low intensity the parallel orbit is clearly observed in the spectrum up to its fourth return ($\bar{S} = 0.84, 1.69, 2.54, 3.38$). At higher scaled actions other vibrator orbits obscure further observation of higher repetitions $V^n$. The divergence of the classical amplitudes close to a bifurcation is clearly visible at the 16th return of the parallel orbit ($V^{16}$) at $\bar{S} = 13.53$. A more careful inspection shows that other returns (i.e., 5th, 6th, 10th, and 11th) of the parallel orbit are also very close to a bifurcation and intensities in principle should be calculated using the uniform approximation. This is beyond the scope of the present paper.

The recurrence peaks at $\bar{S} = 3.93$ and 7.85 in the experimental spectrum are not reproduced by closed-orbit theory calculations. These ghost orbits are observed although the scaled energy is slightly below the bifurcation energy $\varepsilon_{\text{bif}} = -0.697$ for the $R_5^5$ and $R_2^{10}$ orbit. Their positions are exactly at the scaled actions of the zero amplitude $R_5^0$ orbit and its return $R_5^{10}$. Due to the high accuracy in $\varepsilon$ we can attribute these recurrences unambiguously to a quantum effect.

The experimental resolution ($\delta \hbar^{-1}$) limits the highest action orbit that can be deduced from a Fourier transform of the frequency spectrum $[\bar{S}_{\text{max}} = 1/(2 \delta \hbar^{-1})]$. $\bar{S}_{\text{max}}$ varies considerably over the experimental range. At high $n$ peaks broaden due to a higher sensitivity to electric fields as well. Also the measured frequency spectrum is elongated by the transformation to the $\hbar^{-1}$ scale. These effects are apparent in Fig. 3: for $\bar{S} > 10$, experimental recurrence peaks already show a reduced intensity. Due to the better resolution at high magnetic fields and low $n$ ($\delta \hbar^{-1} = 0.002$), recurrence peaks could be resolved up to a scaled action of $\bar{S}_{\text{max}} = 250$. The Heisenberg time $t_H$, for which classical orbits can be extracted from a quantum spectrum, however, is determined by the typical spacing between energy levels. As the mean level spacing ($\hbar^{-1}$) increases at high magnetic fields, attributing recurrence peaks to classical orbits will be limited to an action corresponding to the Heisenberg time. In the low-$n$ experimental range, oscillations corresponding to the mean level spacing are connected to an orbit, the Heisenberg orbit, with action $\bar{S} = 35$. In Fig. 4 (upper part) the action spectrum of our measurement at high magnetic field ($\hbar^{-1} = 91–120$) is reproduced up to $\bar{S} = 35$. Helium $R$-matrix calculations at $\varepsilon = -0.7$ are shown as well in the lower part as a mirror image. Recurrences are found at precisely the same scaled actions and the recurrence intensities match very well up to a scaled action 25. According to closed-orbit theory many orbits are not resolved ($\Delta \bar{S} \approx 0.04$) at higher action and combine coherently. In contrast to positions and amplitudes, phases are extremely sensitive to experimental imperfections. Small variations in the scaled energy introduce phase deviations. As a result combinations of peaks interfere differently, inducing increased deviations between experiment and quantum calculations at higher scaled actions.

V. $\hbar_x$-SCALING EFFECTS IN THE ACTION SPECTRA

The recording of an extended scaled-energy spectrum, as discussed in the preceding section, allows for a subdivision in intervals with significantly different values for the average $\hbar_x$. We opted for a $\hbar_x^{-1}$ interval of 30, which still correlates with a large spectral interval, certainly when compared with existing measurements. When the average $\hbar_x^{-1}$ value is increased by 10 we obtain a set of eight action spectra connected to different quantum conditions. The $R_5^5$ peak, free from interference effects with other orbits, is used to normalize these action spectra. The experimental recurrence peaks show up at the same position, but large intensity variations are found in the different ranges. In Fig. 5 we compare the reduced action spectra of the extreme values of (average) $\hbar_x$: $\hbar_x \sim 0.01$ and $\hbar_x \sim 0.006$. Experimental intensity variations are to a large extent reproduced in quantum calculations of hydrogen (and helium). These are included in Fig. 5 as well, mirror imaged with the experimental spectra in the corresponding range. The striking correspondence in position and strength of theory and experiment in both ranges reflects the high quality of our data. Figure 5 shows that for the measurements on the low-$n$ levels ($\hbar_x \sim 0.01$), where the absolute calibration is less sensitive to stray electric fields, agreement between experiment and quantum calculations is best. The intensity fluctuations of isolated rotator peaks ($\sim 15\%$ in $|D^n_k|^2$) are due to remaining experimental imperfections. The first part of the action spectra ($\bar{S} < 3.5$) is again enlarged by a factor 10. The peaks connected to the parallel orbit up to its fourth return ($V^1 – V^4$) then become visible ($\bar{S} = 0.84, 1.69, 2.54, 3.38$). Comparison of the reduced action spectra in both ranges shows the $\hbar_x$ scaling of recur-
FIG. 5. (a) Squared Fourier transform of the low-$n$ part of the experimental spectrum: $h_z = 91–120$ ($h_x = 0.01$), mirror imaged by quantum hydrogen calculations in the same $h_z^{-1}$ range. (b) The high-$n$ action spectrum: $h_z = 160–189$ ($h_x = 0.006$).

The most pronounced intensity changes are observed for peaks connected to a combination of orbits. Due to the small difference in action ($\Delta S < 0.04$) the $V_7^n$ orbits given in Table I are not resolved from $V^n$, the $n$th return of the parallel orbit. We have measured the intensity of recurrence peaks connected with these orbits in the experimental and calculated action spectra for the eight different values of $h_x$. The variations in intensity of the recurrence peaks can be understood with standard hydrogenic closed-orbit theory. The superposition of sineform oscillations connected to a combination of orbits gives rise to a beat signal with a period determined by the action difference. The additional phase shifts for nonhydrogenic atoms due to the quantum defect cancel out. The Maslov index of the newly created orbit $\mu_{2,k}^n$ ($k = 1, 2, 3$), which is 2 lower than the index of the parallel orbit $\mu_{0,k}^n$, results in an additional phase difference of $3\pi/4$. Standard closed-orbit theory gives for the strength $I$ of such combination peaks

$$I = (D_0^n)^2 + h_x (D_0^n)^2 + 2D_0^n D_0^n \sqrt{\hbar} \cos \left( \frac{2\pi}{\hbar} (S_{0} - S_{k}) + \frac{3}{4} \pi \right).$$

The different scaling of the parallel orbit results in a vanishing amplitude of the oscillation in the classical limit. The most prominent amplitude variation (at $S = 5.07$) is connected with the near coincidence of the $V_7^n$ and $V_6^n$ orbits. According to the classical calculation the action difference between these orbits is only 0.0016. The actions are calculated by a numerical integration of the equations of motion. The step size in the integration routine for this calculation was chosen to be accurate in the sixth digit. Taking Fourier transforms of two ranges of the entire spectrum clearly reveals the interference of these two orbits. If these orbits had to be resolved in a large energy scan a spectrum length (in $h_z^{-1}$) larger than 500 would be required. From a theoretical point of view such a combination of orbits close to their bifurcation energy is difficult to test the validity of Eq. (9) it is better to consider combinations of orbits further away from their bifurcation energy.

Figure 6(a) shows the recurrence strength variation of the pair $V_7^n$, $V_7^n$ for the eight different average $h_x$ values. In Fig. 6(b) the same is plotted for $V_{13}^n$ and $V_{13}^n$. For other pairs of orbits (i.e., $V_7^n$, $V_8^n$) the action differences result in a fast oscillation. Due to the averaging over a spectral length of $h_z^{-1} = 30$ the amplitudes of this oscillation is then suppressed. The slow variation in strength determined by the action differences of 0.0080 ($V_7^n$) and 0.0084 ($V_{13}^n$) is clearly visible in the experiment as well as in the quantum helium calculations. The smooth curves in Fig. 6 are obtained by inserting the calculated actions and amplitudes in Eq. (9), thus producing the semiclassical result. The experimental accuracy in $\epsilon$ will change the action of both orbits by $\pm 0.001$. However, since the orbits lie in the same region of phase space, the variation in their action difference is one order of magnitude smaller. The accuracy in $\epsilon$ will therefore give only a negligible variation in recurrence strength. The accuracy for the rotator orbits (Fig. 6) is a good indication for the error margins for these recurrence peaks. The deviations between quantum calculations and experiment are within these error margins. Standard closed-orbit theory for hydrogen follows the quantum helium calculations for both the $V_7^n$, $V_7^n$ and the $V_{13}^n$, $V_{13}^n$ pairs very well (Fig. 6 smooth closed-orbit theory curves). We find that variations of only 0.0001 in the classical action difference give rise to observable phase shifts. This illustrates the sensitivity of this orbit-interferometric method to detect phase differences. This method obviously can be applied to test calculations of classical actions to a very high degree of accuracy.

<table>
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<th>$n$</th>
<th>$k$</th>
<th>$S_0^n$</th>
<th>$S_1^n$</th>
<th>$D_0^n$</th>
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<td>10.9870</td>
<td>10.9786</td>
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<td>0.52</td>
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</tbody>
</table>
In Fig. 7 we closely examine the action range where the recurrence strength varies most, for $\hbar_e = 0.01$ and 0.006, respectively. In the lower part results from standard closed-orbit theory for hydrogen are given, whereas in the upper part the quantum hydrogen calculations are reproduced. Comparison between quantum theory and semiclassical calculations show good agreement in both regimes. The peak at $\tilde{S} = 7.85$ is connected to the second return of the $R_{1}^{1}$ ghost orbit ($R_{2}^{19}$) which is not incorporated in standard closed-orbit theory. This peak theoretically must show a decreasing intensity approaching the classical limit ($\hbar_e \to 0$). Unfortunately the uncertainty in the experimental recurrence strengths is too large to observe this.

The effect of interfering orbits is clearly observed for the peaks marked with an arrow in Fig. 7. Standard closed-orbit theory clearly fails for the peak at $\tilde{S} = 13.53$. Especially the peak at $\tilde{S} = 14.36$ shows a large variation in strength in the different $\hbar_e$ regimes. Classically this peak is calculated by adding the contributions of three different orbits: $R_{2}^{18}$, $V_{17}$, and $V_{3}$ orbit. The $V_{17}$, $V_{3}$, and $R_{2}^{18}$ orbit also have nearly the same recurrence amplitude. Therefore the visibility of the interference is more pronounced than in the preceding examples. The relative variation in action has a stronger dependence on $\varepsilon$ in the case of the interference of three orbits. Experimentally we can no longer neglect the error caused by a variation in scaled energy. When we compare the recurrence strength at $\tilde{S} = 14.36$ for $\hbar_e \approx 0.01$ with $\hbar_e \approx 0.006$ in Fig. 5 we do observe a clear difference as a result of this interference between three orbits in the experiment. This recurrence peak therefore provides an excellent check for the relative phases of the orbits in closed-orbit theory. Errors which may arise from the evaluation of Maslov indices give rise to large effects in the closed-orbit calculations.

### VI. CONCLUSIONS

We have recorded a long scaled-energy spectrum at $\varepsilon = -0.7$. By subdividing the spectrum in equal-length intervals, recurrence spectra taken under different quantum conditions could be compared. The decreasing recurrence strength of the parallel orbit when approaching the classical limit ($\hbar_e \to 0$) is observed. Largest $\hbar_e$ effects were found for recurrences connected to interferences of classical orbits. Small differences in the scaled actions of adjacent orbits induce a $\hbar_e$-dependent change in the phase difference. An action difference of only $\Delta \tilde{S} = 0.0016$ (between the $V_{17}$ and $V_{6}$ orbit) leads to observable variations in the recurrence strength with $\hbar_e$. The slow variation in recurrence strength of the $V_{17}$, $V_{13}$, and $V_{13}$ orbits deduced from helium experiments and quantum calculation perfectly match with the modulation predicted by standard closed-orbit theory. The most pronounced $\hbar_e$ effect on the recurrence strength is observed at scaled action $\tilde{S} = 14.36$, where three different orbits, $V_{17}$, $V_{13}$, and $R_{2}^{18}$, interfere. We have shown that the scaling properties for the helium case can be reproduced to a large extent in hydrogenic closed-orbit theory. This allows for a direct comparison of our experimental data with experiments at high magnetic field strengths and low-$n$ value, and even with data for more complex atoms at not too high action values. We expect that in combination with such experi-
ments other phenomena, for instance, the scaling of the ghost peak, can be demonstrated experimentally.

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APPENDIX: SPECTRAL RESOLUTION OF THE RECURRENCE SPECTRUM

The frequency spectrum involves a summation over a large number of sinusoidal oscillations. Fourier transformation of \( f(\epsilon, h_\epsilon) \) results in sharp recurrence peaks in the action spectrum. The width of a recurrence peak is inversely proportional to the range of \( h_\epsilon^{-1} \) in the experiment. In a fast Fourier transform (FFT) procedure of spectra over an extended range, the narrow recurrence peaks are represented by a small number of points only. Adding a series of zeroes at the end of a spectrum increases the number of sample points in the action spectrum. In our experiment an oversampling of a factor 8 was needed in order to determine recurrence strengths accurately. When we calculate the Fourier transform of a spectrum sharp edges at the beginning and end of the frequency spectrum will induce sidebands in the action spectrum. The knowledge that a multiplication in the time domain is equivalent to a convolution in the frequency spectrum. The width of a recurrence peak is inversely proportional to the number of sinusoidal oscillations. Fourier transform results in sharp recurrence peaks in the action spectrum.

\[ I_{r}(a,x) = \left[ \frac{2a \sin(ax)}{ax} \right]^2. \quad (A1) \]

where \( x \) and \( a \) are given by \( x = \frac{\tilde{S} - \tilde{S}}{2} \) and \( a = \pi [h_\epsilon^{-1}(\text{end}) - h_\epsilon^{-1}(\text{begin})] \). We can suppress side-lobe effects by multiplying the frequency spectrum with a special window function. In our data analysis we have multiplied the frequency spectrum with a sinc function. The shape of a recurrence peak then is

\[ I_{s}(a,x) = \left[ \frac{4a \pi \cos(ax)}{\pi^2 - 4a^2 \chi^2} \right]^2. \quad (A2) \]

According to the Rayleigh criterion for resolution, orbits in the action spectra are resolved when the position of the first minimum of one peak is exactly at the position of the maximum of the other. Using this criterion the resolution decreases a factor 1.5 using a sinc window, as a result of broadening of the central peak. Inserting the experimental value for \( a \) \( [a = \pi (189 - 91)] \), a comparison with the experimental result is made for both window functions. The width of an experimental recurrence peak matches with that obtained for the quantum recurrence spectra. Side lobes are clearly present for the rectangular window, whereas these features are nicely reduced by the sinc window (see Fig. 8). Despite the broadening of the recurrence peak, the increase in resolution for the sinc window is visible in this figure.